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Algorithmic study of the algebraic parameter estimation problem for a class of perturbations

Maya Chartouny*, Cluzeau Thomas†, Alban Quadrat‡

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Abstract: We consider the algebraic parameter estimation problem for a class of standard perturbations. We assume that the measurement $z(t)$ of a solution $x(t)$ of a linear ordinary differential equation – whose coefficients depend on a set $\theta := \{\theta_1, \ldots, \theta_r\}$ of unknown constant parameters – is affected by a perturbation $\gamma(t)$ whose structure is supposed to be known (e.g., an unknown bias, an unknown ramp), i.e., $z(t) = x(t, \theta) + \gamma(t)$. We investigate the problem of obtaining closed-form expressions for the parameters θ_i's in terms of iterative indefinite integrals or convolutions of z. The different results are illustrated by explicit examples computed using the `NonA` package – developed in `Maple` – in which we have implemented our main contributions.

Key-words: parameter estimation problem, inverse Cauchy problem, algebraic systems, elimination, annihilators, rings of ordinary differential operators

* Sorbonne Université and Université de Paris, CNRS, Inria Paris, IMJ-PRG, F-75005 Paris, France (mayachartouny@hotmail.com)

† University of Limoges ; CNRS ; XLIM UMR 7252, Limoges, France (thomas.cluzeau@unilim.fr, `https://www.unilim.fr/pages_perso/thomas.cluzeau/`)

‡ Sorbonne Université and Université de Paris, CNRS, Inria Paris, IMJ-PRG, F-75005 Paris, France (alban.quadrat@inria.fr, `https://who.rocq.inria.fr/Alban.Quadrat/`)

RESEARCH CENTRE
SOPHIA ANTIPOLIS – MÉDITERRANÉE

2004 route des Lucioles - BP 93
06902 Sophia Antipolis Cedex

Etude algorithmique du problème algébrique d'estimation de paramètres pour une classe de perturbations

Résumé : Nous considérons le problème algébrique d'estimation de paramètres pour une classe classique de perturbations. Nous supposons que la mesure $z(t)$ d'une solution $x(t)$ d'une équation différentielle linéaire − dont les coefficients dépendent d'un ensemble $\theta := \{\theta_1, \ldots, \theta_r\}$ de paramètres constants inconnus − est affectée par une perturbation $\gamma(t)$ dont la structure est supposée connue (par exemple, un biais inconnu, une rampe inconnue), c'est-à-dire, nous supposons que $z(t) = x(t, \theta) + \gamma(t)$. Nous étudions alors le problème d'obtenir des formes closes pour les paramètres θ_i en fonction d'intégrales indéfinies itérées ou de convolutions de z. Les différents résultats sont illustrés par des exemples explicites calculés grâce au package `NonA` − développé en `Maple` − dans lequel nous avons implanté nos principales contributions.

Mots-clés : problème d'estimation de paramètres, problème de Cauchy inverse, systèmes algébriques, élimination, annulateurs, anneaux d'opérateurs différentiels ordinaires

1 Introduction

In many real applications studied in engineering sciences, applied mathematics, mathematical biology, etc., estimating constant parameters of a dynamical system (e.g., a mass, a spring constant, a damping coefficient, a resistance) is a fundamental issue. Hence, parameter estimation has lengthy been studied in different scientific fields. In particular, this problem has been actively studied in both control theory and signal processing. See, e.g., [Kailath et al. (2000), Fliess et al. (2003), Poor (1994), Van Trees (2004)] and the references therein.

In the present paper, we study the algebraic parameter estimation problem following the approach initiated in [Fliess et al. (2003)] and continued in [Belkoura et al. (2009), Mboup (2009), Quadrat (2017), Ushirobira et al. (2016)] (see also the references therein). This problem aims at estimating the constant parameters of solutions of linear Ordinary Differential Equations (ODEs) with polynomial coefficients. More precisely, if we assume that $x(t)$ is a solution of a linear ODE depending on a set $\theta := \{\theta_1, \ldots, \theta_r\}$ of unknown constant parameters, then we look for explicit expressions for the parameters θ_i's in terms of a measured function $z(t) = x(t, \theta) + \gamma(t) + \varpi(t)$, where γ is a perturbation (whose global structure can sometimes be supposed to be known) and ϖ a noise (e.g., a zero-mean Gaussian noise). Moreover, we are interested in closed-form expressions of the parameters θ_i's involving iterative indefinite integrals or convolutions of z which allow to filter a part of the influence of noise ϖ.

The first case to be considered (see also [Chartouny et al. (2021)]) is that of a perfect measurement, i.e., $\gamma(t) = \varpi(t) = 0$. It can be seen as an *inverse Cauchy problem*. Indeed, the *Cauchy problem* characterizes the solutions of an ODE that satisfies fixed initial condition. Conversely, given a function that is known to satisfy a linear ODE of fixed order with polynomial coefficients, the inverse Cauchy problem studies when the constant coefficients of these polynomials as well as the initial conditions can be expressed by means of iterative indefinite integrals or convolutions of the solution. To achieve our goal, we combine *operational calculus* (Laplace transform, convolution product - see Appendix A.1) with *algebraic methods* (elimination methods, linear algebra). It leads to Theorems 1 and 3, where we prove that, if $t = 0$ is an ordinary point of the ODE, then we can obtain explicit expressions for the coefficients of the ODE and for the initial conditions in terms of iterative indefinite integrals of the measured solution.

The main purpose of the present paper is to consider the case where the signal x is corrupted by a perturbation γ so that we measure $z(t) := x(t) + \gamma(t)$. Here, the function γ is a *structured perturbation* of the form $\gamma(t) = \gamma\, t^r\, H(t)$, where γ is an unknown constant, $r \geq 0$ and H is the Heaviside function. These structured perturbations are standard, e.g., in *the disturbance rejection problem* (see, e.g., [Quadrat et al. (2014)] and the references therein). We tackle the problem following the approach of [Fliess et al. (2003)] (see also [Mboup (2009), Quadrat (2017), Ushirobira et al. (2016)], which corresponds to the particular case $r = 0$). Applying the Laplace transform to the ODE, we obtain a new equation in the frequency domain which involves the initial conditions, the coefficients of the ODE, and the unknown constant parameter γ. Using Gröbner basis techniques (see Appendix A.3) to compute the annihilator of a polynomial in the first Weyl algebra (see Appendix A.2), we manage to get rid of the term involving the unknown constant parameter γ. Then, we use elimination techniques to obtain explicit expressions of the initial conditions in terms of the Laplace transform of $z(t)$, its derivatives, and the coefficients of the ODE. Applying the inverse Laplace transform, we can obtain explicit expressions of the initial conditions in terms of $z(t)$, its iterative indefinite integrals (convolutions), and the coefficients of the ODE. The main result of this part is given by Theorem 4. Finally as in the case of a perfect measurement, we show (Theorem 5) that we can use elimination techniques to obtain the desired explicit expressions of the coefficients of the ODE in terms of $z(t)$ and its iterative indefinite integrals (convolutions).

We illustrate our results by explicit examples where the computations were done using the `NonA` package − developed in `Maple` − in which we have implemented the results of this paper. Details on the package as well as a complete worked example are given.

The paper is organized as follows. In Section 2, we recall the main results of [Chartouny et al. (2021)] concerning the case of a perfect measurement. Section 3 contains the main contribution of the present paper for a class of standard structured perturbations. Section 4 contains details concerning the implementation in `Maple` of the results developed in the paper and a worked example that illustrates the main commands. Finally all the mathematical prerequisites (Laplace transform, convolution product, Weyl algebra, Gröbner basis, annihilator) are recalled in Section A.

2 The algebraic parameter estimation problem without perturbation

In this section, we recall the main results obtained in [Chartouny et al. (2021)] for the case without perturbation.

As explained in the introduction, the general *parameter estimation problem* consists in estimating constant parameters $\theta := (\theta_1, \ldots, \theta_r)$ − defining a solution x of a Cauchy problem for an ODE depending on θ − from the measurement of a signal of the form

$$z(t) = x(\theta, t) + \gamma(t) + \varpi(t), \tag{1}$$

where γ is a perturbation (whose global structure can sometimes be supposed to be known) and ϖ a noise (e.g., a zero-mean Gaussian noise). It is natural to first ask whether or not the system parameters θ can always be estimated in the exact case, i.e., when $\gamma = 0$ and $\varpi = 0$. Indeed, if it is not possible, then the algebraic parameter estimation problem cannot be solved. Hence, in this section, we ask whether or not the coefficients of a linear ODE with polynomial coefficients and the initial conditions of the Cauchy problem can be exactly recovered from the knowledge of a "generic solution" and of its iterative indefinite integrals. In other words, we aim at explicitly study the inverse Cauchy problem for linear ODEs with polynomial coefficients.

2.1 Estimation of the initial conditions

Let $\mathbb{K}$ be a field of characteristic 0 and let us suppose that x satisfies the following ODE:

$$\sum_{i=0}^{n} a_i(t) \frac{d^i}{dt^i} x(t) = 0, \quad n \geq 1, \quad \forall\, i = 0, \ldots, n,\, a_i(t) = \sum_{j=0}^{d_i} a_{ij}\, t^j \in \mathbb{K}[t], \quad a_n \neq 0. \tag{2}$$

Let us set $m := \max_{0 \leq i \leq n} d_i$. We first suppose that the coefficients a_{ij}'s are known and we focus on the possibility to recover the initial conditions of the Cauchy problem for (2), i.e., the values $x^{(i)}(0)$ for $i = 0, \ldots, n-1$. In Section 2.2, we shall show how the coefficients a_{ij}'s can be explicitly determined from x and its iterative indefinite integrals.

We first apply the Laplace transform to (2) and using the notation $\widehat{x} = \mathscr{L}(x)$, we obtain:

$$\sum_{i=0}^{n} a_i(-\partial_s) \left(s^i\, \widehat{x}(s) - \sum_{j=0}^{i-1} s^{i-j-1}\, x^{(j)}(0) \right) = 0,$$

which can be rewritten as:

$$\sum_{i=0}^{n} a_i(-\partial_s)\, s^i\, \widehat{x}(s) - \sum_{i=0}^{n} \sum_{j=0}^{i-1} a_i(-\partial_s)\, s^{i-j-1}\, x^{(j)}(0) = 0. \tag{3}$$

Expanding the second term of the above identity, we get:

$$\sum_{i=0}^{n} \sum_{j=0}^{i-1} a_i(-\partial_s)\, s^{i-j-1}\, x^{(j)}(0) = \sum_{j=0}^{n-1} \sum_{i=j+1}^{n} a_i(-\partial_s)\, s^{i-j-1}\, x^{(j)}(0) = \sum_{k=0}^{n-1} \sum_{i=k+1}^{n} a_i(-\partial_s)\, s^{i-k-1}\, x^{(k)}(0).$$

Then, (3) becomes:

$$\sum_{i=0}^{n} a_i(-\partial_s)\, s^i\, \widehat{x}(s) - \sum_{k=0}^{n-1} \sum_{i=k+1}^{n} a_i(-\partial_s)\, s^{i-k-1}\, x^{(k)}(0) = 0.$$

In what follows, we shall note:

$$\begin{cases} P(s,\partial_s) := \sum_{i=0}^{n} a_i(-\partial_s)\, s^i, \\ S_k(s) := -\sum_{i=k+1}^{n} a_i(-\partial_s)\, s^{i-k-1}, \quad \vartheta_k := x^{(k)}(0), \quad k=0,\ldots,n-1, \\ Q(s) := \sum_{k=0}^{n-1} S_k\, \vartheta_k. \end{cases} \tag{4}$$

Notice first that $P(s,\partial_s)$ is a OD operator in ∂_s with polynomial coefficients in s, i.e., is an element of the *first Weyl algebra* $A_1(\mathbb{K}) := \mathbb{K}[s]\langle \partial_s \mid \partial_s\, s = s\, \partial_s + 1\rangle$ (see, e.g., [Coutinho (1995)]), which applies to the function $\widehat{x}(s)$. Moreover, in the term S_k, $a_i(-\partial_s) \in A_1(\mathbb{K})$ applies to s^{i-k-1}, which shows that $S_k \in \mathbb{K}[s]$ and $Q \in \mathbb{K}[\vartheta_0,\ldots,\vartheta_{n-1}][s]$. Equation (3) then becomes:

$$P(s,\partial_s)\,\widehat{x}(s) + Q(s) = 0. \tag{5}$$

Let us now study when the ϑ_k's can be explicitly characterized. If we note $\Theta := (\vartheta_0\ \ldots\ \vartheta_{n-1})^T$, then (5) can be rewritten as:

$$(S_0\ \ldots\ S_{n-1})\,\Theta = -P(s,\,\partial_s)\,\widehat{x}(s). \tag{6}$$

Definition 1. The *valuation* of $a_i \in \mathbb{K}[t]\setminus\{0\}$ at $t=0$, denoted by $v_0(a_i)$, is the degree of the maximum power of t that divides a_i. Moreover, we set $v(0) = -\infty$.

In general, for all $k=0,\ldots,n-1$, we have $\deg_s(S_k) = \max_{i=k+1,\ldots,n}\{i-k-1-v_0(a_i)\}$. However, in what follows, we shall suppose that $v_0(a_n)=0$, i.e., $a_{n0}\neq 0$, which then implies $\deg_s(S_k) = n-k-1$ for $k=0,\ldots,n-1$. The latter assumption $a_{n0}\neq 0$ means that $t=0$ is a *regular point* of (2), and thus, it makes sense to consider the Cauchy problem for (2) at $t=0$.

