Boundedness of the Kalman Filter Revisited

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Abstract: The boundedness of the Kalman filter, as the first cornerstone of its stability analysis, has been proved in the classical literature through upper bounds of non-recursive filters in the sense of the trace of the state estimation error covariance. In this paper, an upper bound of the Kalman filter prediction error covariance is established in the sense of matrix positive definiteness, based on a bounded recursive non-optimal filter. The boundedness of the error covariance is a prerequisite for the definition of a Lyapunov function involved in the state estimation error dynamics stability analysis.

Keywords: Kalman filter, boundedness, stability, prediction error.

1. INTRODUCTION

The Kalman filter is widely applied due to its well known optimal properties and numerical efficiency (Jazwinski, 1970; Anderson and Moore, 1979; Zarchan and Musof, 2005; Kim, 2011; Grewal and Andrews, 2015). While it is no longer necessary to introduce these facts, another important property of the Kalman filter, its *boundedness*, is rarely mentioned in the recent literature. Indeed, as a recursive algorithm, the boundedness of the involved variables is crucial for real time applications. The boundedness of the error covariance matrix is also the first cornerstone of the classical stability analysis of the Kalman filter, since it ensures the well-definedness of a Lyapunov function (Kalman, 1963; Deyst Jr and Price, 1968; Jazwinski, 1970; Deyst Jr, 1973; Moore and Anderson, 1980).

In the classical literature on Kalman filter, as part of stability analysis, the boundedness of the Kalman filter was first studied in Kalman's pioneering work (Kalman, 1963), for the continuous time filter. It was notably established that the state estimation error covariance is bounded. The discrete time case was then addressed in (Deyst Jr and Price, 1968; Jazwinski, 1970; Deyst Jr, 1973). In these results, the established upper bounds were essentially based on the uniform complete observability (UCO [1]) of the considered system. In both the continuous time case and the discrete time case, these results involve the reversed state transition matrix. While the continuous time state transition matrix is naturally invertible, it is not always true in the discrete time case. About a decade later, the requirement on invertible discrete time state transition matrix was relaxed in (Moore and Anderson, 1980).

In these classical results, the key step for establishing an upper bound of the state estimation error covariance consists in building a non-optimal filter, whose error covariance is known to be bounded. Due to the optimality of the Kalman filter, its error covariance cannot be larger than the bounded error covariance of the non-optimal filter. More specifically, in (Deyst Jr and Price, 1968; Jazwinski, 1970; Deyst Jr, 1973; Moore and Anderson, 1980) the non-optimal filters built for this purpose are non-recursive, and the upper bounds have been proved in the sense of the trace of the error covariance matrix. The purpose of this paper is to establish an upper bound of the Kalman filter prediction error covariance in the sense of *matrix positive definiteness*, through a bounded *recursive* non-optimal filter.

To establish the new upper bound of the Kalman filter, the first step consists in building an exponentially stable *recursive* non-optimal filter. The error covariance of this non-optimal filter is bounded, as a result of its exponential stability. The Kalman filter is then bounded by the upper bound of this non-optimal recursive filter, due to the optimality of the Kalman filter. For linear time *invariant* (LTI) systems, it would be easy to build a recursive non-optimal filter with bounded error covariance. Solutions are also well known for linear parameter varying (LPV) systems with some affine parametric structure. However, for general linear time *varying* (LTV) systems, it is a difficult problem. This difficulty due to the time varying nature will be further explained at the beginning of Section 4.

Discrete time LTV systems will be considered in this paper, without assuming the invertibility of the state transition matrix.

2. NOTATIONS AND ASSUMPTIONS

In this paper, For any vector v, $\|v\|$ denotes its Euclidean norm. For any matrix M, $\|M\|$ denotes the matrix norm induced by the Euclidean vector norm, which is equal to the largest singular value of A. For a symmetric real matrix A, the inequality $A > 0$ means that A is positive definite,

[1] In this paper the abbreviation UCO will be used either as a noun for "uniform complete observability" or as an adjective for "uniformly completely observable".

and $A \geq 0$ means positive semidefinite. For two symmetric real matrices A and B of the same size, $A > B$ means $A - B > 0$, and $A \geq B$ means $A - B \geq 0$. For a square matrix A, trace(A) denotes the trace of A. For any positive integer l, the $l \times l$ identity matrix is denoted by I_l.

