To be structured, or unstructured, fifty years of slings and arrows

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TO BE STRUCTURED, OR UNSTRUCTURED, FIFTY YEARS OF SLINGS AND ARROWS

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Abstract: This paper is something like the opposite of a review, telling the personal experience (nearly fifty years) of the author concerning a particular question, the unstructured mesh, well debated during these years.
Keywords: unstructured mesh, finite difference, finite element, finite volume

Introduction In an unstructured mesh, the number $N(i)$ of nodes neighboring a node i is not regularly defined from one node to the other one. Computing structures has lead soon to consider unstructured meshes. Try to mesh the Eiffel tower (in fact Sidney opera was one of the first famous success of unstructured meshes). The static computations allowed to introduce variational formulation and then the finite element method.

In the old time. When I started studying CFD/CSM, the two main enemies were finite differences/volumes on my right and finite elements on my left. FD/FV were applied on structured meshes, and FE on unstructured meshes[1]. Advised by Jean Céa and then recruited by Roland Glowinski, I was necessarily in the FE party. The prefered flow model in aeronautics was the full potential model, a mainly elliptic model and Roland's team and friends (involving Olivier Pironneau) in association with the Dassault team (around Jacques Périaux) had succeeded to compute the flow around a complete aircraft with a finite-element method and an unstructured mesh [6].

Euler flows. But in the eigthies, the challenge was to compute with the Euler model, a first step before (compressible) Navier-Stokes. The mathematical model is hyperbolic and its discretization needs the derivation of a sophisticated stabilization. Indeed,
- already for the convection of a scalar field, finite element computations were limited by Peclet effects (unstability when the diffusion term does not exists or is too small). The answer proposed by the FD/FV camp was "upwinding", that is managing in such a way that the concerned derivatives are not computed in a central manner around the node but upward with respect to the convection velocity. An important activity in finite elements developed in order to upwind this method.
- A second debate in compressible CFD was related to the building of explicit time advancing having a sufficient stability domain. One-step and two-step advancing were initially prefered, like the two-step method of Robert MacCormack [20], with a predictor step with upwinding in one mesh direction, a corrector step upwinded in the opposite direction. Two main advantages were identified. Firstly, such schemes need only one or two flux assemblies for a stable time

[1] FD=Finite Difference, FV=Finite Volume, FE=Finite Element.

step, which was interpreted as a chance for best efficiency, secondly, the schemes were variant of the Lax-Wendroff scheme [16] and enjoyed a built-in dissipation, satisfying to the dissipative Kreiss criterion [18]. The Lax-Wendroff FD method was then extended to P_1 FE independantly by three teams [3, 19, 9].

Lax-Wendroff type time-advancing became definitively unfashioned with the arising of the Jameson-Schmidt-Turkel scheme [15], derived first for structured meshes, then for unstructured ones [14].

A third debate concerned entropy: the satisfaction of a so-called entropy condition is sufficient for ensuring the uniqueness of solution of many nonlinear hyperbolic models. Conversely, the discretization of an hyperbolic model can produce parasitic non-entropic (non-unique) numerical solutions. The entropy issue was observed for full potential models, but it was soon clear that a simple unpwinding of the density permitted to ensure that entropic solutions were obtained. For the Euler equations, most time advancing methods were sufficiently dissipative to ensure entropic solutions.

The above events did not close the debate concerning upwinding. In the late fifties, Sergey Godunov pointed the interest of using Riemann problems solution for upwinding the approximation of nonlinear hyperbolic problems (he tells the story of this in [11]). The option was a little complex for routine computing and Philip Roe proposed his approximate Riemann solver [22] and Stan Osher with Sukumar Chakravarthy a more sophisticated and robust one [21].

First-order accurate upwind schemes were not sufficiently accurate, and second-order schemes not sufficiently stable for shock capturing (although artificial viscosity methods were not so bad). An important progress was carried by the introduction of limiters by Jay Boris and David Book [5]. Bram van Leer [17] brought a clever synthesis between Godunov upwinding and limiters, introducing also the idea of local reconstruction. An illuminating TVD theory was then proposed by Amiram Harten [12].

In a parallel way the stabilization of finite element for hyperbolic CFD models was addressed by Thomas Hughes and Michel Mallet [13].

Many of these ideas (together with implicit time advancing) were integrated in an unstructured method developped for hypersonic flows around the shapes considered in the European project Hermes -Figure 1- [23], project under the direction of Pierre Perrier which gave a favourable impulse to CFD researches in Europe.

