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Abstract

In this paper we recall some general properties of extremal and Barabanov norms and we give a necessary and sufficient condition for a finite-dimensional continuous-time linear switched system to admit a Barabanov norm.

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1 Introduction

The framework of switched systems [16, 19] has been introduced to model dynamics that are governed by non-autonomous fields, whose dependence on time cannot be determined apriori. It can be used to account for time-dependent perturbations, or phenomena whose dynamics are either not precisely known or too impractical to include in the model.

An important feature of switched systems is that they describe a family of non-autonomous dynamics rather than a single one. The properties that one seeks to prove about a switched system are therefore in the same spirit as those of ensemble control [1, 2, 10, 14, 15].

The focus of this paper is on the stability properties of linear switched systems. A linear switched system is characterized by the set $\mathscr{M}$ collecting all the matrices M such that $\dot{x} = Mx$ is one of the possible dynamics among which the system switches. An important quantity capturing the asymptotic behavior of a system is its maximal Lyapunov exponent $\lambda(\mathscr{M})$, which is the largest possible rate of exponential growth of its trajectories (see Section 2 for precise definitions). The sign of $\lambda(\mathscr{M})$ measures the exponential stability or instability of the switched system associated with $\mathscr{M}$. In particular, exponentially stable switched systems correspond to strictly negative Lyapunov exponents.

It is not difficult to check that, if a positively homogeneous Lyapunov function V exists that is common to all the matrices in $\mathscr{M}$, in the sense that there exists $\alpha > 0$ such that $\nabla V(x) \cdot Mx \leq -\alpha V(x)$ for every $M \in \mathscr{M}$, then $\lambda(\mathscr{M}) \leq -\alpha$. Such an observation is usually called a direct Lyapunov property. The main contribution that we present in this note is a converse Lyapunov result for switched system asserting that, under some technical conditions, a tight version of common Lyapunov function, called Barabanov norm, exists. Referring to the next sections for precise definitions, let us say that a norm is extremal for the switched system if it is a common Lyapunov function that is tight in the sense that $\alpha = -\lambda(\mathscr{M})$. A Barabanov norm (a notion first introduced in [3]) is then an extremal norm such that, starting from every initial condition x_0, there exists an admissible trajectory of the switched system saturating the differential bound on the Lyapunov function, i.e., whose norm coincides with $t \mapsto \|x_0\| e^{\lambda(\mathscr{M})t}$.

In our main result (Theorem 14) we provide necessary and sufficient conditions for the existence of a Barabanov norm in terms of the irreducible components of the linear switched system. Our proof adapts some of the ideas introduced in [20], where a construction of Barabanov norm alternative to the one give in [3] was proposed. A remarkable contribution of [20] is also to show how Barabanov norms can be used to prove the local Lipschitz continuity of the maximal Lyapunov exponent in the set of compact irreducible sets of matrices.

This note is organized as follows: in Section 2 we introduce the framework of linear switched systems and we discuss how its stability is measured by the maximal Lyapunov exponent. The notion of extremal and Barabanov norms is recalled with some examples and basic properties in Section 3. Finally, Section 4 contains the characterization of linear systems admitting a Barabanov norm.

2 Linear switched systems and their maximal Lyapunov exponent

Throughout this note we deal with the simplest switched dynamics, namely linear and uncontrolled finite-dimensional switched systems. More precisely, we consider systems $(\Sigma_{\mathcal{S}})$ of the type

$$\dot{x}(t) = A(t)x(t), \qquad t \geq 0, \quad A(\cdot) \in \mathcal{S}, \quad x \in \mathbb{R}^n, \tag{$\Sigma_{\mathcal{S}}$}$$

where $\mathcal{S}$ denotes a set of $\mathscr{M}$-valued functions defined on $[0, +\infty)$, with $\mathscr{M}$ a bounded subset of the space $M_n(\mathbb{R})$ of $n \times n$ real matrices. Then $(\Sigma_{\mathcal{S}})$ is viewed as a family of non-autonomous systems parameterized by $A(\cdot) \in \mathcal{S}$. The elements of $\mathcal{S}$ are called *signals* or *switching laws*. Each matrix $M \in \mathscr{M}$ (and, by extension, each associated autonomous linear system $\dot{x} = Mx$) is referred to as a *mode* of $(\Sigma_{\mathcal{S}})$.

From now on the set $\mathcal{S}$ is assumed to be *positive time-shift-invariant* (*shift-invariant* for short), i.e., for every $h \geq 0$ and $A(\cdot) \in \mathcal{S}$, the signal $A(\cdot + h) : [0, +\infty) \to \mathscr{M}$ belongs to $\mathcal{S}$.