This paper considers general LTV systems of the form

$$x_{k+1} = A_k x_k + w_k, \tag{1a}$$

$$y_k = C_k x_k + v_k, \tag{1b}$$

where k is the discrete time index, $x_k \in \mathbb{R}^n$ is the state, $y_k \in \mathbb{R}^m$ is the output, $w_k \in \mathbb{R}^n$, $v_k \in \mathbb{R}^m$ are zero mean white noises independent of each other and of the initial state x_0, A_k, C_k are matrices of appropriate sizes, and the noise covariance matrices are $\mathrm{E}\left(w_k w_k^T\right) = Q_k$, $\mathrm{E}\left(v_k v_k^T\right) = R_k$. The initial state x_0 has its mean value $\mathrm{E}(x_0) = \bar{x}_0 \in \mathbb{R}^n$, and its covariance $\mathrm{cov}(x_0) = P_0 \in \mathbb{R}^{n \times n}$.

It is possible to add a bounded input term $B_k u_k$ into the state equation. As such an input term has no effect on the stability of the Kalman filter, it is omitted in this paper in order to focus on the boundedness analysis.

For any pair of integers $l > k \geq 0$, the *state transition matrix* $\Phi_{l|k}$ is defined as

$$\Phi_{k|k} = I_n \tag{2a}$$

$$\Phi_{l|k} = A_{l-1} \cdots A_k. \tag{2b}$$

For the results of this paper, $\Phi_{l|k}$ is *not* defined when $l < k$, in which case it would involve the inverse of A_k, which may be singular. In (Deyst Jr and Price, 1968; Jazwinski, 1970; Deyst Jr, 1973) the backward state transition matrix was defined as

$$\Phi_{l|k} = A_l^{-1} A_{l+1}^{-1} \cdots A_{k-1}^{-1}, \quad \text{for } l < k. \tag{3}$$

This case is not required in the present paper.

A basic property of the state transition matrix is, for all integers $l, k, j \geq 0$,

$$\Phi_{l|k} \Phi_{k|j} = \Phi_{l|j}. \tag{4}$$

The observability Gramian is defined as

$$M_{l,k} = \sum_{i=k}^{l} \Phi_{i|k}^T C_i^T C_i \Phi_{i|k} \tag{5}$$

for any pair of integers $l \geq k \geq 0$. In this paper the case $l < k$ involving the inverse system dynamics is not required.

Assumption 1. (Boundedness). There exist positive constants $\overline{a}$, $\overline{c}$, $\overline{q}$, $\underline{r}$, $\overline{r}$, such that, for all $k = 0, 1, 2, \ldots$,

$$\|A_k\| \leq \overline{a} \tag{6a}$$

$$\|C_k\| \leq \overline{c} \tag{6b}$$

$$Q_k \leq \overline{q} I_n \tag{6c}$$

$$0 < \underline{r} I_m \leq R_k \leq \overline{r} I_m. \tag{6d}$$

□

Assumption 2. (UCO). The matrix pair (A_k, C_k) is uniformly completely observable, in the sense that there exist two positive constants $\alpha_2 > \alpha_1 > 0$ and an integer $h > 0$ such that, for all integer $k \geq 0$, the observability Gramian $M_{k+h-1,k}$ defined in (5) satisfies

$$\alpha_1 I_n \leq M_{k+h-1,k} \leq \alpha_2 I_n. \tag{7}$$

□

3. PROBLEM STATEMENT

The well known Kalman filter consists of the following recursive computations for $k = 0, 1, 2, \ldots$, after the initialization $P_{0|0} = P_0$ and $\hat{x}_{0|0} = \bar{x}_0$,

