BOOK REVIEW: *A COURSE ON ROUGH PATHS — WITH AN INTRODUCTION TO REGULARITY STRUCTURES* (PETER K. FRIZ AND MARTIN HAIRER)

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Abstract. We give a review of the book *A course on rough paths — With an introduction to regularity structures* (2nd edition) by P.K. Friz and M Hairer with a brief insight of some current developments in the field as well as their usefulness in the context of stochastic calculus.

1. Why the theory of rough paths?

Stochastic calculus aims at giving meaning to differential systems of the form $\dot{x}_t = f(x_t)\dot{\xi}_t$ driven by signals $\dot{\xi}$ which are random and typically non-differentiable such as the white noise. Such systems are conveniently rewritten in the integral form

$$x_t = x_0 + \int_0^t f(x_s)\,\mathrm{d}\xi_s, \tag{1}$$

where ξ is defined as a Gaussian noise. Eq. (1) generalizes differential equations. One common situation is the case of *stochastic differential equations* (SDE), where ξ is a Brownian motion. When the solution x takes its values in a functional space, ξ is a space-time white-noise, and $f(x_t) = Ax_t$ for an unbounded operator A, then (1) is called a *stochastic partial differential equation* (SPDE).

SDE and SPDE are prevalent in modeling of noisy phenomenon. Itô's and Stratonovich's calculus are the most common tools to give a meaning to the solutions to (1). Although efficient, their use restrict the class of processes ξ to Brownian motion and semi-martingales. Considering other processes, such as the fractional Brownian motion, require *ad hoc* integration.

by their very constructions: the integral $\int f(x_s)\,\mathrm{d}\xi_s$ is defined as a limit in probability of approximations either by Riemann sums or by ODE with a regularized noise. This has various implications that make SDE different from ODE. For example, one should be careful when designing implicit or adaptive numerical schemes because

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they use *anticipative* quantities while Itô calculus is designed for *non-anticipative* integrand, *i.e.*, containing no information about the future.

Revisiting the formalism of K.T. Chen on iterated integrals, T. Lyons provided at the end of the 1990 in [9,10] a *pathwise* construction of SDE with a core feature that the integral is *continuous* with respect to the driving noise, at the price of enhancing it. This continuity property is very useful for modeling and simulating. Besides, the pathwise construction avoid distinction between anticipative and non-anticipative integrands. At last, the construction is no longer restricted to the Brownian motion but may be applied to any other random or deterministic driving signal, provided we suitably enhance the latter.

The core idea of K.T. Chen is to treat ODE as formal algebraic objects (see *e.g.* [1]). If $x : [0,T] \to \mathbb{R}^d$ is a differentiable path and $f_1, \dots, f_d$ a collection of smooth functions from $\mathbb{R}^m$ to $\mathbb{R}^m$, the $\mathbb{R}^m$-valued solution $y = \{y^\ell\}_{\ell=1,\dots,m}$ of the ODE

$$y_t^\ell = a + \int_s^t \sum_{i=1}^d f_i^\ell(y_r)\,\mathrm{d}x_r^i,\ s \leqslant t \leqslant T,\ \ell = 1, \dots, m$$

may be expanded as

$$y_t^\ell = a + \sum_{i=1}^d f_i^\ell(a) \int_s^t \mathrm{d}x_{t_1}^i + \sum_{\substack{i,j=1,\dots,d\\ k=1,\dots,m}} f_j^k(a) \frac{\partial f_i^\ell(a)}{\partial x_k} \int_s^t \int_s^{t_1} \mathrm{d}x_{t_2}^j\,\mathrm{d}x_{t_1}^i + \cdots,$$

the "$\cdots$" involving iterated integrals of higher orders as coefficients of the $\mathbb{R}^m$-valued vector field $\sum_{j=1}^m f_i^j \frac{\partial}{\partial x_j}$ applied to themselves. The family of iterated integrals

$$X_{r,t} = 1 + \int_r^t \mathrm{d}x_s + \int_r^t \int_r^{t_1} \mathrm{d}x_{t_2} \otimes \mathrm{d}x_{t_1} + \cdots$$

is itself best seen as the solution of the linear equation in the Banach tensor (non-commutative) algebra $T(\mathbb{R}^d) = \bigoplus_{k \geqslant 0} (\mathbb{R}^d)^{\otimes k}$ defined by

$$X_{r,t} = 1 + \int_r^t X_{r,s} \otimes \mathrm{d}x_s,\ r \leqslant t \tag{2}$$

as it is easily seen by solving (2) using a Picard principle. In particular, $X = \{X_{r,t}\}_{r \leqslant t}$ satisfies the *Chen relation*: $X_{r,s} \otimes X_{s,t} = X_{r,t}$ for any $r \leqslant s \leqslant t$. This is the algebraic counterpart of the concatenation of the paths $\{x_u\}_{u \in [r,s]}$ and $\{x_u\}_{u \in [s,t]}$. Similarly, time inversion $\{x_{t-u}\}_{u \in [r,t]}$ is translated by inverting $X_{r,t}$ in $T(\mathbb{R}^d)$.

