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Abstract

Usually, singularly perturbed models are used to justify the decomposition of the interconnected systems into the Main Dynamics (MD) and the Parasitic Dynamics (PD). In this paper, the effect of a homogeneous PD on the stability of a homogeneous MD, when Homogeneity Degrees (HD) are possibly different, is studied via ISS approach in the framework of singular perturbations. Thus, the possibilities to reduce the order of the interconnected system considering only Reduced-Order Dynamics (ROD) and neglecting PD are examined. Proposed analysis discovers three kinds of stability in the behavior of such an interconnection by assuming that both, ROD and unforced PD, are Globally Asymptotically Stable (GAS). In the first case, when the HD of both systems coincide and the Singular Perturbation Parameter (SPP) is small enough, GAS of the interconnection can be concluded. In the second case, when the HD of ROD is greater, only local stability of interconnection can be ensured. Moreover, proposed approach allows to estimate the domain of attraction for the trajectories of the interconnection as a function of the SPP. In the third case, when the HD of ROD is smaller than the HD of PD, only practical stability can be concluded, and a kind of chattering phenomenon can arise when the HD of ROD is negative. Furthermore, the asymptotic bound of the system's trajectories is also estimated in terms of the SPP.

***Keywords*—** Homogeneity; Interconnected Systems; Singular Perturbations; Chattering Analysis.

1 Introduction

Homogeneous systems [1, 2, 3] constitute a subclass of nonlinear dynamics admitting special properties, e.g., scalability of solutions or different rates of convergence: rational (or nearly fixed-time), exponential and finite-time. Particularly, homogeneous controllers of negative degree are not Lipschitz continuous, possessing an infinite gain near to the origin and providing finite-time convergence [4, 5, 6, 7]. However, it is a well-known fact that the interconnection of a finite-time convergent dynamics and a linear Parasitic Dynamics (PD) produces the so-called chattering: a high-frequency oscillatory behavior of a system in steady-state. Consequently, the problems of chattering reduction and its analysis attract a lot of attention (see for example [8, 9, 10, 11, 12, 13, 14, 15]).

Commonly, singularly perturbed models are used to justify the decomposition of interconnected systems into the Main Dynamics (MD) and the PD (see the list of reference in [16]). For smooth singularly perturbed systems, methods of stability analysis are based on Klimushchev-Krasvoskii Theorem [17], where asymptotic stability of the interconnection is concluded from uniform exponential stability of the linearized systems. Relaxing the last assumption, [18] has addressed asymptotic and exponential stability by means of quadratic-type Lyapunov Functions

(LF's) and estimated the domain of attraction in terms of the upper bound of the Singular Perturbation Parameter (SPP).

In the framework of *Input-to-State Stability* (ISS), [19] has studied the properties of smooth singularly perturbed systems providing a kind of "total stability" under standard assumptions. The concept of ISS was introduced to analyze the stability of systems affected by external inputs, e.g., exogenous disturbances or measurement noises (see the list of references in [20]). The connection between Lyapunov stability and ISS has permitted the development of many stability concepts, which have been found to be very useful for the analysis and design of nonlinear control systems [21, 22]. For instance, the so-called *Small-Gain Theorem* provides a sufficient condition to guarantee the stability of interconnected systems [21, 23, 24, 25].

Nevertheless, despite the amount of publications on singular perturbations theory, those results do not cover the general case of homogeneous systems. The reasons for this include: the homogeneous systems can be non-smooth, the velocity of convergence of homogeneous systems is parametrized by their Homogeneity Degree (HD), the homogeneous systems of negative degree usually exhibit chattering in the presence of linear PD, and around the origin, the homogeneous systems of positive degree are slower than any exponentially converging dynamics but they are faster in the large.

In order to fill this gap, in [26] we have presented a study of the stability of the interconnection of a homogeneous MD and a linear-like homogeneous PD by means of concepts of singular perturbations, Lyapunov methods, ISS properties and Small-Gain Theorem. That analysis requires just continuity of the MD but continuous differentiability of the PD.

Now, in the present paper we provide a study of the stability of an interconnection where both the MD and the PD are nonlinear, continuous and homogeneous. The latter is an extension of our previous results to a more general class of systems by following the same methodology. The obtained results allow to conclude three kinds of stability for the interconnection according to the HD: Global asymptotic stability when the HD's coincide and the SPP is sufficiently small. Global asymptotic practical stability when the HD of the PD is greater than the HD of the MD (which also describes ROD), where the asymptotic bound of the trajectories is a monotonically increasing function of the SPP. Local asymptotic stability when the HD of the PD is smaller than the HD of the MD/ROD, where the size of the domain of attraction is a monotonically decreasing function of the SPP.

Moreover, this work provides a more complete, detailed and accurate proof of the main theorem than the previous version. In addition, some examples are presented with the aim to illustrate the different cases of the main theorem.

The rest of the paper has the following structure. In Section 2 some useful definitions and preparatory results are presented. Section 3 contains the problem statement and results about the stability of the interconnection of homogeneous systems affected by singular perturbations. In Section 4, some examples are provided in order to illustrate all above mentioned cases. Finally, the conclusions are presented in Section 5.

Notation

- $\mathbb{N}$ and $\mathbb{R}$ are the sets of natural and real numbers, respectively. Moreover, $\mathbb{R}_+$ represents the set of non-negative real numbers, i.e., $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$.

- $|\cdot|$ denotes the absolute value in $\mathbb{R}$, $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$.

- Expressions of the form $|\bullet|^{\gamma} sign(\bullet), \gamma \in \mathbb{R}$ are written as $\lceil \bullet \rfloor^{\gamma}$.

- A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}$ if it is continuous, strictly increasing and $\alpha(0) = 0$. The function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and it is unbounded. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{KL}$ if, for each fixed $t \in \mathbb{R}_+$, $\beta(\cdot, t) \in \mathcal{K}$ and, for each fixed $s \in \mathbb{R}_+$, $\beta(s, \cdot)$ is non-increasing and it tends to zero for $t \to \infty$.

- The space $\mathcal{L}_\infty^m$ is defined as the set of measurable essentially bounded functions $u : [0, \infty) \to \mathbb{R}^m$, such that,

$$\|u\|_{\mathcal{L}_\infty} = \mathit{ess}\sup_{t \geq 0} |u(t)| < \infty.$$

- $s(A)$ represents the eigenvalues of a symmetric matrix $A \in \mathbb{R}^{n \times n}$. Also, $s_{\min}(A)$ and $s_{\max}(A)$ depict the minimum and the maximum eigenvalue of A, respectively.