$$P_{k+1|k} = A_k P_{k|k} A_k^T + Q_k \tag{8a}$$

$$\Sigma_{k+1} = C_{k+1} P_{k+1|k} C_{k+1}^T + R_{k+1} \tag{8b}$$

$$K_{k+1} = P_{k+1|k} C_{k+1}^T \Sigma_{k+1}^{-1} \tag{8c}$$

$$P_{k+1|k+1} = (I_n - K_{k+1} C_{k+1}) P_{k+1|k} \tag{8d}$$

$$\hat{x}_{k+1|k} = A_k \hat{x}_{k|k} \tag{8e}$$

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1}(y_{k+1} - C_{k+1} \hat{x}_{k+1|k}). \tag{8f}$$

The Kalman filter error dynamics stability analyses in (Deyst Jr and Price, 1968; Jazwinski, 1970; Deyst Jr, 1973; Moore and Anderson, 1980; Ni and Zhang, 2016; Zhang, 2017) are based on quadratic Lyapunov functions defined with $P_{k|k}^{-1}$ or $P_{k+1|k}^{-1}$. For well-defined Lyapunov functions, it is essential to ensure that $P_{k|k}$ or $P_{k+1|k}$ is bounded. The purpose of this paper is to study the boundedness of the Kalman filter through a bounded non-optimal recursive filter.

The existing upper bounds of the error covariance of the Kalman filter are based on some specially built non-optimal non-recursive filters. As a matter of fact, if a non-optimal filter has a bounded error covariance, then it provides an upper bound for the Kalman filter, due to its optimality in the sense of minimum error covariance.

In the early Kalman filter stability analysis (Deyst Jr and Price, 1968; Jazwinski, 1970; Deyst Jr, 1973), a non-optimal filter was built in the following way. First notice that, if the state noise w_k was omitted in the state equation (1a), then the state vector at two different instants l and k would be related through the noise-free state equation by $x_l = \Phi_{l|k} x_k$, with the state transition matrix $\Phi_{l|k}$ as defined in (2) and (3). By taking into account the noises w_k and v_k in (1), then x_k is related to $y_k, y_{k-1}, \ldots, y_{k-h+1}$ as follows:

$$\begin{aligned} y_k &= C_k x_k + v_k \\ y_{k-1} &= C_{k-1} x_{k-1} + v_{k-1} = C_{k-1} \Phi_{k-1|k} x_k + \text{noises} \\ &\vdots \\ y_{k-h+1} &= C_{k-h+1} \Phi_{k-h+1|k} x_k + \text{noises} \end{aligned}$$

where "noises" are sums of v_k and linearly transformed w_k. Then it is easy to build a (non-recursive) *least squares estimator* of x_k from $y_k, y_{k-1}, \ldots, y_{k-h+1}$:

$$\hat{x}_k = M_{k-h+1,k}^{-1} \sum_{i=k}^{k-h+1} \Phi_{i|k}^T C_i^T y_i, \tag{9}$$

where $M_{k-h+1,k}$ is the observability Gramian as defined in (5). According to well known properties of the least squares estimation, this estimator is unbiased and has a bounded error covariance matrix, if the inverse matrix $M_{k-h+1,k}^{-1}$ exists and is bounded [2] for all $k \geq h$ with some

[2] This is a UCO assumed in (Deyst Jr and Price, 1968; Jazwinski, 1970; Deyst Jr, 1973), requiring invertible state transition matrices, unlike the Assumption 2 of the present paper.

fixed integer value h. The bounded error covariance of this least squares estimator then provides an upper bound of the Kalman filter.