Example 1. If $t=0$ is a *singular point* for (2), i.e., $a_{n0}=0$, then we have $a_n(t) = p(t)\,t$ for a certain polynomial p, and thus, $S_{n-1} = -a_n(-\partial_s)\,s^0 = p(-\partial_s)\,\partial_s\,1 = 0$ and $Q = \sum_{k=0}^{n-2} S_k\,\vartheta_k$. Hence, $\vartheta_{n-1} = x^{(n-1)}(0)$ cannot be estimated as it does not appear in $\mathbb{Q}$. Similarly, if $v_0(a_n)\geq 2$ and $v_0(a_{n-1})\geq 1$ (e.g., $t^2\,x^{(2)}(t) + t\,x^{(1)}(t) + x(t) = 0$), then $S_{n-1}=0$ and $S_{n-2} = -(a_n(-\partial_s)\,s + a_{n-1}(-\partial)\,1) = 0$, which shows that both ϑ_{n-1} and ϑ_{n-2} cannot be estimated.

If $t=0$ is not an ordinary point of (2), then note that a change of independent variable $t = T+\alpha$, $\alpha\in\mathbb{K}^*$, cannot simply be used to solve our problem. Indeed applying the process described above to the new ODE at $T=0$ will provide the values of $\theta^{(i)}(\alpha)$ but not $\theta^{(i)}(0)$.

Equation (6) is an inhomogeneous linear equation in the ϑ_i's. If we differentiate $n-1$ times (6) with respect to s, we get the following inhomogeneous linear system for the ϑ_i's:

$$U\,\Theta = -\left(P(s,\partial_s) \quad \partial_s\,P(s,\partial_s) \quad \cdots \quad \partial_s^{n-1}\,P(s,\partial_s)\right)^T \widehat{x}(s),$$

where the matrix U is defined by:

$$U = \begin{pmatrix} S_0 & \cdots & S_{n-1} \\ S_0' & \cdots & S_{n-1}' \\ \vdots & \ddots & \vdots \\ S_0^{(n-1)} & \cdots & S_{n-1}^{(n-1)} \end{pmatrix}.$$

The polynomials S_k's are defined in (4) and, under the assumption $a_{n0}\neq 0$, we have that U is an invertible upper "anti-triangular" matrix, namely,

$$U = \begin{pmatrix} S_0 & S_1 & \cdots & S_{n-3} & S_{n-2} & -a_{n0} \\ S_0' & S_1' & \cdots & S_{n-3}' & -a_{n0} & 0 \\ S_0'' & S_1'' & \cdots & -2\,a_{n0} & 0 & 0 \\ \vdots & \vdots & \iddots & \iddots & \vdots & \vdots \\ S_0^{(n-2)} & -(n-2)!\,a_{n0} & 0 & \cdots & \vdots & 0 \\ -(n-1)!\,a_{n0} & 0 & \cdots & \cdots & 0 & 0 \end{pmatrix},$$

and we get:

$$\Theta = \begin{pmatrix} \vartheta_0 \\ \vdots \\ \vartheta_{n-1} \end{pmatrix} = -U^{-1} \begin{pmatrix} P \\ \partial_s P \\ \vdots \\ \partial_s^{n-1} P \end{pmatrix} \widehat{x}(s).$$

Note that we can give an explicit expression for the inverse U^{-1} of the matrix U so that there is no need to compute the inverse of a matrix to get the above expression of Θ. For more details, see [Chartouny et al. (2021)].

To summarize, we have thus proved the following result:

Theorem 1. *If $t = 0$ is a regular point of the ODE (2), then the initial conditions $\{x^k(0)\}_{k=0,\ldots,n-1}$ can be expressed explicitly in terms of $\widehat{x}(s)$, its derivatives, and the coefficients a_{ij}'s of (2).*

Theorem 1 provides formulas of the form

$$\vartheta_i = \underbrace{-\sum_{j=1}^{n} c_{ij} \partial_s^{j-1} P(s, \partial_s)\, \widehat{x}(s)}_{R_i(s, \widehat{x}(s))}, \quad i = 0, \ldots, n-1, \quad c_{ij} \in \mathbb{K}[a_{ij}]_{0 \leq i \leq m,\, 0 \leq j \leq n}, \tag{7}$$

for the initial conditions ϑ_i's. Moreover, in [Chartouny et al. (2021)], it has been shown that, for all $i = 0, \ldots, n-1$, $\deg_s(\vartheta_i) = n + i$.

We shall now apply the inverse Laplace transform to get explicit expressions in terms of $x(t)$ and its indefinite iterative integrals (e.g., convolutions). In order to get formulas for the ϑ_i's involving indefinite iterative integrals of $x(t)$, we first rewrite the right hand side of (7) as

$$\vartheta_i = \frac{N_i}{D_i}, \quad N_i(s, \widehat{x}(s)) := R_i(s, \widehat{x}(s))/s^{n+i+1}, \quad D_i(s) := 1/s^{n+i+1}.$$

Then, we apply the inverse Laplace transform $\mathscr{L}^{-1}$ and since ϑ_i is a constant and $\mathscr{L}^{-1}$ is a linear transformation, we obtain $\mathscr{L}^{-1}(\vartheta_i\, D_i) = \mathscr{L}^{-1}(N_i)$ so that:

$$\vartheta_i = \frac{\mathscr{L}^{-1}(N_i)}{\mathscr{L}^{-1}(D_i)}, \quad i = 0, \ldots, n-1, \quad \mathscr{L}^{-1}(D_i)(t) = \frac{t^{n+i}}{(n+i)!}.$$

Finally, note that the term in the inverse Laplace transform of the right hand side of the following equation

$$\mathscr{L}^{-1}(N_i) = -\sum_{j=1}^{n} \mathscr{L}^{-1}\left(\frac{c_{ij}}{s^{n+i+1}}\, \partial_s^{j-1} P(s, \partial_s)\, \widehat{x}(s)\right), \quad i = 0, \ldots, n-1,$$

is a *strictly proper rational function* in s, namely, the degree in s of its numerator is strictly less than the degree of its denominator. Hence, using the normal forms of OD operators, we have:

$$\frac{c_{ij}}{s^{n+i+1}}\, \partial_s^{j-1} P(s, \partial_s)\, \widehat{x}(s) = \sum_{\substack{0 \leq k \leq n+i+1 \\ 0 \leq l \leq m+j-1}} \frac{d_{kl}}{s^k}\, \partial_s^l\, \widehat{x}(s), \quad d_{kl} \in \mathbb{K}[a_{ij}]_{0 \leq i \leq m,\, 0 \leq j \leq n}.$$

Using $\mathscr{L}\left(\int_0^t y(\tau) d\tau\right)(s) = s^{-1}\, \widehat{y}(s)$, we get that $\mathscr{L}^{-1}(N_i)$ is a finite sum of iterative indefinite integrals of terms of the form $(-t)^l\, x(t)$ – which can also be expressed as a convolution. We have then proved the following result.

Theorem 2. *If $t = 0$ is a regular point of an ordinary differential equation (2), then the initial conditions $\{x^k(0)\}_{k=0,\ldots,n-1}$ can be expressed explicitly in terms of $x(t)$, its iterative indefinite integrals (convolutions), and the coefficients a_{ij}'s of (2).*

2.2 Estimation of the coefficients of the ODE

The closed-forms for the initial conditions $\{x^k(0)\}_{k=0,\dots,n-1}$'s obtained in Theorem 2 above depend on the constant parameters a_{ij}'s of the ODE (2). We now focus on the estimation of the parameters a_{ij}'s as iterative indefinite integrals of $x(t)$.

In the frequency domain, with the notations (4), we recall that the ODE defined by (2) is equivalently defined by (5) which can be rewritten as (6). In Section 2.1 above, we have differentiated $(n-1)^{\text{th}}$ times Equation (6) to get $U\,\Theta = -\,(P(s,\partial_s) \quad \partial_s P(s,\partial_s) \quad \cdots \quad \partial_s^{n-1} P(s,\partial_s))^T\,\widehat{x}(s)$, from which we were able to solve for the ϑ_i's. From the last row of the matrix U, we see that if we differentiate Equation (6) more than n times, then we shall get equations of the form $\partial_s^l\, P(s,\,\partial_s)\,\widehat{x}(s) = 0$ for $l \geq n$ that do not depend on the ϑ_i's. For $l \geq n$, we have

$$(-\partial_s)^l\, P(s,\partial_s)\,\widehat{x}(s) = (-\partial_s)^l \sum_{\substack{0\leq i\leq n\\ 0\leq j\leq m}} a_{ij}\,(-\partial_s)^j\, s^i\,\widehat{x}(s) = \sum_{\substack{0\leq i\leq n\\ 0\leq j\leq m}} (-\partial_s)^{l+j}\, s^i\,(\widehat{x}(s)\,a_{ij}) = 0.$$

Hence, if we set $\vec{a} := (a_{00}\ a_{01}\ \dots\ a_{ij}\ \dots a_{n(m-1)}\ a_{nm})^T$, we get the following system of linear equations for the a_{ij}'s:

$$\forall l \geq n, \quad \underbrace{\begin{pmatrix} (-\partial_s)^n & (-\partial_s)^{n+1} & \dots & (-\partial_s)^{n+j}\, s^i & \dots & (-\partial_s)^{n+m-1}\, s^n & (-\partial_s)^{n+m}\, s^n \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ (-\partial_s)^l & (-\partial_s)^{l+1} & \dots & (-\partial_s)^{l+j}\, s^i & \dots & (-\partial_s)^{l+m-1}\, s^n & (-\partial_s)^{l+m}\, s^n \end{pmatrix}}_{V(s,\partial_s)} \widehat{x}(s)\,\vec{a} = 0. \tag{8}$$

To obtain explicit formulas for the a_{ij}'s in terms of $\widehat{x}(s)$ and its derivatives, we then need to solve the homogeneous linear system (8).

Let us first assume that $a_{nm} \neq 0$ and solve the system for the remaining a_{ij}'s. Let $p := (n+1)\,(m+1) - 1$ be the number of unknown a_{ij}'s and take $l = n + p - 1$ so that (8) provides p equations in p unknowns. Let us denote by $V_p(s,\partial_s)$ the $p \times p$ matrix defined by taking all but the last column of $V(s,\partial_s)$, $b(s,\partial_s)$ the last column of $V(s,\partial_s)$, and $\vec{a_p}$ the column vector of size p formed by all but the last entry of $\vec{a}$. Hence, (8) is equivalent to:

$$(V_p(s,\partial_s)\,\widehat{x}(s))\ \vec{a_p} = -a_{nm}\, b(s,\partial_s)\,\widehat{x}(s),$$

so that if the matrix $V_p(s,\partial_s)\,\widehat{x}(s)$ is invertible, i.e., $\det(V_p(s,\partial_s)\,\widehat{x}(s)) \neq 0$, then we get:

$$\vec{a_p} = -a_{nm}\ (V_p(s,\partial_s)\,\widehat{x}(s))^{-1}\ b(s,\partial_s)\,\widehat{x}(s). \tag{9}$$

From (9), we obtain that each a_{ij} can be written as a fraction $n_{ij}(s,\widehat{x}(s))/d_{ij}(s,\widehat{x}(s))$, where n_{ij} and d_{ij} are polynomials in s, $\widehat{x}(s)$ and its derivatives. We now proceed as we did in Section 2.1 to apply the inverse Laplace transform and obtain formulas depending on $x(t)$ and its iterative indefinite integrals. Setting $q_{ij} := \max\{\deg_s n_{ij},\, \deg_s d_{ij}\} + 1$ and defining $n'_{ij}(s,\widehat{x}(s)) := n_{ij}(s,\widehat{x}(s))/s^{q_{ij}}$ and $d'_{ij}(s,\widehat{x}(s)) := d_{ij}(s,\widehat{x}(s))/s^{q_{ij}}$, we obtain two polynomials in s^{-1} and some derivatives of $\widehat{x}(s)$. Applying the inverse Laplace transform to n'_{ij} and d'_{ij}, and using the fact that the inverse Laplace transform maps a product to a convolution (see Proposition 3 in the appendix), the a_{ij}'s are then ratios of sums of iterative indefinite integrals of terms of the form $(-t)^\alpha\, x(t) = \mathscr{L}^{-1}\,(\partial_s^\alpha\,\widehat{x}(s))\,(t)$, i.e., ratios of two convolutions depending only on $x(t)$.

Now, if $\det(V_p(s,\partial_s)\,\widehat{x}(s)) = 0$, then there exists a non-zero constant vector $\vec{c} := (c_{00}\ c_{01}\ \dots\ c_{0p})^T$ such that $V_p(s,\partial_s)\,\widehat{x}(s)\,\vec{c} = 0$, and thus, we get equations of the form:

$$(-\partial_s)^k \sum_{(i,\,j)\in[\![0,\dots,n]\!]\times[\![0,\dots,m]\!]\setminus\{(n,m)\}} c_{ij}\,(-\partial_s)^j\, s^i\,\widehat{x}(s) = 0, \quad k = n,\dots,l.$$

Using the inverse Laplace transform, the fact that $\mathscr{L}^{-1}(s^i\,\widehat{x}(s)) = x^{(i)}(t) + \sum_{k=0}^{i-1} \delta^{(i-k-1)}\,x^{(k)}(0)$, where $\delta^{(e)}$ denotes the e^{th} derivative of the Dirac distribution at $t = 0$, and $t^j\,\delta^{(e)} = 0$ for $j > e$, we obtain that x then satisfies the following ODE:

$$t^n \sum_{(i,\,j)\in[\![0,\ldots,n]\!]\times[\![0,\ldots,m]\!]\setminus\{(n,m)\}} c_{ij}\,t^j\,x^{(i)}(t) = 0.$$

Hence, (9) holds if x is a *generic* solution of (2), namely, a function which does not satisfy a lower order/degree ODE.

