Spanning eulerian subdigraphs in semicomplete digraphs

Frédéric Havet, Jørgen Bang-Jensen, Anders Yeo

Spanning eulerian subdigraphs in semicomplete digraphs

Jørgen Bang-Jensen* Frédéric Havet† Anders Yeo‡

December 9, 2021

Abstract

A digraph is **eulerian** if it is connected and every vertex has its in-degree equal to its out-degree. Having a spanning eulerian subdigraph is thus a weakening of having a hamiltonian cycle. In this paper, we first characterize the pairs (D, a) of a semicomplete digraph D and an arc a such that D has a spanning eulerian subdigraph containing a. In particular, we show that if D is 2-arc-strong, then every arc is contained in a spanning eulerian subdigraph. We then characterize the pairs (D, a) of a semicomplete digraph D and an arc a such that D has a spanning eulerian subdigraph avoiding a. In particular, we prove that every 2-arc-strong semicomplete digraph has a spanning eulerian subdigraph avoiding any prescribed arc. We also prove the existence of a (minimum) function $f(k)$ such that every $f(k)$-arc-strong semicomplete digraph contains a spanning eulerian subdigraph avoiding any prescribed set of k arcs. We conjecture that $f(k) = k + 1$ and establish this conjecture for $k \leq 3$ and when the k arcs that we delete form a forest of stars.

A digraph D is **eulerian-connected** if for any two distinct vertices x, y, the digraph D has a spanning (x, y)-trail. We prove that every 2-arc-strong semicomplete digraph is eulerian-connected.

All our results may be seen as arc analogues of well-known results on hamiltonian paths and cycles in semicomplete digraphs.
Keywords: Arc-connectivity, Eulerian subdigraph, Tournament, Semicomplete digraph, polynomial algorithm.

1 Introduction

A digraph is **semicomplete** if it has no pair of non-adjacent vertices. A **tournament** is a semicomplete digraph without directed cycles of length 2. Two of the classical results on digraphs are Camion's Theorem and Redéi's theorem (both were originally formulated only for tournaments but they easily extend to semicomplete digraphs).

Theorem 1 (Camion [8]). *Every strong semicomplete digraph has a hamiltonian cycle.*

Theorem 2 (Rédei [13]). *Every semicomplete digraph has a hamiltonian path.*

Thomassen [14] proved the following. (It was originally formulated only for tournaments but the proof works for semicomplete digraphs as it easily follows from Theorem 10.)

Theorem 3 (Thomassen [14]). *In a 3-strong semicomplete digraph, every arc is contained in a hamiltonian cycle.*

The 3-strong assumption in this theorem best possible: Thomassen [14] described an infinite class of 2-strong tournaments containing an arc which is not in any hamiltonian cycle. It is easy to modify his example to show that there is no k such that every k-arc-strong tournament has a hamiltonian

*Department of Mathematics and Computer Science, University of Southern Denmark, Odense DK-5230, Denmark (email:jbj@imada.sdu.dk).

†CNRS, Université Côte d'Azur, I3S and INRIA, Sophia Antipolis, France (email: frederic.havet@inria.fr)

‡Department of Mathematics and Computer Science, University of Southern Denmark, Odense DK-5230, Denmark (email:yeo@imada.sdu.dk).

cycle containing any given arc. No characterization of the set of arcs which belong to a hamiltonian cycle in a semicomplete digraph (or a tournament) is known.

A natural question is whether the 3-strong assumption of Theorem 3 can be relaxed if instead of a hamiltonian cycle, we only require a spanning eulerian subdigraph. In this paper we answer this question by proving the following analogue to Theorem 3.

Theorem 4. *Let $D = (V, A)$ be a 2-arc-strong semicomplete digraph. For every arc $a \in A$ there exists a spanning eulerian subdigraph of D containing a.*

In addition (and contrary to the lack of a known characterization for hamiltonian cycles mentioned above), in Section 5, we characterize the pairs (D, a) such that D is a strong semicomplete digraph containing the arc a and no spanning eulerian subdigraph of D contains the arc a.

In Section 6, we also study spanning eulerian subdigraphs of a semicomplete digraph avoiding a prescribed set of arcs. Fraisse and Thomassen [9] proved the following result on hamiltonian cycles avoiding a set of prescribed arcs. For a strengthening of this result, see [5]. The connectivity requirement of Theorem 5 is best possible as there are k-strong tournaments with vertices of out-degree exactly k.

Theorem 5 (Fraisse and Thomassen [9])**.** *Every $(k+1)$-strong tournament contains a hamiltonian cycle avoiding any prescribed set of k arcs.*

This theorem does not extend to semicomplete digraphs. Indeed the 2-strong semicomplete digraph obtained from a 4-cycle by adding a 2-cycle between each of the two pairs of non-adjacent vertices has a unique hamiltonian cycle, and thus no arc of this cycle can be avoided. Observe however that Theorem 3 implies that every 3-strong tournament contains a hamiltonian cycle avoiding any prescribed arc. Improving a previous bound by Bang-Jensen and Thomassen, Guo [10] proved that every $(3k+1)$-strong semicomplete digraph contains a spanning $(k+1)$-strong tournament. Together with Theorem 5, this implies that every $(3k+1)$-strong semicomplete digraph contains a hamiltonian cycle avoiding any prescribed set of k arcs. We conjecture that a much lower connectivity suffices.

Conjecture 6. Let k be a non-negative integer. Every $(k+2)$-strong semicomplete digraph contains a hamiltonian cycle avoiding any prescribed set of k arcs.

Bang-Jensen and Jordán [6] proved that every 3-strong semicomplete digraph contains a spanning 2-strong tournament. Combining this with Theorem 5 shows that Conjecture 6 holds for $k = 1$.

As an analogue to Theorem 5, we prove that there is a function $f(k)$ such that every $f(k)$-arc-strong semicomplete digraph contains a spanning eulerian subdigraph avoiding any prescribed set of k arcs. In Proposition 28, we show that $f(k) \leq (k+1)^2/4+1$. Since there are k-arc-strong semicomplete digraphs in which one or more vertices have out-degree k, we have $f(k) \geq k+1$. We conjecture that $f(k) = k+1$.

Conjecture 7. For every non-negative integer k, every $(k+1)$-arc-strong semicomplete digraph D has a spanning eulerian subdigraph that avoids any prescribed set of k arcs.

Observe that Camion's Theorem implies this conjecture when $k = 0$, that is $f(0) = 1$. In Corollary 31, we prove Conjecture 7 for $k \leq 2$ and in Theorem 32, we prove it for $k = 3$. Hence $f(1) = 2$, $f(2) = 3$ and $f(3) = 4$. Since this paper has been submitted, it has been proved in [2] that $f(k) \leq \lceil \frac{6k+1}{5} \rceil$. In particular, Conjecture 7 holds for $k \leq 4$.

In Section 7, we characterize the pairs (D, a) such that $D = (V, A)$ is a strong semicomplete digraph, $a \in A$ and every spanning eulerian subdigraph of D contains the arc a (Theorem 35).

A digraph D is (strongly) **hamiltonian-connected** if for any pair of distinct vertices x, y, D has a hamiltonian path from x to y. Thomassen [14] proved the following. (Again it was originally formulated only for tournaments but the proof works for semicomplete digraphs as it easily follows from Theorem 10.)

Theorem 8 (Thomassen [14])**.** *Every 4-strong semicomplete digraph is hamiltonian-connected.*

The 4-strong assumption in this theorem best possible: Thomassen [14] described infinitely many 3-strong tournaments that are not hamiltonian-connected. Again, it is natural to ask whether the connectivity assumption of Theorem 8 can be relaxed if instead of hamiltonian-connected, we only require the digraph to eulerian-connected. A digraph D is **eulerian-connected** if for any two vertices x, y, the digraph D has a spanning (x, y)-trail. We prove that every 2-arc-strong semicomplete digraph is eulerian-connected.

Theorem 9. *Every 2-arc-strong semicomplete digraph is eulerian-connected.*

This theorem can been seen as an analogue of Theorem 8. The 2-arc-strong condition is best possible. In Proposition 17, we describe strong tournaments with arbitrarily large in- and out-degrees in which there is an arc contained in no spanning eulerian subdigraph. Independently from us, Liu et al. [12] also studied the notion of eulerian-connected digraph, which they call *strongly trail-connected.* They proved the restriction of Theorem 9 tournaments.

To prove Theorems 3 and 8. Thomassen [14] gave the following sufficient condition for a semicomplete digraph to contain a hamiltonian (x, y)-path, which implies both results immediately.

Theorem 10 (Thomassen [14])**.** *Let T be a 2-strong semicomplete digraph, and let x and y be two distinct vertices of T. If there are three internally disjoint (x, y)-paths of length greater than 1, then there is a hamiltonian (x, y)-path in D.*

To prove our results, we prove a theorem that can be seen as an arc analogue to Theorem 10.

Theorem 11. *Let D be a strong semicomplete digraph, and let x and y be two vertices of D. If there are two arc-disjoint (x, y)-paths in D, then there is a spanning (x, y)-trail in $D \setminus \{yx\}$.*

This theorem directly implies Theorems 4 and 9.

2 Terminology

Notation generally follows [4, 3]. The digraphs have no parallel arcs and no loops. We denote the vertex set and arc set of a digraph D by $V(D)$ and $A(D)$, respectively and write $D = (V, A)$ where $V = V(D)$ and $A = A(D)$. Unless otherwise specified, the numbers n and m will always be used to denote the number of vertices, respectively arcs, in the digraph in question. We use the notation $[k]$ for the set of integers $\{1, 2, \ldots, k\}$.

Let $D = (V, A)$ be a digraph. The subdigraph **induced** by a set $X \subseteq V$ in a digraph D is denoted by $D\langle X\rangle$. If X is a set of vertices we denote by $D - X$ the digraph $D\langle V \setminus X\rangle$, and if A' is a set of arcs in D, then we denote by $D \setminus A'$ the digraph we obtain by deleting all arcs in A'.

When xy is an arc of D we say that x **dominates** y and write $x \to y$. If $x \to y$ for all $x \in X$ and all $y \in Y$, then we write $X \to Y$ and we write $X \mapsto Y$ when $X \to Y$ and there is no arc from Y to X. For sake of clarity, we abbreviate $\{x\} \to Y$ to $x \to Y$. For a digraph $D = (V, A)$ the **out-degree**, $d^+_D(x)$ (resp. the **in-degree**, $d^-_D(x)$) of a vertex $x \in V$ is the number of arcs of the kind xy (resp. yx) in A. When $X \subseteq V$ we shall also write $d^+_X(v)$ to denote the number of arcs vx with $x \in X$.

A **walk** is an alternating sequence $W = (v_0, a_1, v_1, \ldots, a_p, v_p)$ of vertices and arcs such that $a_i = v_{i-1}v_i$ for all $1 \leq i \leq p$. Its **initial vertex**, denoted by $s(W)$, is v_0 and its **terminal vertex**, denoted by $t(W)$, is v_p. The v_i, $1 \leq i \leq p - 1$, are the **internal vertices** of W. A walk is completely determined by the sequence of its vertices. Therefore for the sake of simplicity, we use the sequence $v_0 v_1 \cdots v_p$ to denote the walk $(v_0, a_1, v_1, \ldots, a_p, v_p)$.

A walk W is **closed** if $s(W) = t(W)$. A **trail** is a walk in which all arcs are distinct, a **path** is a walk in which all vertices are distinct and a **cycle** is a closed walk in which all vertices are distinct except the initial and terminal vertices. Note that, walks, trails, paths and cycles are always directed.

An (s, t)**-walk** (resp. (s, t)**-trail**, (s, t)**-path** is a walk (resp. trail, path) with initial vertex s and terminal vertex t. Observe that if $s \neq t$, then an (s, t)-trail can be seen as a connected digraph such that $d^+(s) = d^-(s) + 1$, $d^-(t) = d^+(t) + 1$ and $d^+(v) = d^-(v)$ for all other vertices. For two sets X, Y of vertices, an (X, Y)-path is a path with initial vertex in X, terminal vertex in Y, and no internal vertices in $X \cup Y$.

Let $P = x_1 \cdots x_p$ be a path. For any $1 \le i \le j \le p$, we denote by $P[x_i, x_j]$ the path $x_i \cdots x_j$, by $P[x_i, x_j)$ the path $x_i \cdots x_{j-1}$, by $P(x_i, x_j]$ the path $x_{i+1} \cdots x_j$, and by $P(x_i, x_j)$ the path $x_{i+1} \cdots x_{j-1}$. Similarly, if C is a cycle and x, y two vertices of C, we denote by $C[x, y]$ the (x, y)-path in C if $x \neq y$ and the cycle C if $x = y$. Denote by x^+ the out-neighbour of x in C and by y^- the in-neighbour of y in C, and let $C(x, y] = C[x^+, y]$, $C[x, y) = C[x, y^-]$ and $C(x, y) = C[x^+, y^-]$.

A digraph D is **eulerian** if it contains an **eulerian tour**, that is a spanning eulerian trail W such that $A(W) = A(D)$. Equivalently, by Euler's theorem, a digraph D is **eulerian** if it is connected and $d^+(v) = d^-(v)$ for all $v \in V(D)$.

