Generic Delay-L Left Invertibility of Structured Systems

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Abstract: This paper studies structured systems, namely linear systems where the state-space matrices have zeros in some fixed positions and free parameters in all other entries. This paper focuses on time-invariant systems in discrete time affected by an unknown input, and their delay-L left invertibility, namely the possibility to reconstruct the input sequence from the output sequence, assuming that the initial state is known, and requiring that the inputs can be reconstructed up to L time steps before the current output. Building upon classical results on linear systems theory and on structured systems, a graphical characterization is obtained of the integers L for which a structured system is generically delay-L left invertible.

Keywords: Network systems, Structured systems, Structural systems, Structural observability, Linear time-invariant systems, Smith-McMillan form, Left invertibility, Unknown input, Delayed observers, Cyber-physical security

1. INTRODUCTION

The study of generic properties of structured systems has been an active research area since the 70's and has received a wide recent attention within the multi-disciplinary community studying complex network systems.

A structured system is a linear system whose state-space matrices have a fixed pattern of zeros (representing known lack of interaction between some states), and the other entries are free parameters. The zero pattern can be equivalently described with a directed graph, where edges represent non-zero entries. The goal is to find graphical conditions ensuring that some system-theoretic property is true generically, where *generically* (or *for almost all parameters*) means *for all parameters except possibly those lying in a proper subvariety of the parameter space.* Since a proper subvariety has zero Lebesgue measure, there is the following probabilistic interpretation: given a property that is generically true, if the parameters are chosen at random (according to any continuous distribution) the probability of the property being true is one.

The early results on structured systems, initiated by the seminal paper Lin (1974), were devoted to controllability and observability. A rich literature was then developed, which is well summarized in the book Murota (2000) and in the survey papers Dion et al. (2003) and Ramos et al. (2020).

This paper focuses on structured systems with an unknown input. Such input may represent a fault, an un-modeled part of the system, or a malicious external attack. The latter interpretation has brought significant attention to systems with an unknown input in the research area of cyber-physical security. In particular, papers Pasqualetti et al. (2013) and Weerakkody et al. (2017b) study perfect attacks, namely attacks that can go completely undetected, since they produce the same output as a legitimate trajectory without input; extensions to near-perfect attacks are proposed in Mo and Sinopoli (2010), to include attacks producing small albeit non-zero residuals in the attack detector. In Pasqualetti et al. (2013), assuming that the initial state is known, the authors characterize perfect attackability as the lack of left invertibility, and then they study generic left invertibility of structured systems, including an extension to descriptor systems. Instead, Weerakkody et al. (2017b) focuses on the optimization problem related to sensor placement, where the goal is to ensure left invertibility with the smallest number of dedicated sensors, each measuring one local state, and furthermore minimizing the communication between sensors.

When the initial state is not known or in the presence of noise, the notion of left invertibility should be accompanied by the one of strong observability, namely the possibility to reconstruct the initial state from the output measurements, despite the presence of the unknown input. This is useful to construct Luenberger-like observers or Kalman-like filters, suitably adapted to cope with the unknown input, as in Sundaram and Hadjicostis (2007), Yong et al. (2016) or Garin et al. (2018).

The simultaneous occurrence of the two properties of strong observability and left invertibility has been studied under the names of state-and-input observability or input-and-state observability. For structured systems, generic input-and-state observability has been characterized in Boukhobza et al. (2007); see also Weerakkody et al. (2017a) and Garin (2017) for a simpler rephrasing of their characterization and for further results, in Weerakkody et al. (2017a) about optimal sensor placement, and in Garin (2017) about the additional property of delay-1 left invertibility, discussed below.

Above-mentioned works on generic state-and-input observability (except Garin (2017)) do not take into account the delay of the left invertibility: For which L can we reconstruct inputs $u(0), \dots, u(k-L)$ from the initial state $x(0)$ and measurements $y(0), \dots, y(k)$?