2 Preliminaries

2.1 Weighted homogeneity

The presentation of this subsection follows [1, 27, 3]. For real numbers $r_i > 0 \ \ (i = 1, \ldots, n)$ called weights and $\lambda > 0$, one can define

- the vector of weights $r = (r_1, \ldots, r_n)^T$, $r_{\max} = \max_{1 \leq j \leq n} r_j$ and $r_{\min} = \min_{1 \leq j \leq n} r_j$;
- the dilation matrix function $\Lambda_r(\lambda) = \mathrm{diag}(\lambda^{r_i})_{i=1}^n$, such that, for all $x \in \mathbb{R}^n$ and for all $\lambda > 0$, $\Lambda_r(\lambda)x = (\lambda^{r_1}x_1, \ldots, \lambda^{r_i}x_i, \ldots, \lambda^{r_n}x_n)^T$ (through the paper the dilation matrix is represented by Λ_r wherever λ can be omitted);
- the r-homogeneous norm of $x \in \mathbb{R}^n$ is given by $||x||_r = \left(\sum_{i=1}^n |x_i|^{\frac{\rho}{r_i}}\right)^{\frac{1}{\rho}}$ for $\rho \geq r_{\max}$ (it is not a norm in the usual sense, since it does not satisfy the triangle inequality);
- for $s > 0$, the sphere and the ball in the homogeneous norm are defined as $S_r(s) = \{x \in \mathbb{R}^n : ||x||_r = s\}$ and $B_r(s) = \{x \in \mathbb{R}^n : ||x||_r \leq s\}$, respectively.

Definition 1 *A function $\phi : \mathbb{R}^n \to \mathbb{R}$ is r-homogeneous with a degree $\mu \in \mathbb{R}$ if for all $\lambda > 0$ and all $x \in \mathbb{R}^n$:*

$$\phi(\Lambda_r(\lambda)x) = \lambda^\mu \phi(x). \tag{1}$$

A vector field $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is r-homogeneous with a degree $\nu \geq -r_{\min}$ if for all $x \in \mathbb{R}^n$ and all $\lambda > 0$,

$$\Phi(\Lambda_r(\lambda)x) = \lambda^\nu \Lambda_r(\lambda)\Phi(x). \tag{2}$$

The system $\dot{x} = \Phi(x)$ is r-homogeneous of degree ν if the vector field Φ satisfies the property (2).

By its definition, $|| \cdot ||_r$ is a r-homogeneous function of degree 1. Moreover, there exists $\underline{\sigma}, \bar{\sigma} \in \mathcal{K}_\infty$, such that,

$$\underline{\sigma}(||x||_r) \leq ||x|| \leq \bar{\sigma}(||x||_r) \qquad \forall x \in \mathbb{R}^n, \tag{3}$$

i.e., there is an equivalence between the norms $|| \cdot ||$ and $|| \cdot ||_r$. Moreover, for $r_{max} = 1$, $|| \cdot ||_r$ is locally Lipschitz continuous (see [2, 1] for more details about homogeneous norms).

2.2 Input-to-state stability

Consider a system

$$\dot{x} = f(x, u), \tag{4}$$

where $x \in \mathbb{R}^n$ is the state vector and $u \in \mathbb{R}^m$ is an input. In addition, $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$ ensures forward existence and uniqueness of the system solutions at least locally in time, and $f(0, 0) = 0$. The next definitions and theorems were introduced by [27, 21, 24].

Definition 2 *The system (4) is said to be input-to-state practically stable (ISpS), if there exist a class $\mathcal{KL}$ function β, a class $\mathcal{K}$ function γ and a constant $c \geq 0$, such that, for any $u \in \mathcal{L}_\infty^m$ and any $x_0 \in \mathbb{R}^n$, the solution $x(t)$ with initial condition $x(0) = x_0$ satisfies*

$$\|x(t)\| \leq \max\{\beta(\|x_0\|, t), \gamma(\|u\|_{\mathcal{L}_\infty}), c\} \tag{5}$$

for all $t \geq 0$. The function γ is called nonlinear asymptotic gain. The system (4) is called ISS if $c = 0$.

If $u(t) = 0$ for all $t \geq 0$, then an ISpS system (4) is called practically GAS (and an ISS system (4) is called GAS); if the estimate (5) is satisfied just for a bounded set of initial conditions x_0, then such a system is called locally ISpS (locally ISS, respectively) [1].

Definition 3 *A smooth function $V : \mathbb{R}^n \to \mathbb{R}$ is called the ISpS-LF for the system (4) if there exist some $c \geq 0$, $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ and $\chi \in \mathcal{K}$, such that, for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$*

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \tag{6}$$

and

$$\|x\| \geq \chi(\max\{\|u\|, c\}) \implies \tfrac{\partial V}{\partial x} f(x, u) \leq -\alpha_3(\|x\|) \tag{7}$$

hold. Moreover, if $c = 0$ then V is called an ISS-LF for the system (4).

Remark 1 *The function $\gamma(\cdot)$ in (5) can be computed from the functions $\alpha_1(\cdot), \alpha_2(\cdot)$ and $\chi(\cdot)$. It is given by*

$$\gamma(s) = \alpha_1^{-1} \circ \alpha_2 \circ \chi(s). \tag{8}$$

Theorem 1 *The system (4) is ISS (ISpS) if it admits an ISS (ISpS) LF.*

2.3 Input-to-state stability of interconnected systems

Consider a system

$$\dot{x} = f(x, y), \qquad \dot{y} = g(x, y, u), \tag{9}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $u \in \mathbb{R}^p$ and the origin $x = y = u = 0$ is an equilibrium point. Assume that both systems are ISpS w.r.t. their corresponding inputs. Therefore, from condition (5) for all $t \geq 0$:

$$\begin{aligned} \|x(t)\| &\leq \max\{\beta_1(\|x_0\|, t), \gamma_1(\|y\|_{\mathcal{L}_\infty}), c_1\}, \\ \|y(t)\| &\leq \max\{\beta_2(\|y_0\|, t), \gamma_2(\|x\|_{\mathcal{L}_\infty}), \gamma_3(\|u\|_{\mathcal{L}_\infty}), c_2\}, \end{aligned}$$

where $x_0 \in \mathbb{R}^n$, and $y_0 \in \mathbb{R}^m$ are the initial conditions for each variable, $u \in \mathcal{L}_\infty^p$, $\beta_1, \beta_2 \in \mathcal{KL}$ and $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{K}$ are some functions from the indicated classes.

Sufficient conditions for ISpS stability of the interconnected system (9) can be found in [23] as follows.

Theorem 2 *Let each subsystem in (9) be ISpS. If there exists some $c_0 \geq 0$, such that,*

$$\gamma_1 \circ \gamma_2(s) < s, \quad \text{for all} \quad s > c_0, \tag{10}$$

then the interconnected system (9) is ISpS. Furthermore, if $c_0 = c_1 = c_2 = 0$ then the system is ISS.

The inequality (10) is commonly refered as the *small-gain condition*. Roughly speaking, the Small-Gain Theorem establishes that the interconnected system (9) is ISS, if the composite function $\gamma_1 \circ \gamma_2(\cdot)$ is a simple contraction.

The counterpart of Theorem 2 for locally ISS systems was presented by [28]. In this case, the small-gain condition (10) is replaced by

$$\gamma_1 \circ \gamma_2(s) < s, \tag{11}$$

for all $0 < s \leq c_3$, and some $c_3 > 0$. In this case, local ISS of the interconnected system (9) is concluded.