The **underlying (multi)graph** of a digraph D, denoted $UG(D)$, is obtained from D by suppressing the orientation of each arc. A digraph $D = (V, A)$ is **connected** if $UG(D)$ is a connected graph. It is **strong** if it contains an (s, t)-path for each ordered pair of distinct vertices $s, t \in V$. It is **k-strong** if $D - W$ is strong for every subset $W \subseteq V$ of at most $k-1$ arcs. It is **k-arc-strong** if $D \setminus A'$ is strong for every subset $A' \subseteq A$ of at most $k - 1$ arcs. The largest k such that D is k-arc-strong is called the **arc-connectivity** of D and is denoted by $\lambda(D)$. A **cut-arc** in D is an arc a such that $D \setminus a$ is not strong.

3 Structure of semicomplete digraphs

Let D be a digraph. A **decomposition** of D is a partition $(S_1, \ldots, S_p)$, $p \ge 1$, of its vertex set. The **index** of vertex v in the decomposition, denoted by $\mathrm{ind}(v)$, is the integer i such that $v \in S_i$. An arc uv is **forward** if $\mathrm{ind}(u) < \mathrm{ind}(v)$, **backward** if $\mathrm{ind}(u) > \mathrm{ind}(v)$, and **flat** if $\mathrm{ind}(u) = \mathrm{ind}(v)$. For sake of clarity, we often abbreviate $S_{\mathrm{ind}(u)}$ into S_u.

A decomposition $(S_1, \ldots, S_p)$ is **strong** if $D\langle S_i \rangle$ is strong for all $1 \le i \le p$. The following proposition is well-known (just consider an acyclic ordering of the strong components of D).

Proposition 12. *Every digraph has a strong decomposition with no backward arcs.*

A **1-decomposition** of a digraph D is a strong decomposition such that every backward arc is a cut-arc and all cut-arcs are either forward or backward.

Proposition 13. *Every strong digraph admits a* 1*-decomposition.*

Proof. Let D be a strong digraph and let C be its set of cut-arcs. If $C = \emptyset$, then the trivial decomposition with only one set $S_1 = V(D)$ is a 1-decomposition, so assume that $C \neq \emptyset$. Observe that $D \setminus C$ is not strong. Thus, by Proposition 12, $D \setminus C$ has a strong decomposition $(S_1, \ldots, S_p)$ with no backward arcs. This decomposition is clearly a 1-decomposition of D. $\square$

Let $(S_1, \ldots, S_p)$ be a decomposition of a digraph. Two backward arcs uv and xy are **nested** if either $\mathrm{ind}(v) \le \mathrm{ind}(y) < \mathrm{ind}(x) \le \mathrm{ind}(u)$ or $\mathrm{ind}(y) \le \mathrm{ind}(v) < \mathrm{ind}(u) \le \mathrm{ind}(x)$. See Figure 1.

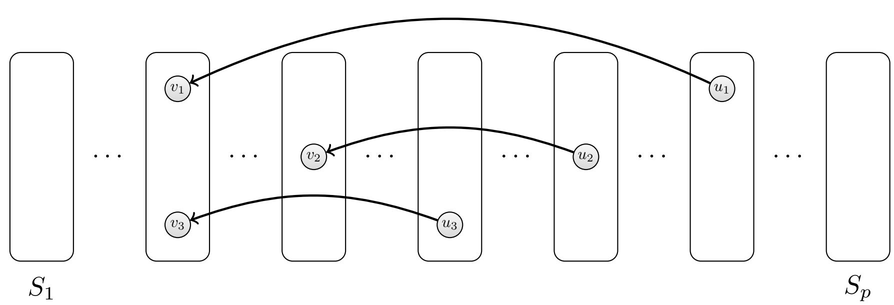

Figure 1: Illustration of nested backwards arcs. The arcs u_1v_1 and u_2v_2 are nested; the arcs u_1v_1 and u_3v_3 are nested; the arcs u_2v_2 and u_3v_3 are not nested.

Proposition 14. *Let $(S_1, \ldots, S_p)$ be a* 1*-decomposition of a strong semicomplete digraph D. The following properties hold:*

(i) If u_1v_1 and u_2v_2 are two cut-arcs, then $\mathrm{ind}(u_1) \neq \mathrm{ind}(u_2)$ *and* $\mathrm{ind}(v_1) \neq \mathrm{ind}(v_2)$.

(ii) There are no nested backward arcs.

(iii) If $|V(D)| \geq 4$ and uv is a forward cut-arc, then $|S_u| = |S_v| = 1$ and $\mathrm{ind}(v) = \mathrm{ind}(u) + 1$.

Proof. (i) Assume for a contradiction that $\mathrm{ind}(u_1) = \mathrm{ind}(u_2)$. Since D is semicomplete, there is an arc between v_1 and v_2. Without loss of generality, we may assume that v_1v_2 is an arc. In $D\langle S_{u_1}\rangle = D\langle S_{u_2}\rangle$, there is a (u_2, u_1)-path P. Note that P avoids u_2v_2 because this arc is not flat. But then $P \cup u_1v_1v_2$ is a (u_2, v_2)-path in $D \setminus u_2v_2$, contradicting that u_2v_2 is a cut-arc.

(ii) Suppose for a contradiction that D contains two nested arcs uv and xy such that $\mathrm{ind}(v) \leq \mathrm{ind}(y) < \mathrm{ind}(x) \leq \mathrm{ind}(u)$. By (i), $\mathrm{ind}(v) < \mathrm{ind}(y)$ and $\mathrm{ind}(x) < \mathrm{ind}(u)$. Moreover by (i), D contains the arcs vy, xu. But now $xuvy$ is an (x, y)-path in $D \setminus xy$, contradicting the fact that xy is a cut-arc.

(iii) Assume $|D| \geq 4$ and let uv be a forward cut-arc.

For any vertex u' in $S_u \setminus \{u\}$, there is a (u, u')-path P in $D\langle S_u\rangle$, and so vu' is a backward arc for otherwise $P \cup u'v$ would be a (u, v)-path in $D \setminus uv$. Hence by (i), $|S_u \setminus \{u\}| \leq 1$, so $|S_u| \leq 2$.

Assume for a contradiction that $|S_u| = 2$, say $S_u = \{u, u'\}$. Let $S = S_{ind(u)+1} \cup \cdots \cup S_v$. If v has an in-neighbour w in S then, by (i), uw is an arc (since vu' is a backward arc), and so uwv is a (u, v)-path, a contradiction to the fact that uv is a cut-arc. Hence, by (i), $S = \{v\}$. Now since $|V(D)| \geq 4$, either $\mathrm{ind}(u) > 1$ or $\mathrm{ind}(v) < p$. By (ii) vu' is the only arc from $S_v \cup \cdots \cup S_p$ to $S_1 \cup \cdots \cup S_u$, and by (i) the only cut-arc with tail in S_u is uv, and the only cut-arc with head in S_v is uv. Therefore, if $\mathrm{ind}(u) > 1$, there is no arc from $S_u \cup \cdots \cup S_p$ to $S_1 \cup \cdots \cup S_{\mathrm{ind}(u)-1}$, and if $\mathrm{ind}(v) < p$, there is no arc from $S_{\mathrm{ind}(v)+1} \cup \cdots \cup S_p$ to $S_1 \cup \cdots \cup S_v$. This is a contradiction to the fact that D is strong.

Hence $|S_u| = 1$. Symmetrically, we obtain $|S_v| = 1$.

Let $W = \{w \mid \mathrm{ind}(u) < \mathrm{ind}(w) < \mathrm{ind}(v)\}$, $X = \{x \mid \mathrm{ind}(x) < \mathrm{ind}(u)\}$, and $Y = \{y \mid \mathrm{ind}(v) < \mathrm{ind}(y)\}$. Observe that for every $w \in W$, either $uw \notin A(D)$ or $wv \notin A(D)$ for otherwise uwv would be a (u, v)-path in $D \setminus uv$ (contradicting that uv is a cut-arc). Since D is semicomplete, this implies that one of the two arcs wu, vw is a backward arc. In particular, $|W| \leq 2$ for otherwise either there would be two backward arcs with tail v or two backwards arcs with head u, contradicting (i).

Assume for a contradiction that $|W| = 2$, say $W = \{w_1, w_2\}$ and $w_1 \rightarrow w_2$. If uw_1 is an arc then the fact that uv is a cut-arc would imply that v would have backwards arcs to each of w_1, w_2, contradicting (i). Hence uw_1 is not an arc and D contains the arcs w_1u (as $uw_1 \notin A(D)$), uw_2 (by (i)), vw_2 (as uv is a cut-arc) and w_1v (by (i)) and does not contain the arcs w_2u, vw_1, w_2w_1. Observe that by (i) w_1w_2 is not a cut-arc and so $\mathrm{ind}(w_2) \geq \mathrm{ind}(w_1)$. Since D is strong, w_1 must have an in-neighbour z, which must be in $X \cup Y$. If $X \neq \emptyset$, then there must be an arc from $W \cup Y \cup \{u, v\}$ to X. By (i) the tail of this arc is not in $\{u, w_1\}$ and so this arc and w_1u are two nested backward arcs, a contradiction to (ii). Similarly, we get a contradiction if $Y \neq \emptyset$. However $X = \emptyset$ and $Y = \emptyset$ is a contradiction to $z \in X \cup Y$.

Assume for a contradiction that $|W| = 1$, say $W = \{w\}$. Since uv is a cut-arc, then uwv cannot be a path, so either uw or wv is not an arc.

Let us assume that uw is not an arc. Then $wu \in A(D)$ because D is semicomplete. Thus $X = \emptyset$, for otherwise wu and any arc from $Y \cup \{u, v, w\}$ to X would be two nested arcs (as by (i) it can not leave $\{u\}$), a contradiction to (ii). Hence $Y \neq \emptyset$, since $|D| \geq 4$. So there must be an arc from Y to $\{u, v, w\}$. By (i), the head of this arc must be w. Let y be its tail. By (i) vw and yu are not backward arcs, so $uywv$ is a (u, v)-path in $D \setminus uv$, a contradiction.

Similarly, we get a contradiction if wv is not an arc. Hence $W = \emptyset$, that is $\mathrm{ind}(v) = \mathrm{ind}(u) + 1$. □

A **nice decomposition** of a digraph D is a 1-decomposition such that the set of cut-arcs of D is exactly the set of backward arcs.

Proposition 15. *Every strong semicomplete digraph of order at least* 4 *admits a nice decomposition.*

Proof. Let D be a strong semicomplete digraph of order at least 4. If uv has a cut-arc, which is forward. By Proposition 14 (iii), $S_u = \{u\}$, $S_v = \{v\}$, and $\mathrm{ind}(v) = \mathrm{ind}(u) + 1$. Inverting S_u and S_v (that is, considering the decomposition $(S_1, \ldots, S_{\mathrm{ind}(u)-1}, \{v\}, \{u\}, S_{\mathrm{ind}(u)+2}, \ldots, S_p)$), we obtain another 1-decomposition with one forward cut-arc less. Doing this for all forward cut-arcs, we obtain a nice decomposition of D. □

Given a semicomplete digraph and a nice decomposition of it, the **natural ordering** of its backward arcs is the ordering in decreasing order according to the index of their tail. Note that this ordering is unique by Proposition 14 (i).

Proposition 16. *Let D be a strong semicomplete digraph of order at least 4, let $(S_1, \ldots, S_p)$ be a nice decomposition of D, and let $(s_1t_1, s_2t_2, \ldots, s_rt_r)$ be the natural ordering of the backward arcs. Then*

(i) $\mathrm{ind}(t_{j+1}) < \mathrm{ind}(t_j) \leq \mathrm{ind}(s_{j+1}) < \mathrm{ind}(s_j)$ *for all* $1 \leq j \leq r-1$ *and* $\mathrm{ind}(t_{j+1}) \leq \mathrm{ind}(s_{j+2}) < \mathrm{ind}(t_j)$ *for all* $1 \leq j \leq r-2$*;*

(ii) $s_1 \in S_p$ *and* $t_r \in S_1$*;*

(iii) If $\mathrm{ind}(t_j) = \mathrm{ind}(s_{j+1}) = i$ *and* $t_j \neq s_{j+1}$*, then there are two arc-disjoint* (t_j, s_{j+1})*-paths in* $D\langle S_i \rangle$*.*

Proof. (i) By Proposition 14 (i), $\mathrm{ind}(s_{j+1}) < \mathrm{ind}(s_j)$, and as D is strong, $\mathrm{ind}(t_j) \leq \mathrm{ind}(s_{j+1}) < \mathrm{ind}(s_j)$. By Proposition 14 (ii), s_jt_j and $s_{j+1}t_{j+1}$ are not nested so $\mathrm{ind}(t_{j+1}) < \mathrm{ind}(t_j)$. Assume for a contradiction that $\mathrm{ind}(t_j) \leq \mathrm{ind}(s_{j+2})$. By Proposition 14 (i), s_js_{j+1} and $t_{j+1}t_{j+2}$ are not arcs, so $s_{j+1}s_j$ and $t_{j+2}t_{j+1}$ are arcs. If $\mathrm{ind}(t_j) < \mathrm{ind}(s_{j+2})$, then $t_js_{j+2} \in A(D)$, and if $\mathrm{ind}(t_j) = \mathrm{ind}(s_{j+1})$, then there is a (t_j, s_{j+2})-path in $D\langle S_{t_j} \rangle$. In both cases, there is a (t_j, s_{j+2})-path P not using the arc $s_{j+1}t_{j+1}$. Now $s_{j+1}s_jt_j \cup P \cup s_{j+2}t_{j+2}t_{j+1}$ is an (s_{j+1}, t_{j+1})-path in $D \setminus s_{j+1}t_{j+1}$, a contradiction.

