The Maker-Breaker Largest Connected Subgraph Game

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Abstract

Given a graph G and an integer $k \in \mathbb{N}$, we introduce the following Maker-Breaker game played in G. Each round, first Alice colours an uncoloured vertex of G red, and then Bob colours an uncoloured vertex blue (if any remain). Once all the vertices have been coloured, Alice wins if there exists a connected red component of order at least k, and if not, then Bob wins. This game is a Maker-Breaker version of the Largest Connected Subgraph game introduced in [Bensmail, Fioravantes, Mc Inerney, and Nisse, WG 2021]. We are interested in computing $c_g(G)$, which is the maximum k such that Alice wins in G, regardless of what Bob does.

Given a graph G and an integer $k \geq 1$, we prove that deciding whether $c_g(G) \geq k$ is PSPACE-complete, even if G is restricted to be in the class of bipartite graphs of diameter 4, split graphs, or planar graphs. Then, we focus on *A-perfect* graphs, namely, graphs for which $c_g(G) = \left\lceil \frac{|V(G)|}{2} \right\rceil$, *i.e.*, the maximum possible value. We show that there exist arbitrarily large A-perfect d-regular graphs for any $d \geq 4$, but, surprisingly, that no cubic graph with order at least 133 is A-perfect. Moreover, we give sufficient conditions, in terms of the number of edges or the maximum and minimum degree, for a graph to be A-perfect.

Finally, we show that $c_g(G)$ can be computed in polynomial time when G is a P_4-sparse graph (a superclass of cographs). We conclude with many open questions. In particular, natural graph classes such as trees and grid-like graphs seem to be difficult to deal with. We only give partial results in the case of some grid-like graphs.

Keywords: maker-breaker game, connection game, largest connected subgraph game, complexity.

1. Introduction

Maker-Breaker games have been vastly studied over the years since the introduction of some of the earliest and most famous Maker-Breaker games such as *Hex*, introduced by Hein in 1942, and independently by Nash in 1948 [13], and the *Shannon switching game*, created by Shannon in the 1950s [14]. As the two previously mentioned games can also be considered as other types of games, *e.g.*, *connection* games (games in which the players want to create connected structures), the actual class of Maker-Breaker games arguably drew more attention after the 1973 paper of Erdös and Selfridge on *positional* games [12], which are a superclass of Maker-Breaker games (see [16] for a survey on positional games).

In Maker-Breaker games, there is a set of elements X and a family of winning sets $\mathcal{F}$, which is a family of subsets of X. The two players, *Maker* and *Breaker*, take turns selecting previously unselected elements of X. Maker wins if she manages to select all the elements of a winning set in $\mathcal{F}$, while Breaker wins if he manages to prevent this, *i.e.*, by selecting at least one element of each winning set in $\mathcal{F}$. Thus, there is no possibility of a draw in Maker-Breaker games. For example, in Hex, X consists of all the tiles of the board, while $\mathcal{F}$ consists of all the "chains" of tiles connecting both of Maker's sides of the board.

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One of the first famous results for Maker-Breaker games is the Erdös-Selfridge Theorem [12] from 1973, which gives sufficient conditions for Breaker to win. In 1978, Schaefer proved that determining the outcome of a Maker-Breaker game is PSPACE-complete, even when each of the winning sets in $\mathcal{F}$ has size at most 11 (or exactly 11) [22]. This result was not improved upon until 2021, when Rahman and Watson proved that determining the outcome of a Maker-Breaker game is PSPACE-complete, even when each of the winning sets in $\mathcal{F}$ has size at most 6 (or exactly 6) [20]. These complexity results are substantial since the problem above (commonly known as POS CNF) is a common problem to reduce from in order to prove PSPACE-hardness, and the size of the largest winning set often has implications on the properties of PSPACE-hard instances of the problem being reduced to.

Apart from general results for Maker-Breaker games, many individual such games have been considered. Let us mention some of the more notable Maker-Breaker games played on graphs. In particular, in 1978, Chvátal and Erdös introduced the following Maker-Breaker games played on the complete graph K_n: the *Hamiltonicity game*, the *Connectivity game*, and the *Clique game* [8]. In each of these games, X consists of the edges of K_n, while $\mathcal{F}$ consists of each Hamiltonian cycle for the former, each spanning tree for the second, and each clique of a given size for the latter. They notably also introduced *biased* Maker-Breaker games, which are those in which Breaker may instead select multiple elements of X on each of his turns, and the goal is to determine the least number he may select, while still guaranteeing him winning [8]. Another Maker-Breaker game on graphs that has attracted a lot of interest is the *Colouring Construction game* [6], reintroduced by Bodlaender in 1991 after Brams had initially introduced it in 1981 as the *Vertex Colouring game* in [15]. In this game, there is a positive integer k given as an input, and X consists of the vertices of the graph, while $\mathcal{F}$ consists of all the proper k-colourings of the graph. There is also the added stipulation that a player must assign a colour in $\{1, \dots, k\}$ to the vertex they select, and that assignment must ensure that the partial colouring remains proper (no two adjacent vertices are assigned the same colour), and if the latter is not possible, then Breaker wins. The *game chromatic number* is the least integer k such that Maker has a winning strategy in this game, and it has been largely studied. Lastly, we mention a recently introduced Maker-Breaker game played on graphs called the *Maker-Breaker Domination game*, conceived by Duchêne et al. in 2020 [11], in which X consists of the vertices of the graph, while $\mathcal{F}$ consists of all the dominating sets of the graph.

Along the same lines, in this paper, we introduce the following Maker-Breaker game, which is a natural game that has, astonishingly, not been considered in the literature to date. In the *Maker-Breaker Largest Connected Subgraph game* played on a given graph G, there is a positive integer k given as an input, and X consists of the vertices of G, while $\mathcal{F}$ consists of all the connected subgraphs of order at least k in G. In particular, for a given graph G, we are interested in the parameter $c_g(G)$, which is the largest integer k such that Maker has a winning strategy in the Maker-Breaker Largest Connected Subgraph game in G.

The other motivation for introducing this game is that it is actually a Maker-Breaker version of the *Largest Connected Subgraph game* introduced in 2021 by Bensmail et al. [4]. The Largest Connected Subgraph game opposes two players, Alice and Bob, through successive turns played on a graph G. Initially, every vertex of G is uncoloured. Each round, Alice starts by colouring an uncoloured vertex of G red, and then, Bob colours an uncoloured vertex blue (if any remain). The game ends once all the vertices of G have been coloured, resulting in a *red subgraph* of G (induced by the vertices coloured red) and a *blue subgraph* of G (induced by the vertices coloured blue). Alice's (Bob's, resp.) score is the order (number of vertices) of the largest connected component of the red subgraph (blue subgraph, resp.). If the players have different scores, then the player with the largest score wins. Otherwise, they have the same score, and the game ends in a draw. In [4], it was proved that Alice always has a strategy to ensure at least a draw, and thus, that Bob can never win if Alice plays optimally. It was also proved that determining the outcome of the game on a given graph is PSPACE-complete, even when restricted to bipartite graphs of diameter 5, but polynomial-time solvable for paths, cycles, and cographs.

The Largest Connected Subgraph game was novel in that it is a very natural game, and there is also a rich background on these types of games. In particular, it is a connection game (see [7] for more on these games) since the players strive to create connected structures, and it is a *scoring* game (see [4] for more examples) since the winner is determined by the scores of the players. Thus, the Maker-Breaker Largest Connected Subgraph game is also a connection game. As it is also related to a scoring game, we mention the works [18, 19], in which a general theory for scoring games has started to be defined.

Now that the context of our Maker-Breaker game and its relation to the Largest Connected Subgraph game are clear, we can better describe our motivation for introducing our game, with regards to the latter game. In the latter game, for certain subgraphs of some graphs, it may not be interesting for Bob to try to increase his score in those subgraphs, but rather just to limit Alice's score in them. In particular, this can be the case in graphs that are not connected. Understanding just how much Bob can limit Alice's score in these subgraphs is equivalent to playing the Maker-Breaker Largest Connected Subgraph game in them. Another motivation is to understand the properties of graphs in which Alice can ensure a single connected red component at the end of the Largest Connected Subgraph game (especially since Alice wins in these graphs if they have odd order). This leads us to define and study *A-perfect graphs*, which are the graphs G for which $c_g(G) = \lceil |V(G)|/2 \rceil$, *i.e.*, the graphs G in which Alice can ensure a single connected red component at the end of the (Maker-Breaker) Largest Connected Subgraph game in G.

This work is organised as follows. Section 2 is a preliminary section in which the main terminology and early observations, to be used throughout, are introduced. In Section 3, we show that, given a graph G and an integer $k \geq 1$, deciding whether $c_g(G) \geq k$ is PSPACE-complete, even if G is restricted to be in the class of bipartite graphs of diameter 4, split graphs, or planar graphs. In Section 4, we derive general bounds on the parameter c_g, of which we study the tightness in the particular case of regular graphs. In the class of d-regular graphs (where $d \geq 3$), we prove an unexpected result, which states that arbitrarily large A-perfect d-regular graphs do not exist only for $d = 3$. In Section 5, we focus further on A-perfect graphs by providing several sufficient conditions (in terms of the minimum and maximum degree, and number of edges) for a graph to be A-perfect. In Section 6, we show that c_g can be determined in polynomial time for $(q, q-4)$-*graphs*, a superclass of cographs. We conclude in Section 7 with a discussion including some perspectives for further work on the topic.

2. Preliminaries

2.1. Graph theory terminology and notation

Throughout this paper, all the graphs we consider are undirected and simple. For a graph G, we denote by $V(G)$ its set of vertices, and by $E(G)$ its set of edges. For a vertex v of G, we denote by $N_G(v)$ its *neighbourhood*, which is the set of vertices that are adjacent to v in G. The *closed neighbourhood* of v, denoted by $N_G[v]$, is the set $\{v\} \cup N_G(v)$. These two notions of neighbourhood extend to sets S of vertices of G, with $N_G(S)$ referring to the subset of vertices of $V(G) \setminus S$ that have a neighbour in S, and $N_G[S]$ referring to the set $S \cup N_G(S)$. For a vertex v of G, we denote by $d_G(v)$ its *degree*, with $d_G(v) = |N_G(v)|$. When the graph G is clear from the context, we will drop the subscript in the parameters N_G and d_G, and write them as N and d instead. The parameters $\delta(G)$ and $\Delta(G)$ refer to the *minimum degree* and *maximum degree*, respectively, of a vertex in G.

For a set S of vertices or edges of G, we denote by $G - S$ the *subgraph* of G resulting from the deletion of the elements in S. If $S = \{x\}$, we will write $G - x$ instead of $G - S$. Similarly, we denote by $G + S$ (or $G + x$, for short, if $S = \{x\}$) the *supergraph* of G obtained by adding the elements (vertices or edges) of S. We denote by $G[S]$ the subgraph of G *induced* by the elements in S.

A vertex u of G *dominates* another vertex v if $v \in N_G(u)$. We say that u is *universal* if $N_G[u] = V(G)$. A set S of vertices of G is *dominating* if $N_G[S] = V(G)$. This dominating set S is *connected* if $G[S]$ is connected. The *distance* between two vertices u and v of G is the length of a shortest path from u to v. The *diameter* of G is the maximum distance between two vertices of G. For any standard notion or terminology on graphs not defined in this work, see [10].

2.2. Additional terminology for the game

Due to the close ties of our game with the Largest Connected Subgraph game, we will refer to Maker as Alice, and Breaker as Bob. To also make the distinction of who selected which vertices easier, we will say that Alice colours a vertex red when she selects a vertex, and that Bob colours a vertex blue when he selects a vertex. Furthermore, we will refer to the score of Alice as the largest connected red component in the graph at the end of the game. Thus, for a given graph G, $c_g(G)$ is the maximum score for which Alice has a strategy ensuring at least this score in G.

A *strategy* for a player P is a function $\mathcal{S}$ taking all the previous moves of both players (and the order of these moves, hence, the history of the game) as an input, and outputting the next move for player P. Given a graph G, an *optimal strategy for Alice* is one that ensures her a score of at least $c_g(G)$, while an *optimal strategy for Bob* is one that ensures Alice's score is at most $c_g(G)$. Since our game is a *parity game* [17], optimal strategies can actually be determined from just the current configuration of coloured vertices, rather than also knowing the order these vertices were coloured in. Thus, for our game, there can also be an equivalent (in terms of optimality) second definition of a strategy for a player P, which is a function $\mathcal{S}$ that takes the current configuration of coloured vertices and outputs the next move for player P. We will interchangeably use both definitions, depending on which one suits us best at the time.

Throughout this paper, several of our proofs rely on the fact that Alice or Bob can reach a certain game configuration (*i.e.*, have a certain set of vertices coloured with their colour) early on. In such cases, to lighten the exposition, we will sometimes allow ourselves to expose only the most important moves of the strategies that Alice or Bob must make in some rounds of the game. In particular, the reader should keep in mind that, in each of the strategies we describe, 1) if Alice or Bob cannot colour a given vertex in a given round because that vertex is already coloured, then they must colour any other uncoloured vertex instead, and 2) if no vertex to colour for Alice or Bob in a given round is specified, then they must colour any uncoloured vertex.

2.3. General results and observations

First, we show that the parameter c_g is closed under taking subgraphs.

Lemma 2.1. *Let H be a subgraph of a graph G. Then, $c_g(H) \leq c_g(G)$.*

Proof. We give a strategy for Alice ensuring her a score of at least $c_g(H)$ in G. Alice first plays in H according to an optimal strategy $\mathcal{S}$ in H. Then, whenever Bob plays in H, Alice responds in H according to $\mathcal{S}$, and if this is not possible (the vertex to be coloured by $\mathcal{S}$ is already coloured or there are no uncoloured vertices in H) or Bob plays in G, then Alice colours any arbitrary uncoloured vertex in G. In particular, whenever Alice is forced to colour an arbitrary vertex in H, she ignores the fact that vertex is coloured when considering her strategy $\mathcal{S}$ in H in the future. The result follows since Alice will colour at least all the vertices in H that she would colour by $\mathcal{S}$, ensuring her a score of at least $c_g(H)$ in G since $\mathcal{S}$ is optimal in H. □

The next result shows that when playing the game on a disconnected graph, Alice should focus on the connected component which is the most favourable for her.