3 Stability analysis of interconnected homogeneous systems affected by singular perturbations

Consider an interconnected system

$$\dot{x} = f(x, y), \tag{12}$$
$$\epsilon\dot{y} = g(x, y), \tag{13}$$

where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ are the state variables, $\epsilon > 0$ is a small parameter (also called the SPP), $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$ and $g : \mathbb{R}^{n+m} \to \mathbb{R}^m$ are continuous vector fields ensuring forward existence and uniqueness of system trajectories. Moreover, for some vectors of weights r and $\tilde{r}$ the vector fields f and g are $(r, \tilde{r})$-homogeneous of degrees ν and μ, respectively, i.e.,

$$f(x, y) = \lambda^{-\nu}\Lambda_r^{-1}(\lambda)f(\Lambda_r(\lambda)x, \Lambda_{\tilde{r}}(\lambda)y) \tag{14}$$
$$g(x, y) = \lambda^{-\mu}\Lambda_{\tilde{r}}^{-1}(\lambda)g(\Lambda_r(\lambda)x, \Lambda_{\tilde{r}}(\lambda)y). \tag{15}$$

Hence, note that $f(0, 0) = 0$ and $g(0, 0) = 0$. Through this paper, the system (12) is called the MD and the system (13) is called the PD.

In the following, three different problems will be addressed: analysis of the stability of the interconnected systems (12)-(13) based on their HD's, estimation of the ultimate bound for the system trajectories, and evaluation of the domain of attraction for the system trajectories.

The most of existing results require sufficient smoothness of the vector fields involved in the analysis (see [16] and references therein). However, in the general case of homogeneous systems such a condition is quite conservative. Thus, in this study just continuity of the vector fields $f,\ g$ is assumed. Moreover, let us introduce the hypotheses for this research.

Assumption 1 *Consider the system (12)-(13):*

1. *The equation $g(\bar{x}, \bar{y}) = 0$ has an isolated and continuously differentiable solution $\bar{y} = h(\bar{x})$.*
2. *The ROD*

$$\dot{\bar{x}} = f(\bar{x}, h(\bar{x})), \tag{16}$$

is GAS at the origin.

3. *Defining $z = y - h(\bar{x})$, the Boundary-Layer (BL) dynamics*

$$\tfrac{dz}{d\tau} = g(\bar{x}, z + h(\bar{x}));\ \ \tau = \tfrac{t}{\epsilon}, \tag{17}$$

is GAS at the origin, uniformly w.r.t. $\bar{x}$. Moreover, there exists a continuous differentiable $(\tilde{r}, r)$-homogeneous LF $W : \mathbb{R}^{n+m} \to \mathbb{R}_+$ satisfying

$$W(\Lambda_{\tilde{r}}(\lambda)z, \Lambda_r(\lambda)\bar{x}) = \lambda^{\iota}W(z, \bar{x}), \tag{18}$$
$$\underline{a}_z\|z\|_{\tilde{r}}^{\iota} \le W(z, \bar{x}) \le \bar{a}_z\|z\|_{\tilde{r}}^{\iota}, \tag{19}$$
$$\tfrac{\partial W(z,\bar{x})}{\partial z}g(\bar{x}, z + h(\bar{x})) \le -b_z\|z\|_{\tilde{r}}^{\mu+\iota}, \tag{20}$$

for all $z \in \mathbb{R}^m$, $\bar{x} \in \mathbb{R}^n$ and for some $\underline{a}_z, \overline{a}_z, b_z > 0$, where $\iota > \max\{0, -\mu\}$ is the HD of W.

As a consequence of (15), the function h admits certain symmetry, i.e., $h(\Lambda_r(\lambda)x) = \Lambda_{\tilde{r}}(\lambda)h(x)$, and consequently, $h(0) = 0$. This implies that the ROD (16) is r-homogeneous of a degree ν, i.e., the ROD inherits the homogeneity properties of the MD.

Moreover, by Assumption 1, the ROD (16) is GAS at the origin and r- homogeneous of degree ν, hence there exists a continuously differentiable LF $V : \mathbb{R}^n \to \mathbb{R}_+$ which satisfies [29]

$$V(x) = \lambda^{-\kappa} V(\Lambda_r(\lambda)x), \tag{21}$$

$$\underline{a}_x \|x\|_r^\kappa \leq V(x) \leq \bar{a}_x \|x\|_r^\kappa, \tag{22}$$

$$\frac{\partial V(x)}{\partial x} f(x, h(x)) \leq -b_x \|x\|_r^{\nu+\kappa}, \tag{23}$$

$$\sup_{x \in B_r(1)} \|\tfrac{\partial V(x)}{\partial x}\| \leq c_x, \tag{24}$$

for all $x \in \mathbb{R}^n$ and for some $\underline{a}_x, \bar{a}_x, b_x, c_x > 0$, where $\kappa > \max\{0, -\nu\}$ is the HD of V. On the other hand, W is continuous differentiable, hence, the inequalities

$$\sup_{\substack{\zeta \in B_r(1) \\ \xi \in B_r(1)}} \left\| \frac{\partial W(\zeta,\xi)}{\partial \zeta} \right\| \leq c_z, \text{ and } \sup_{\substack{\zeta \in B_r(1) \\ \xi \in B_r(1)}} \left\| \frac{\partial W(\zeta,\xi)}{\partial \xi} \right\| \leq d_z, \tag{25}$$

hold, for some $c_z, d_z > 0$.

Furthermore, by the continuity of $f(x, y)$, for any b_x, c_x and $0 < \theta < 1$, there exists $\delta > 0$, such that, if $\|\Lambda_{\tilde{r}}(\|x\|_r^{-1})z\|_{\tilde{r}} \leq \delta$, or just $\|z\|_{\tilde{r}} \leq \delta \|x\|_r$, then

$$\|f(\xi, h(\xi) + \Lambda_{\tilde{r}}(\|x\|_r^{-1})z) - f(\xi, h(\xi))\| \leq \frac{\theta b_x}{c_x}, \tag{26}$$

for all $\xi \in S_r(1)$, also denote

$$\eta = \frac{1}{b_z} \sup_{\substack{\xi \in B_r(1) \\ \zeta \in B_{\tilde{r}}(1)}} \left\{ \left(d_z + c_z \left\| \frac{\partial h(\xi)}{\partial \xi} \right\| \right) \|f(\xi, \zeta + h(\xi))\| \right\}. \tag{27}$$

Under Assumption 1, the next theorem presents the main result of this paper:

Theorem 3 *Let the system (12)-(13) satisfy Assumption 1. There exist $1 > \tilde{\theta}_1, \tilde{\theta}_2 > 0$, such that, the interconnected system (12)-(13) is*

- *Globally asymptotically stable for $\mu = \nu$ and*

$$\epsilon < \min \left\{ \frac{\tilde{\theta}_1}{\eta}, \frac{\tilde{\theta}_2 \left(\delta \frac{\underline{a}_x \underline{a}_z}{\bar{a}_x \bar{a}_z} \right)^{\mu+\iota}}{\eta} \right\}. \tag{28}$$