(ii) Because D is strong, there must be a backward arc with tail in S_p and a backward arc with head in S_1. By the above inequality, necessarily $s_1 \in S_p$ and $t_r \in S_1$.

(iii) Assume for a contradiction that $\mathrm{ind}(t_j) = \mathrm{ind}(s_{j+1}) = i$ and there do not exist two arc-disjoint (t_j, s_{j+1})-paths in $D\langle S_i \rangle$. By Menger's Theorem, there is an arc a such that $D\langle S_i \rangle \setminus \{a\}$ has no (t_j, s_{j+1})-path. But then, there is no (t_j, s_{j+1})-path in $D \setminus \{a\}$, that is a is a cut-arc of D. This contradicts the fact that $(S_1, \ldots, S_p)$ is a nice decomposition. □

4 Eulerian-connected semicomplete digraphs

We first observe that being strong and having large in- and out-degrees are not sufficient to guarantee every arc of a tournament to be in a spanning eulerian subdigraph.

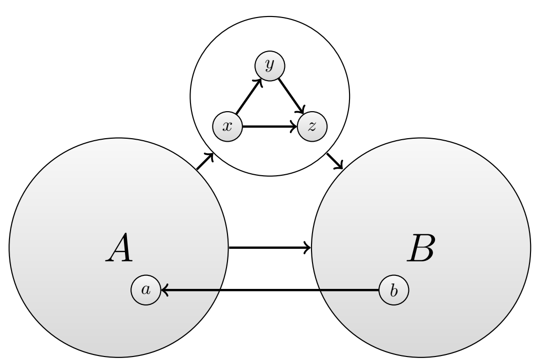

Figure 2: The tournament T in Proposition 17.

Proposition 17. *For every positive integer k, there exist strong tournaments with minimum in- and out-degrees at least k containing an arc which is not in any spanning eulerian subdigraph.*

Proof. Let T (see Figure 2) be a tournament with vertex set $A \cup B \cup \{x, y, z\}$ such that $A \rightarrow \{x, y, z\}$, $\{x, y, z\} \rightarrow B$, $x \rightarrow \{y, z\}$, $y \rightarrow z$, there exists a vertex $a \in A$ and a vertex $b \in B$ such that T contains all arcs from A to B except ab (and so $b \rightarrow a$), and $T\langle A \rangle$ and $T\langle B \rangle$ are strong tournaments with minimum in- and out-degrees at least k. Clearly T is strong and has minimum in- and out-degrees at least k.

Let us now prove that every eulerian subdigraph containing the arc xz does not contain y and is therefore not spanning. Let D be an eulerian subdigraph of Tcontaining xz. Set $S = A \cup \{x\}$. In D, there are as many arcs leaving S (i.e. from S to $V(T) \setminus S$) as arcs entering S (i.e. from $V(T) \setminus S$ to S). Now xz is arc leaving S in D, and ba is the only arc entering S in T. Thus, $ba \in A(D)$ and xz is the unique arc leaving S in D. Therefore y has no in-neighbour in D because all its in-neighbours are in S. So D does not contain y. □

In the remaining of the section, we prove Theorem 11, which we recall.

Theorem 11. *Let D be a strong semicomplete digraph, and let x and y be two vertices of D. If there are two arc-disjoint (x, y)-paths in D, then there is a spanning (x, y)-trail in $D \setminus \{yx\}$.*

Let us start with some useful preliminaries.

A vertex v of a digraph D is an **out-generator** (resp. **in-generator**) if v can reach (resp. be reached by) all other vertices by paths.

The following lemma is easy and well-known.

Lemma 18. *Let D be a non-strong semicomplete digraph. For every out-generator x of D and in-generator y of D, there is a hamiltonian (x, y)-path in D.*

Lemma 18 and Camion's Theorem immediately imply the following.

Corollary 19. *In a semicomplete digraph, every out-generator is the initial vertex of a hamiltonian path.*

We shall now prove a lemma which is a strengthening of Camion's Theorem.

Lemma 20. *Let D be a semicomplete digraph, F a subdigraph of D, and z a vertex in $V(F)$. If $D \setminus A(F)$ is strong, then there is a cycle containing all vertices of $V(D) \setminus V(F)$ and z.*

Proof. Let $D' = D\langle (V(D) \setminus V(F)) \cup \{z\}\rangle$. If D' is strong, then by Camion's Theorem, it has a hamiltonian cycle, which has the desired property.

If D' is not strong, then let X be its set of out-generators and let Y be its set of in-generators. Since $D \setminus A(F)$ is strong, there is a (Y, X)-path P in D. Set $D'' = D' - P(s(P), t(P))$. Clearly, $t(P)$ is an out-generator of D'' and $s(P)$ is an in-generator of D''. Hence, by Lemma 18, D'' has a hamiltonian path Q from $t(P)$ to $s(P)$. The union of P and Q is the desired cycle. □

Proof of Theorem 11. We proceed by induction on the number of vertices, the result holding trivially when $|V(D)| = 3$.

By the assumption there are two arc-disjoint (x, y)-paths P_1, P_2. Let y'_i be the out-neighbour of x in P_i and let x'_i be the in-neighbour of y in P_i. We assume that $P_1 \cup P_2$ has as few arcs as possible and under this assumption that P_1 is as short as possible. In particular, x'_2 and y'_2 are not in $V(P_1)$ and all internal vertices of P_1 except y'_1 dominate x, and all internal vertices of P_1 except x'_1 are dominated by y.

Assume first that $x \rightarrow y$. By our choice of P_1 and P_2, we have $P_1 = xy$. The digraph $D \setminus A(P_1)$ is $D \setminus \{xy\}$ and contains P_2. Hence it is strong, so by Lemma 20, $D \setminus A(P_1)$ contains a cycle C covering all vertices of $V(D) \setminus \{y\}$. The union of C and P_1 is a spanning (x, y)-trail in $D \setminus \{yx\}$.

Assume now that $xy \notin A(D)$. Then $y \rightarrow x$ and P_1 has length at least 2. Let w_1 be the in-neighbour of x'_1 on P_1. Set $D' = D \setminus \{yx\}$.

Assume first that D' is not strong. Since D is strong, by Camion's Theorem, it contains a hamiltonian cycle C. Now C must contain the arc yx, and $C \setminus \{yx\}$ is a hamiltonian (x, y)-path, and so a spanning (x, y)-trail in D'. Henceforth, we assume that D' is strong.

If $D' \setminus A(P_1)$ is strong, then, by Lemma 20, $D' \setminus A(P_1)$ contains a cycle C covering all vertices of $V(D) \setminus V(P_1)$ and a vertex of $V(P_1)$. The union of C and P_1 is a spanning (x, y)-trail in D'. Henceforth we may assume that $D' \setminus A(P_1)$ is not strong. Let (X, Y) be a partition of $V(D)$ such that there is no arc from Y to X in $D' \setminus A(P_1)$ and Y is minimal with respect to inclusion. Then it is easy to see that $D\langle Y\rangle$ is strong. Since D' is strong, there must be an arc of P_1 with tail in Y and head in X. Observe that because P_2 is a path in $D' \setminus A(P_1)$, we cannot have $x \in Y$ and $y \in X$.

Assume for a contradiction that $x \in X$ and $y \in Y$. The vertex x'_1 is the unique vertex of P_1 in X because all other internal vertices of P_1 are dominated by y. Similarly, vertex y'_1 is the unique vertex of P_1 in Y because all others internal vertices of P_1 dominate x. So $P_1 = xy'_1x'_1y$. Consider now P_2 and recall that $x'_2, y'_2 \notin V(P_1)$ and $|V(P_2)| \geq |V(P_1)| = 4$. The vertex y'_2 is dominated by y, so it must be in Y. Similarly, x'_2 dominates x, so it must be in X. But then an arc of $A(P_2)$ must have tail in Y and head in X, a contradiction.

Assume that $x, y \in Y$. The vertex x'_1 is the unique vertex of P_1 in X because all other internal vertices of P_1 are dominated by y. Furthermore $w_1x'_1$ is the unique arc of D from Y to X. Moreover, since D is strong, x'_1 must be an out-generator of $D\langle X\rangle$. Thus, by Corollary 19, there is a hamiltonian path Q_X of $D\langle X\rangle$ with initial vertex x'_1. The terminal vertex of Q_X dominates $Y \setminus \{w_1\}$. Let $D'' = D\langle Y\rangle \cup \{w_1y\}$. This digraph is strong. Observe moreover that w_1y was not in $A(D)$ by our choice of P_1. Therefore $P_1[x, w_1] \cup w_1y$ and P_2 are two arc-disjoint (x, y)-paths in D''. By the induction hypothesis, there is a spanning (x, y)-trail W in D''. Let u be an out-neighbour of w_1 in W. Replacing the arc w_1u by $wx'_1 \cup Q_X \cup t(Q_X)u$, we obtain a spanning (x, y)-trail in D.

By symmetry, we get the result if $x, y \in X$. □

Remark 21.

- Note that in the spanning (x, y)-trail given by the above proof, every vertex has out-degree at most 2.
- The proof of Theorem 11 can easily be translated into a polynomial-time algorithm.

5 Arcs contained in no spanning eulerian subdigraph

The aim of this section is to prove a characterization of the arcs of a semicomplete digraph D that are not contained in any spanning eulerian subdigraph of D. Observe that if the semicomplete digraph is not strong, then there are only such arcs, and if the semicomplete digraph is 2-strong there are no such arcs by Theorem 4.

We first deal with digraphs of order at most 3, before settling the case of digraphs of order at least 4, for which we use structural properties established in Subsection 3.

Let D_3 be the digraph with vertex set $\{x, y, z\}$ and arc set $\{xy, yz, zy, zx\}$. The following easy proposition is left to the reader.

Proposition 22. *Let D be a strong semicomplete digraph D of order at most 3 and let a be an arc of D. The arc a is contained in a spanning eulerian subdigraph unless $D = D_3$ and $a = zy$.*

Let D be a strong semicomplete digraph of order at least 4, $(S_1, \dots, S_p)$ a nice decomposition of D, and $(s_1t_1, s_2t_2, \dots, s_rt_r)$ the natural ordering of the backward arcs. A set S_i is **ignored** if there exists j such that $\mathrm{ind}(s_{j+1}) < i < \mathrm{ind}(t_{j-1})$ or $1 < i < \mathrm{ind}(t_{r-1})$ or $\mathrm{ind}(s_2) < i < p$. An arc uv of D is **regular-bad** if it is forward and there is an integer i such that $\mathrm{ind}(u) < i < \mathrm{ind}(v)$ and S_i is ignored (see Figure 3.) The arc uv is **left-bad** if $S_2 = \{u\}$, $S_1 = \{t_r\}$, $t_r \neq v$, and $t_ru \notin A(D)$. The arc uv is **right-bad** if $S_{p-1} = \{v\}$, $S_p = \{s_1\}$, $s_1 \neq v$, and $vs_1 \notin A(D)$. An arc is **bad** if it is regular-bad, right-bad or left-bad. A non-bad arc is **good**.

Theorem 23. *Let D be a strong semicomplete digraph of order at least 4, $(S_1, \dots, S_p)$ a nice decomposition of D and $(s_1t_1, s_2t_2, \dots, s_rt_r)$ the natural ordering of the backward arcs. An arc is contained in a spanning eulerian subdigraph of D if and only if it is good.*

Proof. Recall that an arc uv is contained in a spanning eulerian subdigraph of D if and only if there is a spanning (v, u)-trail in $D \setminus \{uv\}$.

Let us first prove that a bad arc is not contained in any spanning eulerian subdigraph.

Assume first that uv is a regular-bad arc. Let i_0 be an integer such that $\mathrm{ind}(u) < i_0 < \mathrm{ind}(v)$ and S_{i_0} is ignored. Let j be the integer such that $\mathrm{ind}(s_{j+1}) < i_0 < \mathrm{ind}(t_{j-1})$, or $j = r$ if $1 < i_0 < \mathrm{ind}(t_{r-1})$,

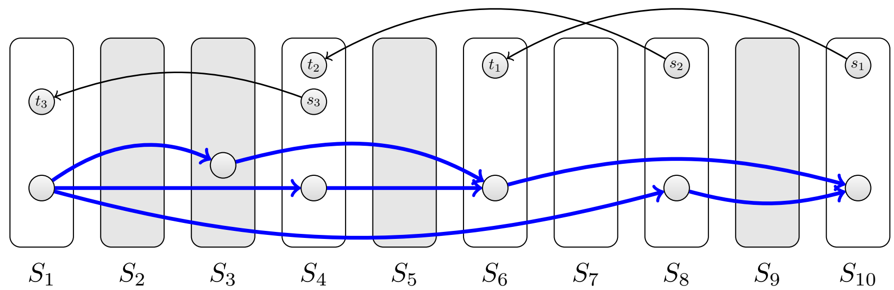

Figure 3: A nice decomposition of a strong semicomplete digraph with three backwards arcs (in thin black). The grey sets (S_2, S_3, S_5, S_9) are ignored. The thick blue arcs are regular-bad.

or $j = 1$ if $\mathrm{ind}(s_2) < i_0 < p$. Set $L = \bigcup_{i=1}^{\mathrm{ind}(s_{j+1})} S_i$ if $j \neq r$ and $L = S_1$ if $j = r$, set $R = \bigcup_{i=\mathrm{ind}(t_{j-1})}^{p} S_i$ if $j \neq 1$ and $R = S_p$ if $j = 1$, and set $M = V(D) \setminus (L \cup R)$. Observe that $M \neq \emptyset$, because $S_{i_0} \subseteq M$. Moreover, by defintion $u \in L$ and $v \in R$. Consider a (v, u)-trail W in D. It must start in R, as $v \in R$, and then use $s_j t_j$, which is the unique arc from R to $L \cup M$. But then W cannot return to $R \cup M$ after using $s_j t_j$, as $u \in L$ and $s_j t_j$ is the unique arc from $R \cup M$ to L. Hence W is not spanning, because it contains no vertex of M. Therefore there is no spanning (v, u)-trail in $D \setminus \{uv\}$.