Supercomputer is coming. Some remarks have to be made concerning the computational time. An interesting question is : in the 80's-90's, did computational aerodynamics reduce "wall clock calculation times" thanks to progress in *algorithms progress* more than thanks to *computer progress*. It is clear that in the 80's we enjoyed the arising of supercomputers like Cray 1 which was somewhat more expensive than mainframes, but more powerful. The arising of "micro killers" (IBM RS6000, etc) set the price of flop/s to a much lower level. The answer to the above question (probably in favor to computers) is difficult to establish since in particular many algorithms were designed for a better use of the pipelined number-crunchers. Meanwhile this evolution was in some manner favourable to structured ("my pipeline prefers structured").

Meanwhile the scientific community was debating on the physical limits to computer speed.

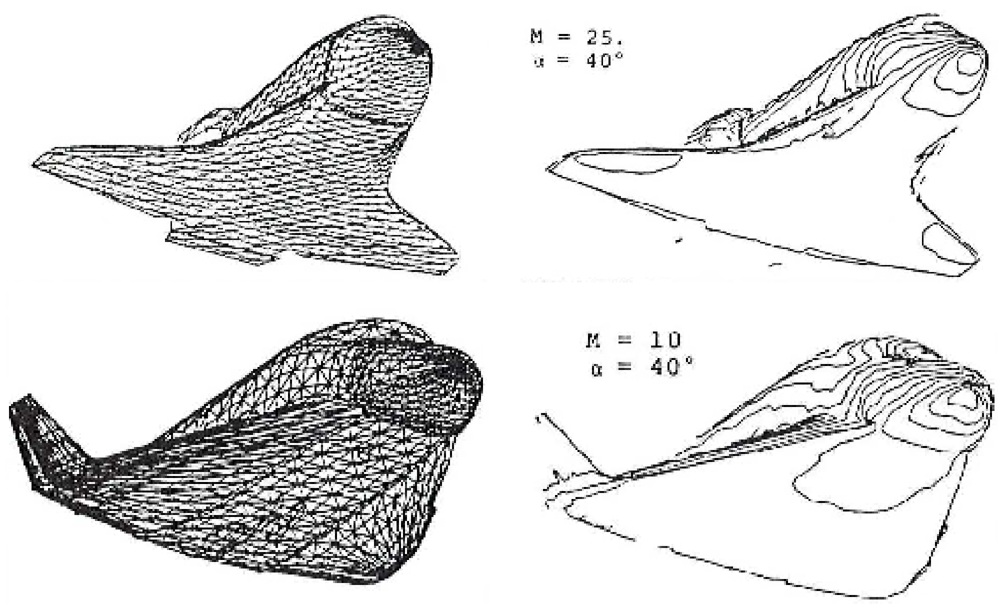

Figure 1: High Mach number high angle of attack Euler computations around the geometry of the spatial shuttle and a geometry of European space aircraft Hermes project. Meshes(skin) and Mach number, from [23].

Grand priests invoked the light velocity, atom size,... In short we were pushed towards the unique issue, parallelism.

Indeed first parallel computers appeared, taking so many forms (network, hypercubes, farms,...), but not so powerful at the beginning as vector computers continuing their progress and where prefered by non academic institutions. Some extra noise were brought by the expectation, starting already before pipelines, of the fifth generation of computers, specialized on Artificial Intelligence. A typical example was the development by the Thinking Machine Corporation of the Connection Machine which became unexpectly a much studied and appreciated number-cruncher.

But algorithms? But let us examine the progress of algorithms, focusing on solution algorithms. Two communities were particularly active.

The practitioners of Multi-Grid had a yearly office, the Copper Mountain MG conference, also good for skyiers. A variant, Full Multi-Grid was the Graal since O(N) complexity was attainable, superior to other methods, as far as the problem size N is sufficiently large. If you follow the advices of grand priest Achi Brandt. Extension to unstructured meshes was not a piece of cake and could not show the miraculous speed up of certain structured demonstrations.

The practitioners of Domain Decomposition Methods were also very active, they produce popular algorithms like Additive-Schwarz (e.g.[7]) or FETI [10], both being definitively adapted to systems coming from unstructured meshes.