When x is a continuous by irregular path, say a α-Hölder path with $\alpha < 1$, then one cannot define iterated integrals such as $\int_r^t \int_r^{t_1} \mathrm{d}x_{t_2} \otimes \mathrm{d}x_{t_1}$ unless $\alpha > 1/2$ where Young integration is used.

The main idea of the theory of rough paths is to consider that above an α-Hölder driving signal x, we are granted $X = \{X_{r,t}\}_{r \leqslant t}$, called a *rough path*, at least in a truncated tensor algebra (in which case, we may reconstruct the whole series) at an order $p = \lfloor 1/\alpha \rfloor$ (the lower the regularity, the higher the order). Such an enhancement X above a given driving signal x is *not* unique. Each extension gives a different integrals. For the Brownian motion (paths of Hölder regularity $1/2 - \epsilon$ almost surely), it requires an extension up to order 2. Itô and Stratonovich integrals are recovered using Itô and Stratonovich iterated integrals.

The construction of integrals in the theory of rough paths is completely deterministic. However, such a theory is well suited in stochastic analysis, not only to deal with the Brownian motion and martingales, but also to many other processes such as the fractional Brownian motions. Getting a full theory of integration only requires showing the existence of the iterated integrals.

2. The theory of rough paths over 20 years

The theory of rough paths is now a vivid field of research that has expanded in many directions and relation with several fields of pure and applied mathematics. It is gathered with related theories in the 2020 AMS classification as **60L**. It would be impossible to give now an exhaustive panorama so that we focus on some main past and present directions of research.

Beyond the original construction of T. Lyons, many alternative constructions have been proposed, which rely on distinct — albeit related — notions of solutions. The one of A.M. Davie [2], based on numerical schemes, and the one of M. Gubinelli [5], the so-called *controlled rough paths* (used in the book under review), are now the most commonly used. Besides, a complete series of works have shown that rough differential equations are natural extensions of ordinary differential equations that share many of their properties: existence, uniqueness, transfer of the regularity of the vector field to the one of the Itô-Lyons map (the map that gives the solution from the driving signal when the solution is unique), generic conditions, existence of global solutions, ... provided that one uses the suitable topology.

Another body of work has been devoted to extending the *sewing lemma*, which is the technical tool at the heart of the rough paths theory. The name appeared in an article from D. Feyel *et al* [4], while the concept generalized ideas proposed by L.C. Young for his integration theory [11], now partially a subcase of the theory of rough paths. For example, an integral is characterized by the Chasles property $\int_r^t \dots = \int_r^s \dots + \int_s^t \dots$. Assuming we are given an "approximation" $I_{s,t}$ of the integral $S_{r,t} = \int_r^t \dots$ even *without* having a formal definition of $S_{s,t}$. With the additive sewing lemma, one may "project" $I_{s,t}$ onto $S_{s,t}$ which satisfies $S_{r,s} + S_{s,t} = S_{r,t}$, provided that one may control $I_{r,s} + I_{s,r} - I_{s,t}$. Hence, S is rigorously defined. The sewing lemma is an effective tool with now many declinations: additive, multiplicative (to construct multiplicative integrals or solutions of linear equations, even involving non-bounded operators), stochastic, non-linear (construction of flows), ...

Another research direction that appeared early in the literature is to apply this theory to perform stochastic integration for a large class of stochastic driving signal. It is perhaps mostly used for fractional Brownian motion or Gaussian processes, where many results similar to the ones known on SDE have been proved despite the lack of the Markov property: Hörmander and support theorem, concentration theorem, ...

The theory of rough paths is also strongly connected with the theory of geometric integrations. Beyond the extension from tensor algebras to trees algebras, the natural and suitable underlying algebraic structures are related to Hopf algebras (shuffle, quasi-shuffle, ...) and tools from renormalization theory.