Lemma 2.2. *If G is a graph with connected components $G_1, \dots, G_k$, then*

$$c_g(G) = \max\{c_g(G_1), \dots, c_g(G_k)\}.$$

Proof. The lower bound follows from Lemma 2.1, and the upper bound holds since the k connected components are pairwise disconnected, so Bob can just respond in the same connected component that Alice just played in each time (when this is not possible, he colours any arbitrary uncoloured vertex in G, which can only be beneficial to him). □

As will be seen later on, Alice can exploit different types of strategies to achieve the best possible score for her. One such strategy, that is particularly relevant in sufficiently dense graphs, is through colouring the vertices of a connected dominating set.

Lemma 2.3. *For a graph G, if, at any point in the game, Alice has coloured all the vertices of a connected dominating set of G, then her score will be $\left\lceil \frac{|V(G)|}{2} \right\rceil$.*

Proof. Assume Alice has coloured the vertices of a connected dominating set S at some point in the game. By the connectivity property of S, there must be, once the game ends, a connected red component containing the vertices of S. Also, by the dominating property of S, all the vertices of G not in S have at least one neighbour in S. This implies that the red subgraph must be connected, and thus, that Alice achieves a score of $\left\lceil \frac{|V(G)|}{2} \right\rceil$. □

3. Computational complexity

Recall that in [4], the Largest Connected Subgraph game was shown to be PSPACE-complete, even when restricted to bipartite graphs of diameter 5. In this section, using a similar reduction scheme, we prove that the Maker-Breaker Largest Connected Subgraph game is also PSPACE-complete, that is, given a graph G and an integer $k \geq 1$, deciding whether $c_g(G) \geq k$ is PSPACE-complete. In fact, we prove that our game is PSPACE-complete, even when restricted to bipartite graphs of diameter 4, split graphs, or planar graphs.

Similarly as in [4], we establish some of our PSPACE-completeness results via reductions from POS CNF, a game for which deciding whether Alice or Bob has a winning strategy was shown to be PSPACE-complete in [22]. This game is a 2-player game where the input (X, ϕ) consists of a set of variables $X = \{x_1, \dots, x_n\}$, and of a formula ϕ in conjunctive normal form (CNF) made up of clauses $C_1, \dots, C_m$ each containing variables of X in their positive forms. Each round, the first player, Alice, sets a variable of ϕ (that is not yet set) to true, before the second player, Bob, sets a variable of ϕ (that is not yet set) to false. Once all the variables of X have been assigned a truth value, Alice wins if ϕ is true, and Bob wins if ϕ is false.

Before we start, note first that when given a graph G and an integer $k \geq 1$, the problem of deciding whether $c_g(G) \geq k$ is in PSPACE since, in the game, there are $\lceil |V(G)|/2 \rceil$ rounds and the number of possible moves for a player in a round is at most $|V(G)|$. Thus, in the upcoming proofs, we focus on proving the PSPACE-hardness of the game.

Theorem 3.1. *Given a graph G and an integer $k \geq 1$, it is PSPACE-complete to decide whether $c_g(G) \geq k$, even when G is restricted to be in the class of bipartite graphs of diameter 4.*

Proof. We prove the PSPACE-hardness via a reduction from POS CNF. Let (X, ϕ) be an instance of POS CNF. Set $X = \{x_1, \dots, x_n\}$ and $\phi = C_1 \wedge \dots \wedge C_m$. By adding a dummy variable in X if needed, we can suppose n is even.

Consider the graph G constructed as follows. For every variable $x_i \in X$, we add a vertex v_i to G. For every clause C_j of ϕ, we add two vertices C_j^1 and C_j^2 to G. For every variable $x_i \in X$ and clause C_j of ϕ, we add the edges $v_iC_j^1$ and $v_iC_j^2$ to G if x_i appears in C_j. Finally, we add two vertices u_1 and u_2 to G, that we make adjacent to all of the v_i's. Note that the resulting G, which is constructed in polynomial time, is bipartite and has diameter at most 4.

Set $k = |V(G)|/2$, and note that $|V(G)|$ is even. We will show that Alice wins in (X, ϕ) if and only if $c_g(G) \geq k$. Let us assume first that Alice has a winning strategy in (X, ϕ). We give a strategy for Alice that ensures a score of at least k when playing the Maker-Breaker Largest Connected Subgraph game in G. In the first round, Alice colours the vertex v_i that corresponds to the variable $x_i \in X$ she would have set to true in the first round of her winning strategy in (X, ϕ). From the second round on, in each round, if the last vertex coloured by Bob is

- some v_i, then Alice colours the vertex v_j corresponding to the variable x_j she would set to true in response to Bob setting x_i to false in her winning strategy in (X, ϕ);
- u_1 (u_2, resp.), then Alice colours u_2 (u_1, resp.);
- some C_j^1 (C_j^2, resp.), then Alice colours C_j^2 (C_j^1, resp.).

Whenever Alice cannot follow the strategy above, she colours any arbitrary vertex. By Alice's strategy, once the game in G ends, exactly one vertex in every pair $\{C_j^1, C_j^2\}$ is red, exactly one vertex in $\{u_1, u_2\}$ is red, and the v_i's corresponding to the x_i's she would have set to true in her winning strategy for (X, ϕ) are also red. Because Alice wins in (X, ϕ) with that strategy, every vertex C_j^ℓ of G coloured red must be adjacent to at least one vertex v_k coloured red corresponding to a variable she would have set to true when playing in (X, ϕ). Since all the v_i's are dominated by u_1 and u_2, and one of these two vertices is red, we deduce that the red subgraph must contain only one connected component. Thus, Alice achieves a score of k and $c_g(G) \geq k$.

Assume now that Bob has a winning strategy in (X, ϕ). We give a strategy for Bob that ensures that Alice's score is strictly less than k when playing the Maker-Breaker Largest Connected Subgraph game in G. In each round, if the last vertex coloured by Alice is

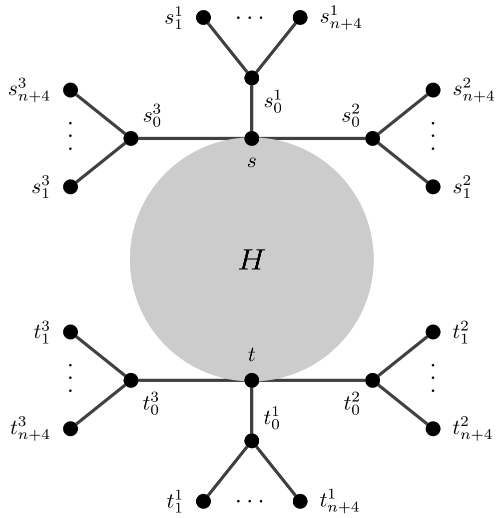


Figure 1: Illustration of the construction in the proof of Theorem 3.3.

- some v_i, then Bob colours the vertex v_j corresponding to the variable x_j he would set to false in response to Alice setting x_i to true in his winning strategy in (X, ϕ);
- u_1 (u_2, resp.), then Bob colours u_2 (u_1, resp.);
- some C_j^1 (C_j^2, resp.), then Bob colours C_j^2 (C_j^1, resp.).

Note that Bob can follow this strategy from start to end, as n is even. By Bob's strategy, once the game in G ends, exactly one vertex in every pair $\{C_j^1, C_j^2\}$ is red. Also, since Bob coloured all the v_i's corresponding to x_i's he would set to false when following a winning strategy in (X, ϕ), there exists a C_q that is not satisfied in (X, ϕ), meaning its variables were all set to false by Bob. In G, this translates to exactly one of C_q^1 or C_q^2 being red while all of their neighbours (the v_i's corresponding to the x_i's that C_q contains), are blue. Thus, the red subgraph contains at least two connected components, and hence, Alice achieves a score of less than k, and $c_g(G) < k$. □

Corollary 3.2. *Given a graph G and an integer $k \geq 1$, it is PSPACE-complete to decide whether $c_g(G) \geq k$, even when G is restricted to be in the class of split graphs.*

Proof. The proof is similar to that of Theorem 3.1, with the slight difference being in the construction of G. Here, neither of the vertices u_1 and u_2 are added, while all the possible edges between the v_i's are added so that they form a clique, thus making G a split graph. The same strategies for Alice and Bob (omitting u_1 and u_2) from the proof of Theorem 3.1 remain applicable by the same arguments, and the result follows. □

From the proofs of Theorem 3.1 and Corollary 3.2, it follows that deciding if a bipartite graph with diameter 4 (resp., a split graph) is A-perfect is PSPACE-complete. To prove that the game is PSPACE-complete, even when restricted to planar graphs, we need a different reduction. This time, we establish the result by a reduction from *Planar Generalised Hex*, which was proved to be PSPACE-complete [21]. Planar Generalised Hex is played on a planar graph G, in which a particular *outside pair* $\{s, t\}$ of vertices, *i.e.*, $st \notin E(G)$ and $G + st$ is planar, is set. Initially, s and t are red. Then, in successive rounds, the first player, Alice, colours an uncoloured vertex red, before the second player, Bob, then colours an uncoloured vertex blue. The game ends once all the vertices of G have been coloured. If the red subgraph contains a path joining s and t, then Alice wins. Otherwise, Bob wins. We can now prove our last result in this section.

Theorem 3.3. *Given a graph G and an integer $k \geq 1$, it is PSPACE-complete to decide whether $c_g(G) \geq k$, even when G is restricted to be in the class of planar graphs.*

Proof. We prove the PSPACE-hardness via a reduction from Planar Generalised Hex. Let (H, s, t) be an instance of Planar Generalised Hex such that H is the planar graph with the outside pair $\{s, t\}$, that the game is being played on. Set $n = |V(H)|$. By adding a degree-1 vertex (a leaf) in H if needed, we can suppose n is even, as this will not change the outcome of (H, s, t). Let G be the graph constructed as follows (see Figure 1). Start from G being the graph H. Then, add three vertices s_0^1, s_0^2, s_0^3 and make each of them adjacent to s, and add another three vertices t_0^1, t_0^2, t_0^3, and make each of those adjacent to t. Finally, to each of these six vertices we have just added, attach $n + 4$ new degree-1 vertices, so that a total of $6(n + 4)$ degree-1 vertices (leaves) are added to G. The construction is achieved in polynomial time, and since H is planar, G is too.

Set $k = n+5$. We will show that Alice wins in (H, s, t) if and only if $c_g(G) \geq k$. Let us assume first that Alice has a winning strategy in (H, s, t). We give a strategy for Alice that ensures a score of at least k when playing the Maker-Breaker Largest Connected Subgraph game in G. In the first round, Alice colours s. In the second round, Alice colours s_0^1 if possible, and if not, then she colours s_0^2. From the third round on,

- if Alice can colour a vertex in $\{s_0^1, s_0^2, s_0^3\}$ in the third round, then she does. If so, then, in each of the next rounds, if possible, Alice colours an uncoloured neighbour of an s_0^i she coloured earlier. At the end of the game, the red subgraph will contain a connected component of order at least $3 + \left\lceil \frac{2(n+4)-3}{2} \right\rceil = n + 6$, and thus, Alice will have a score of at least k;
- otherwise, Bob coloured two vertices in $\{s_0^1, s_0^2, s_0^3\}$ in the first two rounds. Then, Alice colours t in the third round, and she then colours one of t_0^1 and t_0^2 in the fourth round. At this point, for the same reasons as earlier, if Bob has not coloured two vertices in $\{t_0^1, t_0^2, t_0^3\}$ by the end of the fourth round, then Alice can colour a vertex in that set in the fifth round, and, as above, guarantee herself a score of at least k.

Thus, we can suppose that, after four rounds, w.l.o.g., s, t, s_0^1, and t_0^1 are red, while s_0^2, s_0^3, t_0^2, and t_0^3 are blue. From here, Alice's strategy continues as follows. In the fifth round, Alice colours, in G, the vertex of H she would have coloured in the first round of her winning strategy in (H, s, t). From the sixth round on, in each round, if the last vertex coloured by Bob in G is

- some vertex $u \in V(H)$, then Alice colours, in G, the vertex of H she would have coloured in her winning strategy in (H, s, t), as an answer to Bob colouring u;
- a leaf adjacent to some s_0^i or t_0^i, then Alice colours another uncoloured leaf adjacent to the same vertex.

Whenever Alice cannot follow the strategy above, she colours any arbitrary vertex. By this strategy, at the end of the game in G, s and t are red, and all the vertices that Alice would have coloured through her winning strategy in (H, s, t) are also red. Moreover, s_0^1 and t_0^1 are red, and, for each of them, she coloured half of their adjacent leaves. Thus, the red subgraph contains a connected component of order at least $n + 8$. Thus, Alice achieves a score of at least k, and $c_g(G) \geq k$.

Assume now that Bob has a winning strategy in (H, s, t). We give a strategy for Bob that ensures that Alice's score is strictly less than k when playing the Maker-Breaker Largest Connected Subgraph game in G. In each round, if the last vertex coloured by Alice is

- in $\{s, s_0^1, s_0^2, s_0^3\}$, then Bob colours a vertex in $\{s, s_0^1, s_0^2, s_0^3\}$;
- in $\{t, t_0^1, t_0^2, t_0^3\}$, then Bob colours a vertex in $\{t, t_0^1, t_0^2, t_0^3\}$;
- a vertex u of $H - \{s, t\}$, then Bob colours the vertex of H he would have coloured by his winning strategy in (H, s, t), as an answer to Alice colouring u;
- a leaf adjacent to some s_0^i or t_0^i, then Bob colours another uncoloured leaf adjacent to the same vertex.