The above reasoning is based on the fact that the Kalman filter is optimal among all unbiased causal linear state estimators, recursive or not. An alternative and widespread interpretation of the optimality of the Kalman filter is in the sense of the minimum variance estimator among all linear *recursive filters* of the form

$$\hat{x}_{k+1} = A_k\hat{x}_k + L_{k+1}(y_{k+1} - C_{k+1}A_k\hat{x}_k) \tag{10}$$

with any gain matrix $L_{k+1} \in \mathbb{R}^{n\times m}$, where $\hat{x}_{k+1} \in \mathbb{R}^n$ and $\hat{x}_k \in \mathbb{R}^n$ are respectively the estimates of x_{k+1} and x_k. In the following section, a bounded non-optimal *recursive* filter will be built in the form of (10). Its will provide an upper bound for the Kalman filter based on its optimality in the sense recalled above.

The least squares estimator (9), as used in (Deyst Jr and Price, 1968; Jazwinski, 1970; Deyst Jr, 1973), is *not* recursive. It may seem natural to replace the non-recursive estimator (9) by a recursive least squares (RLS) estimator. It would recursively estimate the same state vector x_k at instant k from more and more past observations, but would not recursively deliver an estimate of x_{k+1} based on an estimate of x_k, like the state estimates $\hat{x}_{k+1}$ and $\hat{x}_k$ in (10).

The least squares estimator recalled above involves the state transition matrix $\Phi_{k-1|k} = A_{k-1}^{-1}$, thus requiring invertible A_{k-1}. To avoid this requirement, a slightly different upper bound was proposed in (Moore and Anderson, 1980), based on a UCO which is the same as in Assumption 2 of this paper.

4. BOUNDEDNESS OF A RECURSIVE FILTER

The key step for establishing an error covariance upper bound of the Kalman filter is to find a non-optimal filter whose error covariance is upper bounded. For linear time invariant (LTI) systems, this would be an easy task, as any exponentially convergent state observer would be sufficient. There exist also well known results for affine LPV systems. For general LTV systems, as formulated in (1), it is much more difficult to design such a non-optimal filter or state observer, due to the arbitrary *time varying* nature of the LTV system. In this case the error dynamics is a *time varying* system, whose stability analysis is much more difficult than in the LTI case. As a matter of fact, for a homogeneous *time varying* system $z_{k+1} = F_kz_k$, keeping the eigenvalues of the square matrix F_k inside the unit circle at all instants k is *neither a sufficient nor a necessary* condition for its stability.

4.1 An LTV filter with exponentially stable error dynamics

Consider recursive filters in the form of

$$\hat{\xi}_{k+1|k} = A_k\hat{\xi}_{k|k} \tag{11a}$$

$$\hat{\xi}_{k+1|k+1} = \hat{\xi}_{k+1|k} + L_{k+1}(y_{k+1} - C_{k+1}\hat{\xi}_{k+1|k}), \tag{11b}$$

which is identical to the Kalman filter (8e)-(8f) except the gain matrix $L_{k+1} \in \mathbb{R}^{n\times m}$ instead of the Kalman gain K_{k+1} recursively computed with (8a)-(8d). The notations $\hat{\xi}_{k|k}$ and $\hat{\xi}_{k+1|k}$ denote state estimates in this (non-optimal) recursive filter, whereas $\hat{x}_{k|k}$ and $\hat{x}_{k+1|k}$ are reserved for the Kalman filter.

It is well known that the Kalman filter (8), with the Kalman gain as computed in (8a)-(8d), is optimal (in the sense of minimum variance) among all recursive filters in the form of (11) with any gain matrix $L_{k+1} \in \mathbb{R}^{n\times m}$.

Let $\tilde{\xi}_{k|k-1}$ and $\tilde{\xi}_{k|k}$ denote respectively the *state prediction error* and the *filter error* (also known as update error), defined as

$$\tilde{\xi}_{k|k-1} \triangleq x_k - \hat{\xi}_{k|k-1} \tag{12}$$

$$\tilde{\xi}_{k|k} \triangleq x_k - \hat{\xi}_{k|k}. \tag{13}$$

It is then derived from (1) and (11) that

$$\tilde{\xi}_{k+1|k} = A_k\tilde{\xi}_{k|k} + w_k \tag{14}$$

$$\tilde{\xi}_{k+1|k+1} = (I_n - L_{k+1}C_{k+1})\tilde{\xi}_{k+1|k} - L_{k+1}v_{k+1}. \tag{15}$$

Combining (14) and (15) yields the prediction error dynamics equation

$$\tilde{\xi}_{k+1|k} = A_k\left(I_n - L_kC_k\right)\tilde{\xi}_{k|k-1} - A_kL_kv_k + w_k. \tag{16}$$

The following result, due to Moore and Anderson, provides a particular design of L_k ensuring the exponential stability of this error dynamics.