Assume now that uv is a left-bad arc. Since D is semicomplete, $ut_r \in A(D)$. By Proposition 14 (i), u is the unique in-neighbour of t_r, and u has in-degree 1 in D. Thus any spanning eulerian subdigraph E contains ut_r. Moreover u has in- and out-degree 1 in E and so E does not contain uv. Similarly, if uv is right-bad, we get that there is no spanning eulerian subdigraph containing uv in D.

We shall now prove by induction on $|D|$ that a good arc uv is contained in a spanning eulerian subdigraph. This is equivalent to proving the existence of a spanning (v, u)-trail in $D \setminus \{uv\}$. If $|D| = 4$, the statement can be easily checked. Therefore, we now assume that $|D| > 4$.

For each $1 \leq j < r$, let N_j be a (t_j, s_{j+1})-path in $D\langle S_{t_j}\rangle$ if $\mathrm{ind}(t_j) = \mathrm{ind}(s_{j+1})$ and let $N_j = (t_j, s_{j+1})$ otherwise (that is if $\mathrm{ind}(t_j) < \mathrm{ind}(s_{j+1})$). Let $N = (s_1, t_1) \cup N_1 \cup (s_2, t_2) \cdots \cup N_{r-1} \cup (s_r, t_r)$. Note that N is an (s_1, t_r)-path containing all backward arcs.

We first consider the backward arcs. Let P_1 be a hamiltonian path of $D\langle S_1\rangle$ with initial vertex t_r and let x be its terminal vertex. Let P_p be a hamiltonian path of $D\langle S_p\rangle$ with terminal vertex s_1 and let y be its initial vertex. Then $Q_1 = P_p \cup N \cup P_1$ is a (y, x)-path. Observe that in the semicomplete digraph $D - V(Q_1(y, x))$, x has in-degree zero and y has out-degree zero. Hence, by Lemma 18, there is a hamiltonian (x, y)-path Q_2 in $D - V(Q_1(y, x))$. Thus $Q_1 \cup Q_2$ is a hamiltonian cycle containing all backward arcs.

Assume now that uv is a flat arc. In D, there are two arc-disjoint (v, u)-paths. Indeed, suppose not. By Menger's Theorem, there would be a cut-arc separating v from u. But this cut-arc must be in $D\langle S_u\rangle = D\langle S_v\rangle$, which is strong, contradicting that we have a nice decomposition. Therefore, by Theorem 11, there is a spanning (v, u)-trail in $D \setminus \{uv\}$.

Assume finally that uv is a good forward arc.

Claim 23.1. *If* $\mathrm{ind}(u) \geq 3$ *or* $\mathrm{ind}(u) = 2$ *and* $|S_1| > 1$, *then* D *has a spanning eulerian subdigraph containing* uv.

Proof. Let $L = \{x \mid \mathrm{ind}(x) < \mathrm{ind}(u)\}$, and $R = \{x \mid \mathrm{ind}(x) \geq \mathrm{ind}(u)\}$, and let j be the integer such that $s_j \in R$ and $t_j \in L$. Let D_L be the digraph obtained from $D\langle L\rangle$ by adding a vertex z_L and all arcs from L to z_L and $z_L t_j$. Let D_R be the digraph obtained from $D\langle R\rangle$ by adding a vertex z_R and all arcs from z_R to R and $s_j z_R$. Observe that D_L and D_R are strong. Moreover, $(\{z_R\}, S_{\mathrm{ind}(u)}, \ldots, S_p)$ is a nice decomposition of D_R. Thus uv is neither regular-bad nor a right-bad in D_R for otherwise it would already be regular-bad or right-bad in D, and it is not left-bad in D_R because z_R dominates u in this digraph.

Since $\text{ind}(u) \geq 3$ or $\text{ind}(u) = 2$ and $|S_1| > 1$, then D_R is smaller than D. Observe moreover that if D_R is isomorphic to D_3, then the arc uv is in the spanning eulerian subdigraph uvz_Ru. Therefore, by the induction hypothesis, or this observation, in D_R there is a spanning eulerian subdigraph E_R containing uv. Since z_R has in-degree 1 in D_R, E_R contains the arc s_jz_R and an arc z_Ry_R for some $y_R \in R$. By Camion's Theorem, there is a hamiltonian cycle C_L of D_L. It necessarily contains the arc z_Lt_j because z_L has out-degree 1 in D_L. Let y_L be the in-neighbour of z_L in C_L. Observe that $y_L \neq t_j$, because $|V(D_L)| \geq 3$. Thus $y_L \rightarrow y_R$, and the union of $C_L - z_L$, y_Ly_R, $E_R - z_R$ and s_jt_j is a spanning eulerian subdigraph of D containing uv. $\Diamond$

By Claim 23.1, we may assume that $\text{ind}(u) = 1$ or $\text{ind}(u) = 2$ and $|S_1| = 1$ (that is $S_1 = \{t_r\}$). Similarly, we can assume $\text{ind}(v) = p$ or $\text{ind}(v) = p-1$ and $|S_p| = 1$ (that is $S_p = \{s_1\}$).

Claim 23.2. *If* $\text{ind}(u) = 2$ *and* $|S_1| = 1$*, then* D *has a spanning eulerian subdigraph containing* uv.

Proof. Assume first that $r = 1$ or $\text{ind}(t_{r-1}) > 2$. Let D_1 be the strong semicomplete digraph obtained from D by removing t_r and adding the arc s_ru. Then $|D_1| = |D| - 1 \geq 4$ and $(S_2, \dots, S_p)$ is a nice decomposition of D_1. Consequently, uv is not bad and so, by the induction hypothesis, there is a spanning eulerian subdigraph W_1 of D_1 containing uv. Necessarily, W_1 contains s_ru which is a cut-arc in D_1. Hence $(W_1 \setminus \{s_ru\}) \cup s_rt_ru$ is a spanning eulerian subdigraph of D containing uv.

Assume now that $r \geq 2$ and $\text{ind}(t_{r-1}) = 2$. Consider $D_2 = D - t_r$. As above, one shows that D_2 has a hamiltonian cycle (containing all backward arcs) so D_2 is strong, and $(S_2, \dots, S_p)$ is a nice decomposition of D_2 in which uv is good in D_2 (for otherwise it would not be good for $(S_1, \dots, S_p)$). Therefore, by the induction hypothesis, there is a spanning eulerian subdigraph W_2 of D_2 containing uv. If s_r is the tail of an arc $s_rw \in A(W_2) \setminus \{uv\}$, then $(W_2 \setminus \{s_rw\}) \cup s_rt_rw$ is a spanning eulerian subdigraph of D containing uv. If not, then $s_r = u$ and v is the only out-neighbour of u on W_2. Thus u has a unique in-neighbour z in W_2. Since uv is not left-bad, we have $d_D^-(u) \geq 2$. Thus u has an in-neighbour y distinct from z. If $y = t_r$ then $W_2 \cup ut_ru$ is a spanning eulerian subdigraph of D containing uv, and if $y \neq t_r$ then $W_2 \cup ut_ryu$ is a spanning eulerian subdigraph of D containing uv (Note that $t_ry \in A(D)$ because by Proposition 14 (i), D cannot contain the arc yt_r). $\Diamond$

By Claims 23.1 and 23.2, we may assume that $\text{ind}(u) = 1$ and $\text{ind}(v) = p$.

For every $1 \leq i \leq p$, let C_i be a hamiltonian cycle of $D\langle S_i \rangle$.

Set $t_0 = v$ and $s_{r+1} = u$. For $0 \leq j \leq r$, let $X_j = \{x \mid \text{ind}(t_j) \leq \text{ind}(x) \leq \text{ind}(s_{j+1})\}$. Since each backward arc is a cut-arc the X_j are disjoint. Moreover, as uv is good, there is no ignored set, so every S_j is in some X_j. Hence the X_j, $0 \leq j \leq r$, form a partition of $V(D)$.

Claim 23.3. *For every* $0 \leq j \leq r$*, there is a spanning* (t_j, s_{j+1})*-trail* T_j *in* $D\langle X_j \rangle$.

Proof. Set $i_1 = \text{ind}(t_j)$ and $i_2 = \text{ind}(s_{j+1})$.

If $i_1 < i_2$, then pick a vertex x_i in each set S_i for $i_1 < i < i_2$. Then $t_jx_{i_1+1} \cdots x_{i_2-1}s_{j+1} \cup \bigcup_{i=i_1}^{i_2} C_i$ is a spanning (t_j, s_{j+1})-trail in $D\langle X_j \rangle$.

Assume now that $i_1 = i_2$. There must be two arc-disjoint (t_j, s_{j+1})-paths in $D\langle X_j \rangle$, for otherwise, by Menger's Theorem, there is a partition (T, S) of S_{i_1} with $t_j \in T$, $s_{j+1} \in S$ such that there is a unique arc a with tail in T and head in S. But then a would also be a cut-arc of D, which is impossible because it is a flat arc. Now, by Theorem 11, there is a spanning (t_j, s_{j+1})-trail in $D\langle X_j \rangle = S_{i-1}$. $\Diamond$

Now $\bigcup_{i=0}^{r} T_j \cup \{s_jt_j \mid 1 \leq j \leq r\} \cup \{uv\}$ is a spanning eulerian subdigraph of D containing uv. $\square$

6 Eulerian spanning subdigraphs avoiding prescribed arcs

In this section, we give some support for Conjecture 7. First, in Subsection 6.2, we prove the existence of a minimum function $f(k)$ such that every $f(k)$-arc-strong semicomplete digraph contains a spanning eulerian subdigraph avoiding any prescribed set of k arcs. Conjecture 7 states that $f(k) = k+1$. In Subsections 6.3 and 6.4, we shall verify Conjecture 7 for the cases $k = 1$, $k = 2$ and $k = 3$. The case $k \leq 2$ is obtained in Corollary 31 and the case $k = 3$ is obtained in Theorem 32. Note that, after this paper was written, it was shown in [2] that $f(k) \leq \lceil \frac{6k+1}{5} \rceil$. This proves Conjecture 7 for $k \leq 4$.

However, the proof of the bound on $f(k)$ is long so we have decided to keep our proof of the cases $k = 1, 2, 3$.

We need a number of preliminary results.

6.1 Preliminaries

In this subsection we establish some results for general digraphs that are of independent interest and will be useful in our proofs in the next subsections.

6.1.1 Eulerian factors in semicomplete digraphs

A digraph is **semicomplete multipartite** if it can be obtained from a complete multipartite graph $G = (V, E)$ by replacing each edge $uv \in E$ by either a 2-cycle on u, v or one of the two arcs uv, vu. An **eulerian factor** of a digraph $D = (V, A)$ is a spanning subdigraph $H = (V, A')$ so that $d^+_H(v) = d^-_H(v) > 0$ for all $v \in V$. We need the following theorem.

Theorem 24 (Bang-Jensen and Maddaloni [1])**.** *A strong semicomplete multipartite digraph has a spanning eulerian subdigraph if and only if it is strong and has an eulerian factor. Furthermore, there exists a polynomial-time algorithm for finding a spanning eulerian subdigraph in a strong semicomplete multipartite digraph D or concluding that D has no eulerian factor.*

An **independent set** in a digraph D is a set of pairwise non-adjacent vertices. By a **component** of the eulerian factor H we mean a connected component of the digraph H. Let $d(X, Y)$ denote the number of arcs from X to Y.

Theorem 25. *A digraph D has no eulerian factor if and only if $V(D)$ can be partitioned into R_1, R_2 and Y so that the following hold.*

- *Y is independent.*
- *$d(R_2, Y) = 0$, $d(Y, R_1) = 0$ and $d(R_2, R_1) < |Y|$.*

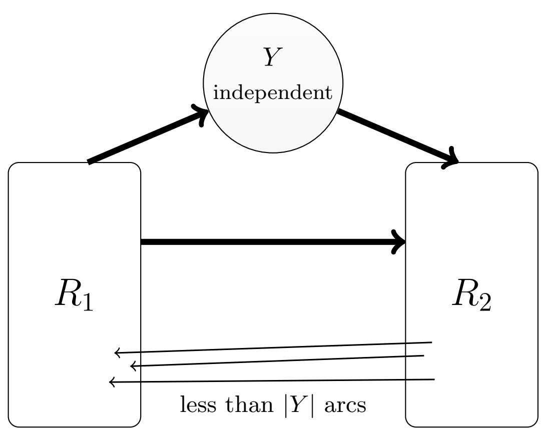

Figure 4: An illustration of Theorem 25. There are no arcs from R_2 to Y and no arcs from Y to R_1 and less than $|Y|$ arcs from R_2 to R_1.

