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RESEARCH ARTICLE

On convergence conditions for generalized Persidskii systems

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Summary

The convergence conditions for a class of generalized Persidskii systems and their discretized dynamics are introduced, which can be checked through linear (matrix) inequalities. The case of almost periodic convergence for this class of systems with almost periodic input is also studied. The proposed results are applied to a Lotka-Volterra model and opinion dynamics.

KEYWORDS:
convergence conditions, generalized Persidskii systems, LMI

1 | INTRODUCTION

A dynamical system is convergent if it admits a unique and bounded asymptotically stable solution [1,2]. So computing only one trajectory is sufficient to determine the asymptotic behavior of all others. This property is similar to the incremental stability [3,4] (without the boundedness requirement) and the contraction theory is frequently used for its analysis [5]. The convergence property has received increased attention due to its usefulness in system analysis and design, including the case of several co-existing solutions for nonlinear output regulation [6], synchronization [7,8,9], steady-state analysis of nonlinear systems [10] and many other control problems.

It is evident that not all nonlinear systems are convergent: the trajectories may be convergent for some inputs (*e.g.*, zero input), and an example of such a behavior is demonstrated by Duffing's model [11], in which multiple limit cycles phenomena and sub-harmonic oscillations may occur. Thus, for nonlinear systems, it is required to consider additional conditions for investigating the convergence property.

We may additionally ask for preservation conditions for the attracting solution of the (almost) periodicity of the input [12,13]. Note that in nonlinear systems, the frequency of entertainment is not necessarily kept by the output [14], and a typical example is again the Duffing's model, which can generate chaotic trajectories under harmonic excitation. The relations between the existence of almost periodic solutions and properties of the right-hand sides of dynamical systems were intensively studied during the last century (see, *e.g.*, Markov's theorem [15]).

The convergence property itself also has a rich history of investigation. For example, Pliss [16] studied convergence of the system $\dot{x} = f(t, x)$ with the periodic right-hand side. Later, Demidovich [17] extended the particular case analyzed by Pliss to a more general f. In the same vein, Zubov [18,19] established a criterion of periodic/almost periodic convergence for nonlinear systems. Further exploration of convergence properties for nonlinear systems includes convergence-like properties by Lyapunov's second method [20], sufficient conditions for the convergence of a system with a scalar nonlinearity satisfying an incremental sector condition [21], the convergence of perturbed nonlinear systems [22,3], and convergence conditions for a class of nonlinear dynamics [23].

Digital revolution highlighted importance and originated discrete-time modeling for many phenomena. In addition, for real-world systems having the models formulated in continuous time, for implementation or simulation, these models have to be discretized. For strongly nonlinear systems the obtained discrete-time representations may loose the stability and convergence properties of the continuous-time counterparts [24,25]. This motivates development of conditions of similar quality performance characteristics for the systems functioning in discrete time.

In this paper, the convergence property and existence of almost periodic trajectories will be studied in continuous and discrete time for a class of Persidskii systems, which was introduced in [26,27] as $\dot{x} = A_0 x + A_1 f(x)$, and recently its generalized version was investigated in [28,29] by taking into account multiple nonlinearities: $\dot{x} = A_0 x + \sum_{i=1}^{M} A_i f_i(x)$ (more detailed definition is given below). The advantage of this class of models is that despite its important nonlinear nature (with $M \geq 1$), a Lyapunov function is suggested in the literature such that the analysis of its properties can be performed by solving linear matrix inequalities (LMIs), which is a constructive approach.

The organization of this paper is as follows. In Section 2, the preliminaries and the definitions of used stability properties are presented. The class of considered systems and the imposed conditions for nonlinearities are introduced in Section 3. The basic conditions of convergence for generalized Persidskii systems, and of almost periodic convergence for those systems with Metzler coefficient matrices are given in Section 4. In Section 5 convergence conditions for two types of discretizations of the considered systems are studied. The Lotka-Volterra model and opinion dynamics are considered to illustrate the efficiency of the proposed results in Section 6.

NOTATION

- $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ represent the sets of natural numbers, integers and real numbers, respectively, $\mathbb{R}_+ = \{s \in \mathbb{R} : s \geq 0\}$. $\mathbb{R}^n$ ($\mathbb{R}^n_+$) and $\mathbb{R}^{m\times n}$ denote the vector spaces of n-tuples of real numbers (nonnegative real numbers) and $m \times n$ real matrices, respectively. The set of $n \times n$ diagonal matrices with nonnegative elements on the diagonal is denoted by $\mathbb{D}^n_+$.
- The symbol $\|\cdot\|$ refers to the Euclidean norm on $\mathbb{R}^n$ (and the induced matrix norm $\|A\|$ for a matrix $A \in \mathbb{R}^{m\times n}$).
- The transpose of a matrix $A \in \mathbb{R}^{n\times n}$ is denoted by $A^\top$.
- For $p, n \in \mathbb{N}$ with $p \leq n$, the notation $\overline{p,n}$ is used to represent the set $\{p, \dots, n\}$. For all $i, j \in \overline{p,n}$, let $(B_{i,j})^n_{i,j=p}$ denote the block matrix $\begin{bmatrix} B_{p,p} & \cdots & B_{p,n} \\ \vdots & \ddots & \vdots \\ B_{n,p} & \cdots & B_{n,n} \end{bmatrix}$.
- The $p \times n$ block diagonal matrix with matrices $v_i \in \mathbb{R}^{p_i \times n_i} (p = \sum_{i=1}^{N} p_i,\ n = \sum_{i=1}^{N} n_i)$ along the main diagonal is denoted by $\mathrm{diag}(v_1, ..., v_N)$. Let I_n stand for the $n \times n$ identity matrix, and denote the vector of dimension n with all elements equal 1 by $\mathbf{1}_n$.
- For a vector $v \in \mathbb{R}^n$, $v > 0$ ($v < 0$) represents $v_i > 0$ ($v_i < 0$) for $i \in \overline{1,n}$. For $r \in \mathbb{R}$, let $|r|$ denote its absolute value.
- For a Lebesgue measurable function $u : \mathbb{R} \to \mathbb{R}^q$, define the norm $\|u\|_{(t_1,t_2)} = \operatorname{ess\,sup}_{t\in(t_1,t_2)} \|u(t)\|$ for $(t_1, t_2) \subseteq \mathbb{R}$. We denote by $\mathscr{L}^q_\infty$ the space of functions u with $\|u\|_\infty := \|u\|_{(-\infty,+\infty)} < +\infty$. For a bounded input sequence $\{\tilde{u}(k)\}_{k\in\mathbb{Z}}$, $\tilde{u}(k) \in \mathbb{R}^q$, define the norm $\|\tilde{u}\|_{\overline{k_1,k_2}} = \sup_{k'\in\overline{k_1,k_2}} \|\tilde{u}(k')\|$ for $k_1, k_2 \in \mathbb{Z}$. Let $\mathscr{U}^q_\infty$ denote the space of functions $\tilde{u}$ with $\|\tilde{u}\|_\infty := \|\tilde{u}\|_{\overline{k_1,k_2}} < +\infty$ for $k_1 \to -\infty, k_2 \to +\infty$.
- A continuous function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathscr{K}$ if it is strictly increasing and $\sigma(0) = 0$; it belongs to class $\mathscr{K}_\infty$ if it is also unbounded. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathscr{KL}$ if $\beta(\cdot, r) \in \mathscr{K}$ and $\beta(r, \cdot)$ is a decreasing to zero function for any fixed $r > 0$.
- For a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$, denote by $\nabla V(v) f(v)$ the Lie derivative of V along the vector field f evaluated at point $v \in \mathbb{R}^n$.

2 | PRELIMINARIES

Definition 1. The matrix $A \in \mathbb{R}^{n\times n}$ is said to be Hurwitz if all its eigenvalues lie in the open left half of the complex plane.

Definition 2. The matrix $A \in \mathbb{R}^{n\times n}$ is Metzler if all its off-diagonal entries are nonnegative.

Proposition 1 ([30]). Let $A \in \mathbb{R}^{n\times n}$ be Metzler. Then A is Hurwitz if and only if there exists a vector $v > 0$ in $\mathbb{R}^n$ with $A^\top v < 0$.

Definition 3 ([15]). A continuous function $\psi : \mathbb{R} \to \mathbb{R}^n$ is called almost periodic if, for every $\varepsilon > 0$ there exists $\tau_\varepsilon > 0$ such that every interval $[r, r + \tau_\varepsilon]$ for any $r \in \mathbb{R}$ contains at least one number η for which

$$\|\psi(t + \eta) - \psi(t)\| < \varepsilon$$

for all $t \in \mathbb{R}$.