Note that Bob always answers to one of Alice's moves by colouring a vertex in a set with even size since n is even. Thus, Bob can follow this strategy from start to end. At the end of the game in G, the largest connected component of the red subgraph cannot contain both s and t, as the moves made by Alice and Bob correspond exactly to the moves that would have been made if they had played in (H, s, t). Moreover, there cannot be two s_0^i's belonging to the same connected red component, as, by the strategy above, Bob must have coloured s in this case. The same holds for the t_0^i's. Also, for any of the s_0^i's and t_0^i's, by Bob's strategy above, Alice can have coloured at most half of the leaves adjacent to it. Thus, because Alice coloured at most half of the vertices in $H - \{s, t\}$, the largest connected red component in G must have order at most $\frac{n-2}{2} + 2 + \frac{n+4}{2} = n+3$. Thus, Alice achieves a score of less than k, and $c_g(G) < k$. □

4. Bounds on c_g and their tightness in regular graphs

In this section, we are mainly interested in the range of possible values that the parameter $c_g(G)$ can take for a graph G, which lies in between the next bounds:

Lemma 4.1. *For every graph G, $\left\lfloor \frac{\Delta(G)}{2} \right\rfloor + 1 \leq c_g(G) \leq \left\lceil \frac{|V(G)|}{2} \right\rceil$.*

Proof. The right-hand side of the inequality follows from the fact that Alice always colours exactly $\left\lceil \frac{|V(G)|}{2} \right\rceil$ vertices. We now give a strategy for Alice that ensures a score of at least $\left\lfloor \frac{\Delta(G)}{2} \right\rfloor + 1$, to prove the left-hand side of the inequality. In the first round, Alice colours a vertex v with degree $\Delta(G)$. Then, in each of the next rounds, if possible, Alice colours an uncoloured neighbour of v. Once the game ends, by the strategy above, Alice must have coloured v and at least half of its neighbours, and the result follows. □

Both bounds in Lemma 4.1 can be reached for arbitrarily large graphs. For the upper bound, recall that a graph G is *A-perfect* if $c_g(G) = \lceil |V(G)|/2 \rceil$. For example, there exist arbitrarily large connected graphs that are A-perfect, since every graph with a universal vertex is A-perfect. Regarding the lower bound, the graph G that is the disjoint union of m copies of the complete graph K_{d+1} (for any $d \in \mathbb{N}$) is d-regular, and $c_g(G) = \left\lfloor \frac{d}{2} \right\rfloor + 1$, while G gets more and more distant from being A-perfect as m increases.

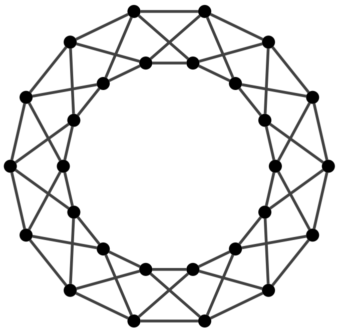
This last remark makes us wonder about the tightness of the bounds in Lemma 4.1 for arbitrarily large regular connected graphs. The next results we prove in this section, establish that there exist arbitrarily large connected d-regular graphs G, with $d \geq 3$, for which $c_g(G)$ is close to the lower bound (Lemma 4.2), while, for every $d \geq 4$, there exist arbitrarily large connected d-regular graphs G that are A-perfect (Lemma 4.3). Surprisingly, the latter result does not hold for every $d \geq 3$, as we prove that any sufficiently large cubic graph is not A-perfect (Theorem 4.4).

Before starting, let us mention the case of 2-regular graphs, *i.e.*, cycles. From a result for paths in [4], it follows that, for every $n \geq 3$, $c_g(C_n) = 2$ (the lower bound is trivial, and for the upper bound, Bob's strategy is to colour a vertex adjacent to the red vertex in the first round, and now, the game is equivalent to one on a path P_{n-1}, with one of its ends initially coloured red). We now show that the lower bound in Lemma 4.1 is almost tight for arbitrarily large connected d-regular graphs, for every $d \geq 3$.

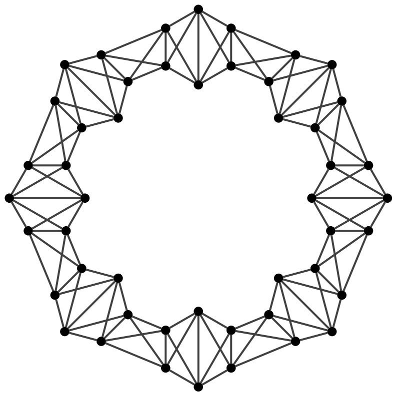
Lemma 4.2. *For every $d \geq 3$, there exist arbitrarily large connected d-regular graphs G such that $c_g(G) \leq \left\lceil \frac{d+3}{2} \right\rceil$.*

Proof. Let G be the graph constructed as follows. Start from $N \geq 2$ disjoint copies $H_0, \dots, H_{N-1}$ of the complete graph on $d+1$ vertices. Now, for every $i \in \{0, \dots, N-1\}$, remove the edge $u_i v_i$, where u_i and v_i are any two distinct vertices of H_i. Finally, add the edge $v_i u_{i+1}$ for every $i \in \{0, \dots, N-1\}$ (where, here and further, operations are understood modulo N). Note that the resulting graph G is d-regular, and, free to consider large values of N, can be as large as desired. For every $i \in \{0, \dots, N-1\}$, every vertex of H_i that is different from u_i and v_i is said to be *internal* (to H_i). Since $d \geq 3$, every H_i has at least two internal vertices.

We give a strategy for Bob that ensures that Alice's score in G is at most $\left\lceil \frac{d+3}{2} \right\rceil$. In each round, if the last vertex coloured by Alice is



(a) $d = 4$



(b) $d = 5$

Figure 2: Examples of d-regular A-perfect graphs constructed in the proof of Lemma 4.3.

- some vertex u_i, then Bob colours v_{i-1};
- some vertex v_i, then Bob colours u_{i+1};
- a vertex internal to some H_i, then Bob colours an uncoloured vertex internal to the same H_i.

By this strategy, once the game ends, every connected red component must be completely contained inside some H_i. This is because this strategy guarantees that any two vertices v_i and u_{i+1} end up coloured either by different players, or by Bob only. It thus follows that the largest connected red component contains, in the worst-case scenario, some u_i, v_i, and half of the other vertices of H_i. In other words, the largest connected red component is of order at most $\lceil \frac{d+3}{2} \rceil$. □

Regarding the upper bound from Lemma 4.1 in the context of arbitrarily large connected d-regular graphs, we prove the following:

Lemma 4.3. *For every $d \geq 4$, there exist arbitrarily large d-regular A-perfect graphs.*

Proof. Let $N > 2$, and let $d \geq 4$ be fixed. To prove the claim, we construct a d-regular graph G, whose order is a function of N, such that $c_g(G) = \left\lceil \frac{|V(G)|}{2} \right\rceil$. We give two possible constructions for G, depending on whether $d = 4$ or $d \geq 5$ (see Figure 2 for an illustration of both constructions).

- For the case $d = 4$, G is the 4-regular graph having two vertices u_1^i and u_2^i for every $i \in \{0, \dots, N-1\}$, and the four edges $u_1^i u_1^{i+1}$, $u_1^i u_2^{i+1}$, $u_2^i u_1^{i+1}$, $u_2^i u_2^{i+1}$ for every $i \in \{0, \dots, N-1\}$ (where, here and further, operations over the superscripts are modulo N).

  To prove that G is A-perfect, we give a strategy for Alice that ensures that, at the end of the game in G, the red subgraph is connected. In the first round, Alice colours u_1^0. Then, in the subsequent rounds, if the last vertex Bob coloured is u_1^j (u_2^j, resp.) for some $j \in \{1, \dots, N-1\}$, Alice responds by colouring u_2^j (u_1^j, resp.). Otherwise, Alice colours any arbitrary uncoloured vertex. By Alice's strategy, at the end of the game, for every $0 \leq i \leq N-1$, exactly one of u_1^i and u_2^i is red, and thus, the red subgraph is connected, and G is A-perfect.

- We now consider the case where $d \geq 5$. Here, G is constructed as follows. Start from N disjoint copies $H_0, \dots, H_{N-1}$ of the complete graph on $d+1$ vertices, where, for every $i \in \{0, \dots, N-1\}$, we denote by $v_1^i, \dots, v_{d+1}^i$ the vertices of H_i. For every $i \in \{0, \dots, N-1\}$, we remove the edges $v_1^i v_3^i$, $v_1^i v_4^i$, $v_2^i v_3^i$ and $v_2^i v_4^i$ from H_i. To finish the construction of G and make it d-regular, we then join the H_i's by adding the edges $v_3^i v_1^{i+1}$, $v_3^i v_2^{i+1}$, $v_4^i v_1^{i+1}$, and $v_4^i v_2^{i+1}$ for every $i \in \{0, \dots, N-1\}$ (again, operations are understood modulo N).

  To prove that G is A-perfect, we give a strategy for Alice that ensures her a score of $\lceil |V(G)|/2 \rceil$. In the first round, Alice colours any vertex. In each of the subsequent rounds, if the last vertex Bob coloured is

– in some pair $\{v_1^i, v_2^i\}$ or $\{v_3^i, v_4^i\}$, then Alice colours the other vertex in that pair;
– some vertex v_j^i with $5 \leq j \leq d+1$, then Alice colours another vertex v_ℓ^i with $5 \leq \ell \leq d+1$ and $j \neq \ell$.

Whenever Alice cannot follow the strategy above, she colours any arbitrary uncoloured vertex. By Alice's strategy, at the end of the game, for every $i \in \{0, \dots, N-1\}$, at least one vertex in $\{v_1^i, v_2^i\}$ is red, at least one vertex in $\{v_3^i, v_4^i\}$ is red, and at least one vertex in $\{v_5^i, \dots, v_{d+1}^i\}$ is red. These vertices form a connected dominating set of G, from which we deduce that $c_g(G) = \lceil |V(G)|/2 \rceil$, by Lemma 2.3. Thus, G is A-perfect. □

As mentioned earlier, the bound on d in the statement of Lemma 4.3 cannot be lowered, as, surprisingly, we prove that A-perfect cubic graphs have bounded order.

Theorem 4.4. *Every A-perfect cubic graph has order at most* 132.

As the proof of Theorem 4.4 is quite technical, we first prove several lemmas to improve its legibility. Before doing so, first recall that any cubic graph has even order (since, in any graph, the sum of the degrees of all the vertices equals twice the number of edges). Also, an edge e of a graph G is called a *bridge* if $G - e$ has more connected components than G. In case no edge of G is a bridge, we call G *bridgeless*.

Now, let us start with the following key lemma. Essentially, it says that, for the red subgraph to be connected at the end of the game, Alice must carefully choose the first vertex she colours.

Lemma 4.5. *Let G be an A-perfect cubic graph of order n. In any optimal winning strategy for Alice, the first vertex coloured by Alice cannot belong to a cycle of length strictly less than $n/2$.*

Proof. Let us consider an optimal strategy for Alice, and let v_0 be the first vertex she colours. Towards a contradiction, let us assume that v_0 belongs to a cycle C of length at most $\frac{n}{2} - 1$. Our goal is to define a strategy for Bob ensuring that, at the end of the game, there are at least two connected red components, contradicting the fact that G is A-perfect. The claimed strategy for Bob we are going to describe below, relies on a particular bipartition $V_1 \cup V_2$ of $V(G)$ that will be gradually built. In short, we want V_1 to contain v_0, and V_2 to be large enough, so that Bob can make sure that Alice colours vertices in both V_1 and V_2. An important property we also need for V_1 and V_2, is that the edges in the cut (V_1, V_2) form a matching M of G. As is going to be seen in what follows, the existence of M makes it possible for Bob to colour vertices so that any eventual connected red component containing vertices in V_1 cannot contain vertices in V_2. With all these prerequisites in hand, we can deduce a strategy for Bob, where, by the end of the game, there must be at least two connected red components. In the upcoming arguments, the existence of C is the main key behind the construction of $V_1 \cup V_2$.

We construct V_1 and V_2 through the following two-step procedure. Initially, let $V_1 = V(C)$ and $V_2 = V(G) \setminus V(C)$. We also need to maintain a certain set $\mathcal{P}$ of pairs of vertices of V_1, which is initially empty, that will be useful in describing the strategy for Bob. The iterative procedure below consists in repeatedly moving vertices from V_2 to V_1, and adding pairs to $\mathcal{P}$, ensuring the following invariants:

1. $|V_1| - |\mathcal{P}| \leq \frac{n}{2} - 1$.
2. For every two pairs $P_1, P_2 \in \mathcal{P}$, we have $P_1 \cap P_2 = \emptyset$.
3. For every pair $\{u, v\} \in \mathcal{P}$, we have $N[u] \cup N[v] \subseteq V_1$.
4. Every vertex of V_1 has at most one neighbour in V_2.

Note that these four invariants are already met by the initial values of V_1 and V_2 above. Indeed, because originally $V_1 = V(C)$ and $|V(C)| \leq \frac{n}{2} - 1$, and $\mathcal{P}$ originally is empty, Invariants 1 to 3 are clearly fulfilled. Invariant 4 is fulfilled because C is a cycle, which means that all the vertices of V_1 have exactly two neighbours in V_1, and thus, exactly one neighbour in $V(G) \setminus V_1 = V_2$. We now start by applying the first step of the procedure:

- **Step 1:** While V_2 has a vertex v whose three neighbours u_1, u_2, u_3 are in V_1, we move v from V_2 to V_1, and add the two pairs $\{v, u_1\}$ and $\{u_2, u_3\}$ to $\mathcal{P}$.

First, note that this first step eventually ends, since moving a vertex $v \in V_2$ with three neighbours in V_1 to V_1 cannot create a new vertex of V_2 satisfying this condition. Moreover, if the four invariants are satisfied before an iteration of Step 1, then they are still satisfied after it. Indeed, first, every iteration of Step 1 adds one vertex to V_1 and two pairs to $\mathcal{P}$, and thus, $|V_1| - |\mathcal{P}|$ decreases, and Invariant 1 still holds. Secondly, recall that Invariant 3 states that all the pairs of $\mathcal{P}$ consist of vertices in V_1 having their neighbourhood in V_1. Because, prior to moving v to V_1, we had $v \in V_2$, it means that no pair of $\mathcal{P}$ could include any of v, u_1, u_2, u_3. Thus, the two pairs $\{v, u_1\}$ and $\{u_2, u_3\}$ cannot intersect with other pairs of $\mathcal{P}$, and Invariant 2 is maintained. Thirdly, since the u_i's satisfy Invariant 4, *i.e.*, v is their only neighbour in V_2 before v is moved to V_1, the pairs $\{v, u_1\}$ and $\{u_2, u_3\}$ fulfill the condition of Invariant 3. Finally, since v is the only vertex that was moved to V_1, and its three neighbours lie in V_1, clearly Invariant 4 remains satisfied as well.

Thus, once Step 1 has completed, Invariants 1 to 4 above remain satisfied, and no vertex of V_2 has its three neighbours in V_1. We then proceed with applying the second step of the procedure:

- **Step 2:** While V_2 has a vertex v with exactly two neighbours u_1, u_2 in V_1, we move v from V_2 to V_1, and add the pair $\{u_1, u_2\}$ to $\mathcal{P}$.