This notion of delay is crucial when one implements observers for recursively reconstructing the input and the state, as highlighted in Sundaram and Hadjicostis (2007). Classical studies by Massey and Sain have found an algebraic condition characterizing delay-L left invertibility as the rank of a matrix involving the matrices of the state-space representation of the system Massey and Sain (1968). Also see Sain and Massey (1969) for a thourough discussion of the counterpart of delay-L left invertibility for continuous-time systems.

Results about generic delay-L left invertibility have only focused on delay 1, with a characterization in Garin (2017) for linear time-invariant systems. A series of papers summarized in the thesis Gracy (2018) have studied various aspects of delay-1 left invertibility together with strong observability.

The novelty of this paper is the study of generic delay-L left invertibility for structured systems, for any given delay L. Such problem was first addressed in Garin and Kibangou (2019), where a graphical characterization was obtained for the particular case where the input is scalar. The current paper solves the problem in the most general case. It builds upon classical results in linear systems theory, together with the literature on the generic structure at infinity of structured systems, to obtain a simple graphical condition which characterizes the values of L for which the structured system is generically left invertible. This result can also be used in combination with the characterization of generic input-and-state observability (Boukhobza et al. (2007), Weerakkody et al. (2017a), Garin (2017)), to obtain the graphical conditions under which the system can generically have a delay-L observer as in Sundaram and Hadjicostis (2007).

This paper is organized as follows. In Section 2 we recall the definition of delay-L left invertibility, and its classical characterization in terms of the rank of suitable matrices involving the (A, B, C, D) matrices of the state-space representation of the system. In Section 3 we show an alternative characterization, involving the Smith-McMillan form at infinity of the system's transfer function. Then, in Section 4 we recall results on the generic zero-orders at infinity of structured systems; such results, together with the ones in Section 3, give us the desired graphical condition that characterizes the values of L for which the structured system is generically left invertible (Theorem 7). Section 5 illustrates the results in an example.

2. DELAY-L LEFT INVERTIBILITY, INHERENT DELAY

Here we recall the definition and the classical algebraic characterization of delay-L left invertibility and of the inherent delay for discrete-time linear time-invariant (LTI) systems, from Massey and Sain (1968); see Sain and Massey (1969) for the continuous-time interpretation of the same.

Consider the discrete-time LTI system

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) + Du(k) \end{cases} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state, $u(k) \in \mathbb{R}^p$ is the unknown input, and $y(k) \in \mathbb{R}^m$ is the output.

Definition 1. For an integer $L \geq 0$, the system (1) is *delay-L left invertible* if the initial unknown input $u(0)$ is uniquely determined by the initial state $x(0)$ and the output sequence $y(0), \dots, y(L)$.

The system (1) is *left invertible* if there exists an integer $L \geq 0$ for which it is delay-L left invertible.

The *inherent delay* of system (1), hereby denoted L_{in}, is the smallest integer $L \geq 0$ such that the system is delay-L left invertible. □

Notice that delay-L left invertibility implies delay-K left invertibility for all $K \geq L$ (by definition). Hence, a system is delay-L left invertible if and only if $L \geq L_{\text{in}}$. For this reason, we will focus our attention on studying the inherent delay.

The characterization of delay-L left invertibility from Massey and Sain (1968) requires to define

$$M_L = \begin{bmatrix} J_0 & 0 & 0 & \dots & 0 \\ J_1 & J_0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ J_L & J_{L-1} & J_{L-2} & \dots & J_0 \end{bmatrix},$$

where $J_0 = D$ and $J_\ell = CA^{\ell-1}B$ for all $\ell \geq 1$ are the Markov parameters of the system.

The key remark is that

$$M_L \begin{bmatrix} u(0) \\ \vdots \\ u(L) \end{bmatrix} = \begin{bmatrix} y(0) \\ \vdots \\ y(L) \end{bmatrix} - \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{L-1} \end{bmatrix} x(0)\,.$$

For this reason, with the above notation, together with the definition $\operatorname{rank} M_{-1} = 0$, delay-$L$ left invertibility is characterized as follows.