<u>*Theorem*</u> *1.* (Moore and Anderson, 1980). Under Assumptions 1 and 2, choose the particular gain matrix L_k as

$$\bar{L}_k \triangleq A_k\Phi_{k,k-h-1}M_{k,k-h-1}^{-1}\Phi_{k,k-h-1}^TC_k^T \tag{17}$$

where $M_{k,k-h-1}$ is the observability Gramian as in (5). Then the homogeneous part of the prediction error dynamics equation (16), namely

$$z_{k+1|k} = A_k\left(I_n - \bar{L}_kC_k\right)z_{k|k-1}, \tag{18}$$

is exponentially stable, in the sense that there exist two positive constants c_1 and c_2 such that, for any integer pairs $l \geq k \geq 0$,

$$\|\bar{\Phi}_{l|k}\| \leq c_2e^{-c_1(l-k)}, \tag{19}$$

where

$$\bar{\Phi}_{l|k} \triangleq A_{l-1}(I_n - \bar{L}_{l-1}C_{l-1})\cdots A_k(I_n - \bar{L}_kC_k). \tag{20}$$

is the state transition matrix of the dynamics equations (16) and (18). □

The proof of this result as given in (Moore and Anderson, 1980, Theorem 3.1) is non trivial, because no valid Lyapunov function is known when A_k is possibly singular. Notice that the notation K_k used in the cited reference corresponds to A_kL_k of the present paper.

In the remaining part of this section, the recursive filter (11) with this particular gain $L_k = \bar{L}_k$ will be analyzed.

4.2 Covariance upper bound of the non-optimal filter

For the recursive filter (11), the covariance matrices of the state prediction error $\tilde{\xi}_{k+1|k}$ and of the filter error $\tilde{\xi}_{k|k}$ will be denoted by

$$\Pi_{k+1|k} \triangleq \text{cov}[\tilde{\xi}_{k+1|k}] \tag{21}$$

$$\Pi_{k|k} \triangleq \text{cov}[\tilde{\xi}_{k|k}]. \tag{22}$$

Remark that $\Pi_{k+1|k}$ and $\Pi_{k|k}$ are the error covariances of the recursive filter (11) with *any* gain matrix L_k, whereas $P_{k+1|k}$ and $P_{k|k}$ are reserved for the Kalman filter with the optimal gain K_k, as in (8).

Lemma 1. Under Assumptions 1 and 2, let the gain of the recursive filter (11) be $L_k = \bar{L}_k$ as defined in (17), and let $\hat{\xi}_{0|0} = \bar{x}_0 = \mathrm{E}(x_0)$. Then the resulting state prediction error covariance matrix, denoted as $\bar{\Pi}_{k+1|k}$, is upper bounded. □

Proof. First notice that $\mathrm{E}[\tilde{\xi}_{0|0}] = 0$ follows from (13), and then $\mathrm{E}[\tilde{\xi}_{1|0}] = 0$ from (14).

By taking the mathematical expectation at both sides of the prediction error dynamics equation (16), it is recursively shown that $\mathrm{E}[\tilde{\xi}_{k|k-1}] = 0$ for $k = 1, 2, 3, \ldots$.