Theorem 3. *Let us consider the ODE (2) with $a_{nm} \neq 0$. If x is a generic solution of (2), then the coefficients a_{ij}'s can be expressed explicitly in terms of $x(t)$ and its iterative indefinite integrals.*

3 The algebraic parameter estimation problem for a class of perturbations

In this section, we consider a signal x which satisfies the following ODE:

$$\sum_{i=0}^{n} a_i(t)\,\frac{d^i}{dt^i}\,x(t) = 0, \quad n \geq 1, \quad \forall\, i = 0,\ldots,n,\; a_i \in \mathbb{K}[t], \quad a_n \neq 0. \tag{10}$$

We further assume that $t = 0$ is an ordinary point of the differential equation, i.e., $a_n(0) \neq 0$. The signal x is now corrupted by a perturbation γ so that we measure $z(t) := x(t) + \gamma(t)$. In what follows, we shall consider the case of a *structured perturbation* of the form

$$\gamma(t) = \gamma\,t^r\,H(t),$$

where γ is an unknown constant, $r \geq 0$ and H is the Heaviside function defined by $H(t) = 1$ for $t \geq 0$ and 0 elsewhere. These structured perturbations are standard, e.g., in *the disturbance rejection problem* (see, e.g., [Quadrat et al. (2014)] and the references therein). For instance, if $r = 0$, then $\gamma(t) = \gamma\,H(t)$ is a *step function* which corresponds to a unknown *bias*. If $r = 1$, then $\gamma(t) = \gamma\,t\,H(t)$ is a *ramp function* and if $r = 2$, then $\gamma(t) = \gamma\,t^2\,H(t)$ is a *parabolic function*.

We shall study how both the initial conditions $x^{(i)}(0)$, for $i = 0,\ldots,n-1$, and the coefficients of the polynomials a_i's can be expressed in terms of the measured function z and its iterative indefinite integrals (e.g., convolutions). To achieve our goal, we first apply the Laplace transform to (10) and denoting by $\widehat{x}(s)$ the Laplace transform of x, where $s \in \mathbb{C}$ is the Laplace variable, we obtain:

$$\sum_{i=0}^{n} a_i(-\partial_s)\left(s^i\,\widehat{x}(s) - \sum_{j=0}^{i-1} s^{i-j-1}\,x^{(j)}(0)\right) = 0,$$

which can be rewritten as:

$$\underbrace{\sum_{i=0}^{n}\left(a_i(-\partial_s)\,s^i\right)}_{R(s,\partial_s)}\widehat{x}(s) + \underbrace{\sum_{k=0}^{n-1}\underbrace{\left(\sum_{i=k+1}^{n} -a_i(-\partial_s)\,s^{i-k-1}\right)}_{S_k(s)}\underbrace{x^{(k)}(0)}_{\vartheta_k}}_{Q(s)} = 0.$$

In terms of $\widehat{z} = \widehat{x} + \widehat{\gamma}$, the latter equation $R(s,\partial_s)\,\widehat{x}(s) + Q(s) = 0$ yields:

$$R(s,\partial_s)\,\widehat{z}(s) + Q(s) - R(s,\partial_s)\,\widehat{\gamma}(s) = 0. \tag{11}$$

Setting $l := r + 1$, we have $\widehat{\gamma}(s) = \gamma/s^l$. As the constant γ is unknown, we shall try to get rid of the last term of (11) by applying a linear differential operator $L(s,\partial_s)$ on the left to Equation (11).

This linear differential operator must satisfy $L(s,\partial_s)\,R(s,\partial_s)\,\widehat{\gamma}(s) = 0$. To compute $L(s,\partial_s)$, we shall first multiply (11) by a suitable power of s to get polynomial expressions. Let us start by computing $R(s,\partial_s)\,\widehat{\gamma}(s)$. We have:

$$R(s,\partial_s)\,\widehat{\gamma}(s) = \sum_{i=0}^{n} a_i(-\partial_s)\,s^i\,\frac{1}{s^l}\,\gamma = \gamma\,\sum_{i=0}^{n} a_i(-\partial_s)\,s^{i-l}.$$

If $l \leq n$, then $R(s,\partial_s)\,\widehat{\gamma}(s) \in \mathbb{K}[s^{-1},s]$, where $\mathbb{K}[s^{-1},s]$ denotes the *ring of Laurent polynomials*, and $R(s,\partial_s)\,\widehat{\gamma}(s)$ can be decomposed uniquely as the sum of the following two terms:

$$\Gamma_{s^{-1}}(s) := \gamma\,\sum_{i=0}^{l-1} a_i(-\partial_s)\,\frac{1}{s^{l-i}} \in \mathbb{K}[s^{-1}]\setminus\mathbb{K}, \quad \Gamma_s(s) := \gamma\,\sum_{i=l}^{n} a_i(-\partial_s)\,s^{i-l} \in \mathbb{K}[s]. \tag{12}$$

Note that the "constant term" $a_l(-\partial_s)\,1$ in s^{-1} has been chosen to contribute to Γ_s. But, we could have chosen to add it to $\Gamma_{s^{-1}}$.

Let us consider the "generic cas"[1]:

$$l \leq n, \quad \Gamma_{s^{-1}} \neq 0, \quad \Gamma_s \neq 0. \tag{13}$$

In order to obtain a polynomial, we shall multiply $R(s,\partial_s)\,\widehat{\gamma}(s)$ by a suitable power of s. Let

$$p := \deg_{s^{-1}}(\Gamma_{s^{-1}}) \in \mathbb{Z}_{>0}, \qquad p' := \deg_s(\Gamma_s) \in \mathbb{Z}_{\geq 0}.$$

Note that if we denote $d_i := \deg_t(a_i)$, then we have $p \leq \max_{i=0,\ldots,l-1}\{l-i+d_i\}$. Moreover, since $t=0$ is an ordinary point, we have $p' = n-l$.

We thus have $s^p\,R(s,\partial_s)\,\widehat{\gamma}(s) \in \mathbb{K}[s]$ and multiplying (11) by s^p, we get that

$$s^p\,R(s,\partial_s)\,\widehat{z}(s) + s^p\,Q(s) - s^p\,R(s,\partial_s)\,\widehat{\gamma}(s) = 0$$

is an ODE with polynomial coefficients. Setting

$$R'(s,\partial_s) := s^p\,R(s,\partial_s), \quad S(s) := s^p\,Q(s) = s^p\,\sum_{k=0}^{n-1} S_k(s)\,\vartheta_k, \quad \overline{Q}(s) := \frac{1}{\gamma}\,s^p\,R(s,\partial_s)\,\widehat{\gamma}(s),$$

we then get that $\widehat{z}$ satisfies:

$$R'(s,\partial_s)\,\widehat{z}(s) + S(s) - \gamma\,\overline{Q}(s) = 0. \tag{14}$$

The last term of (14), which still contains the unknown constant γ, is now a polynomial in s and its degree q verifies:

$$q := \deg_s(\overline{Q}) = p + p' = p + n - l \leq \max_{i=0,\ldots,l-1}\{l-i+d_i\} + n - l. \tag{15}$$

We can now compute the annihilator of the polynomial $\overline{Q}$ which will allow to get rid of the term $-\gamma\,\overline{Q}(s)$ in Equation (14).

Example 2. Let us consider a sinusoidal signal $x(t) = A\,\sin(\omega\,t+\phi)$ satisfying the order $n=2$ ODE $\left(\frac{d^2}{dt^2}+\omega^2\right)\,x(t) = 0$ and corrupted by a perturbation given by a ramp function, i.e., $r=1$ and $\gamma(t) = \gamma\,t\,H(t)$, where γ is a constant. We thus have $l=2$, $\widehat{\gamma}(s) = \gamma/s^2$, and Equation (11) gives:

$$\left(\omega^2+s^2\right)\widehat{z}\,(s) - s\vartheta_0 - \vartheta_1 - \frac{\left(\omega^2+s^2\right)\gamma}{s^2} = 0.$$

We are thus in the "generic case" with:

$$p = \deg_{s^{-1}}(\Gamma_{s^{-1}}) = \deg_{s^{-1}}\left(\frac{\omega^2}{s^2}\right) = 2, \quad p' = \deg_s(\Gamma_s) = \deg_s(1) = 0.$$

[1]See the comments after Theorem 4 where we discuss the non-generic cases.

Equation (14) is thus

$$s^2\left(\omega^2+s^2\right)\widehat{z}(s)-s^3\vartheta_0-s^2\vartheta_1-\left(\omega^2+s^2\right)\gamma=0,$$

and since the constant parameter γ is unknown we need to compute the annihilator of the polynomial $\overline{Q}(s)=\omega^2+s^2$ of degree $q=p+n-l=2$ to cancel the last term in the left hand side of the latter equality.

3.1 Computation of the annihilator

Let us consider the first Weyl algebra $A:=A_1(\mathbb{K})=\mathbb{K}[s]\langle\partial_s \mid \partial_s\, s=s\,\partial_s+1\rangle$ and characterize the *annihilator* of $\overline{Q}$, namely, the left ideal of A defined by:

$$\operatorname{ann}_A(.\overline{Q}):=\{a\in A \mid a\,\overline{Q}=0\}.$$

The annihilator $\operatorname{ann}_A(.\overline{Q})$ can be obtained by considering the polynomial relations between the successive derivatives of $\overline{Q}$, i.e., by considering the following $\mathbb{K}[s]$-module

$$\ker_{\mathbb{K}[s]}(.L):=\left\{\lambda:=(\lambda_0\ \ldots\ \lambda_{q+1})\in\mathbb{K}[s]^{1\times(q+2)} \mid \lambda\,L=0\right\},\quad L:=\begin{pmatrix}\overline{Q}\\ \overline{Q}^{(1)}\\ \vdots\\ \overline{Q}^{(q+1)}\end{pmatrix}\in\mathbb{K}[s]^{(q+2)\times 1}.$$

Equivalently, we can compute a generating set of compatibility conditions of the inhomogeneous linear system $L\,\eta=\zeta$. Let us write $\overline{Q}(s)=\sum_{k=0}^{q} b_k\,s^k$, where the b_k's belong to the field $\mathbb{K}$. We first note that:

$$\overline{Q}^{(j)}(s)=\sum_{k=j}^{q}\frac{k!}{(k-j)!}\,b_k\,s^{k-j}=\sum_{u=0}^{q-j}\frac{(u+j)!}{u!}\,b_{u+j}\,s^u.$$

Setting $j=q-v$, we then get $\overline{Q}^{(q-v)}(s)=\sum_{u=0}^{v} b'_{q-v+u}\,s^u$, where $b'_{q-v+u}:=((q-v+u)!/u!)\,b_{q-v+u}$. We thus obtain:

$$L\,\eta=\zeta \Longleftrightarrow \begin{cases}\overline{Q}\,\eta=\zeta_0,\\ \quad\vdots\\ \overline{Q}^{(q-2)}\,\eta=(b'_q\,s^2+b'_{q-1}\,s+b'_{q-2})\,\eta=\zeta_{q-2},\\ \overline{Q}^{(q-1)}\,\eta=(b'_q\,s+b'_{q-1})\,\eta=\zeta_{q-1},\\ \overline{Q}^{(q)}\,\eta=b'_q\,\eta=\zeta_q,\\ \overline{Q}^{(q+1)}\,\eta=0=\zeta_{q+1}.\end{cases}$$

Since $b'_q\neq 0$, from the last but one equation, we get $\eta=\zeta_q/b'_q$ and substituting this identity into the rest of the equations, we obtain the following compatibility conditions:

$$\frac{\overline{Q}^{(i)}}{\overline{Q}^{(q)}}\,\zeta_q=\zeta_i \quad\Longleftrightarrow\quad \overline{Q}^{(i)}\,\zeta_q-\overline{Q}^{(q)}\,\zeta_i=0,\quad i=0,\ldots,q+1.$$

Hence, we obtain that the rows of the following matrix

$$M:=\begin{pmatrix}-\overline{Q}^{(q)} & 0 & \ldots & 0 & -\overline{Q} & 0\\ 0 & -\overline{Q}^{(q)} & \ldots & 0 & -\overline{Q}^{(1)} & 0\\ 0 & 0 & \ldots & -\overline{Q}^{(q)} & -\overline{Q}^{(q-1)} & 0\\ 0 & 0 & \ldots & \ldots & 0 & 1\end{pmatrix}\in\mathbb{K}[s]^{(q+1)\times(q+2)},$$

generates $\ker_{\mathbb{K}[s]}(.L)$. Then, we have:

$$M \begin{pmatrix} \zeta_0 \\ \vdots \\ \zeta_{q+1} \end{pmatrix} = 0 \iff M \begin{pmatrix} 1 \\ \partial_s \\ \vdots \\ \partial_s^{q+1} \end{pmatrix} \overline{Q}\,\eta = 0 \iff \begin{pmatrix} -\overline{Q}^{(q)} + \overline{Q}\,\partial_s^q \\ -\overline{Q}^{(q)}\,\partial_s + \overline{Q}^{(1)}\,\partial_s^q \\ \vdots \\ -\overline{Q}^{(q)}\,\partial_s^{q-1} + \overline{Q}^{(q-1)}\,\partial_s^q \\ \partial_s^{q+1} \end{pmatrix} \overline{Q}\,\eta = 0,$$

which shows that:

$$\operatorname{ann}_A(.\overline{Q}) = \left\langle -\overline{Q}^{(q)} + \overline{Q}\,\partial_s^q,\, -\overline{Q}^{(q)}\,\partial_s + \overline{Q}^{(1)}\,\partial_s^q,\, \ldots,\, -\overline{Q}^{(q)}\,\partial_s^{q-1} + \overline{Q}^{(q-1)}\,\partial_s^q,\, \partial_s^{q+1} \right\rangle. \tag{16}$$

For instance, the trivial annihilator $\overline{Q}\,\partial_s - \overline{Q}^{(1)}$ of $\overline{Q}$ can be expressed as:

$$\overline{Q}\,\partial_s - \overline{Q}^{(1)} = \frac{1}{\overline{Q}^{(q)}} \left(-\overline{Q}^{(1)} \left(-\overline{Q}^{(q)} + \overline{Q}\,\partial_s^q \right) + \overline{Q} \left(-\overline{Q}^{(q)}\,\partial_s + \overline{Q}^{(1)}\,\partial_s^q \right) \right) \in \operatorname{ann}_A(.\overline{Q}).$$

The left ideal $\operatorname{ann}_A(.\overline{Q})$ of the Weyl algebra A can thus be generated by $q+1$ elements. However, a famous theorem due to Stafford asserts that every left or right ideal of A can be generated by two elements [Stafford (1978)] and we can indeed prove the following result.