Proof. Let $D = (V, A)$ be any digraph and let B be the bipartite digraph obtained from D by splitting every vertex v into an in-going part v^- and an out-going part v^+. Formally, $V(B) = \bigcup_{v \in V(D)} \{v^-, v^+\}$ and $A(B) = \{v^-v^+ \mid v \in V(D)\} \cup \{x^+y^- \mid xy \in A(D)\}$.

Consider the flow network $\mathcal{N} = (B, l, u)$ with l, u, being lower and upper bounds on arcs, respectively, such that

$$l(v^-v^+) = 1, \quad u(v^-v^+) = +\infty \quad l(x^+y^-) = 0, \quad u(x^+y^-) = 1$$

for every $v \in V(D)$, $xy \in A(D)$.

It is easy to check that there is a one-to-one correspondence between feasible integer-valued circulations on $\mathcal{N}$ and eulerian factors of D.

By Hoffman's circulation theorem [11] (see also Theorem 4.8.2 in [3]), there exists a feasible integer circulation of $\mathcal{N}$ if (and only if)

$$u(\bar{S}, S) \geq l(S, \bar{S}) \tag{1}$$

for every $S \subseteq V(B)$.

First assume that D has no eulerian factor, which implies that $u(\bar{S}, S) < l(S, \bar{S})$ for some $S \subseteq V(B)$. Consider the following possiblities for every $x \in V(D)$ and construct Y', R_1' and R_2' as illustrated below.

- If $x^- \in S$ and $x^+ \in \bar{S}$, then the arc x^-x^+ adds 1 to $l(S, \bar{S})$, as $l(x^-x^+) = 1$. Add x to Y'.
- If $x^+ \in S$ and $x^- \in \bar{S}$, then $u(x^-x^+) = +\infty$ which contradicts $u(\bar{S}, S) < l(S, \bar{S})$. So this case cannot happen.
- If $x^- \in S$ and $x^+ \in S$, then add x to R_1'.
- If $x^- \in \bar{S}$ and $x^+ \in \bar{S}$, then add x to R_2'.

Note that R_1', R_2' and Y' is a partition of $V(D)$. We note that $l(S, \bar{S}) = |Y'|$, as the lower bound on all arcs except the x^-x^+, $x \in V(D)$, is 0. We now prove that the following holds.

$$u(\bar{S}, S) = d(R_2', R_1') + d(R_2', Y') + d(Y', R_1') + d(Y', Y') \tag{2}$$

If xy is an arc from R_2' to R_1' then we note that $u(x^+y^-) = 1$ and therefore the arc xy contributes 1 to $u(\bar{S}, S)$. Analogously, if xy is an arc from R_2' to Y' or an arc from Y' to R_1', then it also contributes 1 to $u(\bar{S}, S)$. If y_1y_2 is an arc within Y, then $u(y_1^+y_2^-) = 1$ and therefore the arc y_1y_2 also contributes 1 to $u(\bar{S}, S)$. As all arcs x^+y^- in the cut from $\bar{S}$ to S have been counted, this proves Eq. (2).

Now as $u(\bar{S}, S) < l(S, \bar{S}) = |Y'|$, we have

$$d(R_2', R_1') + d(R_2', Y') + d(Y', R_1') + d(Y', Y') < |Y'| \tag{3}$$

Assume that Y' has minimum size such that Eq. (3) holds. We will first show that $d(R_2', y) = 0$ and $d(Y', y) = 0$ for all $y \in Y'$. Assume for the sake of contradiction that this is not the case and let let $Y^* = Y' \setminus \{y\}$ and let $R_2^* = R_2' \cup \{y\}$ and let $R_1^* = R_1'$. Then $|Y^*| = |Y'| - 1$ and the following holds.

- $d(R_2^*, R_1^*) = d(R_2', R_1') + d(y, R_1')$.
- $d(R_2^*, Y^*) = d(R_2', Y') + d(y, Y') - d(R_2', y)$.
- $d(Y^*, R_1^*) = d(Y', R_1') - d(y, R_1')$.
- $d(Y^*, Y^*) = d(Y', Y') - d(Y', y) - d(y, Y')$.

Summing up the four above equations we obtain the following (as we assumed that $d(R_2', y) \neq 0$ or $d(Y', y) \neq 0$).

$$\begin{aligned}
&d(R_2^*, R_1^*) + d(R_2^*, Y^*) + d(Y^*, R_1^*) + d(Y^*, Y^*) \\
&\qquad = d(R_2', R_1') + d(R_2', Y') + d(Y', R_1') + d(Y', Y') - d(R_2', y) - d(Y', y) \\
&\qquad \leq d(R_2', R_1') + d(R_2', Y') + d(Y', R_1') + d(Y', Y') - 1 \\
&\qquad < |Y'| - 1 \\
&\qquad = |Y^*|
\end{aligned}$$

So we note that the partition (Y^*, R_1^*, R_2^*) is a contradiction to the minimality of Y' and we must have $d(R_2', y) = 0$ and $d(Y', y) = 0$ for all $y \in Y'$. Therefore $d(R_2', Y') = 0$ and $d(Y', Y') = 0$. Analogously if $d(y, R_1') \neq 0$ for some $y \in Y'$, then we can let $R_1'' = R_1' \cup \{y\}$, $Y'' = Y' \setminus \{y\}$ and $R_2'' = R_2'$ and obtain the following (as $d(Y', Y') = 0$).

$$
\begin{aligned}
d(R_2'', R_1'') &+ d(R_2'', Y'') + d(Y'', R_1'') + d(Y'', Y'') \\
&= d(R_2', R_1') + d(R_2', Y') + d(Y', R_1') + d(Y', Y') - d(y, R') \\
&\leq d(R_2', R_1') + d(R_2', Y') + d(Y', R_1') + d(Y', Y') - 1 \\
&< |Y'| - 1 \\
&= |Y''|
\end{aligned}
$$

Therefore $d(Y', R_1') = 0$, which implies that $d(R_2', R_1') < |Y'|$ and $d(Y', R_1') = d(R_2', Y') = d(Y', Y') = 0$. Therefore we have obtained the desired partition of $V(D)$.

This proved one direction of the theorem. Now assume that we can partition the vertices of $V(D)$ into R_1, R_2 and Y such that Y is independent and $d(R_2, Y) = 0$, $d(Y, R_1) = 0$ and $d(R_2, R_1) < |Y|$. In this case we note that to get from one vertex of Y to another vertex of Y (or the same vertex of Y with a path of length at least 1) we need to use at least one arc from R_2 to R_1. However, as $d(R_2, R_1) < |Y|$, this implies that D cannot contain an eulerian factor (which would contain at least $|Y|$ arc-disjoint paths between vertices in Y). □

Lemma 26. *Let k be a non-negative integer and D be a $(k+1)$-arc-strong semicomplete digraph. Then D has an eulerian factor avoiding any prescribed set of k arcs.*

Proof. Let A' be any set of k arcs in a $(k+1)$-arc-strong semicomplete digraph D. Let $D' = D \setminus A'$ and note that D' is strong. For the sake of contradiction, assume that D' can be partitioned into R_1, R_2 and Y such that Y is independent and $d(R_2, Y) = 0$ and $d(Y, R_1) = 0$ and $d(R_2, R_1) < |Y|$. As D' is strong, we must have $R_1 \neq \emptyset$ and $R_2 \neq \emptyset$ and $d(R_2, R_1) \geq 1$. Therefore $|Y| \geq 2$. Note that at least $\binom{|Y|}{2} = |Y|(|Y|-1)/2$ arcs from A' lie completely within Y (as Y is independent in D'). Furthermore at least $k + 1 - (|Y| - 1)$ arcs from A' go from $R_2 \cup Y$ to R_1 as R_1 has at least $k+1$ arcs into it in D (and R_2 has at least $k+1$ arcs out of it in D), as in D' we have $d(R_2, R_1) \leq |Y| - 1$. So the following holds.

$$
|A'| \geq \frac{|Y|(|Y|-1)}{2} + k - |Y| + 2 = k + 2 + |Y|\left(\frac{|Y|-3}{2}\right)
$$

The above implies that $|A'| \geq k+1$ (which can easily be verified when $|Y| = 2$ and $|Y| \geq 3$), a contradiction. Therefore the partition (Y, R_1, R_2) does not exist and D' has an eulerian factor by Theorem 25. □

6.1.2 Merging eulerian subdigraphs

Let $D = (V, A)$ be a digraph and D' an eulerian subdigraph of D which is not spanning. A vertex $x \in V \setminus V(D')$ is **universal to** D' (or just **universal** when D' is clear from the context) if x is adjacent to every vertex of D' and it is **hypouniversal to** D' if it is adjacent to all vertices of D' but at most one. If x has an arc to D' and an arc from D' then we say that x is **mixed to** D'.

Let H be an eulerian factor of a digraph D and let H_1 and H_2 be two distinct components of H. Each H_i has a eulerian tour and, with a slight abuse of notation, for every vertex x of H_i we denote by x^+ (resp. x^-) the successor (resp. predecessor) of x in this eulerian tour. This must be understood as with respect to some fixed occurence of x in the tour.

If there exists a spanning eulerian subdigraph, H^* of $D\langle V(H_1) \cup V(H_2)\rangle$, then we say that H_1 and H_2 can be **merged**, as in H we can substitute H_1 and H_2 by H^* in order to get a eulerian factor of D with fewer components.

Lemma 27. *Let H_1 and H_2 be two components in an eulerian factor of a digraph D that cannot be merged. Then all of the following points hold for all $i \in \{1, 2\}$ and $j = 3 - i$.*

(a): *There is no 2-cycle, uvu, where $u \in H_1$ and $v \in H_2$.*

(b): *For every arc $uv \in A(H_i)$ and every $x \in V(H_j)$ we cannot have $ux, xv \in A(D)$.*

(c): *For every arc $uv \in A(H_i)$ and every arc $xy \in A(H_j)$ we cannot have $uy, xv \in A(D)$.*

(d): *If $x \in V(H_i)$ is universal to H_j, then x is not mixed to H_j.*

That is, $x \mapsto V(H_j)$ or $V(H_j) \mapsto x$.

(e): *If $x \in V(H_i)$ is hypouniversal and mixed to H_j, then there exists a unique $y \in V(H_j)$ such that x and y are not adjacent and $y^- x, xy^+ \in A(D)$.*

Proof. Let D, H_1, H_2 and i, j be defined as in the statement of the lemma. If there was a 2-cycle, uvu, where $u \in H_1$ and $v \in H_2$, then adding this to H_1 and H_2 shows that H_1 and H_2 can be merged, a contradiction. This proves (a).

For the sake of contradiction assume that $uv \in A(H_i)$ and $x \in V(H_j)$ and $ux, xv \in A(D)$. Adding the arcs ux and xv and removing the arc uv from $H_1 \cup H_2$ shows that H_1 and H_2 can be merged, a contradiction. This proves (b).

For the sake of contradiction assume that $uv \in A(H_i)$ and $xy \in A(H_j)$ and $uy, xv \in A(D)$. Adding the arcs uy and xv and removing the arcs uv and xy from $H_1 \cup H_2$ shows that H_1 and H_2 can be merged, a contradiction. This proves (c).

Let $x \in V(H_i)$ be universal to H_j and for the sake of contradiction assume that x is mixed to H_j. Let the eulerian tour in H_2 be $w_1 w_2 w_3 \cdots w_l w_1$ (every arc of H_2 is used exactly once). Without loss of generality we may assume $w_1 x \in A(D)$ (as x is mixed to H_2). Part (b) implies that xw_2 is not an arc in D, so $w_2 x \in A(D)$ (as x is universal). Analogously $w_3 x \in A(D)$. And so on by induction, we get that every vertex of H_2 dominates x, so $V(H_2) \to x$. As there is an arc from x to H_2 in D (as x is mixed) we have a 2-cycle between H_1 and H_2, a contradiction to (a). This proves (d).

We will now prove (e). Let $x \in V(H_i)$ be hypouniversal and mixed to H_j. By (d), vertex x is not universal to H_j, so there exists a unique $y \in V(H_j)$ such that x and y are not adjacent. As x is mixed to H_j there is an arc from x to $V(H_j)$. As $xy \notin A(D)$, we can assume that $w \in V(H_j)$ is chosen such that $xw \in A(D)$ and $xw^- \notin A(D)$. By (b) we note that x and w^- are non-adjacent and therefore $w^- = y$. This implies that $xy^+ \in A(D)$. Analogously, using (b), we can prove that $y^- x \in A(D)$. □

6.2 Avoiding k arcs

Proposition 28. *Every semicomplete digraph $D = (V, A)$ with $\lambda(D) \geq \frac{(k+1)^2}{4} + 1$ has a spanning eulerian subdigraph which avoids any prescribed set of k arcs.*

Proof. Consider a set A' of k arcs and let $X_1, X_2, \ldots, X_r$, $r \leq k$, be the connected components of $D\langle A' \rangle$. Let D^* be the semicomplete multipartite digraph that we obtain by deleting all arcs of A which lie inside some component X_i. It is easy to see that we did not delete more than $\frac{(k+1)^2}{4}$ arcs across any cut of D so D^* is strong. Moreover, every independent set of D^* has size at most $k + 1$. Thus, by Theorem 25, D has an eulerian factor. The claim follows from Theorem 24. □

6.3 Avoiding a collection of stars

If D is a digraph and $A' \subset A(D)$ such that the underlying graph of the digraph induced by A' is a collection of stars, then A' is called a **star-set** in D. Note that a matching in D is also a star-set.