2.1 | Stability properties in continuous time

Let the differential equation

$$\begin{aligned} \dot{x}(t) &= F(x(t), u(t)),\ t \geq t_0 \in \mathbb{R}, \\ y(t) &= H(x(t)) \end{aligned} \tag{1}$$

be given, where $x(t) \in \mathbb{R}^n$ is the state vector; $u(t) \in \mathbb{R}^q$ is the external input, $u \in \mathscr{L}_\infty^q$; $y(t) \in \mathbb{R}^p$ is the output vector. Moreover, $F : \mathbb{R}^{n+q} \to \mathbb{R}^n$ is a nonlinear continuous mapping and $H : \mathbb{R}^n \to \mathbb{R}^p$ is continuously differentiable. For an initial state $x_0 \in \mathbb{R}^n$, initial time $t_0 \in \mathbb{R}$ and $u \in \mathscr{L}_\infty^q$ we denote a corresponding solution of the system (1) by $x(t, x_0, t_0, u)$ for the values of $t \geq t_0$ the solution exists, so the corresponding output is $y(t, x_0, t_0, u) = H\left(x(t, x_0, t_0, u)\right)$. In the case that $t_0 = 0$ (*i.e.*, $t \in \mathbb{R}_+$), we simplify the corresponding denotations to: $x(t, x_0, u)$ and $y(t, x_0, u)$, which are also applicable in the sequel.

Definition 4 ([31]). The system (1) is called forward complete if for all $x_0 \in \mathbb{R}^n$ and $u \in \mathscr{L}_\infty^q$, the solution $x(t, x_0, u)$ is uniquely defined for all $t \in \mathbb{R}_+$.

Definition 5 ([31]). A forward complete system (1) is said to be

1. input-to-output stable (IOS) if there exist $\beta \in \mathscr{KL}$ and $\gamma \in \mathscr{K}$ such that
$$\|y(t, x_0, u)\| \leq \beta\left(\|x_0\|, t\right) + \gamma(\|u\|_\infty),\ \forall t \in \mathbb{R}_+$$
for any $x_0 \in \mathbb{R}^n$ and $u \in \mathscr{L}_\infty^q$. In the special case when $y = x$, the IOS property is called input-to-state stability (ISS).
2. state-independent input-to-output stable (SIIOS) if there exist $\beta \in \mathscr{KL}, \gamma \in \mathscr{K}$ such that
$$\|y(t, x_0, u)\| \leq \beta(\|H(x_0)\|, t) + \gamma(\|u\|_\infty),\ \forall t \in \mathbb{R}_+$$
for any $x_0 \in \mathbb{R}^n$ and $u \in \mathscr{L}_\infty^q$.
3. robustly output stable (ROS) if there exist a smooth function $\alpha \in \mathscr{K}_\infty$ and $\beta \in \mathscr{KL}$ such that the system
$$\dot{x} = \tilde{F}(x, \kappa) := F(x, \kappa\alpha(\|H(x)\|)) \tag{2}$$
is forward complete, and the estimate
$$\|y_\alpha(t, x_0, \kappa)\| \leq \beta(\|x_0\|, t),\ \forall t \in \mathbb{R}_+$$
is satisfied for all $x_0 \in \mathbb{R}^n,\ \kappa \in \mathscr{L}_\infty^q$ with $\|\kappa\|_\infty \leq 1$ and $y_\alpha(t, x_0, \kappa) = H(x(t, x_0, \kappa))$ denotes the output function of the system (2).

Definition 6 ([31]). A forward complete system (1) is said to be uniformly bounded-input-bounded-state stable (UBIBS) if there exists $\beta \in \mathscr{K}$ such that

$$\|x(t, x_0, u)\| \leq \max\{\beta(\|x_0\|), \beta(\|u\|_\infty)\},\ \forall t \in \mathbb{R}_+$$

for all $x_0 \in \mathbb{R}^n$ and $u \in \mathscr{L}_\infty^q$.

2.2 | Convergence definitions

Definition 7 ([1]). The system (1) is convergent if it admits a unique bounded solution for $t \in \mathbb{R}$ that is globally asymptotically stable.

Definition 8 ([19]). The system (1) possesses almost periodic convergence property, if it is convergent and the admitted solution is almost periodic.

2.3 | Stability properties in discrete time

Consider a discrete-time system:

$$\begin{aligned} x(k+1) &= \bar{F}(x(k), u(k)), \\ y(k) &= H(x(k)), \end{aligned} \tag{3}$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^q$ and $y(k) \in \mathbb{R}^p$ are defined as before; $k_0 \leq k \in \mathbb{Z}$ with $k_0 \in \mathbb{Z}$; $\bar{F} : \mathbb{R}^{n+q} \to \mathbb{R}^n$ is a continuous mapping; $u \in \mathscr{U}_\infty^q$, the mapping H and the solution for (3) are defined as in the system (1).

Definition 9 ([32]). The system (3) is called input-to-state stable (ISS) if there exist $\beta \in \mathscr{KL}$ and $\gamma \in \mathscr{K}$ such that

$$\|x(k, x_0, u)\| \leq \beta\left(\|x_0\|, k\right) + \gamma(\|u\|_{\overline{\infty}})$$

for any $k \in \mathbb{N}$, $x_0 \in \mathbb{R}^n$ and $u \in \mathscr{U}_\infty^q$.

2.4 | Lyapunov characterizations

Definition 10 ([31]). For the system (1), a smooth function $V : \mathbb{R}^n \to \mathbb{R}_+$ is

1. an ISS-Lyapunov function if there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathscr{K}_\infty$ and $\chi \in \mathscr{K}$ such that

$$\begin{aligned} &\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \\ &\|x\| \geq \chi(\|u\|) \Rightarrow \nabla V(x) F(x,u) \leq -\alpha_3(\|x\|) \end{aligned} \tag{4}$$

   for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^q$.

2. an IOS-Lyapunov function if there exist $\alpha_1, \alpha_2 \in \mathscr{K}_\infty$, $\chi \in \mathscr{K}$ and $\alpha_3 \in \mathscr{KL}$ such that

$$\begin{aligned} &\alpha_1(\|H(x)\|) \leq V(x) \leq \alpha_2(\|x\|), \\ &V(x) \geq \chi(\|u\|) \Rightarrow \nabla V(x) F(x,u) \leq -\alpha_3(V(x), \|x\|) \end{aligned} \tag{5}$$

   for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^q$.

3. an SIIOS-Lyapunov function if there exist $\alpha_1, \alpha_2 \in \mathscr{K}_\infty$ and $\chi, \alpha_3 \in \mathscr{K}$ such that

$$\begin{aligned} &\alpha_1(\|H(x)\|) \leq V(x) \leq \alpha_2(\|H(x)\|), \\ &V(x) \geq \chi(\|u\|) \Rightarrow \nabla V(x) F(x,u) \leq -\alpha_3(V(x)) \end{aligned} \tag{6}$$

   for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^q$.

4. a ROS-Lyapunov function if there exist $\alpha_1, \alpha_2 \in \mathscr{K}_\infty$, $\chi \in \mathscr{K}$ and $\alpha_3 \in \mathscr{KL}$ such that

$$\begin{aligned} &\alpha_1(\|H(x)\|) \leq V(x) \leq \alpha_2(\|x\|), \\ &\|H(x)\| \geq \chi(\|u\|) \Rightarrow \nabla V(x) F(x,u) \leq -\alpha_3(V(x), \|x\|) \end{aligned} \tag{7}$$

   for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^q$.

Definition 11 ([32]). For the system (3), a smooth function $V : \mathbb{R}^n \to \mathbb{R}_+$ is an ISS-Lyapunov function if there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathscr{K}_\infty$ and $\chi \in \mathscr{K}$ such that

$$\begin{aligned} &\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \\ &\|x\| \geq \chi(\|u\|) \Rightarrow V(\bar{F}(x,u)) - V(x) \leq -\alpha_3(\|x\|) \end{aligned} \tag{8}$$

for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^q$.

Theorem 1 ([33,32]). The system (1) or (3) is ISS if and only if it admits an ISS-Lyapunov function.

Theorem 2 ([31]). A UBIBS system (1) is IOS (ROS) if and only if it admits an IOS (ROS)-Lyapunov function.

Theorem 3 ([31]). A forward complete system (1) is SIIOS if and only if it admits an SIIOS-Lyapunov function.