Let us comment on the assumptions made on $\mathscr{M}$ and $\mathcal{S}$. The boundedness of $\mathscr{M}$ ensures that all trajectories of System $(\Sigma_{\mathcal{S}})$ have at most exponential growth, with a common upper bound on their growth rates, whereas the shift-invariance of $\mathcal{S}$ encodes the fact that the switching laws obey rules which do not change over time.

Notice that $(\Sigma_{\mathcal{S}})$ can be naturally extended to the case of nonlinear systems (considering nonlinear vector fields as modes) or to infinite-dimensional systems, as for instance delay system with switching delay [11] or PDEs with intermittent damping [12].

Several choices of the class $\mathcal{S}$ have been considered in the literature: measurable, piecewise constant, piecewise constant with a prescribed minimal elapsed time between two discontinuities (*dwell time*, in the engineering literature), persistent signals, continuous signals, ... Here, the accent is on worst-case scenario for the stability of $(\Sigma_{\mathcal{S}})$ and therefore we allow for the maximal freedom of the signals with values in $\mathscr{M}$. That is, we set

$$\mathcal{S} = L^\infty([0,+\infty),\mathscr{M})$$

and we denote by $(\Sigma_{\mathscr{M}})$ the system $(\Sigma_{\mathcal{S}})$ corresponding to such a choice of $\mathcal{S}$. Solutions to $(\Sigma_{\mathscr{M}})$ are considered in the sense of Carathéodory and, given $A \in \mathcal{S}$ and $0 \le t_0 \le t_1 < +\infty$, we denote by $\Phi_A(t_1,t_0) \in M_n(\mathbb{R})$ the flow (or *fundamental matrix*) from time t_0 to time t_1. In particular, $\Phi_A(t_0,t_0) = \mathrm{Id}_n$ and $\frac{d}{dt}\Phi_A(t,t_0) = A(t)\Phi_A(t,t_0)$ for almost every $t \in [t_0,t_1]$.

The stability of system $(\Sigma_{\mathscr{M}})$ can be described through the following notions.

Definition 1. *The switched system $(\Sigma_{\mathscr{M}})$ is said to be*

- unbounded *if there exists a trajectory $x(\cdot)$ of $(\Sigma_{\mathscr{M}})$ such that* $\limsup_{t\to+\infty}\|x(t)\| = +\infty$*;*
- bounded *if there exists $C>0$ such that, for every $A \in \mathcal{S}$ and $t \ge 0$, $\|\Phi_A(t,0)\| \le C$;*
- attractive *if every trajectory of $(\Sigma_{\mathscr{M}})$ converges to the origin as time goes to $+\infty$;*
- exponentially stable *if there exist $C,\gamma>0$ such that, for every $A \in \mathcal{S}$ and $t \ge 0$, $\|\Phi_A(t,0)\| \le Ce^{-\gamma t}$.*

Beside the qualitative behavior described by the properties introduced in the previous definition, we are interested in the precise quantitative worst-possible exponential rate, defined as follows: the *maximal Lyapunov exponent of $\mathscr{M}$* is

$$\lambda(\mathscr{M}) = \sup_{A\in\mathcal{S}} \limsup_{t\to\infty} \frac{\log\|\Phi_A(t,0)\|}{t}. \tag{1}$$

It is well known (see, e.g., [7, Proposition 5.4.15]) that an equivalent characterization of the maximal Lyapunov exponent of $\mathscr{M}$ is

$$\lambda(\mathscr{M}) = \limsup_{t\to\infty} \sup_{A\in\mathcal{S}} \frac{\log\|\Phi_A(t,0)\|}{t} \tag{2}$$

and that

$$(\Sigma_{\mathscr{M}}) \text{ attractive} \iff (\Sigma_{\mathscr{M}}) \text{ exponentially stable} \iff \lambda(\mathscr{M}) < 0. \tag{3}$$

For every $\mu \in \mathbb{R}$, let us denote by $\mathscr{M} + \mu \mathrm{Id}_n$ the set $\{M + \mu \mathrm{Id}_n \mid M \in \mathscr{M}\}$. Since Id_n commutes with every other $n \times n$ matrix, it holds

$$\{t \mapsto x(t) \mid \dot{x} = Ax,\, A \in L^\infty([0,+\infty), \mathscr{M} + \mu \mathrm{Id}_n)\}$$
$$= \{t \mapsto e^{\mu t} x(t) \mid \dot{x} = Ax,\ A \in L^\infty([0,+\infty), \mathscr{M})\}$$

and hence

$$\lambda(\mathscr{M} + \mu \mathrm{Id}_n) = \lambda(\mathscr{M}) + \mu.$$

As a consequence, if $\lambda(\mathscr{M}) > 0$ then $(\Sigma_{\mathscr{M}})$ is unstable, since otherwise, if $(\Sigma_{\mathscr{M}})$ were bounded, the switched system corresponding with $\mathscr{M} - (\lambda(\mathscr{M})/2)\mathrm{Id}_n$ would be exponentially stable, which contradict the fact that $\lambda(\mathscr{M} - (\lambda(\mathscr{M})/2)\mathrm{Id}_n) = \lambda(\mathscr{M})/2 > 0$.