- *Locally asymptotically stable for $\mu < \nu$ and all initial condition*

$$\|x_0\|_r < \left(\frac{\tilde{\theta}_2 \left(\delta \frac{\underline{a}_x \underline{a}_z}{\bar{a}_x \bar{a}_z} \right)^{\mu+\iota}}{\epsilon \eta} \right)^{\frac{1}{\nu-\mu}} \tag{29}$$

and

$$\|y_0 - h(x_0)\|_{\tilde{r}} < \min \left\{ \left(\frac{\tilde{\theta}_1}{\epsilon \eta} \right)^{\frac{1}{\nu-\mu}}, \left(\frac{\tilde{\theta}_2 \left(\frac{\underline{a}_z}{\bar{a}_z} \right)^{\mu+\iota}}{\epsilon \eta \left(\delta^{-1} \frac{\bar{a}_x}{\underline{a}_x} \right)^{\nu+\iota}} \right)^{\frac{1}{\nu-\mu}} \right\}. \tag{30}$$

- *Globally asymptotically practically stable for $\mu > \nu$, with asymptotic bounds*

$$\|x(t)\|_r < \left(\frac{\epsilon\eta}{\tilde{\theta}_2 \left(\delta \frac{\underline{a}_x \underline{a}_z}{\bar{a}_x \bar{a}_z} \right)^{\mu+\iota}} \right)^{\frac{1}{\mu-\nu}} \tag{31}$$

and

$$\|y(t) - h(x(t))\|_{\tilde{r}} \leq \max \left\{ \left(\frac{\epsilon\eta}{\tilde{\theta}_1} \right)^{\frac{1}{\mu-\nu}}, \left(\frac{\epsilon\eta \left(\delta^{-1} \frac{\bar{a}_x}{\underline{a}_x} \right)^{\nu+\iota}}{\tilde{\theta}_2 \left(\frac{\underline{a}_z}{\bar{a}_z} \right)^{\mu+\iota}} \right)^{\frac{1}{\mu-\nu}} \right\}. \tag{32}$$

for all $t > T$, with some $T > 0$, and all $x_0 \in \mathbb{R}^n$ and $y_0 \in \mathbb{R}^m$.

For $\nu = \mu$, there exists a critical value ϵ^*, such that, the stability of the system (12)-(13) is ensured for $\epsilon < \epsilon^*$. In this case, the value of ϵ determines the amount of deviations for the trajectories of the system (12)-(13) from the trajectories of the ROD (16). Moreover, this case illustrates the concept of motion separation predicted by classical results on smooth (at least Lipschitz continuous) singularly perturbed systems [16], and it allows to ensure GAS.

For $\mu \neq \nu$, the system (12)-(13) always possesses some kind of stability (local or practical), and by decreasing the value of ϵ it is possible to enlarge the domain of attraction for $\mu < \nu$, or to decrease the size of the asymptotic bounds for $\mu > \nu$. Systems with a smaller HD are faster around the origin and slower in the large, and vice versa. Therefore, the motion separation follows the relations of HD's (contrarily the standard theory of singular perturbations or the case $\nu = \mu$, where it is regulated by SPP) and it only appears near to the origin for $\nu > \mu$ and outside of a neighborhood of the origin for $\nu < \mu$.

3.1 Proof of Theorem 3

Under a hypothesis like Assumption 1, the conventional singular perturbation theory aims to show that the behavior of the system (12)-(13) is pretty similar to the behavior of the ROD (16), such that, the presence of the PD can be neglected (see [16]). Thus, the stability of the system (12)-(13) is guaranteed by proving that the trajectories y of the PD converge to the equilibrium manifold $h(x)$. However, since the initial condition y_0 differs from the initial value $h(x_0)$, there exists a transitory response of the system (13) before it can reach the desired behavior. Let us represent such a transitory by the variable z, such that, $y = z + h(x)$ and the system (12) is rewritten as

$$\dot{x} = f(x, z + h(x)). \tag{33}$$

It can be readily seen that if $z = 0$, the system (33) collapse to the ROD. Therefore, (global, local or practical) asymptotic stability of the origin of the system (33) can be concluded from ISS stability of the system (33) w.r.t. z, plus (global, local or practical) vanishing of z to zero.

Let us take V, described in (21)-(24), as an ISS-LF candidate for the system (33). The derivative of V along the trajectories of the system (33) is given by

$$\begin{aligned}\dot{V} &= \tfrac{\partial V(x)}{\partial x} f(x, z + h(x)) \\ &= \tfrac{\partial V(x)}{\partial x} f(x, h(x)) + \tfrac{\partial V(x)}{\partial x} \left(f(x, z + h(x)) - f(x, h(x)) \right).\end{aligned}$$

From (14) and (21), by means of the dilations $\Lambda_r(\lambda)$ and $\Lambda_{\tilde{r}}(\lambda)$ where $\lambda = \|x\|_r^{-1}$, it is obtained

$$\begin{aligned}\dot{V} &= \tfrac{\partial V(x)}{\partial x} f(x, h(x)) + \|x\|_r^{\nu+\kappa} \tfrac{\partial V(\xi)}{\partial \xi} \left(f(\xi, \Lambda_{\tilde{r}}(\|x\|_r^{-1}) z + h(\xi)) - f(\xi, h(\xi)\right) \\ &\leq \tfrac{\partial V(x)}{\partial x} f(x, h(x)) + \|x\|_r^{\nu+\kappa} \left\| \tfrac{\partial V(\xi)}{\partial \xi} \right\| \left\| f(\xi, \Lambda_{\tilde{r}}(\|x\|_r^{-1}) z + h(\xi)) - f(\xi, h(\xi) \right\|,\end{aligned} \tag{34}$$

where $\xi = \Lambda_r(\|x\|_r^{-1})x$ and $\xi \in S_r(1)$. Therefore, substituting (23), (24) and (26) in (34), it can be concluded that

$$\begin{aligned}\dot{V} \leq & -b_x\|x\|_r^{\nu+\kappa} + \theta b_x\|x\|_r^{\nu+\kappa} \\ \leq & -(1-\theta)b_x\|x\|_r^{\nu+\kappa}, \quad \text{if} \quad \|x\|_r \geq \delta^{-1}\|z\|_{\tilde{r}},\end{aligned} \tag{35}$$

where $0 < \theta < 1$. According to Definition 3, V is an ISS-LF for the system (33) hence it is ISS w.r.t. input z. Moreover, from Definition 2, the solution x of the system (33) satisfies

$$\|x(t)\|_r \leq \max\{\beta_1(\|x_0\|_r, t), \gamma_1(\sup_{\tau\in[0,t)} \|z(\tau)\|_{\tilde{r}})\}, \tag{36}$$

for all $t \geq 0$, where β_1 is a $\mathcal{KL}$ function and, from Definition 3, Remark 1 and inequalities (22) and (35), γ_1 is a $\mathcal{K}$ function given by

$$\gamma_1(s) = \delta^{-1}\frac{\bar{a}_x}{\underline{a}_x}s. \tag{37}$$

Now, let's investigate the scenarios where vanishing of the transitory z to zero can be guarantied, such that, the convergence of the trajectories y to the desired behavior $h(x)$ can be concluded. The dynamics of the variable z is given by.

$$\dot{z} = \tfrac{1}{\epsilon}g(x, z + h(x)) - \tfrac{\partial h(x)}{\partial x}f(x, z + h(x)), \tag{38}$$

where x can be seen as an input.