Note that, due to the conditions, moving a vertex v verifying the condition of Step 2 from V_2 to V_1 cannot create a new vertex verifying the condition of Step 1. However, it can be noted that this can create new vertices verifying the condition of Step 2, which will be considered in later iterations of Step 2. For now, we just assume that, at the end of Step 2, no vertex verifying the condition of the step remains. We will give a bound on the number of vertices that end up in V_1 after this two-step procedure later.

We now show that, if the four invariants are satisfied before an iteration of Step 2, then they are still satisfied after it. Indeed, each time one vertex is added to V_1 in an iteration of Step 2, one pair is added to $\mathcal{P}$, and thus, Invariant 1 remains valid. Moreover, the pair $\{u_1, u_2\}$ added to $\mathcal{P}$ is such that v is the only neighbour of both u_1 and u_2 in V_2 before v is moved to V_1. Hence, Invariant 3 is still valid and, by Invariant 3, none of u_1, u_2 belonged to a pair in $\mathcal{P}$, and so, Invariant 2 is still satisfied. Finally, the vertex v which was moved to V_1 had exactly two neighbours in V_1, so, as G is cubic, it has one neighbour in V_2, and Invariant 4 is still satisfied.

Let V_1, V_2, and $\mathcal{P}$ be the result of the execution of the above two-step procedure. Note that, by the definition of the procedure, every vertex in V_2 has at most one neighbour in V_1. Moreover, by Invariant 4, every vertex in V_1 has at most one neighbour in V_2. Hence, the cut-set M of the cut (V_1, V_2), consisting of the edges with one end in V_1 and one end in V_2, induces a matching. Moreover, by Invariant 3, for every $\{u, v\} \in M$ and $\{x, y\} \in \mathcal{P}$, we have $\{u, v\} \cap \{x, y\} = \emptyset$.

We are now ready to define Bob's strategy, which is well-defined by the remarks above, and recall that, in the first round, Alice coloured $v_0 \in V_1$. In each round, if the last vertex Alice coloured is a vertex u such that $\{u, v\} \in \mathcal{P} \cup M$, then Bob colours v. Otherwise, Bob colours any arbitrary uncoloured vertex of G.

By Bob's strategy, at the end of the game, at least one vertex in every pair of $\mathcal{P}$ is blue. Since Bob colours at least $|\mathcal{P}|$ vertices in V_1, then Alice colours at most $|V_1| - |\mathcal{P}| \leq \frac{n}{2} - 1$ vertices in V_1 (the last inequality is due to Invariant 1). Thus, Alice must colour at least one vertex in V_2 at some point. So, at the end of the game, there exists a red vertex $w_0 \in V_2$. Recall that $v_0 \in V_1$ is also red. These two vertices cannot be in the same connected red component since M is the cut-set of the cut (V_1, V_2), and Bob coloured at least one vertex of each edge in M. Therefore, there cannot be a single connected component in the red subgraph, contradicting that G is A-perfect. □

Lemma 4.5 does not conclude the proof of Theorem 4.4, as there exist (arbitrarily large) connected bridgeless cubic graphs G, for which some of their vertices are only contained in cycles of length more than $|V(G)|/2$. Indeed, consider the following construction. For an $n \geq 1$, let H_n be the graph obtained from a path $(u, x_1, \dots, x_{3n}, v)$ on $3n + 2$ vertices by adding a vertex y_i for every $i \in \{1, \dots, n\}$, as well as the three edges $y_i x_{3(i-1)+1}, y_i x_{3(i-1)+2}, y_i x_{3(i-1)+3}$. Note that H_n is almost cubic, as only u and v do not have degree 3. Also, all paths from u to v in H_n have length at least $3n + 1$. Let G be the graph obtained from the disjoint union of three copies H, H', H'' of

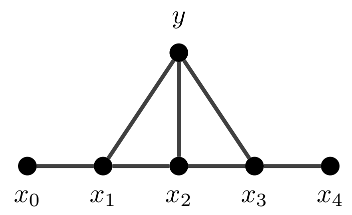


Figure 3: The aqueduct graph.

H_n by identifying their three copies of u to a single vertex (that we still denote u), and similarly identifying their three copies of v to a single vertex (that we still denote v). Note that G is cubic and bridgeless, and that $|V(G)| = 3(4n) + 2 = 12n + 2$. Also, every cycle C of G containing u (or v) is made up of one path from u to v in one of H, H', H'', and of one path from u to v in another of H, H', H''. Thus, C has length at least $2(3n + 1) = 6n + 2 > 6n + 1 = |V(G)|/2$.

The above construction led us to the following gadget. Let the *aqueduct* be the graph depicted in Figure 3. Whenever dealing with an aqueduct later on, we will use the notation introduced in Figure 3. We start by remarking the following:

Lemma 4.6. *No A-perfect cubic graph admits an aqueduct as an induced subgraph.*

Proof. Towards a contradiction, assume G is an A-perfect cubic graph containing an aqueduct A as an induced subgraph. Let us define the following strategy for Bob. In each round, if the last vertex Alice coloured is a vertex

- of A in a pair $\{x_0, x_1\}$, $\{x_2, y\}$ or $\{x_3, x_4\}$, then Bob colours the other vertex of that pair;
- of G not in A, then Bob colours an uncoloured vertex of G not in A.

Note that Bob can always follow this strategy from start to end, as G is of even order. By Bob's strategy, at the end of the game, one vertex in $\{x_2, y\}$ is red, while one vertex in $\{x_0, x_1\}$ and one vertex in $\{x_3, x_4\}$ must be blue. Since $V(G) \setminus V(A)$ is non-empty (as G is cubic), there are at least two connected red components. Hence, G is not A-perfect, a contradiction. □

Let us recall some facts about cubic graphs. It is well-known that every connected bridgeless cubic graph is 2-connected (as a cut-vertex of a cubic graph must be incident to at least one bridge). This implies that any vertex of a connected bridgeless cubic graph is contained in a cycle.

Lemma 4.7. *Every A-perfect cubic graph is* 2*-connected.*

Proof. It is clear that every A-perfect cubic graph must be connected, so let G be a connected A-perfect cubic graph. Towards a contradiction, let us assume that G contains a bridge uv. Let us denote by G_1 and G_2 the two connected components of $G - uv$. Note that G_1 and G_2 each have order at least 3 since u has two neighbours in G_1, while v has two neighbours in G_2. Consider the following strategy for Bob. In each round, if the last vertex Alice coloured is a vertex

- in $\{u, v\}$, then Bob colours the other vertex in $\{u, v\}$;
- in $G_1 - u$ ($G_2 - v$, resp.), then Bob colours an uncoloured vertex of $G_1 - u$ ($G_2 - v$, resp.).

Whenever Bob cannot follow this strategy, he colours any arbitrary uncoloured vertex. Because uv is a bridge of G, both u and v are cut-vertices. Also, since G_1 and G_2 have order at least 3, then, by the strategy for Bob above, Alice must colour vertices in both G_1 and G_2. Since Bob colours at least one of u and v, from all these arguments we deduce that, at the end of the game, there are at least two connected red components, one containing vertices of G_1 only, and another one containing vertices of G_2 only. Thus, this strategy for Bob contradicts that G is A-perfect. Hence, G must be connected, bridgeless, and cubic, and so, as mentioned earlier, it is 2-connected. □

We are now ready to prove Theorem 4.4.

Proof of Theorem 4.4. Towards a contradiction, let G be an A-perfect cubic graph of order $n \geq 133$. Let v_0 be the first vertex coloured by Alice when she follows an optimal strategy in G. By Lemma 4.7, G is 2-connected, and thus, v_0 belongs to some cycle. By Lemma 4.5, any cycle containing v_0 must have length at least $n/2$. The goal is to prove that G contains an aqueduct, which is a contradiction by Lemma 4.6.

Let $\{u, v, w\} = N(v_0)$, and let $G' = G - v_0$. For an integer $r \geq 0$, the *ball of radius r centered at v* is the set of vertices of G' that are at distance at most r from v in G'. Let B_u, B_v, and B_w be the balls of radius $\frac{n}{4} - 2$ centered at u, v, and w, respectively, in G'.

Claim 1. *B_u, B_v, and B_w are pairwise vertex-disjoint.*

Proof of the claim. Towards a contradiction, suppose that two of them intersect, say B_u and B_v. Let $w_1 \in B_u \cap B_v$ be a vertex minimising its distance to u ($w_1 = u$ if $u \in B_v$). Let P be a shortest path from u to w_1 in B_u, and let Q be a shortest path from v to w_1 in B_v. By the minimality of the distance between u and w_1, $V(P) \cap V(Q) = \{w_1\}$. As P and Q are in balls of radius $\frac{n}{4} - 2$ centered at u and v, respectively, they each have length at most $\frac{n}{4} - 2$. Then, the edges in $\{v_0u, v_0v\} \cup E(P) \cup E(Q)$ induce a cycle, containing v_0, of length at most $\frac{n}{2} - 2$ in G. This contradicts the fact that v_0 is not contained in a cycle of length at most $\frac{n}{2} - 1$ (by Lemma 4.5). Thus, B_u, B_v, and B_w are pairwise vertex-disjoint. ◇

Claim 2. *For every $x \in \{u, v, w\}$, $\frac{n}{3} - 2 \leq |B_x| \leq \frac{n}{3} + 3$.*

Proof of the claim. By symmetry, it suffices to prove the claim for $x = u$. Since G is 2-connected, the subgraph G' is connected, and so, let Q be a shortest path between u and v in G'. Let u' be the vertex of $V(Q) \cap B_u$ that is the farthest from u, and let P be the subpath of Q from u to u'. By definition, $V(P) \subseteq B_u$, and also by Claim 1, $|V(P)| = \frac{n}{4} - 1$.

Since P is a shortest path and G is cubic, every internal vertex of P (*i.e.*, those in $Y = V(P) \setminus \{u, u'\}$) has one neighbour in $B_u \setminus V(P)$. Moreover, because G is cubic, the number $d_Y(v)$ of neighbours in Y of every vertex $v \in B_u \setminus V(P)$ is at most 3. Hence,

$$\frac{n}{4} - 3 = |Y| = \sum_{v \in B_u \setminus V(P)} d_Y(v) \leq 3|B_u \setminus V(P)|,$$

and thus, there exist at least $\left\lceil \frac{\frac{n}{4}-3}{3} \right\rceil \geq \frac{n}{12} - 1$ vertices in $B_u \setminus V(P)$. Therefore,

$$|B_u| = |B_u \setminus V(P)| + |V(P)| \geq \frac{n}{12} - 1 + \frac{n}{4} - 1 \geq \frac{n}{3} - 2.$$

By symmetry, $|B_v| \geq \frac{n}{3} - 2$ and $|B_w| \geq \frac{n}{3} - 2$. By Claim 1, and as $v_0 \notin B_u \cup B_v \cup B_w$, we have $|B_u| \leq n - |B_v| - |B_w| - 1$. Therefore, $|B_u| \leq n - (\frac{n}{3} - 2) - (\frac{n}{3} - 2) - 1 = \frac{n}{3} + 3$. ◇

As in the proof of the previous claim, let $u' \in B_u$ be a vertex at distance $\frac{n}{4} - 2$ from u in B_u, and let P be a shortest $\{u, u'\}$-path in G' (so, all its vertices are contained in B_u). By the previous claim, $|B_u| \leq \frac{n}{3} + 3$, so $|B_u \setminus V(P)| = |B_u| - |V(P)| \leq \frac{n}{3} + 3 - (\frac{n}{4} - 1) = \frac{n}{12} + 4$. Let α be the number of vertices in $B_u \setminus V(P)$ adjacent to at most 2 vertices of $Y = V(P) \setminus \{u, u'\}$, and let β be the number of vertices in $B_u \setminus V(P)$ adjacent to 3 vertices of Y. Note that $\alpha + \beta = |B_u \setminus V(P)|$, and that $2\alpha + 3\beta \geq |Y|$. Therefore,

$$\alpha + \beta \leq \frac{n}{12} + 4; \tag{1}$$

$$2\alpha + 3\beta \geq \frac{n}{4} - 3. \tag{2}$$

Computing (2) $- 3 \times$ (1), it follows that $\alpha \leq 15$. Thus, since $n \geq 133$, we have $2\alpha + 3\beta \geq \frac{n}{4} - 3 > 30$, which implies that $\beta \geq 1$. Hence, at least one vertex of $B_u \setminus V(P)$ is adjacent to 3 vertices of P. Since P is a shortest path in B_u, then any vertex adjacent to 3 vertices of P must be adjacent to 3 consecutive vertices of P. This implies that G contains an aqueduct as an induced subgraph, a contradiction by Lemma 4.6. □

5. Sufficient conditions for graphs to be A-perfect

We have already seen a few conditions for graphs to meet the upper bound in Lemma 4.1, *i.e.*, to be A-perfect. In this section, we give two more such sufficient conditions, one is based on particular degree conditions, while the other is based on the number of edges.

5.1. Graphs with large degrees

The next result gives a sufficient condition, in terms of minimum degree and maximum degree, for a graph to be A-perfect.

Theorem 5.1. *If G is a connected graph with $\Delta(G) + \delta(G) \geq |V(G)|$, then G is A-perfect.*

Proof. We give a strategy for Alice ensuring that, at the end of the game, the red subgraph is connected, which implies that G is A-perfect. Let u be any vertex of degree $\Delta(G)$. In the first round, Alice colours u. For every $i \geq 1$, let C_i be the connected component of red vertices at the end of the i^{th} round (we will show that the red vertices always induce a connected subgraph, and so, C_i is well-defined). Let $R_i = V(G) \setminus N[C_i]$, *i.e.*, R_i is the set of (non-red) vertices not dominated by a red vertex at the end of the i^{th} round, and let R_i^U be the subset of uncoloured vertices in R_i at the end of the i^{th} round. Note that $C_1 = \{u\}$ is connected and that

$$|R_1^U| \leq |R_1| = |V(G)| - |N[C_1]| = |V(G)| - \Delta(G) - 1 \leq \delta(G) - 1.$$

Let us show by induction on $i \geq 1$ that, at the end of the i^{th} round, C_i is connected and either $R_i^U = \emptyset$ (in which case we are done) or $|R_i^U| \leq \delta(G) - i$. By the above paragraph, the induction hypothesis holds for $i = 1$. Let $i \geq 1$ and let us assume that the induction hypothesis holds for i. We show it still holds for $i + 1$.