Hence boundedness of $(\Sigma_{\mathscr{M}})$ implies that $\lambda(\mathscr{M}) = 0$. However, the converse implication is not true. It suffices to consider

$$\mathscr{M} = \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\} \tag{4}$$

to get an unbounded switched system for which $\lambda(\mathscr{M}) = 0$.

3 Extremal and Barabanov norms

We start this section by recalling some direct and converse Lyapunov properties of linear switched systems.

It is well known (and it can be deduced from a simple comparison argument) that if there exist a Lipschitz continuous function $V : \mathbb{R}^n \to [0,+\infty)$ and a constant $\alpha > 0$ such that

$$V(x) = 0 \text{ if and only if } x = 0,$$
$$\lim_{\|x\| \to +\infty} V(x) = +\infty,$$

and

$$\nabla V(x) \cdot Ax \leq -\alpha V(x), \quad \text{for almost every } x \in \mathbb{R}^n,\ \forall A \in \mathscr{M}, \tag{5}$$

then $(\Sigma_{\mathscr{M}})$ is exponentially stable. Such a function V is called *common Lyapunov function* for the switched system $(\Sigma_{\mathscr{M}})$ and a *converse Lyapunov* result holds, namely, if $(\Sigma_{\mathscr{M}})$ is exponentially stable then it admits a smooth and convex common Lyapunov function [18]. Moreover, provided that the convex hull of $\mathscr{M}$ is a polyhedron, the common Lyapunov function can be taken polyhedral or polynomial (see [18] and also [5, 9]) albeit with an *a priori* arbitrarily large degree [17].

Geometrically, one can restrict by homogeneity the search of a Lyapunov function to the class of positively homogeneous functions and the existence of a positive constant α as in (5) is equivalent to the fact that each linear vector field $x \mapsto Ax$, $A \in \mathscr{M}$, points strictly inwards the sublevel sets of V. For V convex, and up to setting to 1 its order of positive homogeneity, the converse Lyapunov result ensures that there exists a norm v on $\mathbb{R}^n$ satisfying (5). Notice that the trajectories of $(\Sigma_{\mathscr{M}})$ then satisfy

$$v(x(t)) \leq e^{-\alpha t} v(x(0)),$$

meaning, in particular, that $\lambda(\mathscr{M}) \leq -\alpha$.

For a general set of modes $\mathscr{M}$, by applying the previous observation to $\mathscr{M} - (\lambda(\mathscr{M}) + \varepsilon)\mathrm{Id}_n$ with $\varepsilon > 0$, we deduce that there exist $\alpha > 0$ and a norm $v = v_\varepsilon$ on $\mathbb{R}^n$ such that

$$v(x(t)) \leq e^{(\lambda(\mathscr{M})+\varepsilon-\alpha)t} v(x(0)) \leq e^{(\lambda(\mathscr{M})+\varepsilon)t} v(x(0)).$$

This leads to the question of whether a norm v exists that is uniform with respect to ε in the sense of the following definition.

Definition 2. *A norm v in $\mathbb{R}^n$ is said to be* extremal *for $\mathscr{M}$ if, for every trajectory $x(\cdot)$ of $(\Sigma_{\mathscr{M}})$ and every $t \geq 0$, $v(x(t)) \leq v(x(0)) e^{\lambda(\mathscr{M})t}$.*

The existence of an extremal norm is not guaranteed even in the case where $\mathscr{M}$ is a singleton $\{M\}$, as it follows by considering the case (4) considered above, for which $\lambda(\mathscr{M}) = 0$ and $(\Sigma_{\mathscr{M}})$ has unbounded trajectories.

Let us introduce the following useful notion.

Definition 3. *The set $\mathscr{M}$ is said to be* nondefective *if $(\Sigma_{\mathscr{M}-\lambda(\mathscr{M})\mathrm{Id}})$ is bounded, that is, if there exists $C > 0$ such that $\|\Phi_A(t,0)\| \leq C e^{t\lambda(\mathscr{M})}$ for every $t \geq 0$ and every $A \in \mathcal{S}$.*

The following characterization can be found, e.g., in [13, Theorem 3] for the discrete-time case. The continuous-time equivalent is well known but we were not able to find an explicit proof in the literature and we provide one for completeness.