Remark 1. The UBIBS requirement can be dropped in the sufficient conditions of IOS or ROS if the system (1) is forward complete (or it possesses the unboundedness observability property[34]) and admits an IOS-Lyapunov function satisfying the conditions (5), with

$$V(x) \geq \chi(\|u\|) \Rightarrow \nabla V(x) F(x,u) \leq -\alpha_3(V(x)) \tag{9}$$

or a ROS-Lyapunov function with the conditions (7), and

$$\|H(x)\| \geq \chi(\|u\|) \Rightarrow \nabla V(x) F(x,u) \leq -\alpha_3(V(x)) \tag{10}$$

for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^q$, some $\alpha_1, \alpha_2 \in \mathscr{K}_\infty$ and $\chi, \alpha_3 \in \mathscr{K}$, respectively.

In the rest of the paper, to lighten the notation the time-dependency of variables might remain implicitly understood, for instance we will write x for $x(t)$ or $x(k)$.

3 | PROBLEM STATEMENT

Consider the following system in generalized Persidskii form[27]:

$$\dot{x}(t) = A_0 x(t) + \sum_{j=1}^{M} A_j F_j(x(t)) + \varphi(t),\ t \geq t_0, \tag{11}$$

where $x(t) \in \mathbb{R}^n$ is the state vector; $t_0 \in \mathbb{R}$ denotes the initial time and $x(t_0) = x_0$; $A_s \in \mathbb{R}^{n\times n}$ for $s \in \overline{0, M}$ ($M \in \mathbb{N} \setminus \{0\}$); $F_j(x(t)) = [f_j^1(x_1(t)) \dots f_j^n(x_n(t))]^\top$ for $j \in \overline{1, M}$ are the functions ensuring the existence of the solutions of the system (11) at least locally in time; the disturbance input $\varphi = [\ \varphi_1\ \dots\ \varphi_n\ ]^\top \in \mathscr{L}_\infty^n$.

In this paper, the corresponding term (in sum or a sequence) must be omitted if for an index the upper bound is smaller than the lower one.

The sector restrictions on $f_j^i : \mathbb{R} \to \mathbb{R}$, $j \in \overline{1, M}$, $i \in \overline{1, n}$, are imposed as:

Assumption 1. Assume that for any $i \in \overline{1, n}$ and $j \in \overline{1, M}$,

$$v f_j^i(v) > 0,\ \forall v \in \mathbb{R}\backslash\{0\}.$$

Under Assumption 1, there exists an index $m \in \overline{0, M}$ such that for all $i \in \overline{1, n}$, $a \in \overline{1, m}$

$$\lim_{v\to\pm\infty} f_a^i(v) = \pm\infty,$$

and there exists $\mu \in \overline{m, M}$ such that for all $i \in \overline{1, n}$, $b \in \overline{1, \mu}$

$$\lim_{v\to\pm\infty} \int_0^v f_b^i(r)dr = +\infty.$$

In such a case $m = 0$ implies that all nonlinearities are bounded (at least in positive or negative direction for a component of the state vector).

Assumption 2. Assume that the functions f_j^i are continuous and strictly increasing for all $i \in \overline{1, n}$ and $j \in \overline{1, M}$.

Note that $\mu = M$ under Assumption 2.

Along with (11), consider the system

$$\dot{y}(t) = A_0 y(t) + \sum_{j=1}^{M} A_j F_j(y(t)) + \varphi(t) \tag{12}$$

with the same input but with different initial conditions $y(t_0) \in \mathbb{R}^n$. Define $z = y - x$, then the error dynamics of (11), (12) is

$$\dot{z} = A_0 z + \sum_{j=1}^{M} A_j G_j(x, z), \tag{13}$$

where

$$G_j(x,z) = F_j(z+x) - F_j(x) = [\, g_j^1(x_1,z_1) \;\dots\; g_j^n(x_n,z_n)\,]^\top \in \mathbb{R}^n,$$
$$g_j^r(x_r,z_r) := f_j^r(z_r+x_r) - f_j^r(x_r),\ \forall j \in \overline{1,M}, r \in \overline{1,n}.$$

Note that for any $x \in \mathbb{R}^n$ the functions G_j in the variable $z \in \mathbb{R}^n$ satisfy the properties formulated in assumptions 1 and 2.

We then impose the following assumption used for the stability analysis of the system (13) in Section 4:

Assumption 3. Assume that there exist $S_0^j, S_1^j, S_2^j, S_3^{j,j'}, H_0^j, H_1^j, H_2^j, H_3^{j,j'} \in \mathbb{D}_+^n$ with $j, j' \in \overline{1,M}$ such that

$$G_j(x,z)^\top G_j(x,z) \le z^\top S_0^j z + 2z^\top S_1^j G_j(x,z) + 2z^\top S_2^j F_j(z) + 2\sum_{j'=1}^{M} G_j(x,z)^\top S_3^{j,j'} F_{j'}(z)$$

and

$$F_j(z)^\top F_j(z) \le z^\top H_0^j z + 2z^\top H_1^j G_j(x,z) + 2z^\top H_2^j F_j(z) + 2\sum_{j'=1}^{M} G_j(x,z)^\top H_3^{j,j'} F_{j'}(z)$$

for all $x, y \in \mathbb{R}^n$ and $z = x - y$.

Any globally Lipschitz function F_j satisfies such a condition (moreover, in our setting, the local Lipschitz continuity is enough, as it will be explained in Remark 2 below).

Note that for analysis of the convergence property in (11) we need to consider the system with $t \in \mathbb{R}$. However, the time dependence of the right-hand side of (11) comes through the input $\varphi(t)$ only, then using IOS/ISS framework we can investigate the global stability of that system for $t \in \mathbb{R}_+$ with $\varphi \in \mathcal{L}_\infty^n$, which allows us next to extend the obtained results to $t \in \mathbb{R}$.

4 | MAIN RESULTS

The conditions of (almost periodic) convergence of (11) are given in this section for the general case and for the case of positive systems (all matrices $A_s,\ s \in \overline{0,M}$ are Metzler).

4.1 | General convergence conditions

Theorem 4. Let assumptions 1, 2 and 3 be satisfied. If there exist $0 \le P = P^\top,\ \tilde{P} = \tilde{P}^\top \in \mathbb{R}^{n\times n}$; $\{\Lambda^j = \operatorname{diag}(\Lambda^{j,1},\dots,\Lambda^{j,n})\}_{j=1}^M,\ \{\tilde{\Lambda}^j = \operatorname{diag}(\tilde{\Lambda}^{j,1},\dots,\tilde{\Lambda}^{j,n})\}_{j=1}^M,\ \{\Xi^s\}_{s=0}^M,\ \{\Upsilon_{s,r}\}_{0\le s<r\le M}$, $\{\tilde{\Upsilon}_{j,j'}\}_{j,j'=1}^M,\ \{\Gamma_j\}_{j=1}^M,\ \{\Omega_j\}_{j=1}^M \subset \mathbb{D}_+^n$; $\tilde{\Xi}^0 \in \mathbb{D}_+^n;\ 0 < \Phi = \Phi^\top \in \mathbb{R}^{n\times n}$; $\gamma,\ \theta > 0$ such that

$$P + \sum_{j=1}^{\mu} \Lambda^j > 0;\ Q = Q^\top \le 0, \tag{14}$$

$$\tilde{P} + \sum_{j=1}^{\mu} \tilde{\Lambda}^j > 0;\ \tilde{Q} = \tilde{Q}^\top \le 0, \tag{15}$$

$$\Xi^0 + \sum_{j=1}^{M} \Upsilon_{0,j} + \sum_{s=1}^{m} \Xi^s + \sum_{s=1}^{m}\sum_{r=s+1}^{m} \Upsilon_{s,r} > 0, \tag{16}$$

$$\tilde{\Xi}^0 - \gamma\sum_{j=1}^{M} S_0^j - \theta\sum_{j=1}^{M} H_0^j \ge 0,\ \Gamma_j - \gamma S_1^j - \theta H_1^j \ge 0,\ \Omega_j - \gamma S_2^j - \theta H_2^j \ge 0,\ \tilde{\Upsilon}_{j,j'} - \gamma S_3^{j,j'} - \theta H_3^{j,j'} \ge 0,$$
$$\tilde{\Xi}^0 - \gamma\sum_{j=1}^{M} S_0^j - \theta\sum_{j=1}^{M} H_0^j + \sum_{j=1}^{M}\left(\Gamma_j - \gamma S_1^j - \theta H_1^j + \Omega_j - \gamma S_2^j - \theta H_2^j\right) + \sum_{j=1}^{M}\sum_{j'=1}^{M} \tilde{\Upsilon}_{j,j'} - \gamma S_3^{j,j'} - \theta H_3^{j,j'} > 0, \tag{17}$$