Lemma 29. *Let D be a semicomplete digraph and let $A' \subset A(D)$ be a star-set in D and let $D' = D \backslash A'$. If D' is strongly connected and contains an eulerian factor with two components H_1 and H_2 but no spanning eulerian subdigraph, then the following holds for some $i \in \{1, 2\}$ and $j = 3 - i$.*

(i): *The eulerian tour in H_i can be denoted by $w_1 w_2 w_3 \cdots w_l w_1$, such that w_1 is not adjacent to any vertex in H_j in D'.*

(ii): *There exists a k, such that $R_1 = \{w_2, w_3, \ldots, w_k\}$ and $R_2 = \{w_{k+1}, w_{k+2}, \ldots, w_l\}$ are both non-empty and the only arc in D' from R_1 to R_2 is $w_k w_{k+1}$.*

(iii): *There is no arc from R_1 to w_1 and there is no arc from w_1 to R_2 in D'.*

(iv): *$V(H_j) \mapsto R_1$ and $R_2 \mapsto V(H_j)$ in D'.*

Proof. Let D, A', D', H_1 and H_2 be defined as in the lemma. Assume that D' has no spanning eulerian subdigraph. We now prove the following claims.

Claim 29.1. *There must be a vertex in H_1 which is not mixed to H_2 or a vertex in H_2 which is not mixed to H_1.*

Proof. Suppose there is no such a vertex. Then D contains a cycle C whose vertices alternate between $V(H_1)$ and $V(H_2)$ so taking the union of the arcs of C and those of H_1, H_2 we obtain a spanning eulerian subdigraph of D, contradicting the assumption. This completes the proof of Claim 29.1. $\Diamond$

Definition of x: By Claim 29.1 we may assume without loss of generality that there is a vertex $x \in V(H_1)$ which is not mixed to H_2. Also without loss of generality we may assume that there is no arcs from H_2 to x. As D' is strong we can pick x such that x^+ has an arc into it from H_2 (otherwise consider x^+ instead of x).

Claim 29.2. *$V(H_2) \mapsto x^+$ and x is non-adjacent to every vertex of H_2.*

Proof. Let ux^+ be an arc from H_2 into x^+. By Lemma 27 (b) and (c), we note that $xu \notin A(D')$ and $xu^+ \notin A(D')$. As there is no arc from H_2 to x this implies that x is not adjacent to u or u^+. As A' is a star-set, x^+ and u^+ are adjacent. By Lemma 27 (c), we have $x^+u^+ \notin A(D')$ (as $ux^+ \in A(D')$), which implies that $u^+x^+ \in A(D')$. We have now shown that $ux^+ \in A(D')$ implies that $u^+x^+ \in A(D')$. Analogously we must have $u^{++}x^+ \in A(D')$ and $u^{+++}x^+ \in A(D')$. And so on by induction, we get that every vertex of H_2 dominates xx^+, that is $V(H_2) \rightarrow x^+$. By Lemma 27 (a) there are no 2-cycles between H_1 and H_2, which implies that $V(H_2) \mapsto x^+$. By Lemma 27 (b) and the fact that there is no arc from H_2 to x we get that x is non-adjacent to every vertex of H_2, completing the proof of Claim 29.2. $\Diamond$

Claim 29.3. *Every vertex in H_2 is hypouniversal to H_1. In fact, every vertex in H_2 is universal to $V(H_1) \setminus \{x\}$.*

Proof. This follows from the fact that x is not adjacent to any vertex in H_2 and therefore must be the center of a star in A' (as $|V(H_2)| \geq 2$). Therefore all vertices in H_2 are leaves in a star in A' and therefore have at most one non-neighbour in D'. By the above we note that they have exactly one non-neighbour, which is x. $\Diamond$

Definition: Let $w_1 w_2 w_3 \cdots w_l w_1$ be an eulerian tour of H_1 and let $w_1 = x$.

Claim 29.4. *The vertex x only appears once in the eulerian tour of H_1. That is, in H_1 we have $d^+(x) = d^-(x) = 1$.*

Proof. Assume for the sake of contradiction that x appears more than once in the eulerian tour of H_1. As D' is strong there is an arc from H_1 to H_2, say $w_k u$. Pick u and k such that k is as large as possible. As $w_k u \in A(D')$ we note that by Lemma 27 (b) $uw_{k+1} \notin A(D')$.

By the maximality of k this implies that $k = l$ or u and w_{k+1} are non-adjacent. As $w_{l+1} = w_1 = x$ we note that in both cases u and w_{k+1} are non-adjacent, which by Claim 29.3 implies that $w_{k+1} = x$. So $uw_{k+2} \in A(D')$ by Claim 29.2. Now deleting the arcs $w_k w_{k+1}$ and $w_{k+1} w_{k+2}$ and adding the arcs $w_k u$ and uw_{k+2} we can merge H_1 and H_2 a contradiction. $\square$

Definition of R_1 and R_2: As D' is strong there is an arc from H_1 to H_2, say $w_{k+1}u$. Pick u and k such that k is as small as possible. Note that $k \geq 2$ as by Claim 29.2 we have $V(H_2) \mapsto w_2$. Let $R_1 = \{w_2, w_3, \ldots, w_k\}$ and let $R_2 = \{w_{k+1}, w_{k+2}, \ldots, w_l\}$.

Claim 29.5. *$V(H_2) \mapsto R_1$ and $R_2 \mapsto V(H_2)$ in D'. Note that this proves part (iv) in the lemma.*

Proof.

By Claim 29.3 and the minimality of k we note that $V(H_2) \mapsto R_1$ holds.

As $w_{k+1}u \in A(D')$, by Lemma 27 (b) (and Claim 29.3), we have $w_{k+2}u \in A(D')$ or $w_{k+2} = x$. And so on by induction, one get that every vertex of R_2 dominates u, that is $R_2 \rightarrow u$. By Claim 29.3, u^- is universal to R_2, so by Lemma 27 (b), we have $R_2 \mapsto u^-$. Analogously $R_2 \mapsto u^{--}$. And so on by induction, we get $R_2 \mapsto z$ for every $z \in V(H_2)$. Hence $R_2 \mapsto V(H_2)$. $\Diamond$

Claim 29.6. *$R_1 \cap R_2 = \emptyset$ and the only arc from R_1 to R_2 in D' is $w_k w_{k+1}$.*

Note that this proves part (ii) in the lemma.

Proof. If $y \in R_1 \cap R_2$, then by Claim 29.5 we have $V(H_2) \mapsto y$ and $y \mapsto V(H_2)$, which is not possible since D' has no 2-cycle by Lemma 27 (a). Therefore $R_1 \cap R_2 = \emptyset$.

Now assume for the sake of contradiction that $uv \in A(D')$ is an arc from R_1 to R_2 different from $w_k w_{k+1}$. Note that $uv \notin A(H_1)$ as $R_1 \cap R_2 = \emptyset$ and all arcs in H_1 either lie within R_1 or within R_2 or are incident with w_1 or is the arc $w_k w_{k+1}$. Now let $q \in V(H_2)$ be arbitrary and add the arcs uv, vq, qu to H_1 and H_2 and note that this merges H_1 and H_2, a contradiction. ♢

Claim 29.7. *There is no arc from R_1 to w_1 and there is no arc from w_1 to R_2 in D'.*

Note that this proves part (iii) in the lemma.

Proof. For the sake of contradiction assume that uw_1 is an arc from R_1 to w_1. Let $v \in V(H_2)$ be arbitrary and by Claim 29.5 note that $vu \in A(D')$. We can now merge H_1 and H_2 by taking the union of the tour $vuw_1w_2w_3 \cdots w_l v$ ($w_l v \in A(D')$ by Claim 29.5) and H_2. This contradiction, implies that there is no arc from R_1 to w_1 in D'.

Analogously we can prove that there is no arc from w_1 to R_2 in D'. ♢

The above claims complete the proof of the lemma, as Claim 29.2 implies that part (i) of the lemma holds and parts (ii), (iii) and (iv) follow from the Claims 29.5, 29.6 and 29.7. □

Theorem 30. *Let D be a $(k+1)$-arc-strong semicomplete digraph and let $A' \subset A(D)$ be a star-set of size k. Then D has a spanning eulerian subdigraph which avoids the arcs in A'.*

Proof. Let $D' = D \setminus A'$ and note that D' is strong. By Lemma 26, D' contains an eulerian factor. Let H be an eulerian factor of D' with the minimum number of components. Let $H_1, H_2, \ldots, H_p$ be the components of H, and for every $i \in [p]$ set $D_i = D\langle V(H_i)\rangle$. For the sake of contradiction, assume that $p > 1$.

Let T be the digraph we obtain from D' by contracting each $V(D_i)$, $i \in [p]$, into one vertex, x_i. Since D' is strong, then T is also strong. As every D_i contains at least two vertices, T is a semicomplete digraph. Let G be the graph with $V(G) = V(T)$ and $uv \in E(G)$ if and only if uvu is a 2-cycle in T. We now need the following definitions and claims, which completes the proof of the theorem.

Definition (vital vertex). Assume that $x_i x_j$ is an edge in G, implying that $x_i x_j x_i$ is a 2-cycle in T. By the minimality of p the properties of Lemma 29 hold. By Lemma 29 (i), either there is a vertex in $V(H_i)$ which is not adjacent to any vertex of H_j, in which case we say that x_i is the vital vertex of the edge $x_i x_j$ in G, or a vertex in $V(H_j)$ which is not adjacent to any vertex of H_i, in which case we say that x_j is the vital vertex of the edge $x_i x_j$ in G. Note that x_i and x_j cannot both be vital for $x_i x_j$ as A' is a star-set. If a vertex is vital for any edge in G, then we say that it is a vital vertex in G and otherwise it is non-vital.

Claim 30.1. *G is a (possibly empty) set of vertex-disjoint stars, where the center vertices of the non-trivial (i.e. of order at least 2) stars are exactly the vital vertices of G.*

Proof. Assume that $x_i x_j \in E(G)$ and that x_i is the vital vertex of $x_i x_j$. That is, there is a $w_1 \in V(H_i)$ which is not adjacent to any vertex of H_j. We will now show that $d_G(x_j) = 1$. That is, $x_i x_j$ is the only edge in G touching x_j. Assume for the sake of contradiction that $x_k x_j$ is an edge in G with $k \neq i$. As A' is a star-set we note that x_k cannot be the vital vertex for $x_k x_j$ and x_j also cannot be the vital vertex. This implies that $d_G(x_j) = 1$.

So for every edge in G one endpoint is the vital vertex and the other endpoint has degree one. This implies that G is a vertex-disjoint collection of stars, where the center vertices of the stars are exactly the vital vertices of G, which completes the proof of Claim 30.1. ♢

Claim 30.2. *If there exists a 3-cycle $x_i x_j x_k x_i$ in T such that $x_j x_i \notin A(T)$ and $x_i x_k \notin A(T)$ and there is a vertex $u \in V(H_k)$ that is dominated by all of $V(H_j)$, except for possibly one vertex, then H_i, H_j and H_k can be merged.*

Proof. For the sake of contradiction assume w.l.o.g. that $i = 1$, $j = 2$ and $k = 3$ in the statement of the claim. That is, $x_1x_2x_3x_1$ is a 3-cycle in T and $x_2x_1 \notin A(T)$ and $x_1x_3 \notin A(T)$ and there is a vertex $u \in V(H_3)$ that is dominated by all of $V(H_2)$, except for possibly one vertex. Let $W = H_1 \cup H_2 \cup H_3$.

As A' is a star-set and there is no arc from H_1 to H_3 we note that either u has an arc out of it to H_1 or u^- has an arc out of it to H_1. Consider the two possibilities below.

- If there is an arc uv with $v \in V(H_1)$, then add uv to W.
- Otherwise there exists an arc $u^-v \in A(D')$ with $v \in V(H_1)$ and add the arc u^-v to W and delete the arc u^-u from W.

The new W now has $d^+(a) = d^-(a)$ for all $a \in V(W) \setminus \{u, v\}$ and $d^+(u) = d^-(u) + 1$ and $d^-(v) = d^+(v) + 1$. Analogously to above there is an arc from v to H_2 or from v^- to H_2. Again consider the two possibilities below.

- If there is an arc vw with $w \in V(H_2)$, then let $vw \in A(D')$ be such an arc and add vw to W.
- Otherwise there exists an arc $v^-w \in A(D')$ with $w \in V(H_2)$ and add the arc v^-w to W and delete the arc v^-v from W.

Analogously to above the new W now has $d^+(a) = d^-(a)$ for all $a \in V(W) \setminus \{u, w\}$ and $d^+(u) = d^-(u) + 1$ and $d^-(w) = d^+(w) + 1$. Note that there is an arc from w to u or from w^- to u, as u was dominated by all of $V(H_2)$, except for possibly one vertex.