Proposition 1. *Let $\overline{Q}(s) = \sum_{k=0}^{q} b_k\, s^k$, where the b_k's belong to the field $\mathbb{K}$, $q > 0$ and $b_q \neq 0$. Then, we have:*

$$\operatorname{ann}_A(.\overline{Q}) = \left\langle -\overline{Q}^{(q)} + \overline{Q}\,\partial_s^q,\, \partial_s^{q+1} \right\rangle.$$

Proof. We need to determine two generators for the ideal $\operatorname{ann}_A(.\overline{Q})$ given by (16). Let $P_1 := -\overline{Q}^{(q)} + \overline{Q}\,\partial_s^q$ and $P_2 := \partial_s^{q+1}$. Then, using $\overline{Q}^{(q+1)} = 0$, we have:

$$\partial_s\,P_1 - \overline{Q}\,P_2 = -\overline{Q}^{(q)}\,\partial_s - \overline{Q}^{(q+1)} + \overline{Q}\,\partial_s^{q+1} + \overline{Q}^{(1)}\,\partial_s^q - \overline{Q}\,\partial_s^{q+1} = -\overline{Q}^{(q)}\,\partial_s + \overline{Q}^{(1)}\,\partial_s^q,$$

which shows that the second generator of $\operatorname{ann}_A(.\overline{Q})$ given by (16) belongs to the left ideal $\langle P_1,\, P_2\rangle$ of A generated P_1 and P_2. By induction, let us suppose that $-\overline{Q}^{(q)}\,\partial_s^i + \overline{Q}^{(i)}\,\partial_s^q \in \langle P_1,\, P_2\rangle$ and let us then prove that $-\overline{Q}^{(q)}\,\partial_s^{i+1} + \overline{Q}^{(i+1)}\,\partial_s^q \in \langle P_1,\, P_2\rangle$. Again, using $\overline{Q}^{(q+1)} = 0$, we then have

$$\begin{aligned} \partial_s \left(-\overline{Q}^{(q)}\,\partial_s^i + \overline{Q}^{(i)}\,\partial_s^q \right) - \overline{Q}^{(i)}\,P_2 &= -\overline{Q}^{(q)}\,\partial_s^{i+1} - \overline{Q}^{(q+1)}\,\partial_s^i + \overline{Q}^{(i)}\,\partial_s^{q+1} + \overline{Q}^{(i+1)}\,\partial_s^q - \overline{Q}^{(i)}\,\partial_s^{q+1} \\ &= -\overline{Q}^{(q)}\,\partial_s^{i+1} + \overline{Q}^{(i+1)}\,\partial_s^q \in \langle P_1,\, P_2\rangle, \end{aligned}$$

which proves by induction that all but the first and the last generators of $\operatorname{ann}_A(.\overline{Q})$ belong to $\langle P_1,\, P_2\rangle$. This proves the result. □

If $q = 0$, then $\overline{Q}(s) \in \mathbb{K}$ is a constant and we have $\operatorname{ann}_A(.\overline{Q}) = \langle \partial_s \rangle$. Finally, we note that the first generator $P_1 = -\overline{Q}^{(q)} + \overline{Q}\,\partial_s^q$ of $\operatorname{ann}_A(.\overline{Q})$ depends on the coefficients b_j's of $\overline{Q}$ (and thus of the coefficients of the polynomials a_i's) contrary to the second one $P_2 = \partial_s^{q+1}$.

Example 3. In Example 2, we found $\overline{Q}(s) = \omega^2 + s^2$. Proposition 1 then implies that:

$$\operatorname{ann}_A(.\overline{Q}) = \langle -2 + \left(\omega^2 + s^2\right)\partial_s^2, \partial_s^3 \rangle.$$

3.2 Estimation of the initial conditions

We now investigate when the initial conditions $\vartheta_k = x^{(k)}(0)$, $k = 0, \ldots, n-1$, can be obtained in terms of convolutions of z. Since the perturbation $\gamma(t) = \gamma\, t^r\, H(t)$ is not supposed to be known apriori, from (14), we first have to use the elements of the annihilator of $\overline{Q}$ to delete the term $-\gamma\,\overline{Q}(s)$ containing γ. Then, we have to study whether or not the ϑ_k's can be obtained explicitly in terms of derivatives of $\widehat{z}$. If so, then using the inverse Laplace transform, we can express the ϑ_k's as convolutions of the measured signal z.

With the notations of Section 3.1, applying the operator $P_1(s, \partial_s) = -\overline{Q}^{(q)} + \overline{Q}\,\partial_s^q \in \mathrm{ann}_A(.\overline{Q})$ to (14), we then get:

$$P_1(s, \partial_s)\, S(s) = -P_1(s, \partial_s)\, R'(s, \partial_s)\, \widehat{z}(s). \tag{17}$$

Let us now develop the term $P_1(s, \partial_s)\, S(s)$ to characterize when the initial conditions ϑ_k's can be expressed in terms of $\widehat{z}(s)$. We have:

$$P_1(s, \partial_s)\, S(s, \vartheta) = \sum_{k=0}^{n-1} \underbrace{\left(\left(-\overline{Q}^{(q)} + \overline{Q}(s)\,\partial_s^q\right)\, s^p\, S_k(s)\right)}_{T_k(s) \in \mathbb{K}[s]} \vartheta_k.$$

As in the case without perturbation, a natural ideal to express the ϑ_k's in terms of $\widehat{z}$ is to differentiate (17) a certain number of times to obtain a solvable inhomogeneous linear system in the ϑ_k's. To do that, we first need to study the degrees of the polynomials T_k's. We have:

$$T_k(s) := \left(-\overline{Q}^{(q)} + \overline{Q}(s)\,\partial_s^q\right)\, s^p\, S_k(s), \quad S_k(s) = \sum_{i=k+1}^{n} a_i(-\partial_s)\, s^{i-k-1}.$$

Since $t = 0$ is an ordinary point of the differential equation (10), we have $S_k(s) = a_n(0)\, s^{n-k-1} + \cdots$, where $\cdots$ contains monomials of degree strictly less than $n-k-1$. Moreover, $\overline{Q}(s) = \sum_{k=0}^{q} b_k\, s^k$ with $b_q = a_n(0)$ so that $\overline{Q}^{(q)}(s) = q!\, a_n(0)$, and we further have:

$$\partial_s^q\, s^p\, S_k(s) = a_n(0)\, \frac{(p+n-k-1)!}{(p+n-k-1-q)!}\, s^{p+n-k-1-q} + \cdots,$$

where $\cdots$ contains monomials of degree strictly less than $p+n-k-1-q$. The leading monomial of $-\overline{Q}^{(q)}\, s^p\, S_k(s)$ is thus $-q!\, a_n(0)^2\, s^{p+n-k-1}$ and that of $\overline{Q}(s)\,\partial_s^q\, s^p\, S_k(s)$ is $a_n(0)^2\, \frac{(p+n-k-1)!}{(p+n-k-1-q)!}\, s^{p+n-k-1}$. We then obtain:

$$\begin{array}{rcl} T_k(s) & = & -\overline{Q}^{(q)}\, s^p\, S_k(s) + \overline{Q}(s)\,\partial_s^q\, s^p\, S_k(s), \\ & = & a_n(0)^2\, \left(-q! + \frac{(p+n-k-1)!}{(p+n-k-1-q)!}\right)\, s^{p+n-k-1} + \cdots, \end{array}$$

where $\cdots$ contains monomials of degree strictly less than $p+n-k-1$. We thus have

$$\deg_s(T_k) = p+n-k-1,$$

except if $\frac{(p+n-k-1)!}{(p+n-k-1-q)!} = q!$, i.e., if $q = p+n-k-1$ which, from (15), means $k+1 = l$, i.e., $k = r$.

It thus remains to determine the degree of T_r. Note that, from (12), we have

$$S_r(s) = \sum_{i=r+1}^{n} a_i(-\partial_s)\, s^{i-r-1} = \sum_{i=l}^{n} a_i(-\partial_s)\, s^{i-l} = \frac{1}{\gamma}\, \Gamma_s(s),$$

and as $R(s, \partial_s)\, \widehat{\gamma}(s) = \Gamma_{s^{-1}}(s) + \Gamma_s(s)$, we get:

$$S_r(s) = \frac{1}{\gamma}\, \left(R(s, \partial_s)\, \widehat{\gamma}(s) - \Gamma_{s^{-1}}(s)\right).$$

Consequently, we have

$$T_r(s) = \left(-\overline{Q}^{(q)} + \overline{Q}(s)\,\partial_s^q\right)\, s^p\, S_r(s) = -\frac{1}{\gamma}\left(-\overline{Q}^{(q)} + \overline{Q}(s)\,\partial_s^q\right)\, s^p\, \Gamma_{s^{-1}}(s),$$

since $\left(-\overline{Q}^{(q)} + \overline{Q}(s)\,\partial_s^q\right)\, s^p\, R(s,\partial_s)\,\widehat{\gamma}(s) = \left(-\overline{Q}^{(q)} + \overline{Q}(s)\,\partial_s^q\right)\,\overline{Q}(s) = 0$, and, from (12), we thus have:

$$T_r(s) = -\left(-\overline{Q}^{(q)} + \overline{Q}(s)\,\partial_s^q\right)\, s^p \sum_{i=0}^{r} a_i(-\partial_s)\,\frac{1}{s^{r+1-i}}.$$

Denoting $v_i := v_0(a_i)$ the valuation of a_i, we thus obtain:

$$\deg_s(T_r) \leq p - \min_{i=0,\ldots,r}\{r+1-i+v_i \mid v_i \neq -\infty\}.$$

We have thus proved the following result.

Lemma 1. *With the above notations and assumptions, we have:*

- *For all* $k \in \{0,\ldots,n-1\}\setminus\{r\}$, $\deg_s(T_k) = p+n-k-1$,
- $\deg_s(T_r) \leq p-r-1-\min_{i=0,\ldots,r}\{v_i - i \mid v_i \neq -\infty\}$.

We now state the following corollary of Lemma 1 which implies that (17) provides a *triangular* inhomogeneous linear system for the ϑ_k's.

Corollary 1. *With the above notations and assumptions, we have:*

$$\forall\, k,\, k' \in \{0,\ldots,n-1\},\ k \neq k',\ \deg_s(T_k) \neq \deg_s(T_{k'}).$$

Proof. From Lemma 1, it suffices to prove that $\deg_s(T_r) \neq \deg_s(T_j)$ for all $j \in \{0,\ldots,n-1\}\setminus\{r\}$. If $r < n-1$, then proving that $\deg_s(T_r) < \deg_s(T_{n-1}) = p$ is enough. By contradiction, let us assume $p \leq \deg_s(T_r) \leq p-r-1-\min_{i=0,\ldots,r}\{v_i - i \mid v_i \neq -\infty\}$. Then, there exists $j \in \{0,\ldots,r\}$ such that $p \leq p-r-1-v_j+j$ and $v_j \neq -\infty$. This yields $v_j \leq j-(r+1) < 0$ which is in contradiction with $v_j \geq 0$. Similarly, in the case $r = n-1$, we prove that $\deg_s(T_r = T_{n-1}) < \deg_s(T_{n-2}) = p+1$. This ends the proof. □

Example 4. Let us continue Example 2. We have $S(s,\vartheta) = -s^3\vartheta_0 - s^2\vartheta_1$ and in Example 3, we found $P_1(s,\partial_s) = -2 + \left(\omega^2+s^2\right)\partial_s^2$. This implies:

$$P_1(s,\partial_s)\, S(s,\vartheta) = \left(-6\, s\omega^2 - 4\, s^3\right)\vartheta_0 - 2\,\omega^2\vartheta_1,$$

so that $T_0(s) = -6\, s\omega^2 - 4\, s^3$ and $T_1(s) = -2\,\omega^2$. We recall that in this example $n = p = 2$, $r = 1$ and we can check that:

$$\deg_s(T_0) = 2+2-0-1 = 3 \neq \deg_s(T_1) = 0 \leq 2-1-1-\min_{i=0,1}\{v_i - i \mid v_i \neq -\infty\} = 0,$$

since $v_0 = 0$ and $v_1 = -\infty$.

Let us finally explain how Corollary 1 permits to compute all the ϑ_k's. We start by computing ϑ_j, where j is such that $f_j := \deg_s(T_j) > \deg_s(T_k)$ for $k \neq j$. Indeed differentiating f_j times (17), all the ϑ_k's, for $k \neq j$, disappear and we get a linear equation for ϑ_j. More precisely, if we note $T_j(s) = t_{j f_j}\, s^{f_j} + \cdots$, where $\cdots$ contains monomials of degree strictly less that f_j, then applying $\partial_s^{f_j}$ to (17), we get:

$$\vartheta_j = -\frac{1}{f_j!\, t_{j f_j}}\,\partial_s^{f_j}\, P_1(s,\partial_s)\, R'(s,\partial_s)\,\widehat{z}(s).$$

Note that, if $j \neq r$, then $t_{j f_j} = a_n(0)^2\left(-q! + \frac{(p+n-j-1)!}{(p+n-j-1-q)!}\right) \neq 0$. Knowing ϑ_j, we can then proceed and compute the next ϑ_k's using the same technique.