where

$$Q = (Q_{a,b})_{a,b=1}^{M+2},\ \tilde{Q} = (\tilde{Q}_{a,b})_{a,b=1}^{3}$$

with

$$
\begin{gathered}
Q_{1,1} = A_0^\top P + P A_0 + \Xi^0, \\
Q_{j+1,j+1} = A_j^\top \Lambda^j + \Lambda^j A_j + \Xi^j,\ j \in \overline{1, M}, \\
Q_{1,j+1} = P A_j + A_0^\top \Lambda^j + \Upsilon_{0,j},\ j \in \overline{1, M}, \\
Q_{s+1,r+1} = A_s^\top \Lambda^r + \Lambda^s A_r + \Upsilon_{s,r},\ s \in \overline{1, M-1},\ r \in \overline{s+1, M}, \\
Q_{1,M+2} = P,\ Q_{M+2,M+2} = -\Phi,\ Q_{j+1,M+2} = \Lambda^j,\ j \in \overline{1, M},
\end{gathered}
$$

$$
\begin{gathered}
\tilde{Q}_{1,1} = A_0^\top P + P A_0 + \tilde{\Xi}^0,\ \tilde{Q}_{2,2} = -\gamma I_{nM},\ \tilde{Q}_{1,2} = PA + \Gamma, \\
\tilde{Q}_{1,3} = A_0^\top \tilde{\Lambda} + \Omega,\ \tilde{Q}_{2,3} = A^\top \tilde{\Lambda} + \tilde{\Upsilon},\ \tilde{Q}_{3,3} = -\theta I_{nM}, \\
A = \begin{bmatrix} A_1 & \dots & A_M \end{bmatrix},\ \Gamma = \begin{bmatrix} \Gamma_1 & \dots & \Gamma_M \end{bmatrix}, \\
\tilde{\Lambda} = \begin{bmatrix} \tilde{\Lambda}^1 & \dots & \tilde{\Lambda}^M \end{bmatrix},\ \Omega = \begin{bmatrix} \Omega_1 & \dots & \Omega_M \end{bmatrix},\ \tilde{\Upsilon} = (\tilde{\Upsilon}_{j,j'})_{j,j'=1}^{M},
\end{gathered}
$$

then a forward complete system (11) is convergent.

Proof. Consider a candidate Lyapunov function

$$
V(x) = x^\top P x + 2 \sum_{j=1}^{M} \sum_{i=1}^{n} \Lambda^{j,i} \int_0^{x_i} f_j^i(\tau) d\tau, \tag{18}
$$

which is positive definite and radially unbounded due to Finsler's lemma under the condition (14) and Assumption 1 (the first inclusion property in (4) is verified). Then, consider the time derivative of V:

$$
\begin{aligned}
\dot{V} &= \dot{x}^\top P x + x^\top P \dot{x} + 2 \sum_{j=1}^{M} \dot{x}^\top \Lambda^j F_j(x) \\
&= x^\top \left(A_0^\top P + P A_0 \right) x + \left(\sum_{j=1}^{M} F_j(x)^\top A_j^\top \right) P x + x^\top P \sum_{j=1}^{M} A_j F_j(x) + 2 x^\top P \varphi \\
&\quad + 2 \sum_{j=1}^{M} \left(x^\top A_0^\top \Lambda^j F_j(x) + \varphi^\top \Lambda^j F_j(x) + \left(\sum_{s=1}^{M} F_s(x)^\top A_s^\top \right) \Lambda^j F_j(x) \right).
\end{aligned}
$$

Therefore, under (14) we obtain

$$
\begin{aligned}
\dot{V} &= \begin{bmatrix} x \\ F_1(x) \\ \vdots \\ F_M(x) \\ \varphi \end{bmatrix}^\top Q \begin{bmatrix} x \\ F_1(x) \\ \vdots \\ F_M(x) \\ \varphi \end{bmatrix} - x^\top \Xi^0 x - \sum_{j=1}^{M} F_j(x)^\top \Xi^j F_j(x) - 2 \sum_{j=1}^{M} x^\top \Upsilon_{0,j} F_j(x) \\
&\quad - 2 \sum_{s=1}^{M-1} \sum_{r=s+1}^{M} F_s(x)^\top \Upsilon_{s,r} F_r(x) + \varphi^\top \Phi \varphi \\
&\leq -x^\top \Xi^0 x - \sum_{j=1}^{M} F_j(x)^\top \Xi^j F_j(x) - 2 \sum_{j=1}^{M} x^\top \Upsilon_{0,j} F_j(x) \\
&\quad - 2 \sum_{s=1}^{M-1} \sum_{r=s+1}^{M} F_s(x)^\top \Upsilon_{s,r} F_r(x) + \varphi^\top \Phi \varphi,
\end{aligned}
$$

where the right-hand side of the last inequality is radially unbounded as a function of x under the condition (16) and Assumption 1, then the second relation in (4) can be recovered and the ISS property of the system (11) is guaranteed. Therefore, all solutions of the system are bounded.

Next, we need to establish the asymptotic stability of the dynamics (13) that implies existence of a globally attracting solution in (11), for which a Lyapunov function

$$
\tilde{V}(z) = z^\top \tilde{P} z + 2 \sum_{j=1}^{M} \sum_{i=1}^{n} \tilde{\Lambda}^{j,i} \int_0^{z_i} f_j^i(\tau) d\tau
$$

is considered, and its positive definiteness and radial unboundedness in the variable z follows the same argumentation as before, whose time derivative admits the following representation:

$$\dot{\tilde{V}} = \begin{bmatrix} z \\ G_1(x,z) \\ \vdots \\ G_M(x,z) \\ F_1(z) \\ \vdots \\ F_M(z) \end{bmatrix}^\top \tilde{Q} \begin{bmatrix} z \\ G_1(x,z) \\ \vdots \\ G_M(x,z) \\ F_1(z) \\ \vdots \\ F_M(z) \end{bmatrix} + \gamma \sum_{j=1}^{M} G_j(x,z)^\top G_j(x,z) + \theta \sum_{j=1}^{M} F_j^\top(z) F_j(z)$$
$$-z^\top \tilde{\Xi}^0 z - 2\sum_{j=1}^{M} z^\top \Gamma_j G_j(x,z) - 2\sum_{j=1}^{M} z^\top \Omega_j F_j(z) - 2\sum_{j=1}^{M}\sum_{j'=1}^{M} G_j(x,z)^\top \tilde{\Upsilon}_{j,j'} F_{j'}(z).$$

Since $\tilde{Q} \le 0$ and applying Assumption 3, we get

$$\dot{\tilde{V}} \le -z^\top \left(\tilde{\Xi}^0 - \gamma \sum_{j=1}^{M} S_0^j - \theta \sum_{j=1}^{M} H_0^j \right) z - 2\sum_{j=1}^{M} z^\top \left(\Gamma_j - \gamma S_1^j - \theta H_1^j \right) G_j(x,z)$$
$$-2\sum_{j=1}^{M} z^\top \left(\Omega_j - \gamma S_2^j - \theta H_2^j \right) F_j(z) - 2\sum_{j=1}^{M} G_j(x,z)^\top \sum_{j'=1}^{M} \left(\tilde{\Upsilon}_{j,j'} - \gamma S_3^{j,j'} - \theta H_3^{j,j'} \right) F_{j'}(z).$$

The second and the fourth terms in the above estimate are nonnegative due to Assumption 2. The SIIOS characteristics in (6) are verified due to (17). Therefore, we can substantiate that the system (11), (13) is SIIOS with respect to z, then there exists $\beta \in \mathscr{KL}$ such that

$$\|z(t, z_0, 0)\| \le \beta(\|z_0\|, t), \ \forall t \in \mathbb{R}_+$$

for any $z_0 \in \mathbb{R}^n$. Finally, (11) has all solutions bounded and converging to a unique globally attracting one, which describes the convergence property due to Definition 7. □

Remark 2. Note that Assumption 3 is used only in the second part of the proof of Theorem 4, where the ISS property of the system (11) has been already established. It means that despite the assumption is formulated for all $x, y \in \mathbb{R}^n$, the variable x (that represents the limit motion) can be assumed to be bounded, which enlarges greatly the class of admissible nonlinearities verifying Assumption 3. The current formulation of the assumption is chosen for brevity of exposition.