If $R_i^U = \emptyset$, then C_i is a connected red dominating set of the subgraph of G induced by the vertices of C_i and the remaining uncoloured vertices of G. From now on, Alice may colour any uncoloured vertex, and the induction hypothesis clearly holds for $i + 1$. In particular, the set of red vertices induces a connected subgraph at the end of the game, proving the result.

Otherwise, let $v \in R_i^U$. Since v has at least $\delta(G)$ neighbours (none of which are red since $N[R_i] \cap C_i = \emptyset$) and Bob has coloured i vertices, v has at least $\delta(G) - i$ uncoloured neighbours, and $\delta(G) - i > 0$ since $R_i^U \neq \emptyset$ and $|R_i^U| \leq \delta(G) - i$. Moreover, $|R_i^U \setminus \{v\}| < \delta(G) - i$, so v has at least one uncoloured neighbour w not in R_i, which implies that $w \in N(R_i) = N(C_i)$. In the $(i + 1)^{th}$ round, Alice colours w. Then, $C_{i+1} = C_i \cup \{w\}$ is clearly connected, and $R_{i+1}^U \subseteq R_{i+1} \subseteq R_i \setminus \{v\}$ (since $v \in N(C_{i+1})$), and hence, $|R_{i+1}^U| \leq |R_{i+1}| \leq |R_i| - 1 \leq \delta(G) - (i + 1)$. □

We note that the bound in the statement of Theorem 5.1 is sharp, in the sense that there exists a graph G with $\Delta(G) + \delta(G) = |V(G)| - 1$ that is not A-perfect. For example, consider the graph G consisting of two complete graphs on $d \geq 3$ vertices joined by a single edge e. Then, $\Delta(G) = d$, $\delta(G) = d - 1$, $|V(G)| = 2d$, and thus, $\Delta(G) + \delta(G) = 2d - 1 = |V(G)| - 1$. However, Bob can guarantee that Alice achieves a score of about $|V(G)|/4$, by colouring an uncoloured vertex incident to e in the first round, and then, in each subsequent round, colouring an uncoloured vertex in the same clique that Alice just coloured a vertex in. Thus, G is not A-perfect.

5.2. Graphs with large size

The next result shows that if G has sufficiently many edges, then G is A-perfect.

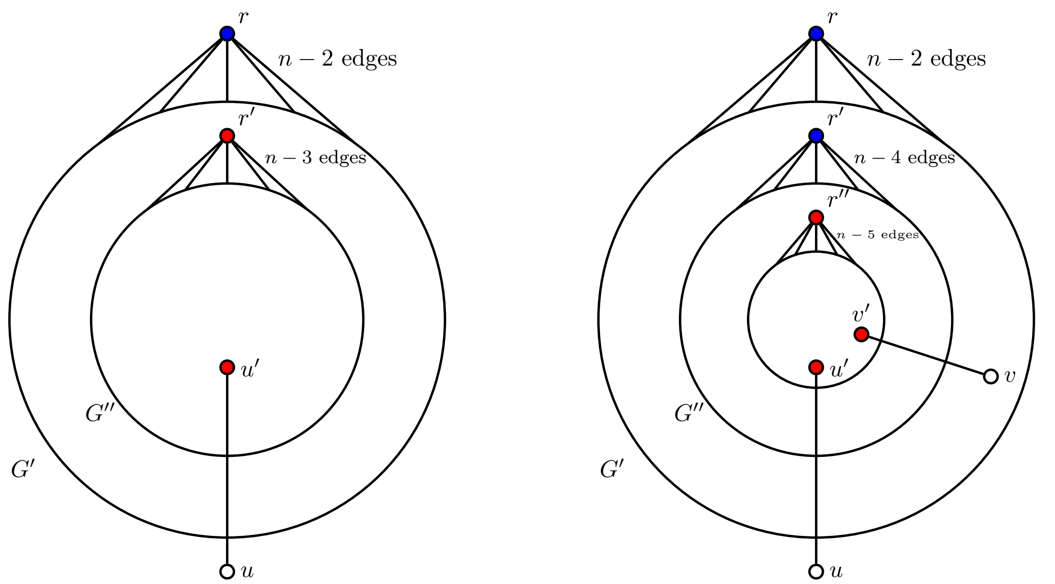
Theorem 5.2. *If G is a connected graph with $|E(G)| > \frac{(|V(G)|-2)(|V(G)|-3)}{2} + 2$, then G is A-perfect.*

Proof. Set $n = |V(G)|$, $m = |E(G)|$, and

$$x = \frac{(n-2)(n-3)}{2} + 2 = \frac{n^2 - 5n + 10}{2}.$$

Note first that $\Delta(G) \geq n - 4$. Indeed, if we had $\Delta(G) \leq n - 5$, then we would deduce that $m \leq \frac{n(n-5)}{2} < x$, which contradicts that $m > x$. Furthermore, if $\Delta(G) = n - 4$, then $\delta(G) \geq 7$. Indeed, if there is a degree-6 vertex, then we have a contradiction since

$$m \leq \frac{(n-1)(n-4) + 6}{2} = \frac{n^2 - 5n + 10}{2} = x.$$



(a) The state of the game in G after Alice's first two turns in Case 1.(a), where $G'' = G'[N(r')]$.

(b) The state of the game in G after Alice's first three turns in Case 1.(b)i.

Figure 4: Cases 1.(a) and 1.(b)i. in the proof of Theorem 5.2.

Thus, if $\Delta(G) = n-4$, then G is A-perfect by Theorem 5.1 since $\delta(G) \geq 7$. Lastly, if $\Delta(G) = n-1$, then $\delta(G) \geq 1$, and thus, G is A-perfect by Theorem 5.1. Hence, in what follows, we assume that $n-3 \leq \Delta(G) \leq n-2$. We give a strategy for Alice that allows her to colour the vertices of a connected dominating set of G within the first four rounds, and so, by Lemma 2.3, G is A-perfect. We treat the two possible values for $\Delta(G)$ independently.

1. $\Delta(G) = n-2$.

   Let $r \in V(G)$ be such that $d(r) = n-2$, and let $G' = G[N(r)]$. Then, $|V(G')| = n-2$. Since $d(r) = n-2$, there is exactly one additional vertex $u \in V(G) \setminus V(G')$ ($u \neq r$). If $d(u) \geq 2$, then Alice colours r in the first round, and then, in the second round, she colours a neighbour of u (this is possible since $d(u) \geq 2$), and these vertices form a connected dominating set of G. Thus, we may assume that $d(u) = 1$, and let $N(u) = \{u'\}$. We have that $\Delta(G') \geq n-4$. Indeed, if $\Delta(G') \leq n-5$, then we have a contradiction since

$$m \leq \frac{(n-2)(n-5)}{2} + n - 2 + 1 = \frac{n^2 - 5n + 8}{2} < x.$$

   We distinguish the following subcases:

   (a) $\Delta(G') = n-3$ (see Figure 4(a) for an illustration).
   Let $r' \in V(G')$ be such that $d_{G'}(r') = n-3$. Alice's strategy is as follows. She starts by colouring u'. Now, if Bob does not colour r, then Alice continues by colouring r, at which point she has coloured the vertices of the connected dominating set $\{u', r\}$ of G. So, we may assume that Bob colours r in the first round. In the second round, Alice colours r'. Observe that $\{u', r'\}$ also forms a connected dominating set of G since $d_{G'}(r') = n-3$, and thus, $u'r' \in E(G)$.

   (b) $\Delta(G') = n-4$.
   Let $r' \in V(G')$ be such that $d_{G'}(r') = n-4$, and let $G'' = G'[N(r')]$. We distinguish cases according to whether $u' \in V(G'')$ or not.

      i. $u' \in V(G'')$.
      Since $d_{G'}(r') = n-4$, there is exactly one additional vertex $v \in V(G') \setminus V(G'')$ ($v \neq r'$). Note that $d_{G'}(v) \geq 1$ because if $d_{G'}(v) = 0$, *i.e.*, $N(v) = r$, then we have

a contradiction since

$$m \leq \frac{(n-3)(n-4)}{2} + n - 2 + 1 = \frac{n^2 - 5n + 10}{2} = x.$$

If $d_{G'}(v) \geq 2$, then Alice's strategy is as follows. She starts by colouring u'. As before, Bob is forced to colour r in the first round. In the second round, Alice colours r'. In the third round, if $u' \notin N(v)$, then Alice colours a neighbour $v' \in V(G')$ (this is possible since $d_{G'}(v) \geq 2$). After three rounds, Alice's vertices form a connected dominating set of G.

Assume now that $d_{G'}(v) = 1$, and let $N_{G'}(v) = \{v'\}$ (see Figure 4(b) for an illustration). Then, $\Delta(G'') = n - 5$. Indeed, if $\Delta(G'') \leq n - 6$ (and so, $n \geq 6$), then we have a contradiction since

$$m \leq \frac{(n-4)(n-6)}{2} + n - 4 + n - 2 + 1 + 1 = \frac{n^2 - 6n + 16}{2} \leq x.$$

Let $r'' \in V(G'')$ be such that $d_{G''}(r'') = n-5$, and observe that $v' \in N(r'')$. Alice's strategy is as follows. She starts by colouring u', forcing Bob to colour r. Then, she colours v' forcing Bob to colour r' (similarly to earlier, if Bob does not colour r', then Alice colours r', and thus, has coloured the vertices of a connected dominating set of G). Finally, Alice colours r''. Observe that the vertices u', v', and r'' form a connected dominating set of G.

ii. $u' \notin V(G'')$.

Observe that u' is the only vertex of G' that is not a neighbour of r', and that $d(u') \geq 3$. Indeed, if $d(u') \leq 2$, then we have a contradiction since

$$m \leq \frac{(n-3)(n-4)}{2} + n - 2 + 1 = \frac{n^2 - 5n + 10}{2} = x.$$

Thus, there is at least one edge $u'u''$ with $u'' \in V(G'')$. If $d(u') \geq 4$, then Alice's strategy is as follows. She starts by colouring u', forcing Bob to colour r. Then, she colours r', and, in the third round, she colours one of the remaining uncoloured neighbours of u' in G'' (which exists since $d(u') \geq 4$). These three vertices form a connected dominating set of G.

Otherwise, $d(u') = 3$, and, as in Case 1.(b)i, there exists $r'' \in V(G'')$ such that $d_{G''}(r'') = n - 5$. Alice's strategy is as follows. She starts by colouring u', forcing Bob to colour r. Then, she colours u'', forcing Bob to colour r'. Finally, Alice colours r''. Note that u', u'', and r'' form a connected dominating set of G.

2. $\Delta(G) = n - 3$.

Observe that G cannot contain two vertices u, v such that $d(u) + d(v) \leq 5$. Indeed, if there are two such vertices, then we have a contradiction since $m \leq \frac{(n-2)(n-3)+5}{2}$, but this is not an integer since $(n-2)(n-3) + 5$ is odd, and thus,

$$m \leq \frac{(n-2)(n-3) + 5 - 1}{2} = \frac{n^2 - 5n + 10}{2} = x.$$

Let r be a vertex of G such that $d(r) = n - 3$, and let $G' = G[N(r)]$. Since $d(r) = n - 3$, there are exactly two additional vertices $u, v \in V(G) \setminus V(G')$ ($u, v \neq r$). We distinguish cases according to the degrees of u and v, and note that $d(u) + d(v) \geq 6$. In what follows, when we say that Alice colours a vertex if needed, it means that if it is not necessary (in the sense that such a vertex has already been coloured), then she either colours the vertex she is supposed to colour in the next round, or she colours any arbitrary uncoloured vertex in that round.

(a) $d(u), d(v) \geq 3$.

Alice's strategy is as follows. She starts by colouring r. In the second round, she colours an uncoloured vertex in $N(v)$ in G' (this is possible since $d(v) \geq 3$). In the third round, if needed, *i.e.*, if Alice has not coloured a vertex in $N(u)$ yet, Alice colours an uncoloured vertex in $N(u)$ in G' if possible, and if not, then $uv \in E(G)$ and Bob coloured $N(u) \setminus \{v\}$ in the first two rounds, and so, she colours v. Then, by the end of the third round, Alice has coloured r, at least one vertex in $N(v)$ in G', and at least one vertex in $N(u)$, and these vertices form a connected dominating set of G.

(b) $d(u) = 2$ and $d(v) \geq 4$.

Alice's strategy is as follows. She starts by colouring r. In the second round, she colours an uncoloured vertex in $N(u)$ in G' if possible, and if not, then $uv \in E(G)$ and Bob coloured $N(u) \setminus \{u\}$ in the first round, and so, she colours v. In the third round, Alice colours an uncoloured vertex in $N(v)$ in G' (this is possible since $d(v) \geq 4$). Then, by the end of the third round, Alice has coloured r, at least one vertex in $N(v)$ in G', and at least one vertex in $N(u)$, and these vertices form a connected dominating set of G.

(c) $d(u) = 1$ and $d(v) \geq 5$.

Let $u' \in N(u)$ be a fixed neighbour of u in $N(u)$. In this case, there exists at least one vertex $r' \in G'$ with $d_{G'}(r') \geq n - 5$, as otherwise, we have a contradiction since

$$m \leq \frac{(n-3)(n-6)}{2} + 2(n-3) + 1 = \frac{n^2 - 5n + 8}{2} < x.$$

Note that v has at least 4 neighbours in G' since $d(v) \geq 5$ and $rv \notin E(G)$. We distinguish the following subcases:

i. $\Delta(G') = n - 4$.

Let $r' \in V(G')$ be such that $d_{G'}(r') = n - 4$, then Alice's strategy is as follows. She starts by colouring u' (it may be that $u' = v$). If Bob colours a vertex in $\{r, r'\}$ (a neighbour $v' \in V(G')$ of v, resp.) in the first round, then, in the second round, Alice colours the other vertex in $\{r, r'\}$ (another neighbour $v^* \in V(G')$ of v, resp.). If Alice coloured a vertex in $\{r, r'\}$ (v^*, resp.) in the second round, then she colours a vertex in $\{v', v^*\}$ ($\{r, r'\}$, resp.) in the third round. After three rounds, Alice's vertices form a connected dominating set of G.

ii. $\Delta(G') = n - 5$.