The stability of the system (38) can be investigated by using W, described in (18)-(20), and (25), as an ISS-LF candidate. So, the derivative of W along the trajectories of the system (38) is given by

$$\begin{aligned}\dot{W} =& \tfrac{\partial W(z,x)}{\partial z}\left(\tfrac{1}{\epsilon}g(x, z+h(x)) - \tfrac{\partial h(x)}{\partial x}f(x, z+h(x))\right) + \tfrac{\partial W(z,x)}{\partial x}f(x, z+h(x)) \\ =& \tfrac{1}{\epsilon}\tfrac{\partial W(z,x)}{\partial z}g(x, z+h(x)) + \left(\tfrac{\partial W(z,x)}{\partial x} - \tfrac{\partial W(z,x)}{\partial z}\tfrac{\partial h(x)}{\partial x}\right)f(x, z+h(x)).\end{aligned} \tag{39}$$

By homogeneity of each component in (39), applying the dilations $\Lambda_r(\lambda^{-1})$ and $\Lambda_{\tilde{r}}(\lambda^{-1})$, where $\lambda = \max\{\|z\|_{\tilde{r}}, \|x\|_r\}$, we obtain:

$$\begin{aligned}\dot{W} =& \tfrac{1}{\epsilon}\tfrac{\partial W(z,x)}{\partial z}g(x, z+h(x)) + \lambda^{\nu+\iota}\left(\tfrac{\partial W(\zeta,\xi)}{\partial \xi} - \tfrac{\partial W(\zeta,\xi)}{\partial \zeta}\tfrac{\partial h(\xi)}{\partial \xi}\right)f(\xi, \zeta+h(\xi)) \\ \leq& \tfrac{1}{\epsilon}\tfrac{\partial W(z,x)}{\partial z}g(x, z+h(x)) + \lambda^{\nu+\iota}\left(\left\|\tfrac{\partial W(\zeta,\xi)}{\partial \xi}\right\| + \left\|\tfrac{\partial W(\zeta,\xi)}{\partial \zeta}\right\|\left\|\tfrac{\partial h(\xi)}{\partial \xi}\right\|\right)\|f(\xi, \zeta+h(\xi))\|,\end{aligned} \tag{40}$$

where $\xi = \Lambda_r^{-1}(\lambda)x$ and $\zeta = \Lambda_{\tilde{r}}^{-1}(\lambda)z$ (*i.e.*, $\xi \in B_r(1)$ and $\zeta \in B_{\tilde{r}}(1)$). Now, substituting (20), and (25) in (40),

$$\dot{W} \leq -\tfrac{b_z}{\epsilon}\|z\|_{\tilde{r}}^{\mu+\iota} + b_z\eta\lambda^{\nu+\iota},$$

where η is given in (27). Then, for any $0 < \tilde{\theta} < 1$, it is obtained that

$$\dot{W} \leq -(1-\tilde{\theta})\tfrac{b_z}{\epsilon}\|z\|_{\tilde{r}}^{\mu+\iota} - \tilde{\theta}\tfrac{b_z}{\epsilon}\|z\|_{\tilde{r}}^{\mu+\iota} + b_z\eta\max\{\|z\|_{\tilde{r}}^{\nu+\iota}, \|x\|_r^{\nu+\iota}\}.$$

Since $\max\{a, b\} \leq a + b$ for any $a, b \in \mathbb{R}_+$, the last expression can be rewritten as

$$\dot{W} \leq -(1-\tilde{\theta})\tfrac{b_z}{\epsilon}\|z\|_{\tilde{r}}^{\mu+\iota} - \tfrac{(\tilde{\theta}_1+\tilde{\theta}_2)b_z}{\epsilon}\|z\|_{\tilde{r}}^{\mu+\iota} + b_z\eta\|z\|_{\tilde{r}}^{\nu+\iota} + b_z\eta\|x\|_r^{\nu+\iota},$$

where $\tilde{\theta}_1, \tilde{\theta}_2 > 0$ are such that $\tilde{\theta}_1 + \tilde{\theta}_2 = \tilde{\theta}$. Hence, if

$$\|z\|_{\tilde{r}}^{\mu+\iota} \geq \tfrac{\eta\epsilon}{\tilde{\theta}_2}\|x\|_r^{\nu+\iota}, \tag{41}$$

then

$$\dot{W} \leq -(1-\tilde{\theta})\tfrac{b_z}{\epsilon}\|z\|_{\tilde{r}}^{\mu+\iota} - \tfrac{\tilde{\theta}_1 b_z}{\epsilon}\|z\|_{\tilde{r}}^{\mu+\iota} + b_z\eta\|z\|_{\tilde{r}}^{\nu+\iota},$$

and therefore the system (38) is

- ISS w.r.t. x, if $\mu = \nu$ and
$$\epsilon \leq \frac{\tilde{\theta}_1}{\eta}. \tag{42}$$
- locally ISS w.r.t. x, if $\mu < \nu$ and
$$\|z\|_{\tilde{r}} \leq \left(\frac{\tilde{\theta}_1}{\epsilon\eta}\right)^{\frac{1}{\nu-\mu}}. \tag{43}$$
- ISpS w.r.t. x, if $\mu > \nu$ and
$$\|z\|_{\tilde{r}} \geq \left(\frac{\epsilon\eta}{\tilde{\theta}_1}\right)^{\frac{1}{\mu-\nu}}. \tag{44}$$

Accordingly, from Definition 2, the trajectories of the system (38) are bounded by

$$\|z(t)\|_{\tilde{r}} \leq \max\{\beta_2(\|z_0\|_{\tilde{r}}, t), \gamma_2(\sup_{\tau\in[0,t)} \|x(\tau)\|_r), \rho\},$$

for all $t \geq 0$, where β_2 is a $\mathcal{KL}$ function, ρ is a constant given by $\rho = 0$ for $\mu \leq \nu$, and $\rho = \frac{\bar{a}_z}{\underline{a}_z}\left(\frac{\epsilon\eta}{\tilde{\theta}_1}\right)^{\frac{1}{\mu-\nu}}$ for $\mu > \nu$, also considering Definition 3, Remark 1 and inequalities (19) and (41), γ_2 is a class $\mathcal{K}$ function given by

$$\gamma_2(s) = \frac{\bar{a}_z}{\underline{a}_z}\left(\frac{\eta\epsilon}{\tilde{\theta}_2} s^{\nu+\iota}\right)^{\frac{1}{\mu+\iota}}. \tag{45}$$

Now, let's analyze the internal stability of the interconnected system (33)-(38) by using Theorem 2 (considering (11) for local behaviors). Note that despite all ISS estimates are obtained with the use of homogeneous norms (which admits (3) but do not verify the triangle inequality), the small-gain arguments stay valid, and it is a straightforward exercise to check this claim. The functions (37) and (45) are the nonlinear asymptotic gains for the systems (33) and (38), respectively. According to the small-gain condition (10) or (11), the stability of the interconnection is insured if the composition

$$\gamma_1(\gamma_2(s)) = \delta^{-1}\frac{\bar{a}_x\bar{a}_z}{\underline{a}_x\underline{a}_z}\left(\frac{\eta\epsilon}{\tilde{\theta}_2} s^{\nu+\iota}\right)^{\frac{1}{\mu+\iota}},$$

is a contraction, i.e., $\gamma_1(\gamma_2(s)) < s$, that is,

$$\frac{\epsilon\eta}{\tilde{\theta}_2\left(\delta\frac{\underline{a}_x\underline{a}_z}{\bar{a}_x\bar{a}_z}\right)^{\mu+\iota}} < s^{\mu-\nu}. \tag{46}$$

Finally, according to the HD's of the systems (12) and (13), there are three different cases of stability. For the cases where $\nu \geq \mu$, vanishing of the transitory z can be concluded (at east locally), which guarantees GAS (or local asymptotic stability) of the interconnected system (12)-(13) at the origin. However, for the case $\nu < \mu$ only practical stability can be proven, but since the system (33) is ISS w.r.t. z then the same property can be concluded for the MD (12) in presence of the PD (13).