Theorem 4. *With the above notations, if $t = 0$ is a regular point for the ODE (10) and if we are in the "generic case" (13), then the initial conditions $\{\vartheta_k = x^{(k)}(0)\}_{k=0,\dots,n-1}$ of (10) can be expressed explicitly in terms of an operator $Q_k(s, \partial_s)$ of $A_1(\mathbb{Q})$ applied to $\widehat{z}(s)$, i.e., $\vartheta_k = Q_k(s, \partial_s)\, \widehat{z}(s)$, for $k = 0, \dots, n-1$.*

Moreover, if $l_k := \deg_s Q_k(s, \partial_s)$, then writing $\vartheta_k = n_k/d_k$, where

$$n_k = \frac{Q_k(s, \partial_s)}{s^{l_k+1}}\, \widehat{z}(s), \quad d_k = \frac{1}{s^{l_k+1}}, \quad k = 0, \dots, n-1,$$

we then have:

$$\vartheta_k = \frac{\mathscr{L}^{-1}(n_k)}{\mathscr{L}^{-1}(d_k)}, \quad \mathscr{L}^{-1}(d_k) = \frac{t^{l_k}}{l_k!}.$$

Hence, the initial conditions can be explicitly expressed as convolutions of z depending on the coefficients of the polynomial coefficients a_i's of the ODE (10).

Let us make some comments on the assumptions of Theorem 4.

1. If $t = 0$ is not an ordinary point of the ODE (10), then, as in the case without perturbation considered in the previous section, we cannot recover all the initial conditions. For instance, $\vartheta_{n-1} = x^{(n-1)}(0)$ cannot be obtained as it does not appear in the equation after applying the Laplace transforn. Indeed, with the above notations, $S_{n-1}(s) = -a_n(-\partial_s)\, 1 = 0$ if $a_n(0) = 0$. For more details, see Example 1.

2. If $\Gamma_{s^{-1}} = 0$, then $T_r(s) = 0$ and we cannot recover the initial condition $\vartheta_r = x^{(r)}(0)$ as it does not appear in the equation after applying the operator $P_1(s, \partial_s)$ to get rid of the unknown constant γ. All the other initial conditions can be obtained as explained above.

3. If $l > n$, then we have $\Gamma_s = 0$ and the strategy developed in this section also permits to obtain all the initial conditions. For more details, see [Chartouny (2021)].

4. If $\overline{Q}(s) \in \mathbb{K}$ is a constant, i.e., $q = 0$, then we have $\mathrm{ann}_A(.\overline{Q}) = \langle \partial_s \rangle$ and the above strategy, where we choose $P_1(s, \partial_s) = -\overline{Q}^{(q)} + \overline{Q}\, \partial_s^q$ instead of ∂_s, allows us to get the initial conditions in a similar manner.

Example 5. Continuing Example 2, we start by computing ϑ_0. Here Equation (17) writes:

$$\left(-6\, s\, \omega^2 - 4\, s^3\right) \vartheta_0 - 2\, \omega^2 \vartheta_1 = -\left(s^2 \left(\omega^2 + s^2\right)^2 \frac{\mathrm{d}^2}{\mathrm{d}s^2} \widehat{z}\,(s) + \left(4\, \omega^4 s + 12\, \omega^2 s^3 + 8\, s^5\right) \frac{\mathrm{d}}{\mathrm{d}s} \widehat{z}\,(s) + \left(2\, \omega^4 + 12\, \omega^2 s^2 + 10\, s^4\right) \widehat{z}\,(s)\right). \tag{18}$$

Differentiating 3 times with respect to s and solving for ϑ_0, we get:

$$\vartheta_0 =$$
$$\tfrac{1}{24} \left(\tfrac{\mathrm{d}^5}{\mathrm{d}s^5} \widehat{z}\,(s)\right) \omega^4 s^2 + \tfrac{1}{12} \left(\tfrac{\mathrm{d}^5}{\mathrm{d}s^5} \widehat{z}\,(s)\right) \omega^2 s^4 + \tfrac{1}{24} \left(\tfrac{\mathrm{d}^5}{\mathrm{d}s^5} \widehat{z}\,(s)\right) s^6 + \tfrac{5}{12} \left(\tfrac{\mathrm{d}^4}{\mathrm{d}s^4} \widehat{z}\,(s)\right) \omega^4 s + \tfrac{3}{2} \left(\tfrac{\mathrm{d}^4}{\mathrm{d}s^4} \widehat{z}\,(s)\right) \omega^2 s^3$$
$$+ \tfrac{13}{12} \left(\tfrac{\mathrm{d}^4}{\mathrm{d}s^4} \widehat{z}\,(s)\right) s^5 + \tfrac{5}{6} \left(\tfrac{\mathrm{d}^3}{\mathrm{d}s^3} \widehat{z}\,(s)\right) \omega^4 + 8 \left(\tfrac{\mathrm{d}^3}{\mathrm{d}s^3} \widehat{z}\,(s)\right) \omega^2 s^2 + \tfrac{55}{6} \left(\tfrac{\mathrm{d}^3}{\mathrm{d}s^3} \widehat{z}\,(s)\right) s^4 + 14 \left(\tfrac{\mathrm{d}^2}{\mathrm{d}s^2} \widehat{z}\,(s)\right) \omega^2 s + 30\, s^3 \tfrac{\mathrm{d}^2}{\mathrm{d}s^2} \widehat{z}\,(s)$$
$$+ 6 \left(\tfrac{\mathrm{d}}{\mathrm{d}s} \widehat{z}\,(s)\right) \omega^2 + 35 \left(\tfrac{\mathrm{d}}{\mathrm{d}s} \widehat{z}\,(s)\right) s^2 + 10\, s \widehat{z}\,(s).$$

Then, applying the inverse Laplace transform, after dividing by s^7 and introducing the denominator $1/s^7$ as explained in Theorem 4, we get:

$$\vartheta_0 =$$
$$30\, \tfrac{1}{t^6} \big(\textstyle\int_0^t - \tfrac{1}{24}\, \omega^4 \left(t - \tau_1\right)^4 {\tau_1}^5 z\,(\tau_1)\; \mathrm{d}\tau_1 + \int_0^t - \omega^2 \left(t - \tau_1\right)^2 {\tau_1}^5 z\,(\tau_1)\; \mathrm{d}\tau_1 + \int_0^t - {\tau_1}^5 z\,(\tau_1)\; \mathrm{d}\tau_1 + \int_0^t \tfrac{1}{12}\, \omega^4 \left(t - \tau_1\right)^5 {\tau_1}^4 z\,(\tau_1)\; \mathrm{d}\tau_1$$
$$+ \textstyle\int_0^t 6\, \omega^2 \left(t - \tau_1\right)^3 {\tau_1}^4 z\,(\tau_1)\; \mathrm{d}\tau_1 + \int_0^t (26\, t - 26\, \tau_1)\, {\tau_1}^4 z\,(\tau_1)\; \mathrm{d}\tau_1 + \int_0^t - \tfrac{1}{36}\, \omega^4 \left(t - \tau_1\right)^6 {\tau_1}^3 z\,(\tau_1)\; \mathrm{d}\tau_1 + \int_0^t - 8\, \omega^2 \left(t - \tau_1\right)^4 {\tau_1}^3 z\,(\tau_1)\; \mathrm{d}\tau_1$$
$$+ \textstyle\int_0^t - 110\, \left(t - \tau_1\right)^2 {\tau_1}^3 z\,(\tau_1)\; \mathrm{d}\tau_1 + \int_0^t \tfrac{14}{5} \omega^2 \left(t - \tau_1\right)^5 {\tau_1}^2 z\,(\tau_1)\; \mathrm{d}\tau_1 + \int_0^t 120\, \left(t - \tau_1\right)^3 {\tau_1}^2 z\,(\tau_1)\; \mathrm{d}\tau_1 + \int_0^t - \tfrac{1}{5}\, \omega^2 \left(t - \tau_1\right)^6 \tau_1 z\,(\tau_1)\; \mathrm{d}\tau_1$$
$$+ \textstyle\int_0^t - 35\, \left(t - \tau_1\right)^4 \tau_1 z\,(\tau_1)\; \mathrm{d}\tau_1 + \int_0^t 2\, \left(t - \tau_1\right)^5 z\,(\tau_1)\; \mathrm{d}\tau_1\big).$$

Now, we can proceed similarly to get ϑ_1 from (18). No differentiation is needed since $\deg_s(T_1) = 0$ and we can replace ϑ_0 by its expression obtained above.

3.3 Estimation of the coefficients of the ODE

Theorem 4 above provides explicit expressions for the initial conditions $\{\vartheta_k = x^{(k)}(0)\}_{k=0,\dots,n-1}$ of (10) in terms of the measured function z and its iterative indefinite integrals (e.g., convolutions). Let the polynomial coefficients of (10) be denoted by $a_i(t) = \sum_{j=v_i}^{d_i} a_{ij}\, t^j$, for $i = 0, \dots, n$, where the a_{ij}'s belong to $\mathbb{K}$. The expressions of the ϑ_k's can depend on the equation parameters, namely, on the a_{ij}'s − some of which could be unknown or unfixed to numerical values. Hence, an important issue, studied here, is to investigate when the coefficients a_{ij}'s of (10) can be found again by means of convolutions of z which are independent on the value γ of the perturbation $\gamma(t)$ (supposed to be unknown).

As noticed at the end of Section 3.1, contrary to the first generator $P_1 = -\overline{Q}^{(q)} + \overline{Q}(s)\,\partial_s^q$ used in Section 3.2, the second generator $P_2 = \partial_s^{q+1}$ of $\mathrm{ann}_A(.\overline{Q})$ does not depend on the coefficients a_{ij}'s. Hence, applying the operator P_2 to (14), i.e., to $s^p\, R(s, \partial_s)\, \widehat{z}(s) + S(s) - \gamma\, \overline{Q}(s) = 0$, we then obtain:

$$\partial_s^{q+1}\, s^p\, R(s, \partial_s)\, \widehat{z}(s) + \partial_s^{q+1}\, S(s) = 0, \tag{19}$$

which is a *linear* expression in the a_{ij}'s.

To obtain linear equations for the a_{ij}'s, we first remove the ϑ_k's from Equation (19). We recall that $S(s) = s^p \sum_{k=0}^{n-1} S_k(s)\, \vartheta_k$, where $S_k \in \mathbb{K}[s]$ and $\deg_s(S_k) = n - k - 1$ so that $\deg_s(S) = p + n - 1$ since we assume that $t = 0$ is an ordinary point of (10)). Then, if $q + 1 > p + n - 1$ which, using (15), yields $l - 1 = r < 1$, i.e., $r = 0$ and the perturbation is a bias, then we have $\partial_s^{q+1}\, S(s) = 0$ and Equation (19) reduces to $\partial_s^{q+1}\, s^p\, R(s, \partial_s)\, \widehat{z}(s) = 0$. Otherwise, for $r > 0$, we still need to apply ∂^r to (19) to remove $S(s)$ and we get: $\partial_s^{r+q+1}\, s^p\, R(s, \partial_s)\, \widehat{z}(s) = 0$. Hence, we always obtain:

$$\partial_s^{r+q+1}\, s^p\, R(s, \partial_s)\, \widehat{z}(s) = 0. \tag{20}$$

Equation (20) depends linearly on the coefficients a_{ij}'s of (10) and on the parameter r defining the perturbation $\gamma(t) = \gamma\, t^r\, H(t)$. In practice, it usually makes sense to consider that r is apriori known.

Using $R(s, \partial_s) = \sum_{i=0}^n a_i(-\partial_s)\, s^i$, where $a_i(-\partial_s) = \sum_{j=v_i}^{d_i} a_{ij}\, (-\partial_s)^j$, Equation (20) becomes:

$$\sum_{i=0}^{n} \sum_{j=v_i}^{d_i} \left(\partial_s^{r+q+1}\, s^p\, (-\partial_s)^j\, s^i\right)\, a_{ij}\, \widehat{z}(s) = 0. \tag{21}$$

Hence, if set $g := \sum_{i=0}^n (d_i - v_i + 1) \geq 0$ and

$$\vec{a} := (a_{0v_0}\ \dots\ a_{0d_0}\ a_{1v_1}\ \dots\ a_{1d_1}\ \dots\ a_{nv_n}\ \dots\ a_{nd_n})^T \in \mathbb{K}^g,$$

then the g coefficients a_{ij}'s of (10) satisfy the following linear equation:

$$\left(\partial_s^{r+q+1}\, s^p\, (-\partial_s)^{v_0} \quad \dots \quad \partial_s^{r+q+1}\, s^p\, (-\partial_s)^{d_n}\, s^n\right)\, \vec{a}\, \widehat{z}(s) = 0.$$

Note that (21) can be differentiated, e.g., w times. Hence, the a_{ij}'s also satisfy the linear system:

$$\begin{pmatrix} \partial_s^{r+q+1}\, s^p\, (-\partial_s)^{v_0} & \dots & \partial_s^{r+q+1}\, s^p\, (-\partial_s)^{d_n}\, s^n \\ \vdots & \dots & \vdots \\ \partial_s^{r+q+1+w}\, s^p\, (-\partial_s)^{v_0} & \dots & \partial_s^{r+q+1+w}\, s^p\, (-\partial_s)^{d_n}\, s^n \end{pmatrix} \vec{a}\, \widehat{z}(s) = 0. \tag{22}$$

We can then repeat what we did in the case without perturbation to obtain $\vec{a}$, i.e., the a_{ij}'s in terms of $\widehat{z}$ and its derivatives by solving the linear system of equations (22). Finally, applying the inverse Laplace transform as before, we get the following result.