Lemma 4. *The set $\mathscr{M}$ admits an extremal norm if and only if it is nondefective.*

Proof. Assume that $\mathscr{M}$ admits an extremal norm v and denote by $\|\cdot\|_v$ the induced matrix norm. Then

$$\|\Phi_A(t+s,s)\|_v \leq e^{\lambda(\mathscr{M})t}, \qquad s,t \geq 0, \quad A \in \mathcal{S}_{\mathrm{arb}(\mathscr{M})}. \tag{6}$$

We conclude that $\mathscr{M}$ is nondefective by equivalence of the norms $\|\cdot\|$ and $\|\cdot\|_v$.

In order to prove the converse, define

$$v(x) = \sup_{A\in\mathcal{S},\, t\geq 0} e^{-t\lambda(\mathscr{M})} \|\Phi_A(t,0)x\|, \qquad x \in \mathbb{R}^n.$$

Then v is finite by the nondefectiveness assumption and positive definite because $v(x) \geq \|x\|$ for every $x \in \mathbb{R}^n$. Let us show that v is a norm. The homogeneity is obvious. The subadditivity follows from the inequality

$$\|\Phi_A(t,0)(x+y)\| \leq \|\Phi_A(t,0)x\| + \|\Phi_A(t,0)y\|, \qquad \forall A \in \mathcal{S},\ t \geq 0.$$

In order to show that v is extremal, consider $B \in \mathcal{S}$, $x_0 \in \mathbb{R}^n$, and notice that

$$\begin{aligned} v(\Phi_B(t,0)x_0) &= \sup_{A \in \mathcal{S},\, s \geq 0} e^{-s\lambda(\mathscr{M})} \|\Phi_A(s,0)\Phi_B(t,0)x_0\| \\ &= e^{t\lambda(\mathscr{M})} \sup_{A \in \mathcal{S},\, s \geq 0} e^{-(t+s)\lambda(\mathscr{M})} \|\Phi_A(s,0)\Phi_B(t,0)x_0\| \\ &\leq e^{t\lambda(\mathscr{M})} \sup_{\tilde{A} \in \mathcal{S},\, \tilde{s} \geq 0} e^{-\tilde{s}\lambda(\mathscr{M})} \|\Phi_{\tilde{A}}(\tilde{s},0)x_0\| \\ &= e^{t\lambda(\mathscr{M})} v(x_0). \end{aligned}$$

□

As stated in the following lemma, a useful linear algebra criterion to check the nondefectiveness of a set $\mathscr{M}$ of matrices is its irreducibility. Recall that $\mathscr{M}$ is said to be *irreducible* if the only subspaces that are preserved by all matrices in $\mathscr{M}$ are $\{0\}$ and $\mathbb{R}^n$. A proof of the lemma can be foundin [20, Proposition 3.2(i)].

Lemma 5. *If $\mathscr{M}$ is irreducible, then it is nondefective.*

It is useful to recall the following result on the reduction of a switched system in irreducible blocks.

Proposition 6. *Let $\mathscr{M}$ be a bounded subset of $M_n(\mathbb{R})$ and consider a finite sequence*

$$\{0\} = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_{r-1} \subsetneq E_r = \mathbb{R}^n \tag{7}$$

such that every E_i is an invariant subspace for $\mathscr{M}$. For every $i = 1, \ldots, r$ denote by n_i the dimension of E_i and assume that there exists no invariant subspace V for $\mathscr{M}$ such that $E_{i-1} \subsetneq V \subsetneq E_i$. Then the integer r is independent on the choices of the sequence (7) and there exists $P \in \mathrm{GL}_n(\mathbb{R})$ such that for every matrix $A \in \mathscr{M}$ one has

$$PAP^{-1} = \begin{pmatrix} A_{11} & A_{12} & \cdots & & \\ 0 & A_{22} & A_{23} & \cdots & \\ 0 & 0 & A_{33} & A_{34} & \cdots \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & \cdots & \cdots & 0 & A_{rr} \end{pmatrix}, \tag{8}$$

where each A_{ij} is a $(n_i - n_{i-1}) \times (n_j - n_{j-1})$ matrix. Moreover, for $i = 1, \ldots, r$, the set of matrices $\mathscr{M}_i := \{A_{ii} \mid A \in \mathscr{M}\}$ is irreducible and unique up to reordering, i.e., if $\mathscr{M}_1, \ldots, \mathscr{M}_r$ and $\mathscr{N}_1, \ldots, \mathscr{N}_r$ are obtained as above, then there exist a permutation γ of $\{1, \ldots, r\}$ and $P_1, \ldots, P_r$ invertible matrices so that $\mathscr{N}_i = P_i \mathscr{M}_{\gamma(i)} P_i^{-1}$ for $i = 1, \ldots, r$.