Remark 3. To prove Theorem 4, in the first part, it is enough to substantiate global stability of trajectories for the system (11), then conditions of ISS can be relaxed. For example, all bounded nonlinearities can be treated as a part of the input φ, or some cross-terms including such nonlinearities, as $\varphi^\top \Lambda^j F_j(x)$ for instance, can be hidden in φ, which may sparse the matrix Q.

Remark 4. It is also possible to use IOS/ROS properties to show the convergence for the system (13) (see Theorem 2 and Remark 1 for proof). Let

$$H(z) = z \tag{19}$$

be the output of (13) and consider IOS or ROS property of the system (11), (13), (19). We see that the system (11), (13), (19) is with zero input, the conditions (7), (9) and (10) can be verified. Thus, all solutions of (13) are bounded by

$$\|z(t, z_0, 0)\| \le \beta\left(\|z_0\|, t\right)$$

for some $\beta \in \mathscr{KL}$, all $z_0 \in \mathbb{R}^n$ and $t \ge 0$.

Corollary 1. Let $\varphi \in \mathscr{L}_\infty^n$ be an almost periodic function for $t \in \mathbb{R}$, the functions F_j, $j \in \overline{1, M}$ be locally Lipschitz continuous and all conditions of Theorem 4 be verified, then the system (11) possesses almost periodic convergence property.

Proof. The claim repeats the arguments given in [18,19] for a convergent system (11). □

4.2 | Case of Metzler matrices

In this subsection, we consider convergence and almost periodic convergence conditions for the system (11) with coefficient matrices satisfying the following assumption:

Assumption 4. Assume that the matrices $A_s \in \mathbb{R}^{n\times n}$, $s \in \overline{0, M}$ are Metzler and there exists a vector $\nu > 0$ in $\mathbb{R}^n$ with $\nu^\top A_s = -c_s^\top \leq 0$ for $s \in \overline{0, M}$ and $\sum_{s'=0}^{m} c_{s'} > 0$.

Note that we do not require that all matrices A_s with $s \in \overline{0, M}$ be Hurwitz, but only its total composition, evaluated by $\sum_{s'=0}^{m} c_{s'}$, has to provide the stability. The proofs of the results of this subsection can be found in [35].

Theorem 5. If assumptions 1, 2 and 4 are fulfilled, then the system (11) is convergent.

In this case, Assumption 4 replaces all LMIs given in the formulation of Theorem 4.

Corollary 2. Let $\varphi \in \mathscr{L}_\infty^n$ be an almost periodic function for $t \in \mathbb{R}$, the functions F_j, $j \in \overline{1, M}$ be locally Lipschitz continuous and assumptions 1, 2 and 4 be fulfilled, then the system (11) possesses almost periodic convergence property.

5 | CONVERGENCE CONDITIONS FOR DISCRETIZED SYSTEMS

In this section, the convergence conditions for two kinds of discretized systems are given, which are obtained from application of two discretization approaches to the system (11), namely the explicit and the implicit Euler methods are studied. Only the convergence property is investigated, the almost periodic one can be obtained by adapting the conditions of [36] and corollaries 1 or 2 (it is skipped for brevity).

Assumption 5. Assume that for every $i \in \overline{1, n}$, $j \in \overline{1, M}$, the function f_j^i is globally Lipschitz continuous, *i.e.*, there exists a number $L_j^i > 0$ such that

$$\left| f_j^i(r_1) - f_j^i(r_2) \right| \leq L_j^i \left| r_1 - r_2 \right|$$

for all $r_1, r_2 \in \mathbb{R}$.

5.1 | Explicit Euler method

Consider the difference system

$$x(k+1) = x(k) + hA_0 x(k) + h\sum_{j=1}^{M} A_j F_j(x(k)) + h\varphi(k), \tag{20}$$

where $h > 0$ is a discretization step; $k_0 \in \mathbb{Z}$ and $k_0 \leq k \in \mathbb{Z}$; $\varphi \in \mathscr{U}_\infty^n$; $x(k)$ and F_j, $j \in \overline{1, M}$ are defined as in the system (11).

It is clear that system (20) is the result of discretization of (11) with the aid of the explicit Euler method.

Theorem 6. Let assumptions 1, 2, 3 and 5 be satisfied. If there exist $0 \leq P = P^\top$, $\tilde{P} = \tilde{P}^\top \in \mathbb{R}^{n\times n}$; $\{\Lambda^j = \operatorname{diag}(\Lambda^{j,1}, \dots, \Lambda^{j,n})\}_{j=1}^M$, $\{\tilde{\Lambda}^j = \operatorname{diag}(\tilde{\Lambda}^{j,1}, \dots, \tilde{\Lambda}^{j,n})\}_{j=1}^M$, $\{\Xi^s\}_{s=0}^M$, $\{\Upsilon_{s,r}\}_{0\leq s<r\leq M}$, $\{\tilde{\Upsilon}_{j,j'}\}_{j,j'=1}^M$, $\{\Gamma_j\}_{j=1}^M$, $\{\Omega_j\}_{j=1}^M \subset \mathbb{D}_+^n$; $\tilde{\Xi}^0 \in \mathbb{D}_+^n$; $0 < \Phi = \Phi^\top \in \mathbb{R}^{n\times n}$, γ, $\theta > 0$ such that the properties (14)-(17) are verified for

$$Q = (Q_{a,b})_{a,b=1}^{M+2},\ \tilde{Q} = (\tilde{Q}_{a,b})_{a,b=1}^{3}$$

with

$$\begin{gathered}
Q_{1,1} = A_0^\top P + PA_0 + hA_0^\top \Pi A_0 + \Xi^0,\\
Q_{j+1,j+1} = hA_j^\top \Pi A_j + \Lambda^j A_j + A_j^\top \Lambda^j + \Xi^j,\ j \in \overline{1, M},\\
Q_{1,j+1} = PA_j + A_0^\top \Lambda^j + hA_0^\top \Pi A_j + \Upsilon_{0,j},\ j \in \overline{1, M},\\
Q_{s+1,r+1} = hA_s^\top \Pi A_r + A_s^\top \Lambda^r + \Lambda^s A_r + \Upsilon_{s,r},\ s \in \overline{1, M-1},\ r \in \overline{s+1, M},\\
Q_{M+2,M+2} = h\Pi - \Phi;\ Q_{j+1,M+2} = hA_j^\top \Pi + \Lambda^j,\ j \in \overline{1, M},\\
Q_{1,M+2} = P + hA_0^\top \Pi,\ \Pi = P + 2\sum_{j=1}^{M} L_j \Lambda^j,\ L_j := \operatorname{diag}(L_j^1, ..., L_j^n),\ j \in \overline{1, M},
\end{gathered}$$

$$\tilde{Q}_{1,1} = A_0^\top P + PA_0 + hA_0^\top \tilde{\Pi} A_0 + \tilde{\Xi}^0,$$
$$\tilde{Q}_{2,2} = hA^\top \tilde{\Pi} A - \gamma I_{nM},\ \tilde{Q}_{1,2} = hA_0^\top \tilde{\Pi} A + PA + \Gamma,$$
$$\tilde{Q}_{1,3} = A_0^\top \tilde{\Lambda} + \Omega,\ \tilde{Q}_{2,3} = A^\top \tilde{\Lambda} + \tilde{\Upsilon},\ \tilde{Q}_{3,3} = -\theta I_{nM},$$
$$\tilde{\Pi} = P + 2\sum_{j=1}^{M} L_j \tilde{\Lambda}^j,\ A = \begin{bmatrix} A_1 & \dots & A_M \end{bmatrix},\ \Gamma = \begin{bmatrix} \Gamma_1 & \dots & \Gamma_M \end{bmatrix},$$
$$\tilde{\Lambda} = \begin{bmatrix} \tilde{\Lambda}^1 & \dots & \tilde{\Lambda}^M \end{bmatrix},\ \Omega = \begin{bmatrix} \Omega_1 & \dots & \Omega_M \end{bmatrix},\ \tilde{\Upsilon} = (\tilde{\Upsilon}_{j,j'})_{j,j'=1}^{M},$$

then the system (20) is convergent.