Theorem 5. *With the above notations, if $t = 0$ is a regular point for the ODE (10), then the coefficients a_{ij}'s of (10) can be expressed explicitly as convolutions of the measured function z.*

Example 6. Let us continue Example 2. The expressions of ϑ_0 and ϑ_1 obtained in Example 5 still depend on the parameter ω^2 appearing as a coefficient of the ODE satisfied by the sinusoidal signal. We can then use Theorem 5 to estimate ω^2. Since $r = 1$ and $q = 2$, Equation (20) writes:

$$24\,\widehat{z}\,(s)+96\,s\frac{\mathrm{d}}{\mathrm{d}s}\widehat{z}\,(s)+12\,\left(\omega^2+s^2\right)\frac{\mathrm{d}^2}{\mathrm{d}s^2}\widehat{z}\,(s)+60\left(\frac{\mathrm{d}^2}{\mathrm{d}s^2}\widehat{z}\,(s)\right)s^2+8\,\left(\omega^2+s^2\right)\left(\frac{\mathrm{d}^3}{\mathrm{d}s^3}z\,(s)\right)s+8\,s^3\frac{\mathrm{d}^3}{\mathrm{d}s^3}\widehat{z}\,(s)+s^2\left(\omega^2+s^2\right)\frac{\mathrm{d}^4}{\mathrm{d}s^4}\widehat{z}\,(s)=0.$$

From the latter equality, we get

$$\omega^2 = -\frac{\left(\frac{\mathrm{d}^4}{\mathrm{d}s^4}\widehat{z}\,(s)\right)s^4+16\,s^3\frac{\mathrm{d}^3}{\mathrm{d}s^3}\widehat{z}\,(s)+72\,\left(\frac{\mathrm{d}^2}{\mathrm{d}s^2}\widehat{z}\,(s)\right)s^2+96\,s\frac{\mathrm{d}}{\mathrm{d}s}\widehat{z}\,(s)+24\,\widehat{z}\,(s)}{s^2\frac{\mathrm{d}^4}{\mathrm{d}s^4}\widehat{z}\,(s)+8\,\left(\frac{\mathrm{d}^3}{\mathrm{d}s^3}\widehat{z}\,(s)\right)s+12\,\frac{\mathrm{d}^2}{\mathrm{d}s^2}\widehat{z}\,(s)},$$

and applying the inverse Laplace transform, we finally obtain:

$$\omega^2 = \frac{\int_0^t -{\tau_1}^4 z\,(\tau_1)\,\mathrm{d}\tau_1+\int_0^t -(-16\,t+16\,\tau_1)\,{\tau_1}^3 z\,(\tau_1)\,\mathrm{d}\tau_1+\int_0^t -36\,(t-\tau_1)^2\,{\tau_1}^2 z\,(\tau_1)\,\mathrm{d}\tau_1+\int_0^t 16\,(t-\tau_1)^3\,\tau_1 z\,(\tau_1)\,\mathrm{d}\tau_1+\int_0^t -(t-\tau_1)^4\,z\,(\tau_1)\,\mathrm{d}\tau_1}{\int_0^t \frac{1}{2}\,(t-\tau_1)^2\,{\tau_1}^4 z\,(\tau_1)\,\mathrm{d}\tau_1+\int_0^t -\frac{4}{3}\,(t-\tau_1)^3\,{\tau_1}^3 z\,(\tau_1)\,\mathrm{d}\tau_1+\int_0^t \frac{1}{2}\,(t-\tau_1)^4\,{\tau_1}^2 z\,(\tau_1)\,\mathrm{d}\tau_1}.$$

4 Implementation in `Maple`

`Maple` is a mathematics-based software and services for education, engineering, and research. For instance, it can manipulate mathematical expressions and find symbolic solutions to certain problems, such as those arising from ordinary and partial differential equations.

4.1 The NonA package

The NonA package is dedicated to the effective study of the algebraic parameter estimation problem, introduced by M. Fliess and H. Sira-Ramirez in [Fliess et al. (2003)] and studied in the former Non-A project-team (Inria Lille-Nord Europe). This `Maple` package, based on the OreModules package [Chyzak et al. (2007)], has been developed by A. Quadrat (Inria Paris, Ouragan project-team). The binary is freely downloadable at:

`https://who.rocq.inria.fr/Alban.Quadrat/Non-A/`.

Let us shortly describe the main commands of this package:

Annihilator	Compute the annihilator of a given polynomial
AnnihilatorOfExpansion	Compute the annihilator of a finite linear combinaison of signals defined by ODEs with polynomial coefficients
ParameterEstimation	State P, Q and $\overline{Q}$ defined in Section 2 and Section 3 The option “bias" treats the particular case of a structured perturbation
ParameterEstimationEq	State $P\,z(s) + Q + \overline{Q}\,\gamma(s)$. The option “bias" treats the particular case of a structured perturbation

4.2 Development of the NonA package

For some examples, `Maple` is not able to calculate the inverse Laplace transform. For instance, if we want to compute the inverse Laplace transform of $Z'(s)^2$ using `Maple`, i.e., the square of the first derivative of $Z(s)$ with respect to s, the output is “`Error, (in collect) invalid 1st argument normal`". We encounter the same problem with expressions of the form $Z(s)^2$, $Z(s)^3$, $Z''(s)^4$, $Z(s)\,Z'(s)^3$. To solve this problem, we have implemented a `convolution` procedure. This function takes as input an expression e that can contain powers of functions in s and the parameter t. The output of the function is the convolution of e. For instance, the command

```
convolution(Z(s)^2,t);
```

yields the following output:

$$\int_0^t Z(t-\tau_1)\ Z(\tau_1)\,d\tau_1.$$

Moreover, if we want to compute the degree in s of an expression that has the following form $e := z(s)^5\,s^9 + s^4 + z''(s)\,s^2$, the function `degree` will display `FAIL`. This result is expected because $z(s)$ is a function in s. Therefore, the degree depends on the expression of z. However, in our study, to calculate the Laplace inverse (as seen in Section 2 and Section 3), we need to know the degree in s of these types of expressions regardless of $z(s)$. Therefore, we implemented a degree function `deg` in order to compute the degree in s of these expressions, i.e., expressions that can contain functions in s as well as their derivatives. For instance, if we consider again the above expression,

```
e := z(s)^5*s^9 + s^4 + diff(z,s,s)*s^2;
deg(e,s);
```

yields 9.

Furthermore, we have implemented a function `ExplicitExpression` that computes the explicit expression of expression given in the frequency domain.

For instance, if we consider

```
v_0 := 2*s*diff(x,s,s) + (s^2 + 2*(k + 2)*s)*diff(x,s) + 2*s*x(s);
ExplicitExpression(v_0,s);
```

we get as output:

$$\frac{2}{t^2}\Big(-\int_0^t \tau\,x(\tau)\,d\tau + 2\int_0^t (t-\tau)\,\tau^2\,x(\tau)\,d\tau + 2\int_0^t (t-\tau)\,x(\tau)\,d\tau - (2+k)\int_0^t (t-\tau)^2\,\tau\,x(\tau)\,d\tau\Big).$$

We have also implemented a function `InvMatrix` that directly computes the inverse of the matrix U defined in Section 2 without using the `Maple` command `MatrixInverse`. As input, the function takes the matrix U and gives U^{-1} as output. This function enables us to save time in the computation process since the explicit expression of U^{-1} can be given (see [Chartouny et al. (2021)]). For instance, when using the regular function of `Maple` to compute the inverse of U for $n = 9$, the function takes 0.183 CPU seconds. On the other hand, the `InvMatrix` only takes 0.005 CPU seconds. Of course, the difference highly increases with the size n of the matrix U to inverse.

More importantly, we are collecting different procedures that make automatic the computation of the initial conditions and the parameters of a signal defined by a linear ODE with polynomial coefficients affected by a structured perturbation of the form given in Section 3. They will be soon added to the NonA package.

Finally, our second contribution to the NonA package is the development of more explicit examples that will be soon added to the NonA library of examples (e.g., sums of exponentials, sums of sinusoidals, sums of orthogonal polynomials).

4.3 Worked example

We illustrate the NonA package to find again the results of Examples 2, 3, 4, 5, and 6. For more complex and therefore longer examples (e.g., sums of exponentials, sums of sinusoidals, sums of orthogonal polynomials), see the library of examples of the NonA package.

We first start by loading the following two packages:

```
> with(OreModules): with(NonA):
```

We then define the OD operator annihilating the signal $x(t) = a\,\sin(\omega\,t + \phi)$ and its order:

```
> L := dt^2+omega^2; n := 2:
```

$$L := dt^2 + \omega^2$$

We can then define the following two Weyl algebras

$$A=\mathbb{Q}(\omega,\vartheta_0,\vartheta_1,\gamma)\langle\partial_t,t\,|\,\partial_t\,t=t\,\partial_t+1\rangle,\quad B=\mathbb{Q}(\omega,\vartheta_0,\vartheta_1,\gamma)\langle\partial_s,s\,|\,\partial_s\,s=s\,\partial_s+1\rangle$$

in which we shall make the computations (for the time-domain and for the frequency-domain):

```
> A := DefineOreAlgebra(diff=[ds,s],polynom=[s],comm=[omega,vartheta[0],
> vartheta[1],gamma]):
> B := DefineOreAlgebra(diff=[dt,t],polynom=[t],comm=[omega,vartheta[0],
> vartheta[1],gamma]):
```

We can now apply the command `ParameterEstimation` of the NonA package that generates the OD operators and polynomials defining the different terms of Equation (11):

```
> ParEst := ParameterEstimation(L,B);
```

$$ParEst:=[\omega^2+s^2,[-s,-1],-\omega^2-s^2]$$

We state again that we consider the case of a ramp perturbation, i.e., $r=1$, so that $\widehat{\gamma}(s)=\gamma/s^2$. Simply denoting the Laplace transform of $z(t)$ by $z(s)$ (instead of $\widehat{z}(s)$ as done in the text), Equation (11) is given by the equation `eq` defined below:

```
> P := ParEst[1]:
> Q := add(ParEst[2][i]*vartheta[i-1],i=1..n):
> eq := ApplyMatrix(P,z(s),A)+Q-ApplyMatrix(P,gamma/s^2,A);
```

$$eq:=\left(\omega^2+s^2\right)z\left(s\right)-s\vartheta_0-\vartheta_1-\frac{\left(\omega^2+s^2\right)\gamma}{s^2}$$

We can define the polynomial $\overline{Q}$

```
> Qbar := 1/gamma*s^2*ApplyMatrix(P,gamma/s^2,A);
```

$$Qbar:=\omega^2+s^2$$

and compute its annihilator using the command `Annihilator` of the NonA package:

```
> AnnQbar := Annihilator(Qbar,A);
```

$$AnnQbar:=\left[\begin{array}{c}\omega^2ds^2+s\,ds-2\\ s\,ds^2-ds\\ ds^3\end{array}\right]$$

We can then define the element P_1 used in Example 4 and check again that it belongs to $\mathrm{ann}_A(\overline{Q})$:

```
> P1 := simplify((-diff(Qbar,s$2)+Qbar*ds^2));
```

$$P1:=-2+\left(\omega^2+s^2\right)ds^2$$

```
> Factorize(Matrix([P1]),AnnQbar,A);
```

$$\left[\begin{array}{ccc}1 & s & 0\end{array}\right]$$

We obtain that P_1 can be expressed as the first entry of the matrix *AnnQbar* plus s times the second entry of *AnnQbar*. Moreover, we can also check again that $P1\,\overline{Q}=0$:

```
> ApplyMatrix(P1,Qbar,A);
```

$$0$$

Next, we apply P_1 to s^2 `eq` to get the equation from which we can compute the initial conditions ϑ_0 and ϑ_1. Note that, as expected, this equation does not depend on the unknown constant γ.

```
> EQ1 := ApplyMatrix(P1,s^2*eq,A);
```

$$EQ1:=s^2\left(\omega^2+s^2\right)^2\frac{\mathrm{d}^2}{\mathrm{d}s^2}z\left(s\right)+\left(4\,\omega^4s+12\,\omega^2s^3+8\,s^5\right)\frac{\mathrm{d}}{\mathrm{d}s}z\left(s\right)+\left(2\,\omega^4+12\,\omega^2s^2+10\,s^4\right)z\left(s\right)$$
$$-4\,s^3\vartheta_0-6\,\omega^2s\vartheta_0-2\,\omega^2\vartheta_1$$

Differentiating 3 times (see Examples 4 and 5 for more details) this equation and solving for ϑ_0 we then get:

```
> vartheta0_s := normal(solve(diff(EQ1,s$3),vartheta[0]));
```

$$vartheta0\_s := 1/24\,\left(\frac{d^5}{ds^5}z(s)\right)\omega^4 s^2 + 1/12\,\left(\frac{d^5}{ds^5}z(s)\right)\omega^2 s^4 + 1/24\,\left(\frac{d^5}{ds^5}z(s)\right)s^6 + \frac{5\left(\frac{d^4}{ds^4}z(s)\right)\omega^4 s}{12}$$
$$+3/2\,\left(\frac{d^4}{ds^4}z(s)\right)\omega^2 s^3 + \frac{13\left(\frac{d^4}{ds^4}z(s)\right)s^5}{12} + 5/6\,\left(\frac{d^3}{ds^3}z(s)\right)\omega^4 + 8\,\left(\frac{d^3}{ds^3}z(s)\right)\omega^2 s^2$$
$$+\frac{55\,s^4\frac{d^3}{ds^3}z(s)}{6} + 14\,\left(\frac{d^2}{ds^2}z(s)\right)\omega^2 s + 30\,s^3\frac{d^2}{ds^2}z(s) + 6\,\left(\frac{d}{ds}z(s)\right)\omega^2 + 35\,\left(\frac{d}{ds}z(s)\right)s^2 + 10\,s\,z(s)$$