The reduction into the block form (8) simply follows from the definition of invariant flag. As for the uniqueness of the $\mathscr{M}_i$ (up to reordering), it is a consequence of the Jordan–Hölder theorem for R-modules (see for instance [8, Theorem 13.7]). It can be used to characterize sets $\mathscr{M}$ which are nondefective (see [6, Theorem 10]).

The relation between the maximal Lyapunov exponent of $\mathscr{M}$ and those of the blocks $\mathscr{M}_i$ is the following:

$$\lambda(\mathscr{M}) = \max_{i=1,\ldots,r} \lambda(\mathscr{M}_i).$$

(See [4, Lemma 2(c)]).

We now recall the notion of Barabanov norm.

Definition 7. *A norm v in $\mathbb{R}^n$ is said to be* Barabanov for $\mathscr{M}$ *if it is extremal for $\mathscr{M}$ and, in addition, for every $x_0 \in \mathbb{R}^n \setminus \{0\}$, there exists a trajectory $x(\cdot)$ of $(\Sigma_{\mathscr{M}})$ starting from x_0 satisfying $v(x(t)) = v(x_0)e^{\lambda(\mathscr{M})t}$, for every $t \geq 0$. Any such trajectory is called an* extremal trajectory for v.

The following remarks are immediate consequences of the above definition.

Remark 8. *If v is a Barabanov norm for $\mathscr{M}$, then, for every $\alpha \in \mathbb{R}$, v is also a Barabanov norm for $\mathscr{M} + \alpha \mathrm{Id}_n$.*

Remark 9. *When $\mathscr{M} = \{M\}$ and $M = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$, we have that $\lambda(\mathscr{M}) = 0$ and the Euclidean norm is clearly extremal, but $\mathscr{M}$ admits no Barabanov norm, since $t \mapsto x(t) = (0, e^{-t})$ is the only trajectory starting from the point $(0, 1)$ and cannot be extremal by homogeneity of any norm.*

Remark 10. *An example of singleton $\mathscr{M} = \{M\}$ for which a Barabanov norm v exists is given by $M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and v equal to the Euclidean norm.*

Remark 11. *It is clear that if v is a Barabanov norm, then νv is a Barabanov norm as well for every positive ν. Therefore, uniqueness of Barabanov norms can only hold up to homogeneity.*

Even in that sense a Barabanov norm is not in general unique. Consider, for instance, $\mathscr{M} = \mathrm{co}\{M_1, M_2, M_3\}$ with

$$M_1 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} -a & 1 \\ -1 & -a \end{pmatrix}.$$

It is easy to see that if $a \geq 1$ then $\lambda(\mathscr{M}) = 0$ and any norm $v_\beta(x) := \max\{|x_1|, \beta|x_2|\}$ with $\beta \in [\frac{1}{a}, a]$ is a Barabanov norm.

Example 12. *Even when a Barabanov norm exists, it cannot in general be taken polyhedral, polynomial, nor smooth, as illustrated by the following example.*

Suppose that $\mathscr{M} = \mathrm{co}\{M_1, M_2\}$ with

$$M_1 = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} a & 3 \\ -0.6 & 0.7 \end{pmatrix},$$

where $a \sim 0.8896$ is chosen in such a way that $\lambda(\mathscr{M}) = 0$. The latter condition is equivalent to the fact that the trajectory of $t \mapsto e^{tM_2}x_0$ with $x_0 = (-1, 0)^T$ touches tangentially the line $x_1 = 1$. In this case, it is easy to see that the closed curve constructed in Figure 1 by gluing together four trajectories of the system is the level set Λ of a Barabanov norm for $\mathscr{M}$.

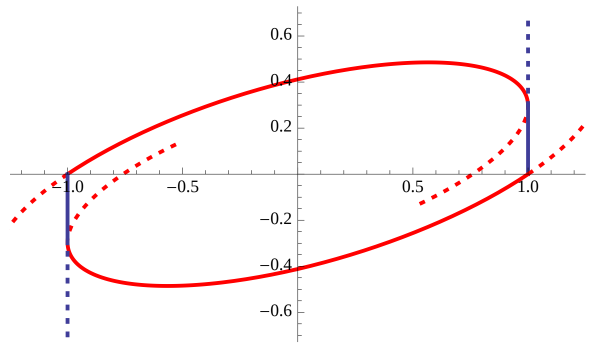

Figure 1: Barabanov norm construction for the switched system of Example 12.

Moreover, the extremal trajectories of the system on Λ *converge either to* $(1,0)^T$ *or to* $(-1,0)^T$*. On the other hand it is possible to construct trajectories of the system starting from* Λ*, turning around the origin an infinite number of times and staying arbitrarily close to* Λ*. One deduces that any Barabanov norm for* $\mathscr{M}$ *must have* Λ *as level set and henceforth is uniquely defined up to a multiplicative constant. Finally, note that the fact that* V *contains segments implies that such a Barabanov norm is not strictly convex.*

We have the following fundamental result due to Barabanov [3], which can be seen as a converse nonstrict Lyapunov result (for the system $(\Sigma_{\mathscr{M}-\lambda(\mathscr{M})\mathrm{Id}_n})$).