Notice that, in (16), $\tilde{\xi}_{k|k-1}$ is independent of the white noises w_k and v_k (though $\tilde{\xi}_{k|k-1}$ does depend on noises prior to instant k), then

$$\mathrm{cov}[\tilde{\xi}_{k+1|k}] = A_k (I_n - L_k C_k) \mathrm{cov}[\tilde{\xi}_{k|k-1}] (I_n - L_k C_k)^T A_k^T + A_k L_k \mathrm{cov}[v_k] L_k^T A_k^T + \mathrm{cov}[w_k]. \quad (23)$$

Choose $L_k = \bar{L}_k$, then $\mathrm{cov}[\tilde{\xi}_{k+1|k}] = \bar{\Pi}_{k+1|k}$, and (23) is more compactly rewritten as

$$\bar{\Pi}_{k+1|k} = A_k \left(I_n - \bar{L}_k C_k\right) \bar{\Pi}_{k|k-1} \left(I_n - \bar{L}_k C_k\right)^T A_k^T + \bar{\Lambda}_k, \quad (24)$$

with

$$\bar{\Lambda}_k \triangleq A_k \bar{L}_k R_k \bar{L}_k^T A_k^T + Q_k. \quad (25)$$

The particular gain $\bar{L}_k$ specified in (17) is bounded, because of Assumptions 1 and 2:

$$\|\bar{L}_k\| \leq \|A_k\| \|\Phi_{k,k-h-1}\|^2 \|M_{k,k-h-1}^{-1}\| \|C_k\| \leq c_3, \quad (26)$$

with some constant $c_3 > 0$. Then $\bar{\Lambda}_k$ as defined in (25) is also bounded:

$$\|\bar{\Lambda}_k\| \leq \|A_k\|^2 \|\bar{L}_k\|^2 \|R_k\| + \|Q_k\| \leq c_4, \quad (27)$$

with some constant $c_4 > 0$.

With the state transition matrix $\bar{\Phi}_{l|k}$ as defined in (20), recursively applying (24) yields

$$\bar{\Pi}_{k+1|k} = \bar{\Phi}_{k+1|1} \bar{\Pi}_{1|0} \bar{\Phi}_{k+1|1}^T + \sum_{j=1}^{k} \bar{\Phi}_{k|j} \bar{\Lambda}_j \bar{\Phi}_{k|j}^T. \quad (28)$$

According to (14),

$$\bar{\Pi}_{1|0} = A_0 \mathrm{cov}[\tilde{\xi}_{0|0}] A_0^T + \mathrm{cov}[w_k] = A_0 P_0 A_0^T + Q_0, \quad (29)$$

then

$$\|\bar{\Pi}_{1|0}\| \leq \|A_0\| \|P_0\| \|A_0\| + \|Q_0\| \leq c_5, \quad (30)$$

with some constant $c_5 > 0$.

Based on Theorem 1, $\bar{\Phi}_{l|k}$ satifies (19). It then follows from (28) that

$$\|\bar{\Pi}_{k+1|k}\| \leq \|\bar{\Phi}_{k+1|1}\| \|\bar{\Pi}_{1|0}\| \|\bar{\Phi}_{k+1|1}^T\| + \sum_{j=1}^{k} \|\bar{\Phi}_{k|j}\| \|\bar{\Lambda}_j\| \|\bar{\Phi}_{k|j}^T\| \quad (31)$$

$$\leq c_2^2 e^{-2c_1 k} c_5 + \sum_{j=1}^{k} c_2^2 e^{-2c_1(k-j)} c_4 \quad (32)$$

$$\leq c_2^2 c_5 + c_2^2 c_4 e^{-2c_1 k} \sum_{j=1}^{k} e^{2c_1 j} \quad (33)$$

$$= c_2^2 c_5 + c_2^2 c_4 e^{-2c_1 k} \frac{e^{2c_1 k} - 1}{e^{2c_1} - 1} e^{2c_1} \quad (34)$$

$$\leq c_2^2 c_5 + \frac{c_2^2 c_4 e^{2c_1}}{e^{2c_1} - 1}. \quad (35)$$

An upper bound of $\bar{\Pi}_{k+1|k}$ is thus established. □

5. COVARIANCE UPPER BOUND FOR THE OPTIMAL FILTER

It is well known that the Kalman gain K_k defined in (8c) leads to the optimal filter minimizing the trace of the filter error covariance, *i.e.*,

$$\mathrm{Trace}(P_{k|k}) \leq \mathrm{Trace}(\Pi_{k|k}) \quad (36)$$

where $P_{k|k}$ is the error covariance of the Kalman filter as recusively computed in (8), and $\Pi_{k|k}$ is the error covariance of the recursive filter (11) with *any* gain $L_{k+1} \in \mathbb{R}^{n\times m}$.