The estimations (28), (29) and (31) are derived from inequality (46), where (42) is also considered. Furthermore, since $z = y - h(x)$ and $h(0) = 0$, the estimations (30) and (32) can be readily obtained from the composition $\gamma_2(\gamma_1(s))$, where (43) and (44) are considered, too. Thus, Theorem 3 is proven. ■

4 Illustrative Examples

The following examples have the purpose to illustrate the different kinds of stability predicted by Theorem 3. To this end, some simplifications are introduced in order to exhibit that results nicely.

4.1 Case $\nu < \mu$

Consider the system

$$\dot{x} = -\lceil y_1 \rfloor^{\frac{2}{3}} \tag{47}$$

$$\epsilon \dot{y}_1 = y_2, \quad \epsilon \dot{y}_2 = -y_1 - 2y_2 + \alpha^{\frac{3}{2}} x, \tag{48}$$

where x is the state of the MD (47), y_1, y_2 are the states of the PD (48), and ϵ is a small parameter. For $\epsilon = 0$, the solution $h(x)$ is given by $y_1 = \alpha^{\frac{3}{2}} x$ and $y_2 = 0$, such that, the ROD

$$\dot{x} = -\alpha \lceil x \rfloor^{\frac{2}{3}} \tag{49}$$

is continuous and r-homogeneous of degree $\nu = -\frac{1}{2}$ for the weight $r = \frac{3}{2}$. Also, for any $\alpha > 0$, it is finite-time stable at the origin. On the other hand, for $\epsilon \approx 0$ define $z_1 = y_1 - \alpha^{\frac{3}{2}} x$, $z_2 = y_2$ and $\tau = \epsilon^{-1} t$, such that, the BL

$$\tfrac{dz_1}{d\tau} = z_2, \quad \tfrac{dz_2}{d\tau} = -z_1 - 2z_2, \tag{50}$$

is continuous, $\tilde{r}$-homogeneous of degree $\mu = 0$ for the weights $\tilde{r} = [1, 1]$, and exponentially stable for any $\epsilon > 0$. Then, $\nu < \mu$ and according to Theorem 3 the system (47)-(48) is globally asymptotically practically stable as it is depicted in Figure 1.

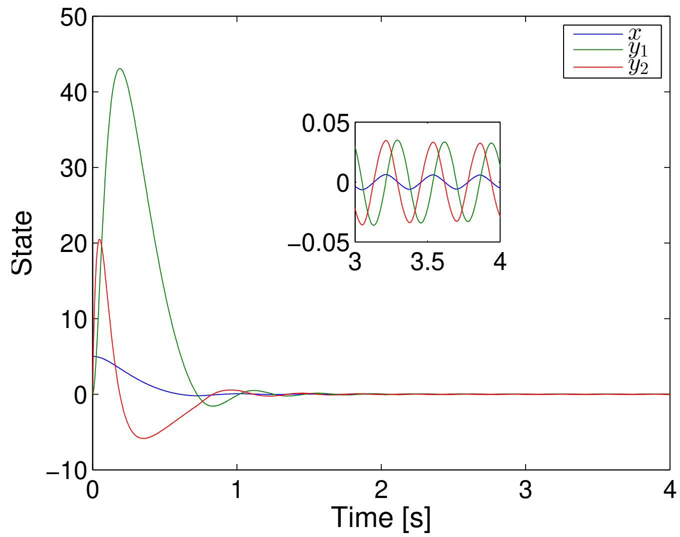

Figure 1: Behavior of the interconnected system (47)-(48) where $\alpha = 5$ and $\epsilon = 0.05$.

Note that the states of the system (47)-(48) exhibit oscillations in the steady state. To the intuition of the authors, this behavior is due to the PD is not fast enough to reach the quasi-stationary state $h(x) = [\alpha^{\frac{3}{2}} x, 0]^\top$. According to Theorem 3, the amplitude of the oscillations depends on the parameter ϵ, and it is illustrated by Figure 2.

Moreover, from Theorem 3 the chattering level for the trajectories $x(t)$ is computed as follows. The stability of the ROD (49) can be proven by the LF $V(x) = \frac{1}{2}x^2$, which fulfills the inequalities (21)-(24) with the constants $\kappa = 2$, $\underline{a}_x = \bar{a}_x = 0.5$, $b_x = \alpha$ and $c_x = 1$. On the other hand, a LF for the BL (50) is given by $W(z) = \frac{1}{2} z^\top P z$, where $P = P^\top > 0$ is a solution of the equation $\bar{A}^\top P + P\bar{A} = -Q$ with $Q > 0$. Selecting

$$P = \begin{bmatrix} 3 & 0.5 \\ 0.5 & 3 \end{bmatrix}, \quad \text{and} \quad Q = \begin{bmatrix} 1 & 1 \\ 1 & 11 \end{bmatrix},$$

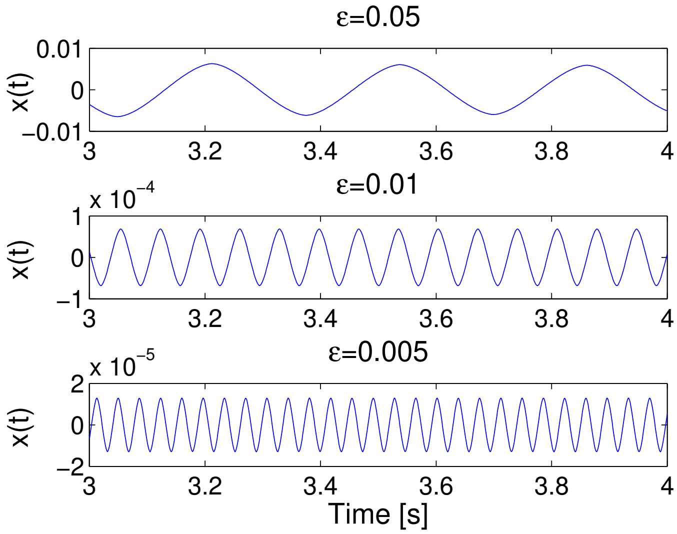

Figure 2: Chattering in the output of the closed-loop system (47)-(48) with $\alpha = 5$.