Theorem 13. *Let* $\mathscr{M}$ *be convex, irreducible, and compact. Then there exists a Barabanov norm for* $\mathscr{M}$*.*

4 Necessary and sufficient conditions ensuring the existence of a Barabanov norm

Our main result, presented in this section, is a the following characterization of linear systems admitting a Barabanov norm, which generalizes Theorem 13.

Theorem 14. *Let* $\mathscr{M}$ *be compact and convex. Then the following three conditions are equivalent:*

- (C_1) *There exists a Barabanov norm for* $\mathscr{M}$*;*
- (C_2) $\mathscr{M}$ *is nondefective and for every proper invariant subspace* V *of* $\mathbb{R}^n$ *for* $\mathscr{M}$*,* $\lambda(\mathscr{M}|_V) = \lambda(\mathscr{M})$*;*
- (C_3) $\mathscr{M}$ *is nondefective and for every proper invariant subspace* V *of* $\mathbb{R}^n$ *for* $\mathscr{M}$ *such that* $\mathscr{M}|_V$ *is irreducible,* $\lambda(\mathscr{M}|_V) = \lambda(\mathscr{M})$*.*

Proof. Without loss of generality we can assume that $\lambda(\mathscr{M}) = 0$.

Assume that (C_1) holds true and let us prove (C_2). Denote by v a Barabanov norm for $\mathscr{M}$. The extremality of v implies, by Lemma 4, that $\mathscr{M}$ is nondefective. Assume by contradiction that $\lambda(\mathscr{M}|_V) < 0$ for some proper invariant subspace V for $\mathscr{M}$. Then for every nonzero initial condition x_0 in V and every trajectory $x(\cdot)$ of $(\Sigma_{\mathscr{M}})$ starting from x_0, $\lim_{t\to+\infty} x(t) = 0$. This contradicts the existence of an extremal trajectory starting from x_0.

Notice that (C_2) trivially implies (C_3).

Assume now that (C_3) holds true and let us prove (C_1). Let us associate with the set $\mathscr{M}$ the semigroup of matrices

$$\mathscr{S} = \cup_{t\geq 0}\mathscr{S}_t, \qquad \text{where } \mathscr{S}_t = \{\Phi_A(t,0) \mid A \in \mathcal{S}\} \quad \forall t \geq 0.$$

Following the approach proposed by Wirth in [20], we define, moreover, the *limit semigroup* $\mathscr{S}_\infty \subset M_n(\mathbb{R})$ as

$$\mathscr{S}_\infty = \{M \mid M = \lim_{k\to\infty} M_k \text{ with } M_k \in \mathscr{S}_{t_k}\, \forall k \in \mathbb{N} \text{ and } t_k \to +\infty\}.$$

Notice that $\mathscr{S}_\infty$ is indeed a semigroup, since for every $A \in \mathcal{S}$ and every $\tau, t \geq 0$, $\mathscr{S}_\tau \Phi_A(t,0) \subset \mathscr{S}_{\tau+t}$, which leads to

$$M\Phi_A(t,0) \in \mathscr{S}_\infty \qquad \forall M \in \mathscr{S}_\infty,\ A \in \mathcal{S},\ t \geq 0 \tag{9}$$

and to the semigroup property of $\mathscr{S}_\infty$. We claim, moreover, that $\mathscr{S}_\infty$ is compact and nonempty. Closedness is easily checked by rewriting $\mathscr{S}_\infty$ as

$$\mathscr{S}_\infty = \cap_{s>0}\overline{\{\Phi_A(t,0) \mid A \in \mathcal{S},\ t > s\}},$$

while boundedness and nonemptiness follow from the boundedness of $\mathcal{S}$, which is a consequence of Lemma 5.

Define v by

$$v(x) = \max_{M\in\mathscr{S}_\infty} \|Mx\|, \qquad x \in \mathbb{R}^n. \tag{10}$$

We conclude the proof by showing that v is a Barabanov norm. Its homogeneity is obvious, while its subadditivity follows by noticing that

$$v(x+y) \leq \max_{M\in\mathscr{S}_\infty} (\|Mx\| + \|My\|) \leq \max_{M\in\mathscr{S}_\infty} \|Mx\| + \max_{M\in\mathscr{S}_\infty} \|My\| = v(x) + v(y).$$