Both methodologies give parallelisable algorithms. A common paradox is the problem posed by the coarse grid. In MG the coarse grid is too small to be efficiently solved on the so many processors reserved by the user for the fine grid problem. In DDM, the use of deflation to accelerate the global coupling also sets the question of an efficient solution of the coarse grid with one degree of freedom by processor [2].

Towards high order. In the old time there was first-order methods and higher order ones, *i.e.* second order. High order methods (order > 2) are also fascinating. A graal also was identified in Spectral methods with a convergence to continuous better than polynomial. However, the increasing interest to industrial issues lead to concentrate investigations on unstructured high order methods. The Discontinuous Galerkin (DG) for CFD was popularized by Bernardo Cockburn [8], Francesco Bassi and Stephano Rebay [4]. It should be noted that for many higher-order approximations, unstructuration involves in fact a regular substructure inside each element [2].

Mesh generation and adaptation. The issue of easy mesh generation was the main motivation for choosing the unstructured option. Indeed, after many works, the unstructured mesh generation became fast and robust (at least for tetrahedra). Then the unstructured option won the competition on the preprocessing, in terms of engineer time. But the important capabilities of unstructured mesh *adaptation* was early identified. An active competition in unstructured mesh adaptation started no later than in 1986 at the Reading Conference. Unstructured mesh adaptation took a strong acceleration with the use of mesh metrics, and goal oriented strategies. A recent impressing accomplishment [1] is the use of goal oriented metrics for computing high-lift flows around an aircraft geometry, Figure 2.

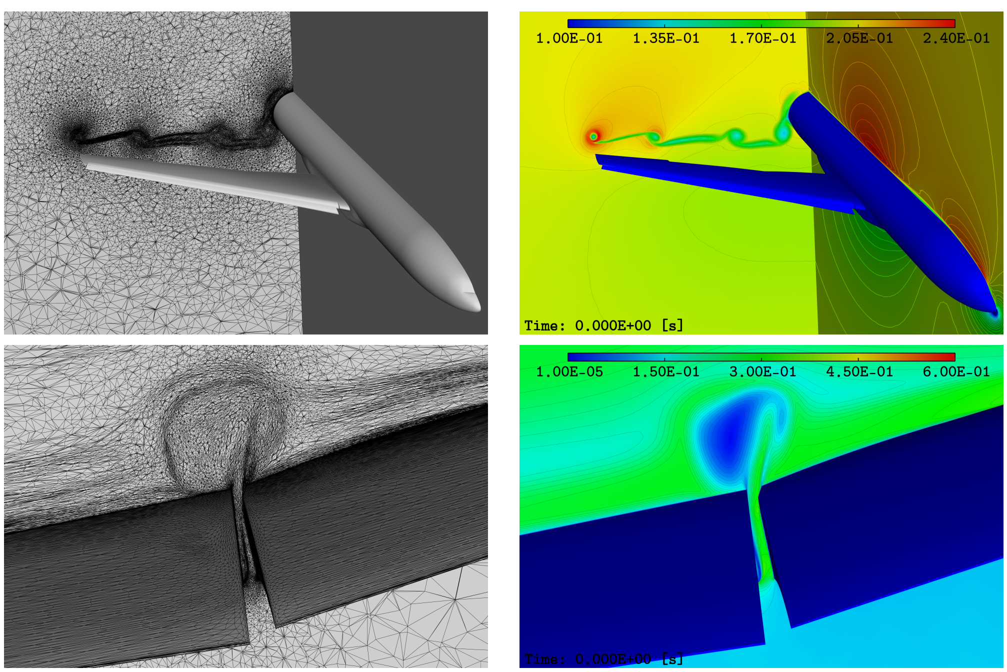

Figure 2: HL-CRM 16° high lift calculation (Courtesy of F. Alauzet). The Mach number values demonstrate the accurate capture of the wake, with a first vortex from wing tip, a second one from the gap between flaps, a third one starting where the wing is craked, and a fourth one from wing root. A zoom at flaps gap shows the initial capture of the second vortex [1].

Concluding remarks. The debate between unstructured and structured for compressible

Navier-Stokes is at least 40 years old. It has been more or less a debate about *when* unstructured would be definitively better than structured. But if we examine High-lift or drag reduction workshops, we observe that block-structured codes enjoy still higher performances for the same number of cells, and can then be used with higher numbers of cells. Regarding engineering time saving, however, almost all main software vendors in CFD have migrated towards unstructured.

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