the function $W(z)$ satisfies (18)-(20), and (25) with $\iota = 2$, $\underline{a}_z = s_{\min}(P) = 2.5$, $\bar{a}_z = c_z = s_{\max}(P) = 3.5$, $b_z = s_{\min}(Q) \approx 0.9$. Furthermore, for the system (47)-(48), $\delta = 0.875\alpha^{\frac{3}{2}}$, $\eta = 5.13\alpha^{\frac{3}{2}}(\alpha^{\frac{3}{2}} + 1)^{\frac{2}{3}}$, $\theta = 0.75$ and $\tilde{\theta}_2 = 0.75$ satisfy (26) and (27). Finally, substituting all the parameters in equation (31), $\lim_{t\to\infty} \|x(t)\|_r \leq 39.53\epsilon^2$. The following table provides the estimation of chattering level for different values of ϵ. Note that the

ϵ	$\lvert x\rvert$
0.05	0.0988
0.01	0.00395
0.005	0.000988

Table 1: Estimation of ultimate bounds for different values of ϵ.

results presented by Figure 2 satisfy the estimations provided by Table 1.

4.2 Case $\nu = \mu$

Now, let us consider the following interconnection

$$\dot{x} = -\alpha\lceil y_1\rfloor^{\frac{2}{3}} \tag{51}$$

$$\epsilon\dot{y}_1 = y_2, \quad \epsilon\dot{y}_2 = -\lceil y_1 - x\rfloor^{\frac{1}{3}} - 2\lceil y_2\rfloor^{\frac{1}{2}}, \tag{52}$$

where x is the state of the system (51), y_1, y_2 are the states of the PD (52), and ϵ is a parameter. For $\epsilon = 0$, $y_1 = x$ and $y_2 = 0$ give the expression of $h(x)$, hence, the reduced order system (49) is recovered, which is continuous and r-homogeneous of degree $\nu = -\frac{1}{2}$ for the weight $r = \frac{3}{2}$, and for any $\alpha > 0$, finite-time stable at the origin. On the other hand, for $\epsilon \approx 0$ define $z_1 = y_1 - x$, $z_2 = y_2$ and $\tau = \epsilon^{-1}t$, such that, the BL

$$\frac{dz_1}{d\tau} = z_2, \quad \frac{dz_2}{d\tau} = -\lceil z_1\rfloor^{\frac{1}{3}} - 2\lceil z_2\rfloor^{\frac{1}{2}}, \tag{53}$$

is continuous, $\tilde{r}$-homogeneous of degree $\mu = -\frac{1}{2}$ for the weights $\tilde{r} = [\frac{3}{2}, 1]$, and finite-time stable at the origin, for any $\epsilon > 0$. In this case, $\nu = \mu$, hence by Theorem 3, the system (51)-(52) is expected to be globally finite-time stable as it is illustrated in Figure 3.

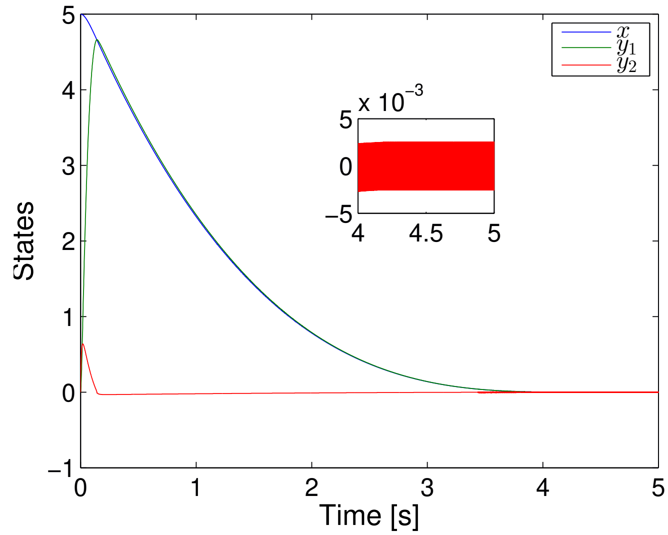

Figure 3: Behavior of the closed-loop system (51)-(52) where $\alpha = 1.2$ and $\epsilon = 0.01$.

A critical value ϵ^* is computed as follows. The stability of the ROD (49) can be proven by the LF $V(x) = \frac{1}{2}x^2$, which fulfills the inequalities (21)-(24) with the constants $\kappa = 2$, $\underline{a}_x = \bar{a}_x = 0.5$, $b_x = \alpha$ and $c_x = 1$. On the other hand, a LF for the BL (50) is given by $W(z,x) = 10.5|z_1|^{\frac{5}{3}} + 8.5|z_2|^{\frac{5}{2}} + 3.5z_1z_2$, which satisfies (18)-(20) and (25) with $\iota = \frac{5}{2}$, $\underline{a}_z = 6.462$, $\bar{a}_z = 10.809$, $b_z = 2.654$ and $c_z = 22.11$. Furthermore, solving (26) and (27) for the system (54)-(55), $\delta = 0.875$ and $\eta = 13.22\alpha$, where $\theta = 0.75$, $\tilde{\theta}_1 = 0.25$ and $\tilde{\theta}_2 = 0.75$ were used. Then, substituting all the parameter in (28) a critical value $\epsilon^* = 0.0129$ is obtained, hence, the stability of the interconnection (51)-(52) is guaranteed for any $\epsilon < 0.0129$. In this case the value of ϵ determines how the trajectories of (51)-(52) deviate from the trajectories of the reduced order dynamics (49) as it is shown in Figure 4.

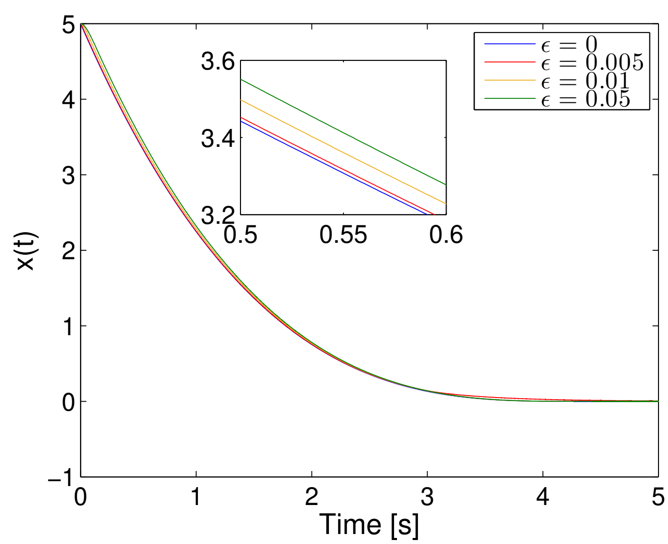

Figure 4: Comparison of the behavior of the closed-loop system (51)-(52) for different values of ϵ.