Let $r' \in V(G')$ be such that $d_{G'}(r') = n-5$, and let $G'' = G'[N(r')]$. We distinguish cases according to whether $u' \in V(G'')$ or not.

A. $u' \in V(G'')$.

As $u' \in V(G'')$, $uv \notin E(G)$. Since $d(r') = n - 5$, there is exactly one additional vertex $w \in V(G') \setminus V(G'')$ ($w \neq r'$). Note that $d_{G'}(w) \geq 1$ because if $d_{G'}(w) = 0$, *i.e.*, $N(w) = r$, then we have a contradiction since

$$m \leq \frac{(n-4)(n-5)}{2} + 2(n-3) + 1 = \frac{n^2 - 5n + 10}{2} = x.$$

If $d_{G'}(w) \geq 2$, then Alice's strategy is as follows. She starts by colouring u'. As before, Bob is forced to colour r in the first round. Indeed, if he does not, then Alice will colour r in the second round, and then she will colour an uncoloured neighbour $v' \in V(G')$ of v in the third round (this is possible since $d(v) \geq 5$), and her vertices form a connected dominating set of G. In the second round, Alice colours r'. In the third round, if needed, *i.e.*, if $u' \notin N(w)$, Alice colours an uncoloured neighbour $w' \in V(G')$ of w (this is possible since $d_{G'}(w) \geq 2$). In the fourth round, Alice colours an uncoloured neighbour $v' \in V(G')$ of v (this is possible since v has at least 5 neighbours in G' as $uv \notin E(G)$ and $d(v) \geq 5$). At the end of the fourth round, Alice's vertices form a connected dominating set of G.

Assume now that $d_{G'}(w) = 1$, and let $N_{G'}(w) = \{w'\}$. Then, $\Delta(G'') = n - 6$. Indeed, if $\Delta(G'') \leq n - 7$ (and so, $n \geq 7$), then we have a contradiction since

$$m \leq \frac{(n-5)(n-7)}{2} + n - 5 + 2(n-3) + 1 + 1 = \frac{n^2 - 6n + 17}{2} \leq x.$$

Let $r'' \in V(G'')$ be such that $d_{G''}(r'') = n-6$, and observe that $w' \in N(r'')$. Alice's strategy is as follows. She starts by colouring u'. As before (when $d_{G'}(w) \geq 2$), Bob is forced to colour r in the first round. In the second round, Alice colours w'. Analogously to why Bob was forced to colour r in the first round, Bob is forced to colour r' in the second round. In the third round, Alice colours r''. In the fourth round, Alice colours an uncoloured neighbour $v' \in V(G')$ of v (this is possible since v has at least 5 neighbours in G'). At the end of the fourth round, Alice's vertices form a connected dominating set of G.

B. $u' \notin V(G'')$.

First, assume that $uv \notin E(G)$. Then, u' is the only vertex of G' that is not a neighbour of r', and $d(u') \geq 3$. Indeed, if $d(u') \leq 2$, then we have a contradiction since

$$m \leq \frac{(n-4)(n-5)}{2} + 2(n-3) + 1 = \frac{n^2 - 5n + 10}{2} = x.$$

Thus, there is at least one edge $u'u''$ with $u'' \in V(G'') \cup \{v\}$. If $d(u') \geq 4$, then Alice's strategy is as follows. She starts by colouring u', forcing Bob to colour r, as before. Then, she colours r', and in the third round, she colours one of the remaining uncoloured neighbours of u' in G'' (which exists since $d(u') \geq 4$). In the fourth round, Alice colours an uncoloured neighbour $v' \in V(G')$ of v (this is possible since v has at least 5 neighbours in G'). At the end of the fourth round, Alice's vertices form a connected dominating set of G.

Otherwise, $d(u') = 3$, and, as in Case 2.(c)iiA, there exists $r'' \in V(G'')$ such that $d_{G''}(r'') = n-6$. Let $r'' \in V(G'')$ be such that $d_{G''}(r'') = n-6$. Alice's strategy is as follows. She starts by colouring u', forcing Bob to colour r, as before. Then, she colours u'', forcing Bob to colour r', as before. In the third round, Alice colours r''. In the fourth round, Alice colours an uncoloured neighbour $v' \in V(G')$ of v (this is possible since v has at least 5 neighbours in G'). At the end of the fourth round, Alice's vertices form a connected dominating set of G.

Now, assume that $uv \in E(G)$. Then, $u' = v$ and there is exactly one additional vertex $w \in V(G') \setminus V(G'')$ ($w \neq r'$). Note that $d_{G'}(w) \geq 2$ because if $d_{G'}(w) = 1$, then we have a contradiction since

$$m \leq \frac{(n-4)(n-5)}{2} + 2(n-3) + 1 = \frac{n^2 - 5n + 10}{2} = x.$$

Alice's strategy is as follows. She starts by colouring $u' = v$, forcing Bob to colour r, as before. In the second round, she colours r'. In the third round, Alice colours an uncoloured neighbour $w' \in V(G')$ of w (this is possible since $d_{G'}(w) \geq 2$). In the fourth round, if needed, *i.e.*, if Alice has not yet coloured a vertex in $N(v)$ that is not u, Alice colours an uncoloured neighbour $v' \in V(G')$ of v (this is possible since $d(v) \geq 5$). At the end of the fourth round, Alice's vertices form a connected dominating set of G. □

We note that the bound in the statement of Theorem 5.2 is sharp, in the sense that there exists a graph G with $\frac{(|V(G)|-2)(|V(G)|-3)}{2} + 2$ edges that is not A-perfect. For example, consider, as G, any graph obtained from a complete graph on an odd number $N \geq 3$ of vertices, by taking any of its vertices u, and attaching at u a pending path (u, v, w) of length 2. Note that $|V(G)| = N + 2$ and that

$$|E(G)| = \frac{N(N-1)}{2} + 2 = \frac{(|V(G)|-2)(|V(G)|-3)}{2} + 2.$$

Now, to see that G is not A-perfect, consider the following strategy for Bob. Bob colours a vertex in $\{u, v\}$ in the first round, and then, in each of the subsequent rounds, he colours any uncoloured vertex different from w. Since $|V(G)|$ is odd, Alice is forced to colour w at some point, which, by the end of the game, cannot be part of a single connected red component due to Bob having coloured u or v in the first round. Thus, G is not A-perfect.

6. Graphs with few P_4's

In this section, we consider the Maker-Breaker Largest Connected Subgraph game played in cographs, and, more generally, in $(q, q-4)$-graphs. For both classes of graphs, we prove that c_g can be determined in linear time. It is worth recalling that the outcome of the Largest Connected Subgraph game can be decided in linear time in cographs [4], while this was not proved for the more general class of $(q, q-4)$-graphs.

6.1. Cographs

Let us first recall what cographs are. The main definition we give needs the following two graph operations. Given two graphs G and H, we denote the *disjoint union* of G and H by $G+H$, where $V(G+H) = V(G) \cup V(H)$ and $E(G+H) = E(G) \cup E(H)$, and we denote the *join* of G and H by $G \oplus H$, where $V(G \oplus H) = V(G) \cup V(H)$ and $E(G \oplus H) = E(G) \cup E(H) \cup \{uv : (u, v) \in V(G) \times V(H)\}$. One way to define cographs is through the following recursive definition:

- K_1, the graph with a single vertex, is a cograph.
- If G and H are two cographs, then $G + H$ and $G \oplus H$ are cographs.

Also, cographs are the class of P_4-free graphs [9], *i.e.*, the graphs that do not have the path of order 4 as an induced subgraph. By Lemma 2.2, when studying c_g for a given class of graphs, we only need to focus on its connected members. Thus, we are interested in connected cographs, hence, the join of two cographs.

Lemma 6.1. *For every two graphs G and H, the join $G \oplus H$ is A-perfect.*

Proof. Consider the strategy for Alice where she aims at having coloured a vertex $u \in V(G)$ and a vertex $v \in V(H)$ by the end of the second round. Note that this is always possible to achieve, unless $|V(G)| = |V(H)| = 1$, in which case the statement is clearly true. In the other cases, note that $\{u, v\}$ is a connected dominating set of $G \oplus H$, and thus, the result follows by Lemma 2.3. $\square$

Theorem 6.2. *For a cograph G, determining $c_g(G)$ can be done in linear time.*

Proof. Since every connected cograph is the join of two cographs, then, by Lemmas 2.2 and 6.1, we have that $c_g(G) = \left\lceil \frac{|V(G_i)|}{2} \right\rceil$, where G_i is the connected component of G with the largest order. $\square$

6.2. $(q, q-4)$-graphs

We now turn to $(q, q-4)$-graphs, which are a generalisation of cographs defined as graphs having a limited number of P_4's [1]. Let us start by recalling the formal definitions. Let $G = (S, K, R, E)$ be a graph with $V(G) = S \cup K \cup R$ and $E(G) = E$. Consider the following properties:

1. $S \cup K \cup R$ is a partition of $V(G)$ and R can be the empty set.
2. $G[K \cup R] = K \oplus R$ (*i.e.*, for all $u, v \in V(G)$ such that $u \in K$ and $v \in R$, we have that $uv \in E$), and K separates S from R (*i.e.*, for all $u \in S$ and $v \in R$, we have that $uv \notin E$).
3. S is an independent set, K is a clique, $|S| = |K| \geq 2$, and there exists a bijection $f : S \to K$ such that, either, for every vertex $s \in S$, $N(s) \cap K = K \setminus \{f(s)\}$, or, for every vertex $s \in S$, $N(s) \cap K = \{f(s)\}$. In the former case, we say that f is an *antimatching*, with the vertices s and $f(s)$ being *antimatched*, and in the latter case, we say that f is a *matching*, with the vertices s and $f(s)$ being *matched*.

If $G = (S, K, R, E)$ verifies all the properties above, it is called a *spider*. In that case, if f is a matching (antimatching, resp.), we say that G is a *matched spider* (*antimatched spider*, resp.). Also, if G only verifies Properties 1. and 2. above, it is called a *pseudo-spider*. In this case, for any fixed $q \geq 0$ such that $|V(S \cup K)| \leq q$, we say that G is a *q-pseudo-spider*.

For a fixed $q \geq 0$, a graph G is a $(q, q-4)$-*graph* if every subset $S \subseteq V(G)$ of at most q vertices of G induces at most $q-4$ paths on 4 vertices. Note that a cograph is a $(q, q-4)$-graph when $q = 4$. A graph G is a $(q, q-4)$-graph if one of the following is satisfied [2]:

1. G is the graph K_1.

2. $G = G_1 + G_2$, where G_1 and G_2 are $(q, q-4)$-graphs.

3. $G = G_1 \oplus G_2$, where G_1 and G_2 are $(q, q-4)$-graphs.

4. G is the spider (S, K, R, E), where $G[R]$ (if R is not empty) is a $(q, q-4)$-graph. Note that, by the definition of a spider, $G[S \cup K]$ induces a $(q, q-4)$-graph.

5. G is the q-pseudo-spider (S, K, R, E), where $G[R]$ (if R is not empty) is a $(q, q-4)$-graph.

The above definition is actually a recursive definition. In particular, for every $(q, q-4)$-graph G, there exists a decomposition-tree (not necessarily unique) representing G. The internal nodes of such a decomposition correspond to subgraphs of G that are $(q, q-4)$-graphs, and its leaves either correspond to a single vertex or to a subgraph with at most q vertices. The root corresponds to G, and every internal node has exactly two children (describing the four cases 2 to 5 above). Such a decomposition-tree can be computed in linear time [3]. We are now ready to prove the main result in this section:

Theorem 6.3. *Let $q \geq 0$. For a $(q, q-4)$-graph G, determining $c_g(G)$ and an optimal strategy for Alice can be done in linear time.*

Proof. Let us first compute (in linear time) a decomposition-tree T of G. Now, let us describe the algorithm that proceeds bottom-up from the leaves to the root of T. Every leaf of T corresponds to a subgraph G' with a bounded number of vertices, and therefore, $c_g(G')$ and an optimal strategy for Alice can be computed in time $O(1)$. For every internal node v (corresponding to a subgraph G' of G) of T, $c_g(G')$ and a corresponding strategy for Alice are computed from what has already been computed for the two subgraphs corresponding to the children of v.

Precisely, let G_1 and G_2 be the two subgraphs of G corresponding to the children of the root of T, and assume by induction that $c_g(G_1)$, $c_g(G_2)$, and optimal strategies for Alice in G_1 and G_2 have been computed in linear time. We now describe how the algorithm proceeds for G, and we set $|V(G)| = n$. There are 4 cases depending on how G is obtained from G_1 and G_2.

1. If $G = G_1 + G_2$, then $c_g(G) = \max\{c_g(G_1), c_g(G_2)\}$ by Lemma 2.2. W.l.o.g., $c_g(G) = c_g(G_1)$. By induction, $c_g(G_1)$ and a strategy for Alice have already been computed.

2. If $G = G_1 \oplus G_2$, then $c_g(G) = \lceil \frac{n}{2} \rceil$ by Lemma 6.1. Moreover, a corresponding strategy for Alice is also given in the proof of Lemma 6.1.

3. Assume that $G = (S, K, R, E)$ is a spider with $G_1 = G[S \cup K]$ and $G_2 = G[R]$. Note that if $|R|$ is odd (even, resp.), then n is odd (even, resp.), as $|S| = |K|$. There are two subcases:

   (a) G is an antimatched spider.
   Assume that $|K| \geq 3$ since G is a matched spider if $|K| = 2$. Then, $c_g(G) = \lceil \frac{n}{2} \rceil$. Indeed, consider any strategy for Alice where she colours two uncoloured vertices $v_1, v_2 \in K$ in the first two rounds (this is possible since $|K| \geq 3$). Since G is an antimatched spider, for every vertex $v \in S$, at least one of the edges in $\{vv_1, vv_2\}$ is in E. Thus, since K is also a clique and $G[K \cup R] = K \oplus R$, the set $\{v_1, v_2\}$ forms a connected dominating set of G, and we get the result by Lemma 2.3.