Proposition 1. (Massey and Sain (1968), Thm. 4). For any integer $L \geq 0$, the system (1) is delay-L left invertible if and only if

$$\operatorname{rank} M_L - \operatorname{rank} M_{L-1} = p.$$

□

This immediately implies the following characterization of the inherent delay.

Proposition 2. The inherent delay L_{in} of system (1) is the smallest integer $L \geq 0$ such that

$$\operatorname{rank} M_L - \operatorname{rank} M_{L-1} = p.$$

□

The inherent delay L_{in} is infinite if the system is not left invertible, and otherwise it is at most n, as shown in Massey and Sain (1968), Coroll. 2. This implies that the system (1) is left invertible if and only if $\operatorname{rank} M_n - \operatorname{rank} M_{n-1} = p$.

3. INHERENT DELAY AND SMITH-MCMILLAN FORM AT INFINITY

The characterization described in Sect. 2 of (delay-L) left invertibility and of the inherent delay is based on the state-space description of the system. However, it is also useful to study characterizations involving the transfer function $G(z) = C(zI - A)^{-1}B + D$.

Left invertibility is equivalent to the fact that $G(z)$ is a left-invertible rational matrix, which explains the name 'left invertibility' (and also the name 'TFM left invertibility' used in some literature for the same notion, where 'TFM' stands for 'Transfer Function Matrix'). This equivalence, stated below as Prop. 3, can be found for example in the book Trentelman et al. (2001), where it is proved for the continuous-time case, but the same proof argument can be easily applied also to the discrete-time case.

Proposition 3. (Trentelman et al. (2001), Thm. 8.8). The system (1) is left invertible if and only if its transfer function $G(z)$ is a left-invertible rational matrix. □

Recall that $G(z)$ is left invertible if and only if its normal rank is equal to p (see e.g. Trentelman et al. (2001), Sect. 2.8).

It turns out that also inherent delay can be determined with properties of the transfer function $G(z)$, as we will show in the remainder of this section with the use of the Smith-McMillan form at infinity. We only consider proper transfer functions, which correspond to causal state-space representations as in (1).

We recall here the well-known Smith-McMillan factorization at infinity of a rational function $G(z)$, which corresponds to the Smith-McMillan factorization at zero of $H(z)$ defined as $H(z) = G(z^{-1})$, see e.g. Kailath (1980), Chapter 6. Given a proper rational function $G(z)$, there exist two biproper [1] matrices $U(z)$ and $V(z)$ such that

$$G(z) = U(z)\Lambda(z)V(z)$$

with

$$\Lambda(z) = \left[\begin{array}{cccc|c} z^{-n_1} & & & & \\ & z^{-n_2} & & & 0 \\ & & \ddots & & \\ & & & z^{-n_r} & \\ \hline & & 0 & & 0 \end{array}\right]$$

where $n_1, \ldots n_r$ are integers, $0 \leq n_1 \leq \cdots \leq n_r$. This factorization is not unique for what concerns $U(z)$ and $V(z)$, but $\Lambda(z)$ is unique, and is called the Smith-McMillan form at infinity of $G(z)$. The non-negative integers $n_1, \ldots, n_r$ are called the zero-orders at infinity of $G(z)$.

We will use the results in Van Dooren et al. (1979), where a characterization of the zero-orders at infinity is obtained from the coefficients of the Laurent series at infinity, i.e., the Laurent series of $H(z) = G(z^{-1})$ at zero.

Since $(zI - A)^{-1} = z^{-1}\sum_{h\geq 0}(z^{-1}A)^h$, the Laurent series at infinity of $G(z)$ is the following:

$$G(z) = D + \sum_{h\geq 1} CA^{h-1}B\, z^{-h},$$

i.e.,

$$G(z) = \sum_{h\geq 0} J_h z^{-h}, \tag{2}$$

where J_h's are the Markov parameters defined in Sect. 2.