Proof. Consider a candidate Lyapunov function

$$V(x(k)) = x(k)^\top P x(k) + 2\sum_{j=1}^{M}\sum_{i=1}^{n} \Lambda^{j,i} \int_0^{x_i(k)} f_j^i(\tau) d\tau$$

for the system (20), so the first condition in (8) is verified as before. Then, consider the difference of V:

$$\begin{aligned}
\Delta V \ :=\ & x(k+1)^\top P x(k+1) - x(k)^\top P x(k) + 2\sum_{j=1}^{M}\sum_{i=1}^{n} \Lambda^{j,i} \int_{x_i(k)}^{x_i(k+1)} f_j^i(\tau) d\tau \\
=\ & \left[x(k+1)^\top P x(k+1) - x(k)^\top P x(k)\right] + 2h\left[\sum_{j=1}^{M} F_j(x(k))^\top \Lambda^j T(k)\right] \\
& + 2\left[\sum_{j=1}^{M} G_j(x(k), \varsigma(k) - x(k))^\top \Lambda^j \left(x(k+1) - x(k)\right)\right] \\
\le\ & \left[x(k+1)^\top P x(k+1) - x(k)^\top P x(k)\right] + 2h\left[\sum_{j=1}^{M} F_j(x(k))^\top \Lambda^j T(k)\right] \\
& + 2h^2\left[T(k)^\top \left(\sum_{j=1}^{M} L_j \Lambda^j\right) T(k)\right] \\
=\ & h\begin{bmatrix} x(k) \\ F_1(x(k)) \\ \vdots \\ F_M(x(k)) \\ \varphi(k) \end{bmatrix}^\top Q \begin{bmatrix} x(k) \\ F_1(x(k)) \\ \vdots \\ F_M(x(k)) \\ \varphi(k) \end{bmatrix} - hx(k)^\top \Xi^0 x(k) \\
& - h\sum_{j=1}^{M} F_j(x(k))^\top \Xi^j F_j(x(k)) - 2h\sum_{j=1}^{M} x(k)^\top \Upsilon_{0,j} F_j(x(k)) \\
& - 2h\sum_{s=1}^{M-1}\sum_{r=s+1}^{M} F_s(x(k))^\top \Upsilon_{s,r} F_r(x(k)) + h\varphi(k)^\top \Phi \varphi(k)
\end{aligned}$$

by Assumption 5 and the Mean Value Theorem, where $T(k) = A_0 x(k) + \sum_{j=1}^{M} A_j F_j(x(k)) + \varphi(k)$ and $\varsigma_i(k) \in [\min(x_i(k), x_i(k+1)), \max(x_i(k), x_i(k+1))]$, $i \in \overline{1,n}$ for all $k \ge k_0$. Therefore, with the condition (14) we obtain

$$\begin{aligned}
\Delta V \ \le\ & -hx^\top \Xi^0 x - h\sum_{j=1}^{M} F_j(x)^\top \Xi^j F_j(x) - 2h\sum_{j=1}^{M} x^\top \Upsilon_{0,j} F_j(x) \\
& - 2h\sum_{s=1}^{M-1}\sum_{r=s+1}^{M} F_s(x)^\top \Upsilon_{s,r} F_r(x) + h\varphi^\top \Phi \varphi,
\end{aligned}$$

so that under assumptions 1, 2 and the condition (16) the system (20) is ISS.

In a similar way, we then study the asymptotic stability of

$$z(k+1) = z(k) + hA_0 z(k) + h\sum_{j=1}^{M} A_j G_j(x(k), z(k)) \tag{21}$$

with $y(k+1) = y(k) + hA_0 y(k) + h\sum_{j=1}^{M} A_j F_j(y(k)) + h\varphi(k)$ and $z(k) = y(k) - x(k)$ for all $k \geq k_0$, for which we consider a Lyapunov function

$$\tilde{V}(z(k)) = z(k)^\top \tilde{P} z(k) + 2\sum_{j=1}^{M}\sum_{i=1}^{n} \tilde{\Lambda}^{j,i} \int_0^{z_i(k)} f_j^i(\tau) d\tau$$

for the dynamics (21). Under the repetition of previous development in the proof for Theorem 4, it follows that under assumptions 1, 2, 3, 5 and the conditions (15), (17)

$$\begin{aligned}\Delta\tilde{V} \;\leq\; & -hz^\top \left(\tilde{\Xi}^0 - \gamma\sum_{j=1}^{M} S_0^j - \theta\sum_{j=1}^{M} H_0^j\right) z - 2h\sum_{j=1}^{M} z^\top \left(\Gamma_j - \gamma S_1^j - \theta H_1^j\right) G_j(x,z) - 2h\sum_{j=1}^{M} z^\top \left(\Omega_j - \gamma S_2^j - \theta H_2^j\right) F_j(z) \\ & -2h\sum_{j=1}^{M} G_j(x,z)^\top \sum_{j'=1}^{M} \left(\tilde{\Upsilon}_{j,j'} - \gamma S_3^{j,j'} - \theta H_3^{j,j'}\right) F_{j'}(z) \\ <\; & 0\end{aligned}$$

for all $z \neq 0$ and $x \in \mathbb{R}^n$. Therefore, (21) is globally asymptotically stable (GAS). This completes the proof. □

Note that if Assumption 5 holds only locally, i.e., the nonlinear functions in (20) are locally Lipschitz, then local convergence can be established in the same way.

The result of Theorem 6 can be reformulated as follows:

Corollary 3. Let all conditions of Theorem 4 be satisfied with Assumption 5. Then there exists $h_0 > 0$ such that for all $h \in (0, h_0]$ the system (20) is convergent.

Proof. It is easy to check that the difference between matrices Q and $\tilde{Q}$ given in theorems 4 and 6 is proportional to h, and all other conditions are the same, except Assumption 5 that is additionally imposed in the last theorem. This substantiates the claim. □

5.2 | Implicit Euler method

In this subsection the convergence conditions for a discretization of (11) with the aid of implicit Euler method is investigated. The considered discretized system is

$$x(k+1) = x(k) + hA_0 x(k+1) + h\sum_{j=1}^{M} A_j F_j(x(k+1)) + h\varphi(k+1), \tag{22}$$

where $x(k),\ h,\ k,\ F_j,\ j \in \overline{1, M}$ and $\varphi(k+1)$ are defined as in the system (20), and it has to be resolved with respect to $x(k+1)$.

Remark 5. In the sequel we assume that for any $h > 0$ and $\zeta,\ \varphi \in \mathbb{R}^n$, the equation

$$(I_n - hA_0)\xi - h\sum_{j=1}^{M} A_j F_j(\xi) = \zeta + h\varphi \tag{23}$$

admits a unique solution $\xi = \xi(h, \zeta, \varphi)$ that continuously depends on $h,\ \zeta,\ \varphi$. Hence, for any $k_0 \in \mathbb{Z}$ and $x(k_0)$, the system (22) admits a unique solution $x(k, x(k_0), k_0, \varphi)$ that is defined for all $k_0 \leq k \in \mathbb{Z}$. Sufficient conditions of such a property (under assumptions 1 and 2 with $M = 1$) were obtained in [37,38].

Theorem 7. Let assumptions 1 and 2 be satisfied. If there exist $0 < P = P^\top \in \mathbb{R}^{n\times n}$; $\{\Xi^s\}_{s=0}^{M}$, $\{\Upsilon_{s,r}\}_{0\leq s<r\leq M} \subset \mathbb{D}_+^n$ and $0 < \Phi = \Phi^\top \in \mathbb{R}^{n\times n}$ such that

$$Q = Q^\top \leq 0, \tag{24}$$

$$\Xi^0 + \sum_{j=1}^{M} \Upsilon_{0,j} + \sum_{s=1}^{m} \Xi^s + \sum_{s=1}^{m}\sum_{r=s+1}^{m} \Upsilon_{s,r} > 0, \tag{25}$$

where

$$Q = (Q_{a,b})_{a,b=1}^{M+2}$$

with

$$\begin{gathered}
Q_{1,1} = A_0^\top P + PA_0 - hA_0^\top PA_0 + \Xi^0,\\
Q_{j+1,j+1} = -hA_j^\top PA_j + \Xi^j,\ j \in \overline{1,M},\\
Q_{1,j+1} = PA_j - hA_0^\top PA_j + \Upsilon_{0,j},\ j \in \overline{1,M},\\
Q_{s+1,r+1} = -hA_s^\top PA_r + \Upsilon_{s,r},\ s \in \overline{1,M-1},\ r \in \overline{s+1,M},\\
Q_{1,M+2} = P - hA_0^\top P,\\
Q_{M+2,M+2} = -hP - \Phi;\ Q_{j+1,M+2} = -hA_j^\top P,\ j \in \overline{1,M},
\end{gathered}$$

then the system (22) is convergent.