In order to show that v is positive definite (and, hence, a norm), assume by contradiction that there exists a nonzero vector $x \in \mathbb{R}^n$ such that $v(x) = 0$. The space V spanned by all the vectors $\Phi_A(t,0)x$ for $A \in \mathcal{S}$ and $t \geq 0$ is invariant for all matrices in $\mathscr{S}$. Hence V is invariant for $\mathscr{M}$. Moreover v is 0 at any point $\Phi_A(t,0)x$ for $A \in \mathcal{S}$ and $t \geq 0$, thus its restriction to V vanishes identically. By Proposition 6, there exists an invariant subspace W of V such that $\mathscr{M}|_W$ is irreducible. By assumption (C_3), we then have that $\lambda(\mathscr{M}|_W) = 0$. On the other hand, $v|_W \equiv 0$, which means, by definition of v and nondefectiveness of $\mathscr{M}$, that every trajectory of $(\Sigma_{\mathscr{M}})$ with initial condition in W converges to 0. It follows from (3) that $\lambda(\mathscr{M}|_W) < 0$, leading to a contradiction. This concludes the proof that v is positive definite.

We conclude the proof by checking that v is a Barabanov norm on $\mathbb{R}^n$. Indeed, v is extremal as it follows from its definition and property (9).

Pick now x_0 in $\mathbb{R}^n \setminus \{0\}$. Consider $(A_k)_{k\in\mathbb{N}} \subset \mathcal{S}$ and $(t_k)_{k\in\mathbb{N}} \subset [0,\infty)$ going to infinity such that $v(x) = \|Mx\|$ with $M = \lim_{k\to\infty} \Phi_{A_k}(t_k, 0)$. By the Banach–Alaoglu theorem, up to extracting a subsequence, A_k weak-$\star$ converges to some $A \in \mathcal{S}$.

We claim that for every $t \geq 0$ and $k \in \mathbb{N}$ the matrix $E_k(t) = \Phi_{A_k}(t,0) - \Phi_A(t,0)$ tends to zero as k goes to infinity. Indeed,

$$\dot{E}_k(s) = A_k(s)E_k(s) + (A_k(s) - A(s))\Phi_A(s,0),$$

for almost every $s \in [0,t]$, and $E_k(0) = 0$. Then,

$$\|E_k(t)\| \leq C \int_0^t \|E_k(s)\| ds + \|h_k\|,$$

where $C = \sup_{k\in\mathbb{N}} \|A_k\|_\infty$ and $h_k = \int_0^t (A_k(s) - A(s))\Phi_A(s,0)ds$. The sequence $(h_k)_{k\in\mathbb{N}}$ tends to zero by weak-$\star$ convergence of A_k to A. The convergence of $\Phi_{A_k}(t,0)$ to $\Phi_A(t,0)$ follows from Gronwall's lemma.

We claim now that $v(\Phi_A(t,0)x) \geq v(x)$ for every $t \geq 0$ (hence, by extremality of v, $v(\Phi_A(t,0)x) \equiv v(x)$). Indeed,

$$\begin{aligned}
\lim_{k\to\infty} \Phi_{A_k(\cdot+t)}(t_k - t, 0)\Phi_A(t,0) &= \lim_{k\to\infty} \Phi_{A_k}(t_k, t)\Phi_A(t,0) \\
&= \lim_{k\to\infty} \Phi_{A_k}(t_k, 0)\Phi_{A_k}(t,0)^{-1}\Phi_A(t,0) \\
&= M,
\end{aligned}$$

which implies that $M\Phi_A(t,0)^{-1}$ is in $\mathscr{S}_\infty$. It follows from the definition of v that $v(\Phi_A(t,0)x) \geq \|M\Phi_A(t,0)^{-1}\Phi_A(t,0)x\| = v(x)$ and this concludes the proof of the theorem. □

Let us conclude by reformulation in terms of graphs of the statement of Theorem 14. Given $\mathscr{M}$, let us consider the directed graph $\mathscr{G}$ whose vertices are the subsystems $\mathscr{M}_1, \dots, \mathscr{M}_r$ introduced in Proposition 6 and for which the vertex $\mathscr{M}_i$ is connected by an edge to the vertex $\mathscr{M}_j$ if and only if there exists $A \in \mathscr{M}$ such that, using the notation of equation (8), $A_{ji} \neq 0$ (independently of the linear system of coordinates). Then we can restate Theorem 14 by saying that there exists a Barabanov norm for $\mathscr{M}$ if and only if $\mathscr{M}$ is nondefective and for every sink $\mathscr{M}_i$ of $\mathscr{G}$ it holds $\lambda(\mathscr{M}_i) = \lambda(\mathscr{M})$. Indeed, a subsystem $\mathscr{M}_i$ corresponds to a proper invariant subspace for $\mathscr{M}$ if and only if it is a sink of $\mathscr{G}$, i.e., there is no edge exiting from $\mathscr{M}_i$ (see Figure 2).