We then apply the command `ExplicitExpression` of the NonA package to obtain the following expression of ϑ_0 in the time-domain:

```
> vartheta0_t := ExplicitExpression(vartheta0_s,s);
```

$$vartheta0\_t := 30\,\frac{1}{t^6}\Big(\int_0^t -1/24\,\omega^4(t-\tau_1)^4\tau_1^5 z(\tau_1)\,d\tau_1 + \int_0^t -\omega^2(t-\tau_1)^2\tau_1^5 z(\tau_1)\,d\tau_1 + \int_0^t -\tau_1^5 z(\tau_1)\,d\tau_1$$
$$+\int_0^t 1/12\,\omega^4(t-\tau_1)^5\tau_1^4 z(\tau_1)\,d\tau_1 + \int_0^t 6\,\omega^2(t-\tau_1)^3\tau_1^4 z(\tau_1)\,d\tau_1 + \int_0^t (26\,t-26\,\tau_1)\tau_1^4 z(\tau_1)\,d\tau_1$$
$$+\int_0^t -1/36\,\omega^4(t-\tau_1)^6\tau_1^3 z(\tau_1)\,d\tau_1 + \int_0^t -8\,\omega^2(t-\tau_1)^4\tau_1^3 z(\tau_1)\,d\tau_1 + \int_0^t -110\,(t-\tau_1)^2\tau_1^3 z(\tau_1)\,d\tau_1 + \int_0^t \frac{14\,\omega^2(t-\tau_1)^5\tau_1^2 z(\tau_1)}{5}\,d\tau_1$$
$$+\int_0^t 120\,(t-\tau_1)^3\tau_1^2 z(\tau_1)\,d\tau_1 + \int_0^t -1/5\,\omega^2(t-\tau_1)^6\tau_1 z(\tau_1)\,d\tau_1 + \int_0^t -35\,(t-\tau_1)^4\tau_1 z(\tau_1)\,d\tau_1 + \int_0^t 2\,(t-\tau_1)^5 z(\tau_1)\,d\tau_1\Big)$$

Using this formula for ϑ_0, we can compute an expression for ϑ_1. We do not print the result here.

```
> vartheta1_s := normal(subs(vartheta[0]=vartheta0_s,solve(EQ1,vartheta[1]))):
> vartheta1_t := ExplicitExpression(vartheta1_s,s):
```

The above expressions involved the unknown coefficient ω^2 of the ODE satisfied by $x(t)$. We shall now estimate it. To achieve this task, we first apply ∂_s^4 to the equation s^2 `eq` (see Example 6):

```
> P3 := ds^4:
> EQ2 := ApplyMatrix(P3,s^2*eq,A);
```

$$EQ2 := \left(\frac{d^4}{ds^4}z(s)\right)\omega^2 s^2 + \left(\frac{d^4}{ds^4}z(s)\right)s^4 + 8\,\left(\frac{d^3}{ds^3}z(s)\right)\omega^2 s + 16\,\left(\frac{d^3}{ds^3}z(s)\right)s^3$$
$$+12\,\left(\frac{d^2}{ds^2}z(s)\right)\omega^2 + 72\,\left(\frac{d^2}{ds^2}z(s)\right)s^2 + 96\,s\frac{d}{ds}z(s) + 24\,z(s)$$

and we then solve the result for ω^2 to obtain:

```
> omega2_s := solve(EQ2,omega^2);
```

$$omega2\_s := -\frac{\left(\frac{d^4}{ds^4}z(s)\right)s^4 + 16\,\left(\frac{d^3}{ds^3}z(s)\right)s^3 + 72\,\left(\frac{d^2}{ds^2}z(s)\right)s^2 + 96\,s\frac{d}{ds}z(s) + 24\,z(s)}{\left(\frac{d^4}{ds^4}z(s)\right)s^2 + 8\,s\frac{d^3}{ds^3}z(s) + 12\,\frac{d^2}{ds^2}z(s)}$$

Then, we use again the `ExplicitExpression` command of the NonA package to obtain the following expression of ω^2 in the time-domain:

```
> omega2_t := ExplicitExpression(omega2_s,s);
```

$$omega2\_t :=$$
$$\frac{\int_0^t -\tau_1^4 z(\tau_1)\,d\tau_1 + \int_0^t -(-16\,t+16\,\tau_1)\tau_1^3 z(\tau_1)\,d\tau_1 + \int_0^t -36\,(t-\tau_1)^2\tau_1^2 z(\tau_1)\,d\tau_1 + \int_0^t 16\,(t-\tau_1)^3\tau_1 z(\tau_1)\,d\tau_1 + \int_0^t -(t-\tau_1)^4 z(\tau_1)\,d\tau_1}{\int_0^t 1/2\,(t-\tau_1)^2\tau_1^4 z(\tau_1)\,d\tau_1 + \int_0^t -4/3\,(t-\tau_1)^3\tau_1^3 z(\tau_1)\,d\tau_1 + \int_0^t 1/2\,(t-\tau_1)^4\tau_1^2 z(\tau_1)\,d\tau_1}$$

Finally, if we replace $z(t)$ by $x(t) = a\,\sin(\omega\,t+\phi)$ in the formulas obtained for ω^2, ϑ_0, and ϑ_1, then we can check that we find again ω^2, $a\,\sin(\phi)$, and $\omega\,a\,\cos(\phi)$ as expected:

```
> z := t -> a*sin(omega*t+phi):
> simplify(convert(omega2_t,int));
> simplify(convert(vartheta0_t,int));
> simplify(convert(vartheta1_t,int));
```

$$\omega^2$$
$$a\sin(\phi)$$
$$\omega\,a\cos(\phi)$$

5 Conclusion

In this paper, we have studied mathematical and computer algebra aspects of the algebraic estimation problem initiated in [Fliess et al. (2003)] and further developed in [Belkoura et al. (2009), Mboup (2009), Quadrat (2017), Ushirobira et al. (2016)] (see also the references therein). In particular, we have shown how the results obtained in [Chartouny et al. (2021)] on the inverse Cauchy problem for linear ODEs with polynomial coefficients can be extended to handle the case of a signal corrupted by a standard structured perturbation of the form $\gamma(t) = \gamma\, t^r\, H(t)$, where $r \in \mathbb{Z}_{\geq 0}$. To our knowledge, the results obtained in this paper are the first general ones that characterize the possibility to estimate the initial conditions and the constant parameters of a signal defined by a linear ODE with polynomial coefficients. These results are implemented in the NonA package − developed in `Maple` − dedicated to the effective study of the algebraic estimation problem.

In the future, we shall incorporate constant delays in the class of structured perturbations to cover the main perturbations classically considered in practice. To do that, we shall use methods and results developed in [Belkoura et al. (2009)]. Moreover, using the general results obtained in this paper based on closed-form solutions, we can now analyze different numerical aspects of the algebraic estimation method developed in [Fliess et al. (2003)] and study of the effects of noise that also corrupts the measurement. See [Fliess et al. (2003), Mboup (2009)] for a precise study of these important problems in the case of particular classes of signals.

Finally, based on the recent effective study of rings of ordinary integro-differential operators (see [Quadrat et al. (2020)] and the references therein) and its implementation in `Maple`, we want to develop a purely time-domain approach to the algebraic estimation problem. Within this new approach, the closed-form solutions for the initial conditions and the constant parameters defining the signal would be obtained directly in the time-domain, thus avoiding the transformations back and forth from time-domain to frequency-domain by means of the (inverse) Laplace transform.

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A Appendix: mathematical prerequisites

A.1 Laplace transform

The Laplace transform is an integral transform which has been proved to be an important tool for solving ordinary or partial differential equations.

Definition 2. The *Laplace transform*, denoted $\mathscr{L}$, maps a real-valued integrable Lebesgue measurable function f to a complex-valued function $\mathscr{L}(f)$ defined by:

$$\mathscr{L}\,(f)\,(s) := \int_0^{+\infty} e^{-st}\, f(t)\, dt.$$

We shall simply denote $\mathscr{L}\,(f)\,(s)$ *by* $\widehat{f}(s)$.

If we do not make any hypothesis on f, then the integral $\int_0^{+\infty} e^{-st}\, f(t)\, dt$ does not necessarily exist. In Definition 2, we assume that f is Lebesgue integrable but the Laplace transform can be extended to *temperate distributions* [Schwartz (1966)]. Hence, if W denotes the *Heaviside distribution* then $\mathscr{L}(W)(s) = s^{-1}$. Similarly, if we denote by δ the *Dirac distribution*, defined as the derivative of W in the sense of the theory of distributions, i.e., $\delta = \dot{W}$, then we have $\mathscr{L}(\delta) = 1$. See [Schwartz (1966)].

In the following table, where H denotes the Heaviside function defined by $H(t) = 1$ for $t \geq 0$ and 0 elsewhere, we recall a few standard examples of Laplace transforms.

function	Laplace transform
$H(t)$	$1/s$
$\cos(t)\, H(t)$	$s/\left(s^2+1\right)$
$\sin(t)\, H(t)$	$1/\left(s^2+1\right)$
$e^t\, H(t)$	$1/(s-1)$
$t^n\, H(t)$	$n!/s^{n+1}$

We following three classical results are used in the present paper.

Proposition 2. *Let f be a function. The Laplace transform satisfies the following identities:*

1) $\mathscr{L}\left(f^{(n)}\right)(s) = s^n\,\mathscr{L}(f)(s) - \sum_{i=0}^{n-1} s^{n-i-1}\, f^{(i)}(0)$, *where* $f^{(n)}$ *denotes the* n^{th} *derivative of* f,

2) $\mathscr{L}(t^n\, f(t))(s) = (-\partial_s)^n\,(\mathscr{L}(f)(s))$, *where* $\partial_s^n := \dfrac{d^n}{ds^n}$ *denotes the* n^{th} *derivative with respect to* s,

3) $\mathscr{L}(a_i(t) f(t))(s) = a_i(-\partial_s)\,\mathscr{L}(f)(s)$, *where the* a_i*'s are polynomials in* t.

Proposition 3. *Let* $(f \star g)(t) := \int_0^t f(t-\tau)\, g(\tau)\, d\tau$ *denotes the* convolution product *of a function* f *by a function* g. *The Laplace transform maps a convolution product to a product, i.e.,* $\mathscr{L}(f \star g) = \widehat{f}\,\widehat{g}$. *Conversely, the* inverse Laplace transform *maps a product to a* convolution, *i.e.,* $\mathscr{L}^{-1}\left(\widehat{f}\,\widehat{g}\right) = f \star g$.

A.2 Weyl algebra

Let k be a field of characteristic 0, $k[s]$ the commutative ring formed by all the polynomials in s with coefficients in k, and $\mathrm{End}_k(k[s])$ the ring formed by all the k-*endomorphisms* of $k[s]$, namely, $f \in \mathrm{End}_k(k[s])$ if f is a k-linear map from $k[s]$ to $k[s]$.

Definition 3 ([Coutinho (1995)])**.** Let A be the smallest k-sub-algebra of $\mathrm{End}_k(k[s])$ generated by the following two endomorphisms of $\mathrm{End}_k(k[s])$

$$\begin{array}{rcl} \partial_s : k[s] & \longrightarrow & k[s] \\ p & \longmapsto & \dfrac{dp}{ds}, \end{array} \qquad \begin{array}{rcl} s : k[s] & \longrightarrow & k[s] \\ p & \longmapsto & s\,p, \end{array}$$

which satisfy the identity $\partial_s \circ s = s \circ \partial_s + 1$ in $\mathrm{End}_k(k[s])$, where $\circ$ denotes the composition of endomorphisms and 1 is the identity endomorphism of $k[s]$. The k-algebra A is called the *first Weyl algebra*, also denoted by $A_1(k)$.

To simplify the notations, in what follows, we shall remove the composition symbol $\circ$. For instance, the above identity between endomorphisms of $k[s]$ will be written as $\partial_s\, s = s\,\partial_s + 1$.

Let us consider the *free associative* k*-algebra* $k\langle D, S\rangle$ formed by all finite k-linear combinations of *words* constructed with the two *letters* D and S (e.g., $S\,S\,D$ and $S\,D\,S$ are two different words) [Cohn (1971)]. Let $J = \langle D\,S - S\,D - 1\rangle \subset k\langle D, S\rangle$ be the two-sided ideal of $k\langle D, S\rangle$ generated by $D\,S - S\,D - 1$, namely, the set of elements defined as finite k-linear combinations of words of the form $W_1\,(D\,S - S\,D - 1)\,W_2$, where W_1 and W_2 are two words. If we note by s (resp., ∂) the *residue class* of S (resp., D) in $k\langle D, S\rangle/J$, then we have $k\langle D, S\rangle/J = k\langle \partial, s \,|\, \partial\, s = s\,\partial + 1\rangle = A$.

A.3 Gröbner bases

In the present paper, we merely use Gröbner bases methods to compute annihilators. For completeness, we briefly recall here the basics of Gröbner bases using the standard commutative setting, i.e., for the case of a polynomial ring in several commuting variables.

A.3.1 Gröbner bases for ideals over a commutative polynomial algebra

Let $x := x_1, \ldots, x_n$ be a collection of variables, $D := k[x]$ the ring of multivariate polynomials in $x_1, \ldots, x_n$ with coefficients in the field k, $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$.

Definition 4. A *monomial order* on D is any relation $\prec$ on the set $\{x^\alpha \mid \alpha \in \mathbb{N}^n\}$ satisfying:

1. $\prec$ is a total order on $\mathbb{N}^n$, i.e., all the elements of the set are comparable to each other,
2. $\prec$ is compatible with multiplication in D, i.e, if $\alpha \prec \beta$, then $\alpha + \gamma \prec \beta + \gamma$, for $\alpha, \beta, \gamma \in \mathbb{N}^n$,
3. $\prec$ is a well-ordering, i.e., any nonempty subset of $\mathbb{N}^n$ has a smaller element for $\prec$.