Proof. Consider a candidate Lyapunov function

$$V(x(k)) = x(k)^\top Px(k)$$

for (22), again the first condition in (8) is recovered. Then, the increment of such a V for (22) takes the form:

$$\begin{aligned}
\Delta V &= x(k+1)^\top Px(k+1) - x(k)^\top Px(k)\\
&= h\begin{bmatrix} x(k+1)\\ F_1(x(k+1))\\ \vdots\\ F_M(x(k+1))\\ \varphi(k+1)\end{bmatrix}^\top Q\begin{bmatrix} x(k+1)\\ F_1(x(k+1))\\ \vdots\\ F_M(x(k+1))\\ \varphi(k+1)\end{bmatrix} - hx(k+1)^\top\Xi^0x(k+1)\\
&\quad -h\sum_{j=1}^{M}F_j(x(k+1))^\top\Xi^jF_j(x(k+1)) - 2h\sum_{s=1}^{M-1}\sum_{r=s+1}^{M}F_s(x(k+1))^\top\Upsilon_{s,r}F_r(x(k+1))\\
&\quad -2h\sum_{j=1}^{M}x(k+1)^\top\Upsilon_{0,j}F_j(x(k+1)) + h\varphi(k+1)^\top\Phi\varphi(k+1).
\end{aligned}$$

Therefore, under (24) we have

$$\begin{aligned}
\Delta V \le\ & -hx(k+1)^\top\Xi^0x(k+1) - h\sum_{j=1}^{M}F_j(x(k+1))^\top\Xi^jF_j(x(k+1))\\
& -2h\sum_{s=1}^{M-1}\sum_{r=s+1}^{M}F_s(x(k+1))^\top\Upsilon_{s,r}F_r(x(k+1)) - 2h\sum_{j=1}^{M}x(k+1)^\top\Upsilon_{0,j}F_j(x(k+1))\\
& +h\varphi(k+1)^\top\Phi\varphi(k+1),
\end{aligned}$$

hence, with the condition (25) the system (22) is ISS (recall Remark 5).

Similarly, for the dynamics

$$z(k+1) = z(k) + hA_0z(k+1) + h\sum_{j=1}^{M}A_jG_j(x(k+1), z(k+1)) \tag{26}$$

with $y(k+1) = y(k) + hA_0 y(k+1) + h\sum_{j=1}^{M} A_j F_j(y(k+1)) + h\varphi(k+1)$ and $z(k) = y(k) - x(k)$ for all $k \geq k_0$, by considering $\tilde{V}(z) = z^\top \tilde{P} z$ and following the previous proof lines we obtain

$$\begin{aligned}\Delta\tilde{V} \leq\ & -hz(k+1)^\top \Xi^0 z(k+1) - h\sum_{j=1}^{M} G_j(x(k+1), z(k+1))^\top \Xi^j G_j(x(k+1), z(k+1)) \\ & -2h\sum_{j=1}^{M} z(k+1)^\top \Upsilon_{0,j} G_j(x(k+1), z(k+1)) \\ & -2h\sum_{s=1}^{M-1}\sum_{r=s+1}^{M} G_s(x(k+1), z(k+1))^\top \Upsilon_{s,r} G_r(x(k+1), z(k+1)) \\ <\ & 0\end{aligned}$$

for all $z(k+1) \neq 0$ and $x(k+1) \in \mathbb{R}^n$. Thus, the dynamics (26) is GAS. Therefore, we conclude that the system (22) is convergent. □

6 | EXAMPLES

6.1 | Application to a modified Lotka-Volterra model

In this subsection, a modified Lotka-Volterra (LV) dynamics is considered, which has been widely investigated in infectious disease, biology, finance, to mention a few subjects[39]. However, the earliest basic model does not reflect some important phenomena, *e.g.*, stable coexistence and predator preference. Therefore, there are many modified LV models, among them the following one considers additionally the mutualistic interactions in different species[40]:

$$\dot{x}(t) = \operatorname{diag}(x(t))(r_0 + r(t) + Ax(t) + A_2\phi(x(t))),\ t \in \mathbb{R}, \tag{27}$$

where $x(t) \in \mathbb{R}^n_+$ is the population sizes of n species; $r_0 \in \mathbb{R}^n$ denotes the birth or mortality rates of the species and $r(t) \in \mathbb{R}^n$ is introduced to model the deviations of the rates from the nominal quantities r_0; $A \in \mathbb{R}^{n\times n}$ represents the community matrix and $A_2 \in \mathbb{R}^{n\times n}$ refers to the net effect of the mutualism (A and A_2 are Metzler); the function $\phi : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is diagonal, where $\phi_i(x_i) = \frac{x_i}{b_i + x_i}$ with constants $b_i > 0$ for $i \in \overline{1, n}$.

Assuming existence of a unique equilibrium point $x_e \in \mathbb{R}^n_+ \setminus \{0\}$ for (27) with $r(t) = 0$, and defining

$$\rho(t) = \ln(x(t)) - \ln(x_e),$$

we obtain

$$\dot{\rho}(t) = A_1 F_1(\rho(t)) + A_2 F_2(\rho(t)) + r(t), \tag{28}$$

where

$$A_1 := A\operatorname{diag}(x_e),\ F_1(\rho) = e^\rho - \mathbf{1}_n,\ F_2(\rho) = \phi\left(\operatorname{diag}(x_e)e^\rho\right) - \phi(x_e).$$

Note that for such functions F_1 and F_2 the requirement of Assumption 1 for $\nu \to -\infty$ is not satisfied. However, as in[23], due to assumed existence of the global equilibrium x_e it is possible to show that for $r_0 + r(t) \geq r_{\min} > 0$ all trajectories converge to a neighborhood of the steady state, where the analysis can be next performed without taking into account the unbounded deviations of the state.

For illustration, let

$$A = \begin{bmatrix} -0.8 & 0.3 \\ 0.4 & -0.6 \end{bmatrix},\ A_2 = \begin{bmatrix} -0.5 & 0.2 \\ 0.3 & -0.9 \end{bmatrix},\ b = \begin{bmatrix} 0.1 \\ 0.4 \end{bmatrix},\ r_0 = \begin{bmatrix} 0.2 \\ 0.9 \end{bmatrix},\ r(t) = \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix},$$

where $r(t)$ is periodic and represents the external input for (28), then the value of x_e can be obtained as previously mentioned. Assumptions 3 and 5 are satisfied for

$$S_0^1 = S_0^2 = \begin{bmatrix} 0.0041 & 0 \\ 0 & 0.0070 \end{bmatrix},\ S_1^1 = \begin{bmatrix} 0.1177 & 0 \\ 0 & 0.1617 \end{bmatrix},\ S_1^2 = \begin{bmatrix} 0.2171 & 0 \\ 0 & 0.2907 \end{bmatrix},$$

$$S_2^1 = S_2^2 = 10^{-5} \times \begin{bmatrix} 0.0109 & 0 \\ 0 & 0.0117 \end{bmatrix},\ S_3^{1,1} = S_3^{1,2} = \begin{bmatrix} 0.1092 & 0 \\ 0 & 0.2105 \end{bmatrix},\ S_3^{2,1} = S_3^{2,2} = \begin{bmatrix} 0.1545 & 0 \\ 0 & 0.3244 \end{bmatrix},$$

$$h_1 = 0.386,\ h_2 = 0.557,\ L_1 = \begin{bmatrix} 40 & 0 \\ 0 & 40 \end{bmatrix},\ L_2 = \begin{bmatrix} 0.0588 & 0 \\ 0 & 0.3694 \end{bmatrix}.$$

The LMIs proposed in Theorem 4 are verified by

$$P = \begin{bmatrix} 0.8054 & 0.2870 \\ 0.2870 & 0.7936 \end{bmatrix},\ \tilde{P} = 10^{-5} \times \begin{bmatrix} 0.1751 & 0.0762 \\ 0.0762 & 0.1153 \end{bmatrix},$$

$$\Lambda^1 = 10^8 \times \begin{bmatrix} 0.9572 & 0 \\ 0 & 1.7737 \end{bmatrix},\ \Lambda^2 = 10^8 \times \begin{bmatrix} 1.5435 & 0 \\ 0 & 2.4658 \end{bmatrix},\ \tilde{\Lambda}^1 = \tilde{\Lambda}^2 = 10^{-5} \times \begin{bmatrix} 0.1057 & 0 \\ 0 & 0.1144 \end{bmatrix},$$

and the ones in theorems 6 and 7 under the selection $h = 10^{-4}$ are checked for

$$P = \begin{bmatrix} 0.9939 & 0.3007 \\ 0.3007 & 0.9362 \end{bmatrix},\ \tilde{P} = 10^{-4} \times \begin{bmatrix} 0.1840 & 0.0802 \\ 0.0802 & 0.1072 \end{bmatrix},$$

$$\Lambda^1 = 10^8 \times \begin{bmatrix} 0.9949 & 0 \\ 0 & 1.8803 \end{bmatrix},\ \Lambda^2 = 10^8 \times \begin{bmatrix} 1.6589 & 0 \\ 0 & 2.6251 \end{bmatrix},$$

$$\tilde{\Lambda}^1 = 10^{-4} \times \begin{bmatrix} 0.5388 & 0 \\ 0 & 0.6215 \end{bmatrix},\ \tilde{\Lambda}^2 = 10^{-4} \times \begin{bmatrix} 0.5388 & 0 \\ 0 & 0.6217 \end{bmatrix},$$

and $P = 10^{-7} \times \begin{bmatrix} 0.2932 & 0.0983 \\ 0.0983 & 0.1739 \end{bmatrix}$, respectively.