The results in Van Dooren et al. (1979) involve block-Toeplitz matrices T_i's defined as follows:

$$T_i = \begin{bmatrix} J_0 & J_1 & J_2 & \ldots & J_i \\ 0 & J_0 & J_1 & \ldots & J_{i-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & J_0 \end{bmatrix},$$

where matrices J_h's are the ones involved in the Laurent expansion at infinity (2), and hence they are the Markov parameters of the system.

The property stated below as Prop. 4 reveals the relation between the zero-orders at infinity and the rank indexes, where the ith rank index is defined as $\operatorname{rank} T_i - \operatorname{rank} T_{i-1}$. Such property is proved in Van Dooren et al. (1979), where it is used as the main tool for an efficient and stable algorithm to compute the Smith-McMillan form. More specifically, Van Dooren et al. (1979) gives this property in Sect. III for the Smith-McMillan form at a finite α, and then Sect. IV explains how to extend this result to $\alpha = \infty$ by studying $H(z) = G(z^{-1})$ at $\alpha = 0$.

Proposition 4. (Van Dooren et al. (1979)). Given a proper rational matrix $G(z)$, denote by r its normal rank and by $n_1, \ldots, n_r$ its zero-orders at infinity and consider the above-defined block-Toeplitz matrices T_i's, letting $\operatorname{rank} T_{-1} = 0$.

For any integer $i \geq 0$, the ith rank index is equal to the number of zero-orders whose value is less or equal to i, i.e.,

$$\operatorname{rank} T_i - \operatorname{rank} T_{i-1} = \#\Big\{j \in \{1, \ldots, r\} \text{ s.t. } n_j \leq i\Big\}.$$

□

Since $n_1 \leq \cdots \leq n_r$, the expression for the rank index in Prop. 4 can be equivalently re-written as follows: if $i < n_1$, then $\operatorname{rank} T_i - \operatorname{rank} T_{i-1} = 0$; if $i \geq n_1$, then

$$\operatorname{rank} T_i - \operatorname{rank} T_{i-1} = \max\Big\{j \in \{1, \ldots, r\} \text{ s.t. } n_j \leq i\Big\}.$$

It is clear that matrices T_i involved in Prop. 4 and matrices M_i involved in Prop. 1 are similar, since M_i can be obtained from T_i by permuting blocks of columns and then blocks of rows. Hence, $\operatorname{rank} T_i = \operatorname{rank} M_i$. For this reason, from Prop. 4, together with the characterization of delay-L left invertibility in Prop. 1, we easily obtain the following result.

Proposition 5. The inherent delay of system (1), with transfer function $G(z)$, is infinite if the normal rank of $G(z)$ is smaller than p, and else is equal to n_p, its largest zero-order at infinity. □

Proof. By Prop. 1, together with the remark $\operatorname{rank} T_j = \operatorname{rank} M_j$, the inherent delay is the smallest integer L such that $\operatorname{rank} T_L - \operatorname{rank} T_{L-1} = p$. From Prop. 4, we obtain that $\operatorname{rank} T_L - \operatorname{rank} T_{L-1} < r$ for all $L < n_r$ and $\operatorname{rank} T_L - \operatorname{rank} T_{L-1} = r$ for all $L \geq r$, where r is the normal rank of $G(z)$. This ends the proof of the claim. □

[1] a biproper rational matrix is a square rational matrix which is proper, invertible, and whose inverse is proper

4. GENERIC INHERENT DELAY OF A STRUCTURED SYSTEM

A structured system is a family of LTI systems sharing a same imposed zero-pattern for their A, B, C, D matrices, while all entries that are not fixed to zero are free parameters, that is, distinct real-valued parameters, which can be chosen arbitrarily. A structured system is usually represented by a directed graph (digraph) $\mathcal{G}$, see e.g. Dion et al. (2003). The vertex set of $\mathcal{G}$ is $U \cup X \cup Y$, where $U = \{u_1, \ldots, u_p\}$ is the set of input vertices, $X = \{x_1, \ldots, x_n\}$ is the set of state vertices, and $Y = \{y_1, \ldots, y_m\}$ is the set of output vertices. Edges of $\mathcal{G}$ correspond to the entries of matrices A, B, C, D that are free parameters: a parameter a_{ij} in position (i, j) in matrix A corresponds to an edge (x_j, x_i), representing the influence of state $x_j(k)$ on the state $x_i(k+1)$; a parameter b_{ij} in position (i, j) in matrix B corresponds to an edge (u_j, x_i); a parameter c_{ij} in position (i, j) in matrix C corresponds to an edge (x_j, y_i); and a parameter d_{ij} in position (i, j) in matrix D corresponds to an edge (u_j, y_i). For an example, see Sect. 5.