   (b) G is a matched spider.
   Let us show that

$$c_g(G) = \begin{cases} \lceil \frac{n}{2} \rceil - \left\lceil \frac{\lfloor \frac{|K|}{2} \rfloor}{2} \right\rceil & \text{if } n \text{ and } \left\lfloor \frac{|K|}{2} \right\rfloor \text{ are odd.} \\ \lceil \frac{n}{2} \rceil - \left\lfloor \frac{\lfloor \frac{|K|}{2} \rfloor}{2} \right\rfloor & \text{otherwise.} \end{cases}$$

Bob's strategy. First, we give a strategy for Bob to prove the upper bound on $c_g(G)$ in both cases. Bob first plays exhaustively in K (*i.e.*, until every vertex in K is coloured), then he plays exhaustively in R, then he colours the vertices of S that are matched to red vertices of K, and finally, he colours any remaining uncoloured vertices (the vertices of S that are matched to blue vertices of K). By Bob's strategy, at the end of the game, any red vertex in S that is matched to a blue vertex of K forms a one-vertex connected red component. Let r_S^* be the number of such red vertices. Then, $c_g(G) \leq \lceil \frac{n}{2} \rceil - r_S^*$.

Let us first show that $r_S^* \geq \left\lfloor \frac{\lfloor \frac{|K|}{2} \rfloor}{2} \right\rfloor$. Let b_K be the number of blue vertices in K once all the vertices of K are coloured. Since Bob first exhaustively colours the vertices in K, we have that $b_K \geq \left\lfloor \frac{|K|}{2} \right\rfloor$. Then, while it is possible, Bob colours vertices that are not vertices of S matched to blue vertices in K. Consider the very first point of the game where no such vertex exists (this can occur after a move made by Alice or Bob). Let $r_S \geq 0$ be the number of vertices in S that, at this point, are red and matched to a blue vertex in K. Now, Bob colours the uncoloured vertices of S matched to blue vertices, and thus, Bob colours at most $\lceil \frac{b_K - r_S}{2} \rceil$ such vertices. Hence, Alice colours at least $\lfloor \frac{b_K - r_S}{2} \rfloor$ such vertices. We get that $r_S^* \geq r_S + \lfloor \frac{b_K - r_S}{2} \rfloor \geq \lfloor \frac{b_K}{2} \rfloor \geq \left\lfloor \frac{\lfloor \frac{|K|}{2} \rfloor}{2} \right\rfloor$.

Now, let us consider the particular case where n and $\left\lfloor \frac{|K|}{2} \right\rfloor$ are odd, and let us refine the above analysis to show that, in this case, $r_S^* \geq \left\lceil \frac{\lfloor \frac{|K|}{2} \rfloor}{2} \right\rceil$. First, if $b_K > \left\lfloor \frac{|K|}{2} \right\rfloor$, then, since $\left\lfloor \frac{|K|}{2} \right\rfloor$ is odd, we get that $\lfloor \frac{b_K}{2} \rfloor > \left\lfloor \frac{\lfloor \frac{|K|}{2} \rfloor}{2} \right\rfloor$, and so, $\lfloor \frac{b_K}{2} \rfloor \geq \left\lceil \frac{\lfloor \frac{|K|}{2} \rfloor}{2} \right\rceil$, implying that $r_S^* \geq \left\lceil \frac{\lfloor \frac{|K|}{2} \rfloor}{2} \right\rceil$. Hence, we may assume that $b_K = \left\lfloor \frac{|K|}{2} \right\rfloor$, and so, b_K is odd. Since n is odd, Alice is the last player to colour a vertex in G. Hence, just before Bob colours his first vertex of S matched to a blue vertex in K, there are an even number of such uncoloured vertices remaining. Since b_K is odd, this implies that $r_S \geq 1$. Hence,

$$r_S^* \geq 1 + \left\lfloor \frac{b_K - 1}{2} \right\rfloor = 1 + \left\lfloor \frac{\left\lfloor \frac{|K|}{2} \right\rfloor - 1}{2} \right\rfloor = 1 + \left\lfloor \frac{\left\lfloor \frac{|K|}{2} \right\rfloor}{2} \right\rfloor \geq \left\lceil \frac{\left\lfloor \frac{|K|}{2} \right\rfloor}{2} \right\rceil.$$

Thus, we have proved the upper bound on $c_g(G)$ in both cases.

Alice's strategy. Now, we give a strategy for Alice to prove the lower bound on $c_g(G)$ in both cases. Alice first plays exhaustively in K, then she plays exhaustively in R, then she colours the vertices of S that are matched to red vertices of K, and finally, she colours any remaining uncoloured vertices (the vertices of S that are matched to blue vertices of K). Let r_K be the number of red vertices in K once all the vertices of K are coloured. Since Alice first exhaustively colours the vertices in K, we have that $r_K \geq \left\lceil \frac{|K|}{2} \right\rceil$. Let $b_K = |K| - r_K \leq \left\lfloor \frac{|K|}{2} \right\rfloor$ be the number of blue vertices in K once all the vertices of K are coloured. Let u_S be the number of vertices of S that are matched to blue vertices in K. Obviously, $u_S \leq b_K$. Alice's strategy ensures that, at the end of the game, the red vertices induce one connected component X and (if Bob plays optimally) some isolated vertices in S that are matched to blue vertices in K. By Alice's strategy, there are at most $\lceil \frac{u_S}{2} \rceil$ such isolated red vertices. Hence, $|X| \geq \lceil \frac{n}{2} \rceil - \lceil \frac{u_S}{2} \rceil \geq \lceil \frac{n}{2} \rceil - \lceil \frac{b_K}{2} \rceil$. Thus, $|X| \geq \lceil \frac{n}{2} \rceil - \left\lceil \frac{\lfloor \frac{|K|}{2} \rfloor}{2} \right\rceil$, which matches the upper bound when n and $\left\lfloor \frac{|K|}{2} \right\rfloor$ are odd. Also, if $\left\lfloor \frac{|K|}{2} \right\rfloor$ is even, then $\left\lceil \frac{\lfloor \frac{|K|}{2} \rfloor}{2} \right\rceil = \left\lfloor \frac{\lfloor \frac{|K|}{2} \rfloor}{2} \right\rfloor$, and so, $|X| \geq \lceil \frac{n}{2} \rceil - \left\lfloor \frac{\lfloor \frac{|K|}{2} \rfloor}{2} \right\rfloor$.

The last case to consider is when n is even. Then, Bob is the last player to colour a vertex. This implies that Alice colours at most $\lfloor \frac{u_S}{2} \rfloor$ vertices of S matched to blue vertices in K. So, $|X| \geq \lceil \frac{n}{2} \rceil - \lfloor \frac{u_S}{2} \rfloor \geq \lceil \frac{n}{2} \rceil - \lfloor \frac{b_K}{2} \rfloor \geq \lceil \frac{n}{2} \rceil - \left\lfloor \frac{\lfloor \frac{|K|}{2} \rfloor}{2} \right\rfloor$.

4. Finally, let us assume that $G = (S, K, R, E)$ is a q-pseudo-spider with $G_1 = G[S \cup K]$ (with $|V(G_1)| \leq q$) and $G_2 = G[R]$. By Lemma 2.2, we may assume that G is connected.

First, let us consider the case where $|V(G_2)| \leq 2q$, and so, $|V(G)| \leq 3q$. An exhaustive search allows to compute $c_g(G)$ and a corresponding strategy for Alice in time $O(1)$. Roughly, the set of all games in G can be described by one rooted tree with maximum degree at most $3q$ and depth $3q$. A classical dynamic-programming algorithm on this execution-tree can be used to compute the result in time $O(1)$.

From now on, let us assume that $|V(G_2)| > 2q$. Note that, in this setting, as soon as Alice colours a vertex of G_2 (and she will always be able to do that in the strategies below because $|V(G_2)| > 2q$), all the red vertices of K will belong to the same connected red component (since $G[K \cup R] = K \oplus R$). Moreover, in what follows, Alice will always colour at least $\left\lfloor \frac{|V(G_2)|}{2} \right\rfloor \geq q$ vertices in G_2, connected by a vertex of K, ensuring that the largest connected red component is always this one (the one containing all the red vertices of G_2) since $|V(G_1)| \leq q$.

In what follows, we make use of the following slight variation of the Maker-Breaker Largest Connected Subgraph game. Consider the following game that takes a graph H and $X \subseteq V(H)$ as inputs. The game proceeds as the Maker-Breaker Largest Connected Subgraph game does, *i.e.*, Alice and Bob take turns colouring vertices of G starting with Alice and with all the vertices being initially uncoloured. The difference lies in the objective of Alice. At the end of the game, the score achieved by Alice is the total number of red vertices that belong to the connected red components containing vertices of X. Intuitively, we see all the connected red components with at least one vertex in X as a single connected red component. Let $c_g(H, X)$ be the largest integer k such that Alice has a strategy to ensure a score of at least k with input (H, X), regardless of how Bob plays. Note that, by arguments similar to those near the beginning of this proof, if $|V(H)| = O(1)$, then $c_g(H, X)$ (and a corresponding strategy for Alice) can be computed in time $O(1)$ for all $X \subseteq V(H)$.

By the previous remark, $c_g(G_1, K)$ (and a corresponding strategy $\mathcal{S}_a^1$ for Alice) can be computed in time $O(1)$. By an exhaustive computation in constant time (since $|V(G_1)| = O(1)$), it is actually possible to consider all the strategies for Alice and Bob, including the ones where they may each skip one of their turns. If (in the variant game with input (G_1, K)) there exists a strategy for Alice guaranteeing her a score of at least $c_g(G_1, K)$, in which she skips one of her turns, and such that, if Bob skips a turn before Alice, then Alice can score at least $c_g(G_1, K) + 1$ without skipping any of her turns, then let $\mathcal{S}_a^2$ be such a strategy for Alice. On the other hand, if (in the variant game with input (G_1, K)) there exists a strategy for Bob guaranteeing that Alice cannot score more than $c_g(G_1, K)$, in which he skips one of his turns, and such that, if Alice skips a turn before Bob, then Bob can guarantee that Alice scores at most $c_g(G_1, K) - 1$ without skipping any of his turns, then let $\mathcal{S}_b^2$ be such a strategy for Bob. Note that, by definition, $\mathcal{S}_a^2$ and $\mathcal{S}_b^2$ cannot both exist simultaneously.

Now, let us consider the following strategy $\mathcal{S}_b$ for Bob. Whenever Alice colours a vertex in G_1, Bob plays in G_1 following a strategy that ensures that Alice scores at most $c_g(G_1, K)$ in the variant game with input (G_1, K). Whenever Alice colours a vertex in G_2, Bob colours any vertex of G_2 (if no such move is possible, Bob colours any arbitrary uncoloured vertex in G). This ensures that the largest connected red component is of order at most $c_g(G_1, K) + \left\lceil \frac{|V(G_2)|}{2} \right\rceil$. That is, $c_g(G) \leq c_g(G_1, K) + \left\lceil \frac{|V(G_2)|}{2} \right\rceil$.

Let us also define the following strategy $\mathcal{S}_a$ for Alice. First, Alice colours the first vertex in G_1 that ensures her a score of at least $c_g(G_1, K)$ in the variant game with input (G_1, K) (following strategy $\mathcal{S}_a^1$). Then, whenever Bob colours a vertex in G_1, Alice colours the vertex of G_1 following her strategy $\mathcal{S}_a^1$ to ensure a score $c_g(G_1, K)$ in the variant game with input (G_1, K). Whenever Bob colours a vertex in G_2, Alice colours any vertex in G_2. If no such move is possible, Alice colours any arbitrary uncoloured vertex. This ensures that the largest connected red component is of order at least $c_g(G_1, K) + \left\lfloor \frac{|V(G_2)|}{2} \right\rfloor$ (recall that, since $|V(G_2)| \geq 2$, Alice colours at least one vertex in G_2). That is, $c_g(G) \geq c_g(G_1, K) + \left\lfloor \frac{|V(G_2)|}{2} \right\rfloor$.

Note that the upper and lower bounds above match when $|V(G_2)|$ is even. Assume now that $|V(G_2)|$ is odd. We distinguish three cases in what follows. In all of the strategies below, the first player to colour a vertex in G_2 will colour at least $\left\lceil \frac{|V(G_2)|}{2} \right\rceil$ vertices in G_2.

- First, let us assume that the strategy $\mathcal{S}_a^2$ for Alice in G_1 defined above exists. In that case, let us define Alice's strategy for G as follows. Alice plays her first turns in G_1 following $\mathcal{S}_a^2$ until she can skip a turn in G_1 (*i.e.*, the first time she can skip a turn in G_1 while still guaranteeing a score of at least $c_g(G_1, K)$ in the variant game with input (G_1, K)).
  - If, in one of these rounds, Bob plays in G_2, then Alice first plays an extra turn in G_1 (following $\mathcal{S}_a^2$ that ensures her a score of at least $c_g(G_1, K) + 1$ in the variant game with input (G_1, K)), and then, each time Bob plays in G_1, she plays in G_1 according to $\mathcal{S}_a^2$ in the variant game with input (G_1, K), and each time Bob plays in G_2, she plays in G_2.
  - Otherwise, Bob also plays in G_1 until Alice can skip a turn in G_1. Then, once she can skip a turn in G_1 according to $\mathcal{S}_a^2$, Alice colours a vertex in G_2. From then, whenever Bob colours a vertex in G_1, she colours a vertex in G_1 following $\mathcal{S}_a^2$. Otherwise, she colours any arbitrary vertex in G_2.

  In both cases, this guarantees Alice a score of at least $c_g(G_1, K) + \left\lceil \frac{|V(G_2)|}{2} \right\rceil$, matching the upper bound.

- Second, let us assume that the strategy $\mathcal{S}_b^2$ for Bob in G_1 defined above exists. Note that $\mathcal{S}_a^2$ does not exist, so Alice cannot skip one turn in G_1 without decreasing her score in the variant game with input (G_1, K). Bob plays his first turns in G_1 following $\mathcal{S}_b^2$ until he can skip a turn in G_1.
  - If, in one of these rounds, Alice plays in G_2, then Bob first plays an extra turn in G_1 (following $\mathcal{S}_b^2$ that ensures him that Alice will score at most $c_g(G_1, K) - 1$ in the variant game with input (G_1, K)). Then, whenever Alice plays in G_1, he continues to follow $\mathcal{S}_b^2$ in the variant game with input (G_1, K), and when Alice plays in G_2, Bob plays in G_2.
  - Otherwise, Alice also plays in G_1 until Bob can skip a turn in G_1. Then, once he can skip a turn in G_1 according to $\mathcal{S}_b^2$, Bob colours a vertex in G_2. From then, whenever Alice colours a vertex in G_1, he colours a vertex in G_1 following $\mathcal{S}_b^2$. Otherwise, he colours any arbitrary vertex in G_2.