- If there is an arc from w to u, then add wu to W.
- Otherwise $w^-u \in A(D')$ and add the arc w^-u to W and delete the arc w^-w from W.

Now W is a spanning eulerian subdigraph of $D\langle V(H_1) \cup V(H_2) \cup V(H_3)\rangle$, contradicting the minimality of p, and thereby proving Claim 30.2. $\Diamond$

Claim 30.3. *There is no induced* 3*-cycle in* T.

Proof. For the sake of contradiction assume $x_1x_2x_3x_1$ is an induced 3-cycle in T. That is, $x_1x_3, x_3x_2, x_2x_1 \notin A(T)$. If there is no vertex in H_3 that is hypouniversal to H_2, then all vertices in H_2 are hypouniversal to H_3, as A' is a star-set. So we can assume without loss of generality that there is a vertex $u \in V(H_3)$ that is hypouniversal to H_2 (otherwise reverse all arcs and rename H_1, H_2 and H_3). Claim 30.2 now implies that H_1, H_2 and H_3 can be merged, a contradiction. This proves Claim 30.3. $\Diamond$

Claim 30.4. *There is no vertex in* G *of degree* $p - 1$ *(that is,* G *does not consist of one spanning star).*

Proof. Assume for the sake of contradiction that $x \in V(G)$ has degree $p-1$ in G. By Claim 30.1 and Claim 30.3 we note that $T - x$ is a transitive tournament, so without loss of generality assume that $x = x_1$ and $x_2, x_3, \ldots, x_p$ are named such that if $2 \le i < j \le p$ then $x_ix_j \in A(T)$ (and $x_jx_i \notin A(T)$). By Claim 30.1 we may assume that x_1 is the vital vertex for all edges x_1x_i, $i \in \{2, 3, \ldots, p\}$ in G.

Consider the 2-cycle $x_1x_2x_1 \in T$. By Lemma 29 the following holds.

(i): The eulerian tour in H_1 can be denoted by $w_1w_2w_3 \cdots w_lw_1$, such that w_1 is not adjacent to any vertex in H_2 in D'.

(ii): There exists a k, such that $R_1 = \{w_2, w_3, \ldots, w_k\}$ and $R_2 = \{w_{k+1}, w_{k+2}, \ldots, w_l\}$ are both non-empty and the only arc in D' from R_1 to R_2 is w_kw_{k+1}.

(iii): There is no arc from R_1 to w_1 and there is no arc from w_1 to R_2 in D'.

(iv): $V(H_2) \mapsto R_1$ and $R_2 \mapsto V(H_2)$ in D'.

In H_1 we note that the only arc into R_2 is $w_k w_{k+1}$. We now consider the cases when there is an arc into R_2 in $D' \setminus w_k w_{k+1}$ and when there is no such arc.

Case 1. There is an arc into R_2 in $D' \setminus w_k w_{k+1}$. In this case assume that uv is such an arc and note that $u \in H_j$ for some $j \in \{3, 4, \ldots, p\}$ and $v \in R_2$. Let $q \in V(H_2)$ be arbitrary and note that $vq \in A(D')$ (as $v \in R_2$ and $R_2 \mapsto V(H_2)$, by (iv) above) and $qu \in A(D')$ (as $V(H_2) \mapsto V(H_j)$, as A' is a star-set). Therefore, $uvqu$ is a 3-cycle in D' and adding this 3-cycle to H_1, H_2 and H_j merges them, a contradiction to the minimality of p.

Case 2. There is no arc into R_2 in $D' - w_k w_{k+1}$. In this case consider A'', which consists of all the arcs in A' except the arcs between w_1 and $V(H_2)$. As there are at least two arcs between w_1 and $V(H_2)$ in A' (as $|V(H_2)| \geq 2$) we have $|A''| \leq |A'| - 2 \leq (k+1) - 2 = k - 1$. Furthermore $w_k w_{k+1}$ is the only arc into R_2 in $D \setminus A''$, which implies that D is at most k-arc-connected, a contradiction. $\diamond$

Claim 30.5. *There exists a 3-cycle, say $x_1 x_2 x_3 x_1$, in T such that $x_2 x_1 \in A(T)$ and $x_3 x_2 \notin A(T)$ and $x_1 x_3 \notin A(T)$.*

Proof. If $|V(G)| = 2$, then as D' is strong and therefore also T, we note that T consists of a 2-cycle. However this is a contradiction to Claim 30.4. Therefore we may assume that $|V(G)| = |V(T)| \geq 3$. As by Claim 30.3 T does not contain an induced 3-cycle, G must contain a star S, and by Claim 30.4 a vertex $x \in V(T) \setminus V(S)$. Let y be the center of the star S (if $|E(S)| = 1$ let $y \in V(S)$ be arbitrary) and without loss of generality assume that $yx \in A(T)$. Let $P = p_1 p_2 \ldots p_l$ be a shortest path from x ($x = p_1$) to y ($p_l = y$) in T. (Such a path exists because T is strong.) By the minimality of l note that $y \mapsto \{p_1, p_2, \ldots, p_{l-2}\}$ and as $x \notin V(S)$ we have $l \geq 3$.

Therefore $C = p_{l-2} p_{l-1} p_l p_{l-2}$ is a 3-cycle in T and $p_{l-2} \notin V(S)$ (as $p_{l-2} p_l \notin A(T)$). Therefore $p_l p_{l-2}$ is not an edge in G. If $p_{l-1} \in V(S)$ then $p_{l-2} p_{l-1} \notin E(G)$ and if $p_{l-1} \notin S$ then $p_{l-1} p_l \notin E(G)$. So, in both cases C, has at most one arc belonging to a 2-cycle. By Claim 30.3 we note that there is exactly one arc belonging to a 3-cycle, thereby proving Claim 30.5. $\diamond$

One can now prove the theorem. By Claim 30.5, we may let $x_1 x_2 x_3 x_1$ be a 3-cycle in T such that $x_2 x_1 \in A(T)$ and $x_3 x_2 \notin A(T)$ and $x_1 x_3 \notin A(T)$. By Lemma 29 either there is a vertex in H_2 that is not adjacent to any vertex in H_1 or there is a vertex in H_1 that is not adjacent to any vertex in H_2. By reversing all arcs if necessary, we may assume that $x \in V(H_2)$ is not adjacent to any vertex of H_1. This implies that $x^- \mapsto V(H_1)$ and $V(H_1) \mapsto x^+$, by Lemma 29. As $x_2 x_3, x_3 x_1 \in A(T)$ and $x_3 x_2, x_1 x_3 \notin A(T)$ and $V(H_1) \mapsto x^+$ it follows from Claim 30.2 that H_1, H_2 and H_3 can be merged, a contradiction.

This completes the proof. $\square$

Since a set of at most two arcs always form a star-set, we have the following corollary.

Corollary 31. *Every 2-arc-strong semicomplete digraph has a spanning eulerian digraph which avoids any prescribed arc and every 3-arc-strong semicomplete digraph has a spanning eulerian digraph which avoids any set of two prescribed arcs.*

6.4 Avoiding three arcs

Theorem 32. *Every 4-arc-strong semicomplete digraph has a spanning eulerian digraph which avoids any set of three prescribed arcs.*

Proof. Let $D = (V, A)$ be a 4-arc-strong semicomplete digraph, let $F = \{a, a', a''\} \subset A$ be a set of three arcs and let $D' = D \setminus F$. A
it non-edge of D is a pair $\{x, y\}$ such that x and y are not adjacent, that is neither xy nor yx are arcs. By Theorem 30 we may assume that the graph N induced by non-edges of D' is either a triangle or the path P_4 on four vertices. If N is a triangle, then D' is semicomplete multipartite and the claim follows from Theorem 24, so the only remaining case is that N is a P_4.

By Lemma 26, D' contains an eulerian factor. Let $\mathcal{E}$ be an eulerian factor of D' with the minimum number of components. Let $H_1, H_2, \ldots, H_p$ be the components of $\mathcal{E}$, and for every $i \in [p]$ let W_i be a closed spanning trail of H_i. For the sake of contradiction, assume that $p > 1$. If D' contains a cycle C all of whose arcs go between different components of $\mathcal{E}$, then by adding the arcs of C we obtain a better eulerian factor, contradicting the choice of $\mathcal{E}$. Hence we may assume w.l.o.g. that H_1 contains a vertex v with no arc into it from any other H_j. As D' is strong we can furthermore assume that the successor v^+ of v on W_1 has an arc into it from another H_j and by renumbering if necessary we can assume that there is a vertex u of H_2 such that uv^+ is an arc of D'. Let u^+ be the successor of u on W_2. Since $D'\langle V(H_1) \cup V(H_2)\rangle$ has no spanning closed trail it follows from Lemma 27 (c) that v is non-adjacent to both u and u^+.

If v^+ and u^+ are adjacent in D', then we must have $u^+v^+ \in A(D')$ by Lemma 27 (b), and now since v dominates $V(H_2) - \{u, u^+\}$ we have $V(H_2) = \{u, u^+\}$ for otherwise the arcs vu^{++}, u^+v^+ contradict Lemma 27 (c). If v^+ and u^+ are not adjacent in D', then $V(N) = \{u, u^+, v, v^+\}$.

Suppose first that $p > 2$. It follows from the minimality of p and the fact that N is a P_4 that we must have $V(H_1) \mapsto V(H_3) \cup \ldots \cup V(H_p)$. Suppose there is an arc $zw \in A(D')$ from $V(H_i)$ to $V(H_2)$ for some $i > 2$. If $u^+v^+ \in A(D')$, then $w \in \{u, u^+\}$ by the argument above and thus $vzwv^+$ is a path in D' which shows that H_1, H_2, H_i can be replaced by one eulerian subdigraph, contradicting the choice of $\mathcal{E}$. So u^+ and v^+ must be non-adjacent as otherwise there is no arc entering $V(H_2)$, contradicting that D' is strong. As remarked above , this means that $V(N) = \{u, u^+, v, v^+\}$ and hence every vertex of $V(H_i)$ is adjacent to every vertex of $V(H_2)$ so by Lemma 27 and the choice of $\mathcal{E}$ we must have $V(H_i) \mapsto V(H_2)$. Now v^+zuv^+ is a 3-cycle in D' which shows that we can merge W_1, W_2, W_i, contradicting the minimality of p. So we must have $p = 2$.

Suppose first that $|V(H_2)| > 2$. By the remark above, $V(N) = \{u, u^+, v, v^+\}$. Hence v^+ is adjacent to all vertices of $R = V(H_2) \setminus \{u, u^+\}$ and since v dominates all of these, we also conclude from Lemma 27 and the minimality of p that $v^+ \mapsto R$ and we see that $V(H_1) \mapsto R$. Let u^{++} be the successor of u^+ on W_2. If H_2 has a spanning (u^{++}, u)-trail T then we can insert $V(H_2)$ in W_1 by deleting the arc vv^+ and adding the arcs of the trail $vu^{++}T[u^{++}, u]uv^+$, contradicting the minimality of $\mathcal{E}$. Thus there is no spanning (u^{++}, u)-trail in H_2 and, by Theorem 11 and Menger's theorem, we can partition $V(H_2)$ into two sets Z_1, Z_2 such that $u^{++} \in Z_1, u \in Z_2$ and there is precisely one arc from Z_1 to Z_2 in D_2. But then there are at most three arcs leaving Z_1 in D, contradicting that D is 4-arc-strong.

Henceforth $V(H_2) = \{u, u^+\}$. As D is 4-arc-strong this implies that $|V(H_1)| > 2$. Note that if $u^+v^+ \in A(D')$, then we may assume, by renaming u, u^+ if necessary, that u is adjacent to all vertices of $V(H_1) \setminus \{v, v^+\}$ in D'. This holds automatically if $V(N) = \{u, u^+, v, v^+\}$.

If uv^- is an arc of D (and hence of D'), then it follows from Lemma 27 (b) that $u \mapsto V(H_1) \setminus \{v\}$, contradicting that the in-degree of u is at least 4 in D.

Hence $v^-u \in A(D')$. This implies that either u^+ and v^- are non-adjacent or $v^- \mapsto u^+$ by Lemma 27 (b). As D is 4-arc-strong the vertex u has at least two in-neighbours and two out-neighbours in $V(H_1)$ in D'. This and the minimality of p imply that there exists a vertex $w \in V(H_1)$ such that $u \mapsto Y$ and $X \mapsto u$, where $Y = V(W_1[v^+, w^-])$ and $X = V(W_1[w, v^-])$. It is easy to see that we also have $u^+ \mapsto Y \setminus \{v^+\}$ and if u^+ is not adjacent to v^- then $u^+ \mapsto Y$. Now we conclude that H_1 has no spanning (v^+, v^-)-trail T' as otherwise either $uv^+T'[v^+, v^-]v^-u^+u$ or $u^+v^+T'[v^+, v^-]v^-uu^+$ would be a closed spanning trail of D. As $v^-v, vv^+ \in A(D')$ and v is adjacent to all vertices of $V(H_1) \setminus v$ and cannot be inserted in the trail $W_1[v^+, v^-]$, there exists a vertex $z \in V(H_1) \setminus v$ such that $v \mapsto W_1[v^+, z^-]$ and $W_1[z, v^-] \mapsto v$. By symmetry we can assume that $z \in X$ and thus $v \mapsto Y$.