By choosing a periodic and oscillating input signal, we can see a convergence phenomenon that is not caused by asymptotic stability of the considered system. The trajectories with distinct initial states

$$x^1(0) = \begin{bmatrix} 2.4 \\ 0.7 \end{bmatrix},\ x^2(0) = \begin{bmatrix} 5.3 \\ 0.8 \end{bmatrix},\ x^3(0) = \begin{bmatrix} 0.1 \\ 3.7 \end{bmatrix}$$

are shown in Fig. 1. These results demonstrate the convergence phenomenon in (28).

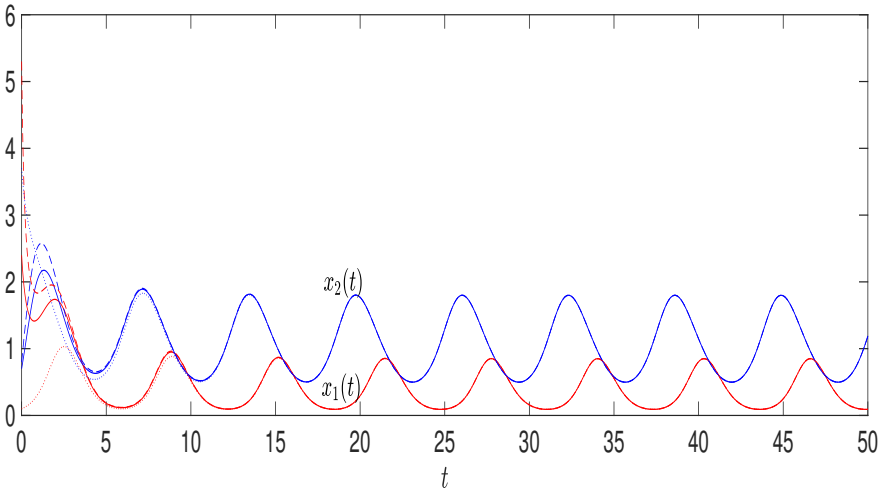


FIGURE 1 The state trajectories of (27) versus the time t

6.2 | Application to opinion dynamics

For opinion dynamics among a network of agents, the following model can be considered[41,42,43]:

$$\dot{x}(t) = -x(t) + \sum_{j=1}^{M} K_j A_j \tanh(\alpha_j x(t)) + Bu(t) + \varphi(t), \tag{29}$$

where $x(t) \in \mathbb{R}^n$ is the opinion variable of n agents, and the sign of $x_i(t)$, $i = 1, \dots, n$ describes the qualitative stance toward a binary choice (the larger $|x_i(t)|$, more extreme is the opinion of the agent i); $M \geq 1$ is the number of social networks connecting the agents; $K_j > 0$ denotes the social interaction strength among agents in the network $j \in \overline{1, M}$, $A_j \in \mathbb{R}^{n\times n}$ is the adjacency matrix in that network, and $\alpha_j > 0$ characterizes controversialness of the issue for j^{th} media (for small α_j, the social influence of moderate individuals on others is weak, while for a large α_j, everybody have a strong social impact); $B \in \mathbb{R}^{n\times q}$; $u(t) \in \mathbb{R}^q$ is a controlling input that can be applied to modify the network connections among the agents; $\varphi(t) \in \mathbb{R}^n$ can be used to model the off-network influences on orientations of agents (*e.g.*, governmental communication). The detailed motivation for this model (for the case $M = 1$ and time-varying matrix A_1) can be found in[42,43]. Clearly, the system under feedback control is presented in the form of (11), and assumptions 1, 2, 3 and 5 are verified. If Assumption 4 is also valid, then in the resulting closed-loop system, the opinion of an agent can be changed by the external manipulation φ only.

Since all nonlinearities are bounded and $A_0 = -I_n$, the system always has bounded solutions for bounded inputs. Hence, the conditions of Theorem 4 can be simplified, as mentioned in Remark 3, by focusing only on the asymptotic stability of (13).

Let $n = 4$, $M = 2$,

$$A_1 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 2 & 1 & 1 & 0 \\ -1 & 0.5 & 0 & -0.1 \\ 4 & -0.2 & 3 & 1 \\ 2 & 0.1 & 4 & 2 \end{bmatrix}, \; K_1 = 2, \; K_2 = 1, \; \alpha_1 = 0.5, \; \alpha_2 = 0.1, \; \varphi(t) = \begin{bmatrix} 0.25 \\ 0.6 \\ 0.3 \\ 0.8 \end{bmatrix},$$

$$u(t) = Z_1 \tanh(\alpha_1 x(t)) + Z_2 \tanh(\alpha_2 x(t)),$$

$$Z_1 = \begin{bmatrix} 1.1107 & 1.2711 & -0.3044 & -1.0206 \\ 3.1045 & 0.7848 & -0.4515 & -2.8485 \\ -5.4459 & -3.667 & 0.0203 & 3.4097 \\ 10.6159 & 6.4136 & -0.4136 & -6.6364 \end{bmatrix}, \; Z_2 = \begin{bmatrix} -1.3417 & 0.5825 & 1.0519 & -0.1570 \\ -2.8894 & 1.922 & 2.2015 & 0.3864 \\ 4.6529 & -2.2669 & -3.6353 & 0.5077 \\ -8.2295 & 4.0852 & 6.2886 & -0.9977 \end{bmatrix}.$$

The trajectories with different initial conditions are presented in Fig. 2, and the LMIs in Theorem 4 are verified. As we can conclude from these results, the opinions of all agents are directed to the predetermined by φ direction. Simulations of the same system with $u = 0$ and $\varphi = 0$ demonstrate pluralism of opinions in the uncontrolled network depending on the initial conditions.

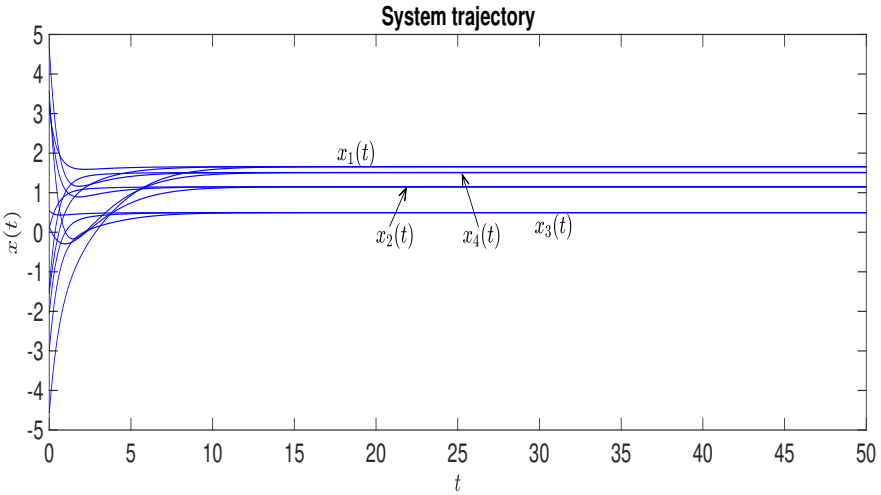


FIGURE 2 The state trajectories of (29) versus the time t

7 | CONCLUSION

In this paper, broad convergence conditions for generalized Persidskii systems and their discretizations, and the auxiliary results of convergence and almost periodic convergence conditions for those systems, with Metzler coefficient matrices, were proposed. The formulated conditions were obtained in the form of LMIs (linear inequalities in the Metzler case), which can be constructively verified. The simulations of Lotka-Volterra models and opinion dynamics were presented for the examination of the proposed results.

Future directions of research include the analysis of more sophisticated Lyapunov functions and the investigation of robust convergence properties, and the influence of delays as well.

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CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

DATA AVAILABILITY STATEMENT

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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