  In both cases, this guarantees that Alice's score is at most $c_g(G_1, K) + \left\lfloor \frac{|V(G_2)|}{2} \right\rfloor$, matching the lower bound.

- Finally, if none of the strategies $\mathcal{S}_a^2$ and $\mathcal{S}_b^2$ exist, the result depends on the parity of $|V(G_1)|$. Indeed, if Alice skips one turn in G_1, then Bob can ensure she scores at most $c_g(G_1, K) - 1$ in the variant game with input (G_1, K). On the other hand, if Bob skips one turn in G_1, Alice can score at least $c_g(G_1, K) + 1$ in the variant game with input (G_1, K). For Alice to ensure her upper bound and for Bob to ensure the lower bound, both of them will play in priority in G_1. That is, the first vertex of G_2 is coloured after all the vertices of G_1 have been coloured (and Alice has achieved a score of $c_g(G_1, K)$ in the variant game with input (G_1, K)). If $|V(G_1)|$ is even, Alice is the first player to colour a vertex in G_2, which allows her to score the upper bound $c_g(G_1, K) + \left\lceil \frac{|V(G_2)|}{2} \right\rceil$. Otherwise, Bob is the first player to colour a vertex in G_2, which implies that Alice can score at most the lower bound $c_g(G_1, K) + \left\lfloor \frac{|V(G_2)|}{2} \right\rfloor$. □

7. Discussion and directions for further work

A certain number of directions for further work on the Maker-Breaker Largest Connected Subgraph game seem particularly appealing to us. While some of the ones we mention are about tightening some of our results from the previous sections, others are original ones that are discussed only in this section.

7.1. Differences between the two versions of the Largest Connected Subgraph game

One direction for research could be to try to establish the significant differences between the Maker-Breaker Largest Connected Subgraph game and the Largest Connected Subgraph game. Some of our results in this work are already a step in that direction. For instance, in Section 3, we showed that the Maker-Breaker version remains PSPACE-complete when restricted to various classes of graphs, but we do not know whether the same holds for the Largest Connected Subgraph game in those classes of graphs. Lemma 2.2 draws another neat difference between the two versions of the game, as the outcome of the Largest Connected Subgraph game in a disconnected graph cannot be established as simply as in the Maker-Breaker version. This is because, in the latter version, Bob does not care about the structure induced by the blue vertices. However, in the Largest Connected Subgraph game, there are scenarios in which it is more favourable for Bob to play in a connected component G_2 different from the one G_1 that Alice just played in. This would be like skipping a turn in G_1, but playing an extra turn in G_2 (or playing first in G_2). Thus, to establish a result similar to Lemma 2.2 for the Largest Connected Subgraph game, one has to deal with the effects of skipping and playing extra turns, as well as Bob playing first, which seems like a tricky, yet interesting, aspect to study.

7.2. Types of strategies for the Maker-Breaker Largest Connected Subgraph game

As we have seen in some graphs, notably in Section 5, some optimal strategies for Alice ensure that the red subgraph is connected at all times. We believe it would be interesting to study a connected variant of the Maker-Breaker Largest Connected Subgraph game, in which Alice is always (except on her first turn) constrained to colour a neighbour of another red vertex, and the game ends when she cannot. Consequently, we could define $c_g^c(G)$ as the maximum score Alice can achieve in G when obliged to play in such a connected way. Clearly, $c_g^c(G) \leq c_g(G)$. We were able to observe that it is far from true that these two parameters are equal in general, even sometimes in quite simple graphs. As an illustration, this is true for *king's grids* (strong products of two paths) with only two rows (denoted by $P_2 \boxtimes P_m$).

Lemma 7.1. *For any $m \geq 1$, $c_g^c(P_2 \boxtimes P_m) = O(1)$ and $c_g(P_2 \boxtimes P_m) = m$.*

Proof. In the connected case, it is sufficient for Bob to colour the 4 vertices at distance 4 from the first vertex coloured by Alice. In the non-connected case, each time Bob colours a vertex v, Alice colours the neighbour of v in the other row (we say that $P_2 \boxtimes P_m$ has two rows and m columns). At the end of the game, the red subgraph is connected, and so, Alice achieves a score of m. □

7.3. Other classes of graphs

Some of our results on particular classes of graphs leave open questions. Since the Maker-Breaker Largest Connected Subgraph game is PSPACE-complete in split graphs by Corollary 3.2, and split graphs have diameter at most 3, there is the question of whether it is hard to compute c_g for graphs of diameter 2. Regarding the results from Section 6, recall that [4] provides a result similar to Theorem 6.2 for the Largest Connected Subgraph game. One question is whether this result can be extended to $(q, q-4)$-graphs, as we did in Theorem 6.3 for the Maker-Breaker version.

Regarding determining c_g for other graph classes, an appealing direction could be to consider standard graph classes such as trees. From [4], we have that $c_g(P_n) = 2$ for any path P_n of order $n \geq 3$, and we believe that understanding the game in larger subclasses of trees such as caterpillars and subdivided stars is not so difficult, but requires a lot of work to prove, for a not so substantial result. Thus, we think it would be most interesting to study the class of trees rather than its subclasses. Other natural graph classes to be investigated are graph products. For instance, we wonder whether $c_g(Q_n)$ can be easily determined for a hypercube Q_n (where, recall, Q_2 is the

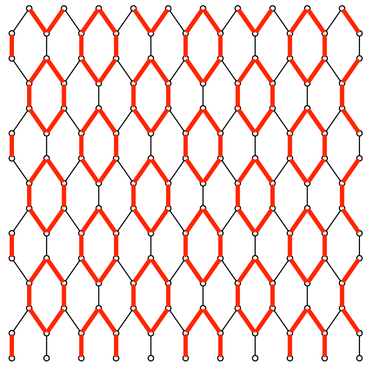


Figure 5: Illustration of the infinite Hexagonal grid G^H_∞ in the proof of Lemma 7.2. The connected red subgraphs are vertex-disjoint 6-cycles covering all the vertices of G^H_∞. The black edges induce a matching of G^H_∞.

cycle C_4 of length 4, and, for every $n > 2$, the hypercube Q_n is the Cartesian product $Q_{n-1} \square P_2$ of Q_{n-1} and the path P_2 of order 2). We also wonder about different types of grids, which, in the Largest Connected Subgraph game, seem hard to comprehend [4]. Another point for considering such graphs is that grids are natural structures to play on in several types of games, as illustrated by Hex. To give some insight into what can be done in the Maker-Breaker version, we finish off with some partial results on grids in the rest of this section. We first consider *hexagonal grids.*

Lemma 7.2. *If G is any finite subgraph of the infinite hexagonal grid, then $c_g(G) \leq 6$.*

Proof. Let G^H_∞ be the infinite hexagonal grid as partially shown in Figure 5. By Lemma 2.1, it is sufficient to show that $c_g(G^H_\infty) \leq 6$. Let $(C_i)_{i \in \mathbb{N}}$ be the set of vertex-disjoint subgraphs of G^H_∞ depicted in red in Figure 5. Note that, for any $i \in \mathbb{N}$, C_i induces a cycle of order 6 and $(V(C_i))_{i \in \mathbb{N}}$ is a partition of $V(G^H_\infty)$. Furthermore, $M = E(G^H_\infty) \setminus (\bigcup_{i \in \mathbb{N}} E(C_i))$ (black edges in Figure 5) is a matching of G^H_∞. Note also that, for any $i \neq j$, every path from a vertex of C_i to a vertex of C_j contains an edge in M (since, for every subgraph C_i, the edges adjacent to a vertex of $V(C_i)$, but not in $E(C_i)$, are by definition in M).

Let us consider the following strategy for Bob. First, note that, for any vertex $v \in V(G)$, there is at most one edge $uv \in M$ incident to v since M is a matching. Thus, each time Alice colours a vertex v, Bob colours the vertex u such that $uv \in M$, if it exists and it is uncoloured, and if not, then he colours any arbitrary uncoloured vertex in G. Let us show that Bob's strategy ensures that Alice cannot create a connected red component of order more than 6. Towards a contradiction, let us assume that Alice creates a connected red component S of order at least 7. Then, there exist $u, v \in S$ and $i \neq j$ such that $u \in V(C_i)$ and $v \in V(C_j)$ (because the C_k's partition the vertex-set of G^H_∞ and each C_k has order 6). As was mentioned above, every path between u and v must contain an edge of M, and so, by Bob's strategy, a vertex of this path has been coloured by Bob, contradicting that u and v belong to the same connected red component. □

Through a tedious case analysis, it might be possible to prove that $c_g(G^H_\infty) = 6$. However, the case of other classic types of grids seems trickier, and we are only able to prove partial results. Recall that we mentioned the case of king's grids earlier, for which we have provided bounds on c_g when there are two rows and m columns. Other classical grids to consider are *Cartesian grids*, which are the Cartesian product $P_n \square P_m$ of P_n and P_m. For these grids, we provide the following upper bound.

Lemma 7.3. *Let $n \leq m$. Then, $c_g(P_n \square P_m) \leq 2n$.*

Sketch of the proof. Let us consider an $n \times m$ grid $P_n \square P_m$ with n rows and m columns (with left and right being defined naturally). Let us consider the following strategy for Bob. When Alice colours a vertex v, if the right neighbour u of v exists and is uncoloured, then Bob colours u, otherwise, Bob colours the left neighbour of v if it exists and is uncoloured, and otherwise, Bob colours any arbitrary uncoloured vertex.

The above strategy for Bob is well-defined and ensures that no three consecutive vertices in a row are ever red (see the case of paths in [4] for more details). This ensures that, for any strategy of Alice, any connected red component has at most 2 vertices in each row, hence, proving the lemma. Indeed, consider a largest connected red component S at the end of the game. Towards a contradiction, assume that there exists a row whose intersection with S contains strictly more than 2 vertices. Then, the restriction of S to this row induces at least two connected red components X and Y (since there cannot be three consecutive red vertices in a same row). Let P be any red path from X to Y (that exists since S is connected). It can be shown that P must contain 3 consecutive red vertices in a same row, a contradiction. $\diamond$

Regarding Lemma 7.3, we would be interested in knowing the precise value of $c_g(P_n \Box P_m)$ in general. One issue we ran into is the fact that Alice can play in a non-connected way (recall the notion of connected moves discussed earlier), and it is not clear how Bob should anticipate to prevent connected red components to merge later on. Let us mention, however, that if Alice plays in a connected way in a Cartesian or king's grid, then the game becomes quite similar to the Angel and Devil Problem of Conway [5]. Optimal strategies for the devil in that game [5] allow to prove that $c_g^c(P_n \Box P_m)$ is bounded above by an absolute constant.

References

[1] L. Babel. Triangulating graphs with few P_4's. *Discrete Applied Math.*, 89(1-3):45–57, 1998.

[2] L. Babel and S. Olariu. On the isomorphism of graphs with few P4s. In *Proceedings of the 21st International Workshop on Graph-Theoretic Concepts in Computer Science (WG 1995)*, volume 1017 of *Lecture Notes in Computer Science*, pages 24–36. Springer, 1995.

[3] L. Babel and S. Olariu. On the p-connectedness of graphs - A survey. *Discrete Applied Math.*, 95(1-3):11–33, 1999.

[4] J. Bensmail, F. Fioravantes, F. Mc Inerney, and N. Nisse. The largest connected subgraph game. In *Proceedings of the 47th International Workshop on Graph-Theoretic Concepts in Computer Science (WG 2021)*, volume 12911 of *Lecture Notes in Computer Science*, pages 296–307. Springer, 2021.

[5] E. R. Berlekamp, J. H. Conway, and R. K. Guy. *Winning Ways for Your Mathematical Plays, Volume 3.* A K Peters/CRC Press, 2018.

[6] H. L. Bodlaender. On the complexity of some coloring games. *Internat. J. Found. Comput. Sci.*, 2:133–147, 1991.

[7] C. Browne. *Connection Games: Variations on a Theme.* AK Peters, 2005.

[8] V. Chvátal and P. Erdös. Biased positional games. *Annals of Discrete Math. 2*, pages 221–229, 1978.

[9] D. G. Corneil, H. Lerchs, and L. S. Burlingham. Complement reducible graphs. *Discrete Applied Math.*, 3(3):163–174, 1981.

[10] R. Diestel. *Graph Theory, 4th Edition*, volume 173 of *Graduate texts in mathematics.* Springer, 2012.

[11] E. Duchêne, V. Gledel, A. Parreau, and G. Renault. Maker-breaker domination game. *Discrete Math.*, 343(9), 2020.

[12] P. Erdös and J. L. Selfridge. On a combinatorial game. *Journal of Combinatorial Theory, Series A*, 14(3):298–301, 1973.

[13] M. Gardner. *The Scientific American Book of Mathematical Puzzles & Diversions.* Simon and Schuster, N.Y., 1959.

[14] M. Gardner. *The Second Scientific American Book of Mathematical Puzzles and Diversions*. Simon and Schuster, N.Y., 1961.

[15] M. Gardner. Mathematical games. *Scientific American*, 244:23–26, 1981.

[16] D. Hefetz, M. Krivelevich, M. Stojaković, and T. Szabó. *Positional Games*, volume 44 of *Oberwolfach Seminars*. Springer, Basel, 2014.

[17] R. Küsters. *Memoryless determinacy of parity games*, volume 2500 of *Lecture Notes in Computer Science*. Springer, 2002.

[18] U. Larsson, R. J. Nowakowski, J. P. Neto, and C. P. Santos. Guaranteed scoring games. *The Electronic Journal of Combinatorics*, 23(3), 2016.

[19] U. Larsson, R. J. Nowakowski, and C. P. Santos. Games with guaranteed scores and waiting moves. *International Journal of Game Theory*, 47(2):653–671, 2018.

[20] M. L. Rahman and T. Watson. 6-uniform maker-breaker game is PSPACE-complete. In *Proceedings of the 38th International Symposium on Theoretical Aspects of Computer Science (STACS 2021)*, pages 57:1–57:15, 2021.

[21] S. Reisch. Hex ist pspace-vollständig. *Acta Inf.*, 15(2):167–191, 1981.

[22] T. J. Schaefer. On the complexity of some two-person perfect-information games. *Journal of Computer and System Sciences*, 16(2):185–225, 1978.