We implicitly set up an ordering on the variables x_i's in $k[x_1, ..., x_n]$: $x_n \prec x_{n-1} \prec \ldots \prec x_1$. With this choice, there are still many ways to define monomial orders. Some of the most important are given in the following example.

Example 7. 1. The *lexicographic order* on x-monomials is defined by $\alpha \prec_{\text{lex}} \beta$ whenever the first nonzero entry of $\beta - \alpha$ is positive. For instance, if we consider $\mathbb{Q}[x_1, x_2]$, then we have:

$$1 \prec_{\text{lex}} x_2 \prec_{\text{lex}} x_2^2 \prec_{\text{lex}} x_1 \prec_{\text{lex}} x_1 x_2 \prec_{\text{lex}} x_1^2.$$

2. The *degree reverse lexicographic order* on x-monomials is defined by $\alpha \prec_{\text{tdeg}} \beta$ whenever $|\alpha| \prec |\beta|$ or if we have $|\alpha| = |\beta|$, then the last nonzero entry of $\beta - \alpha$ is negative. For instance, if we consider $\mathbb{Q}[x_1, x_2]$, then we have:

$$1 \prec_{\text{tdeg}} x_2 \prec_{\text{tdeg}} x_1 \prec_{\text{tdeg}} x_2^2 \prec_{\text{tdeg}} x_1 x_2 \prec_{\text{tdeg}} x_1^2.$$

3. Let $y := y_1, \ldots, y_m$. Assume that an admissible monomial order $\prec_x$ (resp., $\prec_y$) on x-monomials (resp., on y-monomials) is given. An *elimination order* is then defined by:

$$u\,v \prec w\,t \iff u \prec_x w \ \text{ or } \ u = w \ \text{ and } v \prec_y t,$$

where u, w (resp., v, t) are x-monomials (resp., y-monomials). An elimination order serves to eliminate the x_i's. The elimination order, which will be used in what follows, is the one induced by the total degree orders on x-monomials and y-monomials. This is a very common order called *lexdeg*. For instance, if we consider $\mathbb{Q}[x_1, x_2, x_3]$, $\mathbf{x} = x_1, x_2$, $\mathbf{y} = x_3$, $\prec_X = \prec_{\text{tdeg}}$ and $\prec_Y = \prec_{\text{tdeg}}$, then we have:

$$1 \prec_{\text{lexdeg}} x_3 \prec_{\text{lexdeg}} x_3^2 \prec_{\text{lexdeg}} x_2 \prec_{\text{lexdeg}} x_2 x_3 \prec_{\text{lexdeg}} x_1 \prec_{\text{lexdeg}} x_1 x_3 \prec_{\text{lexdeg}} x_2^2 \prec_{\text{lexdeg}} x_1 x_2 \prec_{\text{lexdeg}} x_1^2.$$

Definition 5. Let $\prec$ be a monomial order of D and $P \in D$. We can then define:

1. The *leading monomial* $\text{lm}_\prec(P)$ of P to be the $\prec$-maximal monomial that appears in P with nonzero coefficient.
2. The *leading coefficient* $\text{lc}_\prec(P)$ of P to be the coefficient of $\text{lm}_\prec(P)$.
3. The *leading term* $\text{lt}_\prec(P)$ of P to be the product $\text{lc}_\prec(P)\,\text{lm}_\prec(P)$.

Definition 6. Fix a monomial order $\prec$ on D and let $I \subset D$ be an ideal. A *Gröbner basis* for I (with respect to $\prec$) is a finite collection of polynomials $G = \{Q_1, ..., Q_t\} \subset I$ with the property that for every nonzero $P \in I$, $lt(P)$ is divisible by $lt(Q_i)$ for some i.

Nowadays, Gröbner bases are widely use in computer algebra and in many applications. In particular, we dispose of Buchberger's algorithm to compute a Gröbner basis of an ideal. Note that the algorithm initially due to Buchberger has been well improved in the last decades. In `Maple`, we have an implementation in the `Groebner` package.

A.3.2 Gröbner bases for modules

Let us state again the definition of a module.

Definition 7. Let D be a commutative ring. A *D-module* M is an abelian group $(M, +)$ equipped with a scalar multiplication

$$\begin{array}{rcl} D \times M & \longrightarrow & M \\ (d,\, m) & \longmapsto & d\,m, \end{array}$$

which satisfies the following properties: for all $d_1, d_2 \in D$ and for all $m_1, m_2 \in M$:

1. $d_1\,(m_1 + m_2) = d_1\,m_1 + d_1\,m_2$,
2. $(d_1 + d_2)\,m_1 = d_1\,m_1 + d_2\,m_1$,

3. $(d_2\, d_1)\, m_1 = d_2\, (d_1\, m_1)$,
4. $1\, m_1 = m_1$.

Note that the definition of a D-module is similar to the one of a vector space but where the scalars belong to a ring D instead of a field.

Definition 8. A D-module M is said to be *finitely generated* if M admits a finite set of generators, namely there exists a finite set $S := \{m_i\}_{i=1,\ldots,r}$ of elements of M such that for every $m \in M$, there exist $d_i \in D$ for $i = 1, \ldots, r$ such that $m = \sum_{i=1}^{r} d_i\, m_i$. Then, S is called a *set of generators* of M.

In what follows, we consider D to be the polynomial Ore algebra $A = k[x_1, \ldots, x_n]$. Let $\operatorname{Mon}(A)$ be the set of monomials of A and $\{f_j\}_{j=1,\ldots,p}$ the *standard basis* of the free finitely generated left A-module $A^{1\times p} := \{(\lambda_1, \ldots, \lambda_p) \,|\, \lambda_i \in A,\ i = 1, \ldots, p\}$, namely the k^{th} component of f_j is 1 if $k = j$ and 0 otherwise. First, we can extend a monomial order $\prec$ from $\operatorname{Mon}(A)$ to the set of monomials of the form $u\, f_j$, where $u \in \operatorname{Mon}(A)$ and $j = 1, \ldots, p$. This extension is also denoted by $\prec$ and it has to satisfy the following two conditions:

1. $\forall\, w \in \operatorname{Mon}(A) :\ u\, f_i \prec v\, f_j \implies w\, u\, f_i \prec w\, v\, f_j$.
2. $u \prec v \implies u\, f_j \prec v\, f_j$ for $j = 1, \ldots, p$.

Without loss of generality, we let $f_p \prec f_{p-1} \prec \cdots \prec f_1$. There are two natural extensions of a monomial order to $\operatorname{Mon}(A^{1\times p})$.

Definition 9. Let $\prec$ be an admissible monomial order on $\operatorname{Mon}(A)$ and $\{f_j\}_{j=1,\ldots,p}$ the standard basis of the left A-module $A^{1\times p}$.

1. The *term over position order* on $\operatorname{Mon}(A^{1\times p})$ induced by $\prec$ is defined by:

$$u\, f_i \prec v\, f_j \iff u \prec v \text{ or } u = v \text{ and } f_i \prec f_j.$$

2. The *position over term order* on $\operatorname{Mon}(A^{1\times p})$ induced by $\prec$ is defined by:

$$u\, f_i \prec v\, f_j \iff f_i \prec f_j \text{ or } f_i = f_j \text{ and } u \prec v.$$

The term over position order is of more computational value with regard to efficiency. The position over term order can be used to eliminate components.

A.3.3 Computation of a left kernel

We now give an algorithm that computes the left kernel of a matrix $R \in A^{q\times p}$, namely, the set of row vectors $\lambda \in A^{1\times q}$ which are such that $\lambda\, R = 0$. Basically, the idea is to consider the inhomogeneous linear system $R\, \eta = \zeta$ to eliminate the η_i's from these equations, and to select the equations in the ζ_i's only. In other words, computing the left kernel of R is equivalent to computing a generating set of compatibility conditions for the inhomogeneous linear system $R\, \eta = \zeta$.

Algorithm: Computation of the left kernel of $R \in A^{q\times p}$, i.e., find $S \in A^{r\times q}$ such that:

$$\ker_A(.R) := \{\lambda \in A^{1\times q} \,|\, \lambda\, R = 0\} = A^{1\times r} S := \{\mu\, S \,|\, \mu \in A^{1\times r}\}.$$

- **Input:** $R \in A^{p\times q}$.
- **Output:** A matrix $S \in A^{r\times q}$ such that $\ker_A(.R) = A^{1\times r}\, S$.

1. Introduce the indeterminates $\eta_1, \ldots, \eta_p, \zeta_1, \ldots, \zeta_q$ over A and define the set:

$$P := \Big\{ \sum_{j=1}^{p} R_{ij}\, \eta_j - \zeta_i \mid i = 1, \ldots, q \Big\}.$$

2. Compute a Gröbner basis G of P in the free A-module generated by the η_j's and the ζ_i's for $j = 1, \ldots, p$ and $i = 1, \ldots, q$, namely, $\bigoplus_{j=1}^{p} A\eta_j \oplus \bigoplus_{i=1}^{q} A\,\zeta_i$ with respect to a term order which eliminates the η_j's.
3. Compute $G \cap \bigoplus_{i=1}^{q} A\,\zeta_i = \sum_{i=1}^{q} S_{ki}\,\zeta_i$, $k = 1, \ldots, r$ by selecting the elements of G containing only the ζ_i's, and return the matrix $S := (S_{ij}) \in A^{r\times q}$.

We finally illustrate how the above algorithm is used here to compute the annihilator of a polynomial over the Weyl algebra. We consider the first Weyl algebra $A := A_1(k) = k[s]\langle \partial_s \mid \partial_s\, s = s\,\partial_s + 1\rangle$ and characterize the *annihilator* of a polynomial $Q \in k[s]$, namely, the left ideal of A defined by:

$$\mathrm{ann}_A(.Q) := \{a \in A \mid a\,P = 0\}.$$

If $d := \deg_s(Q)$, then this annihilator can be obtained by considering the polynomial relations between the $d+1$ first derivatives of Q, i.e., by considering the following left kernel:

$$\ker_{k[s]}(.L) := \{\lambda := (\lambda_0\ \ldots\ \lambda_{d+1}) \in k[s]^{1\times(d+2)} \mid \lambda\, L = 0\}, \quad L := \begin{pmatrix} Q \\ Q^{(1)} \\ \vdots \\ Q^{(d+1)} \end{pmatrix} \in k[s]^{(d+2)\times 1}.$$

More precisely, if $\ker_{k[s]}(.L) = k[s]^{1\times r}\, S$ with $S \in k[s]^{r\times(d+2)}$ and $r \in \mathbb{N}$, then we have:

$$\mathrm{ann}_A(.Q) = S \begin{pmatrix} 1 \\ \partial_s \\ \vdots \\ \partial_s^{d+1} \end{pmatrix}.$$

Example 8. Let us compute the annihilator of the polynomial $Q(s) = \omega^2 + s^2 \in k[s]$, where $k = \mathbb{Q}[\omega]$. Then, we have $d = 2$. We consider the following matrix:

$$L := \begin{pmatrix} \omega^2 + s^2 \\ 2s \\ 2 \\ 0 \end{pmatrix} \in k[s]^{4\times 1}.$$

Using the algorithm described above, we can compute the left kernel $\ker_{k[s]}(.L)$. To do that, we first consider λ and $\mu = (\mu_1, \ldots, \mu_4)^T$ and write the entries of $L\,\lambda - \mu$, namely:

$$P := \{\left(\omega^2 + s^2\right)\,\lambda - \mu_1,\ 2\,s\,\lambda - \mu_2,\ 2\,\lambda - \mu_3,\ -\mu_4\}.$$

Computing a Gröbner basis for P with respect to a (lexdeg) term order which eliminates λ (we can consider $\mu_4 \prec \cdots \prec \mu_1 \prec \lambda$), we get $G := \{\mu_4,\ s\,\mu_3 - \mu_2,\ \omega^2\,\mu_3 + s\,\mu_2 - 2\,\mu_1,\ 2\,\lambda + \mu_3\}$. Then, in G, we can consider the elements which only contain the μ_i's, i.e., $\{\omega^2\,\mu_3 + s\mu_2 - 2\mu_1,\ s\,\mu_3 - \mu_2,\ \mu_4\}$. Hence, we have

$$\underbrace{\begin{pmatrix} -2 & s & \omega^2 & 0 \\ 0 & -1 & s & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{S} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{pmatrix} = \begin{pmatrix} \omega^2\,\mu_3 + s\mu_2 - 2\mu_1 \\ s\,\mu_3 - \mu_2 \\ \mu_4 \end{pmatrix},$$

which shows that $\ker_{k[s]}(.L) = k[s]^{1\times 3}\, S$. Finally, we obtain:

$$\mathrm{ann}_{A_1(k)}(.Q) = S \begin{pmatrix} 1 \\ \partial_s \\ \partial_s^2 \\ \partial_s^3 \end{pmatrix} = \begin{pmatrix} \omega^2\,\partial_s^2 + s\partial_s - 2 \\ s\,\partial_s^2 - \partial_s \\ \partial_s^3 \end{pmatrix}.$$

In `Maple`, we can compute the kernel of L using the `OreModules` package that contains the `SyzygyModule` command which computes this kernel as follows:

```
A := DefineOreAlgebra(diff=[ds,s],polynom=[s],comm=[omega]):
L := Vector[column](4, [omega^2 + s^2, 2*s, 2, 0]):
S:= SyzygyModule(L,A);
```

$$S = \begin{bmatrix} -2 & s & \omega^2 & 0 \\ 0 & -1 & s & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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