References

[1] A. Agrachev and A. Sarychev. Control in the spaces of ensembles of points. *SIAM J. Control Optim.*, 58(3):1579–1596, 2020.

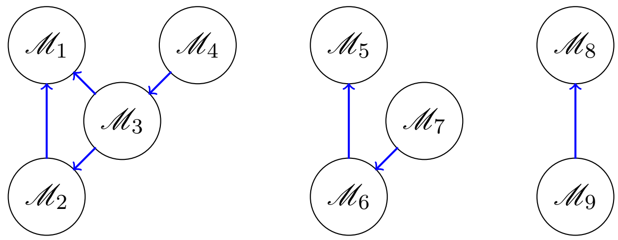

Figure 2: *Representation of a graph $\mathscr{G}$ with 9 vertices. The sinks of the graph are $\mathscr{M}_1$, $\mathscr{M}_5$, and $\mathscr{M}_8$. A Barabanov norm exists, according to Theorem 14, if and only if $\mathscr{M}$ is nondefective and $\lambda(\mathscr{M}_1) = \lambda(\mathscr{M}_5) = \lambda(\mathscr{M}_8) = \lambda(\mathscr{M})$.*

[2] N. Augier, U. Boscain, and M. Sigalotti. Adiabatic ensemble control of a continuum of quantum systems. *SIAM J. Control Optim.*, 56(6):4045–4068, 2018.

[3] N. E. Barabanov. An absolute characteristic exponent of a class of linear nonstationary systems of differential equations. *Sibirsk. Mat. Zh.*, 29(4):12–22, 222, 1988.

[4] M. A. Berger and Y. Wang. Bounded semigroups of matrices. *Linear Algebra Appl.*, 166:21–27, 1992.

[5] F. Blanchini and S. Miani. A new class of universal Lyapunov functions for the control of uncertain linear systems. *IEEE Trans. Automat. Control*, 44(3):641–647, 1999.

[6] Y. Chitour, P. Mason, and M. Sigalotti. On the marginal instability of linear switched systems. *Systems Control Lett.*, 61(6):747–757, 2012.

[7] F. Colonius and W. Kliemann. *The dynamics of control.* Systems & Control: Foundations & Applications. Birkhäuser Boston, Inc., Boston, MA, 2000. With an appendix by Lars Grüne.

[8] C. Curtis and I. Reiner. *Representation theory of finite groups and associative algebras.* Pure and Applied Mathematics, Vol. XI. Interscience Publishers, a division of John Wiley & Sons, New York-London, 1962.

[9] W. P. Dayawansa and C. F. Martin. A converse Lyapunov theorem for a class of dynamical systems which undergo switching. *IEEE Trans. Automat. Control*, 44(4):751–760, 1999.

[10] G. Dirr and M. Schönlein. Uniform and L^q-ensemble reachability of parameter-dependent linear systems. *J. Differential Equations*, 283:216–262, 2021.

[11] I. Haidar, P. Mason, and M. Sigalotti. Converse Lyapunov-Krasovskii theorems for uncertain retarded differential equations. *Automatica J. IFAC*, 62:263–273, 2015.

[12] F. M. Hante, M. Sigalotti, and M. Tucsnak. On conditions for asymptotic stability of dissipative infinite-dimensional systems with intermittent damping. *J. Differential Equations*, 252(10):5569–5593, 2012.

[13] V. S. Kozyakin. Algebraic unsolvability of a problem on the absolute stability of desynchronized systems. *Avtomat. i Telemekh.*, (6):41–47, 1990.

[14] J.-S. Li. Ensemble control of finite-dimensional time-varying linear systems. *IEEE Trans. Automat. Control*, 56(2):345–357, 2011.

[15] J.-S. Li and N. Khaneja. Ensemble control of Bloch equations. *IEEE Trans. Automat. Control*, 54(3):528–536, 2009.

[16] D. Liberzon. *Switching in systems and control*. Systems & Control: Foundations & Applications. Birkhäuser Boston Inc., Boston, MA, 2003.

[17] P. Mason, U. Boscain, and Y. Chitour. Common polynomial lyapunov functions for linear switched systems. *SIAM J. Control Optimization*, 45(1):226–245, 2006.

[18] A. P. Molchanov and Y. S. Pyatnitskiy. Criteria of asymptotic stability of differential and difference inclusions encountered in control theory. *Systems Control Lett.*, 13(1):59–64, 1989.

[19] Z. Sun and S. S. Ge. *Stability theory of switched dynamical systems*. Communications and Control Engineering Series. Springer, London, 2011.

[20] F. Wirth. The generalized spectral radius and extremal norms. *Linear Algebra Appl.*, 342:17–40, 2002.