An indirect by-product of the theory of rough paths is the theory of regularity structures [7], one of the contributions for which M. Hairer was awarded the Fields medal. As for the theory of rough paths, this theory mixes both analysis (a formal Taylor expansion of distributions at each point) and algebra (an algebraic relationship between the objects) to describe the solution of PDE in presence of singularities. The *reconstruction lemma* is the counterpart of the sewing lemma. Within this framework, M. Hairer was able to give a rigorous meaning to the Kardar-Parisi-Zhang (KPZ) equation [8] and other singular (S)PDE arising in mathematical physics, such as the Φ^3_4 equation of quantum field theory. This topic is now strongly connected to algebraic approaches in renormalization theory such as the Bogoliubov-Parasiuk-Hepp-Zimmermann (BPHZ) approach and Connes and Kraimer Hopf algebra.

Finally, a recent trend of research concerns machine learning: rough paths encode some non-linear features of a signal in a low dimensional space. Using iterated integrals to classifications and other learning procedures leads to state-of-art techniques, for example in hand-writing recognition.

Many more directions are now explored, let us cite for example applications to mean-field differential equations, dynamical systems, filtering, Monte Carlo techniques, volatility models (a model in finance that may be tackled with tools from regularity structures), model free finance, and other various theoretical and applications fields.

3. The content of the book

The book [3] introduces the theories of rough paths and regularity structures. Basically, its content may be divided in 4 categories:

- Construction of rough integrals and rough differential equations (RDE), using mostly the framework of *controlled rough paths* [5]. Chapters 2, 4, 7, and 8 are essential to apprehend the core of the theory: existence, uniqueness, continuity with respect to the driving signal, the impact of the regularity of the driving path, flows, numerical schemes, and many of the properties on RDE that are similar to those on ODE.

- Applications to stochastic analysis (Chapters 3, 5, 6, 9, 10, and 11) regarding Brownian motion seen as a rough path, martingales and Gaussian rough paths. In particular, parallels are drawn between RDE and SDE, the recent stochastic sewing lemma is presented, and several recent and original extensions of stochastic calculus : support and Hörmander theorems for Gaussian rough paths, extension to Burkhölder-Davis-Gundy inequality to rough martingales, ... Together with their exercises, these chapters contain a grand tour to many of the possible usages of rough analysis to stochastic analysis.
- SPDE using ideas from rough paths theory (Chapter 12). The basic idea is to use the continuity to pass from a regularized noise to a "true" noise (in time, but also in space and times) to deal with PDE perturbed by noises. Several types of PDE are thus covered by this chapter: rough transport equation for measures; second-order PDE with linear noise through the Feynman-Kac formula, and then semi-linear PDE using mild notions of solutions which requires to define some convolutional rough integrals; fully non-linear equations for which the solutions are rough viscosity solutions in presence of space-time white noise; stochastic heat equations with space-time white noise. This chapter ends up with several exercises with applications of such SPDE to filtering.
- SPDE and regularity structures to deal with singular situations. Chapters 13 and 14 contain some introduction to the analytic and algebraic aspects of the regularity structures with examples on the so-called polynomials and rough paths models. A new proof of the reconstruction lemma is given. Afterward, operations such as product, differentiation and integration are presented, as they are useful to write down some Schauder estimates, which are estimates on the solutions of the SPDE. Some applications to rough volatility models are given. Chapter 15 is devoted to the KPZ equation and to show how the regularities structures give meaning to such equations despite diverging terms. The renormalization strategy is carefully presented.

As we see, the applicative side is focused on stochastic analysis, a topic in which several original and recent results are collected. However, the general sections are of interest for any mathematician willing to apprehend this theory.

As underlined above, the theory of rough paths now evolves in many directions. The book uses the framework of controlled rough paths, albeit there are several alternatives such as the original construction from T. Lyons or the flow *à la Davie* and their extensions. An alternative to regularity structure, the approach through *paracontrolled distributions* [6] is not presented. Results related to the fractional Brownian motion are shortly covered, as well as the treatment of lower order regularity. Application to data science (*signature methods*), which is a topic by itself is not mentioned. The authors never had the aim to be exhaustive on the subject but to “complement the existing literature on the subject”. An exhaustive book will run over many more pages and would be impossible to write so fast is the pace.

4. Conclusion

Written by two leading experts of the field, this book offers a nice and smooth introduction to the theory of rough paths and regularity structures, with an insight into their applications to stochastic calculus and the resolution of singular SPDEs. With detailed proofs and up-to-date developments come many examples and applications. Each chapter ends with a list of exercises. This book is well suited both for newcomers in the field, as well as experts. It may be recommended to any mathematician willing to learn more about this fascinating topic.

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