In order to establish an upper bound for the Kalman filter, it is possible to continue with the aforementioned trace optimality, but it will be more direct to proceed with a less well known optimality result about the prediction error covariance $P_{k+1|k}$, in accordance with the result of Lemma 1. This optimality result is stated in the following lemma.

Lemma 2. Consider a recursive filter (11) specified with any gain matrix $L_{k+1} \in \mathbb{R}^{n\times m}$, with the corresponding prediction error covariance matrix $\Pi_{k+1|k}$ defined through (11), (12) and (21). The Kalman filter is optimal in the sense that its prediction error covariance $P_{k+1|k}$ satisfies

$$P_{k+1|k} \leq \Pi_{k+1|k}, \quad (37)$$

i.e., the matrix difference $\Pi_{k+1|k} - P_{k+1|k}$ is positive semidefinite, for any gain matrix $L_k \in \mathbb{R}^{n\times m}$ behind $\Pi_{k+1|k}$. □

This result was shown in Theorem 11.5 of (Åström and Wittenmark, 2011). A proof is given in Appendix A for completeness.

Lemmas 1 and 2 then lead to the following result.

Theorem 2. Under Assumptions 1 and 2, the state prediction error covariance $P_{k+1|k}$ of the Kalman filter is upper bounded. □

Proof. According to Lemma 2, $P_{k+1|k} \leq \Pi_{k+1|k}$ for *any* gain matrix $L_k \in \mathbb{R}^{n\times m}$ behind $\Pi_{k+1|k}$. In particular, let $L_k = \bar{L}_k$, then $P_{k+1|k} \leq \bar{\Pi}_{k+1|k}$. Under Assumptions 1 and 2, Lemma 1 ensures that $\bar{\Pi}_{k+1|k}$ is upper bounded, hence $P_{k+1|k}$ is also upper bounded. □

Based on this main result, the following corollary then ensures the boundedness of all the other auxiliary variables involved in the Kalman filter (8), namely $P_{k|k}$, K_k and Σ_k.

Corollary 1. Under Assumptions 1 and 2, the filter error covariance $P_{k|k}$, the Kalman gain K_k and the innovation covariance matrix Σ_k are all bounded. □

Proof. The upper bound of Σ_{k+1} follows from (8b) and the upper bound of $P_{k+1|k}$. The inverse matrix Σ_{k+1}^{-1} is also upper bounded due to the strictly positive lower bound of R_{k+1} assumed in Assumption 1. K_{k+1} is then upper bounded following (8c), so is $P_{k+1|k+1}$ following (8d). □

6. CONCLUSION

Among the properties of the Kalman filter, boundedness is of crucial importance for real time applications. It is also a prerequisite for its error dynamics stability analysis based on the definition of a Lyapunov function. In classical results upper bounds of the Kalman filter have been shown through the upper bounds of non-recursive filters. In this paper, an upper bound of the Kalman filter prediction error covariance is directly established in the sense of matrix positive definiteness, through a bounded recursive non-optimal filter.