4.3 Case $\nu > \mu$

Finally, consider an interconnection given by

$$\dot{x} = -y_1, \tag{54}$$

$$\epsilon \dot{y}_1 = y_2, \quad \epsilon \dot{y}_2 = -\lceil y_1 - \alpha x \rfloor^{\frac{1}{3}} - 2\lceil y_2 \rfloor^{\frac{1}{2}}, \tag{55}$$

where x is the state of the system (54), y_1, y_2 are the states of the PD (55), and ϵ is a parameter. For $\epsilon = 0$, $y_1 = \alpha x$ and $y_2 = 0$, and the reduced order dynamics is given by

$$\dot{x} = -\alpha x, \tag{56}$$

which is r-homogeneous of degree $\nu = 0$ for the weight $r = 1$, and also, asymptotically stable at the origin for any $\alpha > 0$. On the other hand, for $\epsilon \approx 0$ define $z_1 = y_1 - \alpha x$, $z_2 = y_2$ and $\tau = \epsilon^{-1} t$, such that, the BL (53) is obtained and it is continuous, $\tilde{r}$-homogeneous of degree $\mu = -\frac{1}{2}$ for the weights $\tilde{r} = [\frac{3}{2}, 1]$, and finite-time stable at the origin for any $\epsilon > 0$. In this case, $\nu > \mu$, hence, the interconnected system (54)-(55) is locally asymptotically stable at the origin as it is predicted by Theorem 3 and confirmed in Figure 5, where for an initial condition $x_0 = 3$ and $y_0 = [0, 0]$ the states converge to zero but for an initial condition $x_0 = 5$ and $y_0 = [0, 0]$ the stability cannot be ensured.

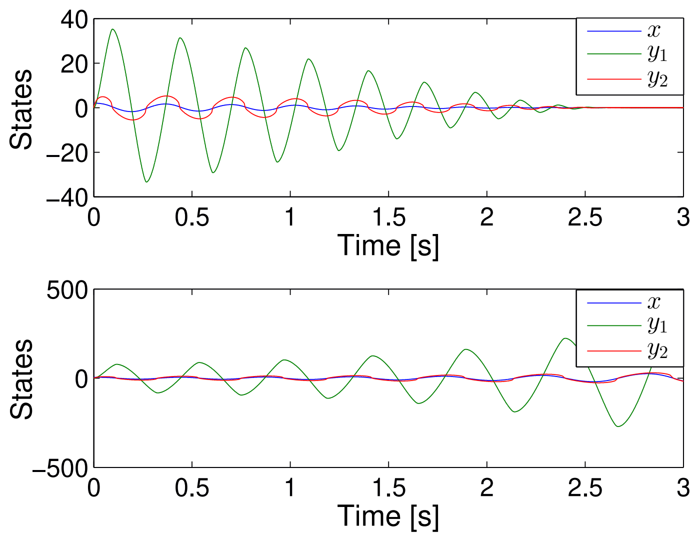

Figure 5: Behavior of the closed-loop systems (54)-(55) where $\alpha = 60$, $\epsilon = 0.01$, $x_0 = 2$ and $y_0 = [0, 0]$ (top), or $x_0 = 5$ and $y_0 = [0, 0]$ (bottom).

The domain of attraction for the trajectories of the system (54)-(55) can be evaluated from Theorem 3 as follows. The stability of the ROD (56) can be proven by the LF $V(x) = \frac{1}{2}x^2$, which fulfills the inequalities (21)-(24) with the constants $\kappa = 2$, $\underline{a}_x = \bar{a}_x = 0.5$, $b_x = \alpha$ and $c_x = 1$. On the other hand, a LF for the BL (53) is proposed as $W(z, x) = 10.5|z_1|^{\frac{5}{3}} + 8.5|z_2|^{\frac{5}{2}} + 3.5 z_1 z_2$, which satisfies (18)-(20), and (25) with $\iota = \frac{5}{2}$, $\underline{a}_z = 6.462$, $\bar{a}_z = 10.809$, $b_z = 2.654$ and $c_z = 22.11$. Furthermore, by solving (26) and (27) for the system (54)-(55), we obtain $\delta = 0.75\alpha$ and $\eta = 8.33\alpha(\alpha + 1)$, where $\theta = 0.75$ and $\tilde{\theta}_2 = 0.75$ were used.

So, substituting all the parameters in equation (29), $\|x_0\|_r \le 0.000317\epsilon^{-2}$. Therefore, for $\epsilon = 0.01$, it is obtained $\|x_0\|_r \le 3.17$ supporting the simulation results shown in Figure 5. Now, if the value of the parameter ϵ decreases then the domain of attraction for trajectories $x(t)$ increases, such that, for $\epsilon = 0.005$ it turns out that $\|x_0\|_r \le 12.68$. Accordingly, the stability of the interconnected system (54)-(55) is ensured for $x_0 = 5$ and $y_0 = [0, 0]$ as Figure 6 shows.

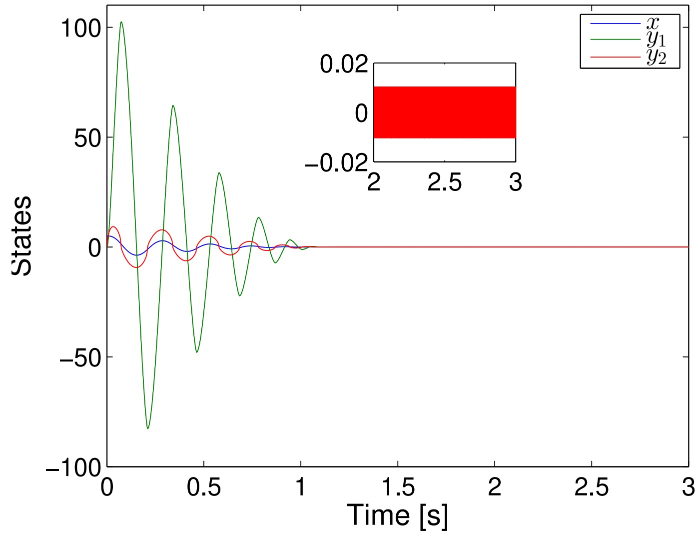

Figure 6: Behavior of the closed-loop systems (54)-(55) where $\alpha = 60$, $\epsilon = 0.005$, $x_0 = 5$ and $y_0 = [0, 0]$.

5 Conclusions

This paper presented a study of the effect of a stable homogeneous PD on the stability of a homogeneous MD, which based on classical concepts: ISS and Small-Gain Theorem by assuming only continuity of the considered vector fields. Three types of stability for such an interconnection were discovered depending on the relation between HDs of PD and MD:

- Global asymptotic stability when both dynamics have the same HD and the SPP is sufficiently small.
- Global asymptotic practical stability, when the PD has a greater HD than the MD. The estimation of the asymptotic bound of the trajectories is provided and its size grows with the SPP. In this case, the chattering may appear if a finite-time convergent MD is considered.
- Local asymptotic stability with an estimation of the domain of attraction, when the PD has a smaller HD than the MD. The size of the domain of attraction decreases if the SPP is increased.

The first case can be interpreted as a validation of the concept of motion separation, predicted by classical results on smooth (at least Lipschitz continuous) singularly perturbed systems, for a wider class of homogeneous systems. On the other hand, such a concept of motion separation is only valid outside of a neighborhood of the origin for the second case, and near to the origin for the third one, hence, just local and practical stability can be concluded, respectively.

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