As D is 4-arc-strong, by Menger's theorem, there are at least four arcs with tail in Y and head in $V(D) \setminus Y$. As we have $\{u, u^+\} \mapsto Y - v^+$, $u \mapsto v^+$ and $v \mapsto Y$ the head of at least three of those arcs must be in X. Consequently, in H_1 there are at least three arcs with tail in Y and head in X. In particular, there are $y \in Y, x \in X$ such that yx is not the arc v^+v^- and there are two arc-disjoint (y, x)-paths in H_1. Thus by Theorem 11, there exists a spanning (y, x)-trail T_1 in H_1. Now either $T_1[y, x]xu^+uy$ or $T_1[y, x]xuu^+y$ (or both) is a spanning eulerian trail of D', a contradiction.

$\square$

7 Unavoidable arcs in semicomplete digraphs

Let D be a strong semicomplete digraph with at least one cut-arc (so $\lambda(D) = 1$) An arc a is **unavoidable** if it is contained in all spanning eulerian subdigraphs of D (so $D \setminus a$ has no spanning closed trail). Observe that every cut-arc is unavoidable.

The following is a direct consequence of Theorem 24. Note that if $D \setminus a$ is semicomplete then it has a hamiltonian cycle and we can find such a cycle in polynomial time in any semicomplete digraph.

Corollary 33. *There is a polynomial-time algorithm that, given a semicomplete digraph D and an arc a, decides whether a is unavoidable in D and returns a spanning eulerian subdigraph avoiding a when one exists.*

We believe that Corollary 33 can be generalized to the following.

Conjecture 34. For each fixed positive integer k, there exists a polynomial-time algorithm which, given a semicomplete digraph $D = (V, A)$ and $A' \subset A$ with $|A'| = k$, decides whether $D \setminus A'$ has a spanning eulerian subdigraph.

The analogous conjecture for hamiltonian cycles was posed in [3, Conjecture 7.4.14] and is still open for $k \geq 2$. For $k = 1$ a polynomial-time algorithm follows from [7].

7.1 A classification of the set of unavoidable arcs

In this subsection we give a complete characterization of the pairs (D, a) such that D is a semicomplete digraph in which a is an unavoidable arc. We shall prove the following theorem.

Theorem 35. *Let D be a semicomplete digraph and let $uv \in A(D)$ be arbitrary and let $D' = D\setminus\{uv\}$. If D' is strong, then D' contains a spanning eulerian subdigraph if and only if $V(D')$ cannot be partitioned into R_1, R_2 and $Y = \{u, v\}$ such that Y is independent, $d(R_2, Y) = 0$, $d(Y, R_1) = 0$ and $d(R_2, R_1) = 1$.*

Proof. Let D' be defined as in the theorem. If D' can be partitioned into R_1, R_2 and Y such that Y is independent and $d(R_2, Y) = 0$ and $d(Y, R_1) = 0$ and $d(R_2, R_1) < |Y|$, then we must have $Y = \{u, v\}$ since we only deleted one arc from a semicomplete digraph and $d(R_2, R_1) > 0$ as D' is strong. Now it follows from Theorem 25 that D' contains no eulerian factor and therefore also no spanning eulerian subdigraph. So assume that D' cannot be partitioned in this way, which by Theorem 25 implies that D' contains an eulerian factor. D' is clearly a semicomplete multipartite digraph so it follows from Theorem 24 that D' has a spanning eulerian subdigraph. □

We first observe that the backward arcs with respect to a nice decomposition are unavoidable since they are cut-arcs.

Proposition 36. *Let D be a strong semicomplete digraph of order at least 4 and let $(S_1, \ldots, S_p)$ be a nice decomposition of D. Every backward arc is unavoidable.*

If D is a semicomplete digraph with vertex set $\{a, b, c, d\}$ such that $\{ab, bc, cd, ad, ca, db\} \subseteq A(D) \subseteq \{ab, bc, cd, ad, ca, db, cb\}$, then the arc ad is **exceptional**. See Figure 5.

Figure 5: The two digraphs having an exceptional arc (ad in thick blue).

Let D be a semicomplete digraph of order at least 4 and let $(S_1, \ldots, S_p)$ be a nice decomposition of D. A forward arc uv is **regular-compulsory** if there is an index i such that $1 < i < p - 1$, $S_i = \{u\}$, $S_{i+1} = \{v\}$, and both S_i and S_{i+1} are ignored. If $|S_i| = 1$ for all $1 \leq i \leq 3$, say $S_i = \{v_i\}$,

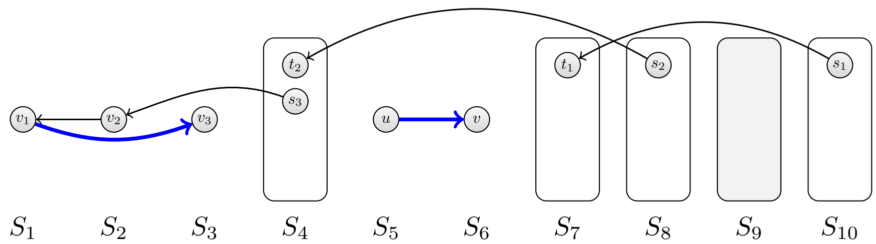

Figure 6: A nice decomposition of a strong semicomplete digraph with four backwards arcs (in thin black). The arc uv is regular-compulsory. The arc v_1v_3 is left-compulsory.

and $v_2v_1 \in A(D)$, $v_1v_2 \notin A(D)$, $v_3v_2 \notin A(D)$ and $N^-(v_3) = \{v_1, v_2\}$, then the arc v_1v_3 is **left-compulsory**. If $|S_i| = 1$ for all $p-2 \leq i \leq p$, say $S_i = \{v_i\}$, and $v_pv_{p-1} \in A(D)$, $v_{p-1}v_p \notin A(D)$, $v_{p-1}v_{p-2} \notin A(D)$, $N^+(v_{p-2}) = \{v_{p-1}, v_p\}$, then $v_{p-2}v_p$ is **right-compulsory**. See Figure 6.

Theorem 37. *Let D be a strong semicomplete digraph of order at least 4 and let $(S_1, \ldots, S_p)$ be a nice decomposition of D. An arc is unavoidable if and only if it is either a cut-arc, regular-compulsory, left-compulsory, right-compulsory, or exceptional.*

Proof. If an arc ad is exceptional, then Theorem 35 implies that it is unavoidable ($R_1 = \{c\}$ and $R_2 = \{b\}$). If v_1v_3 is left-compulsory, then, again by Theorem 35, v_1v_3 is unavoidable ($R_1 = \{v_2\}$ and $R_2 = V(D) \setminus \{v_1, v_2, v_3\}$). Analogously, if $v_{p-2}v_p$ is right-compulsory then it is unavoidable. If uv is regular-compulsory, then again by Theorem 35, uv is unavoidable ($R_1 = S_1 \cup \cdots \cup S_{i-1}$ and $R_2 = S_{i+2} \cup S_p$).

Let us now prove the reciprocal: if uv is not a cut-arc (backward arc) and not exceptional, left-compulsory, right-compulsory or regular-compulsory, then it is not unavoidable.

Note that Theorem 35 implies that if D is a strong semicomplete digraph and $uv \in A(D)$ is not a cut-arc then the following holds, where $D' = D \setminus uv$. The arc uv is unavoidable if and only if $N^+_{D'}(u) = N^+_{D'}(v)$ and $N^-_{D'}(u) = N^-_{D'}(v)$ and $N^+_{D'}(u) \cap N^-_{D'}(u) = \emptyset$ and there is only one arc from $N^+_{D'}(u)$ to $N^-_{D'}(u)$. In particular if there is a path of length 2 between u and v then uv is not unavoidable (unless it is a cut-arc). We shall use this observation several times below.

First assume that uv is an unavoidable forward arc where $u \in S_i$ and $v \in S_j$. If $|S_i| > 1$, then let $w \in S_i$ be an out-neighbour of u. If $wv \in A(D)$, then uwv is a path of length 2 so uv is not unavoidable, a contradiction. So $vw \in A(D)$ and vw is a backward arc. In D there must be two arc-disjoint paths, say P_1 and P_2, from w to u as otherwise there would be a cut-arc separating w from u which, as S_i is strong, must belong to S_i, a contradiction. Therefore there must be at least two arcs from $N^+(u)$ (as $w \in N^+(u)$) to $N^-(u)$, so uv is not unavoidable, a contradiction. So $|S_i| = 1$ and analogously $|S_j| = 1$.

Assume that there is a backward arc ru into u, where $r \in S_k$. As there is no a path of length 2 from v to u, $rv \in A(D)$ and therefore $i < k < j$ (as S_k cannot have two backward arcs out of it). Assume that $i > 1$ and let xy be a backward arc from $S_i \cup \cdots \cup S_p$ to $S_1 \cup \cdots \cup S_{i-1}$. As backward arcs are not nested (Proposition 14) we see that x must belong to $S_i \cup \cdots \cup S_{k-1}$. If $x = u$ then uyv is a path, a contradiction, so $x \in S_{i+1} \cup \cdots \cup S_{k-1}$. This implies that $xv \in A(D)$ (as otherwise ru wouldn't be a cut-arc) and uxv is a path, a contradiction. Therefore $i = 1$. Analogously if there is a backward arc out of v then $j = p$.

Now assume that there is a backward arc vx out of v and a backward arc yu into u. Then $i = 1$ and $j = p$. Note that $x \neq y$ as otherwise vxu is a path. If there is any vertex in $w \in S_2 \cup \cdots \cup S_{p-1} \setminus \{x, y\}$ then uwv is a path, so $V(D) = \{u, v, x, y\}$ and it is easy to see that uv is an exceptional arc.

So now assume that u has a backward arc, yu, into it and v has no backward arc out of it. Then $i = 1$ and $S_1 = \{u\}$, $S_2 = \{y\}$ and $S_3 = \{v\}$ as otherwise we could find a path of length 2 from u to v. It is now easy to see that uv is left-compulsory. Analogously if there is a backward arc out of v but no backward arc into u, then uv is right-compulsory.

Finally assume that there is no backward arc into u and no backward arc out of v. In this case $j = i + 1$ as otherwise it is easy to find a path of length 2 from u to v. We now see that uv must be regular-compulsory.

The remaining case is that uv is a flat arc and $u, v \in S_i$. Then there are two arc-disjoint paths from v to u in $D \setminus uv$ as otherwise there would be a cut-arc in S_i. But this implies that there are at least two arcs from $N^+(v)$ to $N^-(v)$, implying that uv is not unavoidable by the above characterization of unavoidable arcs derived from Theorem 35. $\square$

Acknowledgements

This research was supported by the Danish research council under grant number DFF 7014-00037B, by the DISCO project, PICS, CNRS, and by the french Agence Nationale de la Recherche under contract Digraphs ANR-19-CE48-0013-01.

We are grateful to the referees for their comments that helped improving our paper.

References

[1] J. Bang-Jensen and A.Maddaloni. Sufficient conditions for a digraph to be supereulerian. *J. Graph Theory*, 79(1):8–20, 2015.

[2] J. Bang-Jensen, H. Déprés, and A. Yeo. Spanning eulerian subdigraphs avoiding k prescribed arcs in tournaments. *Discrete Mathematics*, 343(12):112129, 2020.

[3] J. Bang-Jensen and G. Gutin. *Digraphs: Theory, Algorithms and Applications*. Springer-Verlag, London, 2nd edition, 2009.

[4] J. Bang-Jensen and G. Gutin. *Classes of Directed Graphs*. Springer Monographs in Mathematics. Springer Verlag, London, 2018.

[5] J. Bang-Jensen, G. Gutin, and A. Yeo. Hamiltonian cycles avoiding prescribed arcs in tournaments. *Combin. Prob. Comput.*, 6(3):255–261, 1997.

[6] J. Bang-Jensen and T. Jordán. Spanning 2-strong tournaments in 3-strong semicomplete digraphs. *Discrete Math.*, 310:1424–1428, 2010.

[7] J. Bang-Jensen, Y. Manoussakis, and C. Thomassen. A polynomial algorithm for Hamiltonian-connectedness in semicomplete digraphs. *J. Algor.*, 13(1):114–127, 1992.

[8] Paul Camion. Chemins et circuits hamiltoniens des graphes complets. *C. R. Acad. Sci. Paris*, 249:2151–2152, 1959.

[9] P. Fraisse and C. Thomassen. Hamiltonian dicycles avoiding prescribed arcs in tournaments. *Graphs Combin.*, 3(3):239–250, 1987.

[10] Y. Guo. Spanning local tournaments in locally semicomplete digraphs. *Discrete Appl. Math.*, 79(1-3):119–125, 1997.

[11] A.J. Hoffman. Some recent applications of the theory of linear inequalities to extremal combinatorial analysis. In R. Bellman and M. Hall, editors, *Combinatorial Analysis*, pages 113–128. American Mathematical Society, Providence, RI, 1960.

[12] J. Liu, Q. Liu, X. Zhang, and X. Chen. Trail-connected tournaments. *Applied Mathematics and Computation*, 389(C):S0096300320305506, 2021.

[13] L. Rédei. Ein kombinatorischer Satz. *Acta. Litt. Sci. Szeged*, 7:39–43, 1934.

[14] C. Thomassen. Hamiltonian-connected tournaments. *J. Combin. Theory Ser. B*, 28(2):142–163, 1980.