Appendix A. PROOF OF LEMMA 2

Proof. Rewrite (23) as

$$\Pi_{k+1|k} = A_k \left(I_n - L_k C_k\right) \Pi_{k|k-1} \left(I_n - L_k C_k\right)^T A_k^T + A_k L_k R_k L_k^T A_k^T + Q_k. \tag{A.1}$$

The right hand side of this equation is a symmetric matrix expression with two quadratic terms involving L_k. Rearrange it for a single square form of L_k,

$$\begin{aligned}&\Pi_{k+1|k}\\ &= A_k(L_k - \Pi_{k|k-1}C_k^T\Omega_k^{-1})\Omega_k(L_k - \Pi_{k|k-1}C_k^T\Omega_k^{-1})^T A_k^T\\ &+ A_k\Pi_{k|k-1}A_k^T - A_k\Pi_{k|k-1}C_k^T\Omega_k^{-1}C_k\Pi_{k|k-1}A_k^T\\ &+ Q_k,\end{aligned} \tag{A.2}$$

where

$$\Omega_k \triangleq C_k\Pi_{k|k-1}C_k^T + R_k \tag{A.3}$$

is an invertible matrix, because $C_k\Pi_{k|k-1}C_k^T \geq 0$ and $R_k \geq \underline{r}I_m > 0$ according to Assumptions 1.

Notice that $\Pi_{k|k-1}$ does not depend on L_k (though it depends on L_{k-1}), nor A_k, C_k, R_k and Ω_k appearing at the right hand side of (A.2). The fact that Ω_k is positive definite implies

$$A_k(L_k - \Pi_{k|k-1}C_k^T\Omega_k^{-1})\Omega_k(L_k - \Pi_{k|k-1}C_k^T\Omega_k^{-1})^T A_k^T \geq 0. \tag{A.4}$$

In view of (A.2), among all the possible gain matrices $L_k \in \mathbb{R}^{n\times m}$, the one annihilating the quadratic expression in (A.4) then minimizes $\Pi_{k+1|k}$. More specifically, the particular gain

$$L_k^* = \Pi_{k|k-1}^* C_k^T \Omega_k^{-1} \tag{A.5}$$

leads to the minimum

$$\Pi_{k+1|k}^* = A_k\Pi_{k|k-1}^*A_k^T - A_k\Pi_{k|k-1}^*C_k^T\Omega_k^{-1}C_k\Pi_{k|k-1}^*A_k^T + Q_k \tag{A.6}$$

which is recursively defined for $k = 1, 1, 2, \ldots$, with $\Pi_{1|0}^* = A_0P_0A_0^T + Q_0$.

Now it remains to show that $\Pi_{k+1|k}^*$ is equal to $P_{k+1|k}$, the one as defined in (8).

Decompose (A.6) into two steps:

$$\Pi_{k+1|k}^* = A_k\Pi_{k|k}^*A_k^T + Q_k \tag{A.7}$$

$$\Pi_{k|k}^* = \Pi_{k|k-1}^* - \Pi_{k|k-1}^*C_k^T\Omega_k^{-1}C_k\Pi_{k|k-1}^*, \tag{A.8}$$

and rearrange (A.7), (A.3), (A.5), (A.8) as

$$\Pi_{k+1|k}^* = A_k\Pi_{k|k}^*A_k^T + Q_k \tag{A.9a}$$

$$\Omega_{k+1} = C_{k+1}\Pi_{k+1|k}^*C_{k+1}^T + R_{k+1} \tag{A.9b}$$

$$L_{k+1}^* = \Pi_{k+1|k}C_{k+1}^T\Omega_{k+1}^{-1} \tag{A.9c}$$

$$\Pi_{k+1|k+1}^* = (I_n - L_{k+1}^*C_{k+1})\Pi_{k+1|k}^*. \tag{A.9d}$$

The recursion in (A.9) is indeed identical to the one defined by (8a)-(8d). By choosing the same initial value $\Pi_{0|0}^* = P_0 = P_{0|0}$, the equalities

$$L_k^* = K_k \tag{A.10}$$

$$\Pi_{k+1|k}^* = P_{k+1|k} \tag{A.11}$$

$$\Pi_{k|k}^* = P_{k|k} \tag{A.12}$$

hold for all $k = 0, 1, 2, \ldots$.

It is thus concluded that $P_{k+1|k} = \Pi_{k+1|k}^* \leq \Pi_{k+1|k}$ for all $k = 0, 1, 2, \ldots$. □

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