Given a structured system, we say that some property is true *generically* if it is true for all systems in the family, except possibly for some systems that correspond to a proper subvariety of the parameters space. The goal of this paper is to find graphical conditions on $\mathcal{G}$ that characterize whether or not the corresponding structured system is generically delay-L left invertible. The main results will follow from merging the system-theoretic properties described in Sections 2 and 3 with results from the literature on structured systems, as described below. Such results will involve the so-called linkings, defined as follows.

Definition 2. A U-Y *linking* in $\mathcal{G}$ is a collection of vertex-disjoint paths in $\mathcal{G}$, each originated in a vertex in U and ending in a vertex in Y. The *size* of the linking is the number of paths. $\square$

The two results given below in Prop. 6 are stated in the survey paper Dion et al. (2003), see Theorem 2. The first result, about left invertibility, has been proved in van der Woude (1991a) and the second one, about the zero-orders at infinity, has been independently proved in Commault et al. (1991) and van der Woude (1991b). All such proofs were given for the case without direct feedthrough of the input to the output, i.e., $D = 0$. However, Theorem 2 in Dion et al. (2003) is stated for a system as in (1), and a careful examination of the proofs in van der Woude (1991a) and Commault et al. (1991) confirms this result, since all proof arguments remain valid also in the presence of D. Also, above-cited results are given for continuous-time systems, but they are based on the transfer function, which is the same in the discrete-time case.

Proposition 6. (Dion et al. (2003), Thm. 2). Given a structured system with digraph $\mathcal{G}$, let $G(z) = C(zI - A)^{-1}B + D$ be the corresponding transfer function. Denote by s the largest size of a U-Y linking in $\mathcal{G}$. For $h = 1, \ldots, s$, denote by α_h the minimum number of X-vertices in a U-Y linking of size h in $\mathcal{G}$, and set $\alpha_0 = 0$. Then

(1) Generically, $G(z)$ has normal rank equal to s.
(2) Generically, the zero-orders at infinity of $G(z)$ are n_1, $\ldots$, n_s given by $n_h = \alpha_h - \alpha_{h-1}$. $\square$

From Prop. 6, together with Prop. 5 from Sect. 3, we immediately obtain the following characterization of generic inherent delay, which is the main result of this paper.

Theorem 7. Consider a structured system with digraph $\mathcal{G}$. Denote by s the largest size of a U-Y linking in $\mathcal{G}$. For $h \in \{1, \ldots, s\}$, denote by α_h the minimum number of X-vertices in a U-Y linking of size h in $\mathcal{G}$, and set $\alpha_0 = 0$. Generically, the inherent delay of the structured system is $L_{\text{in}} = \infty$ if $s < p$ and $L_{\text{in}} = \alpha_p - \alpha_{p-1}$ if $s = p$. $\square$

This means that, if $s = p$, then the structured system is generically delay-L left invertible if and only if $L \geq \alpha_p - \alpha_{p-1}$.

Remark 8. The statements of Proposition 6 and Theorem 7 do not discuss what happens for those particular parameters for which the properties are different from the generic case.

Concerning item (1) in Proposition 6, the following result is true: for any choice of parameters, the normal rank of $G(z)$ is smaller than or equal to s. Indeed, Theorem 1 in van der Woude (1991a) shows that the normal rank can be smaller than the maximum normal rank only over a proper subvariety; equivalenty, the generic normal rank is equal to the maximum normal rank, no choice of parameters allows to exceed the generic normal rank.

The above remark on the normal rank implies the following one on left invertibility: if $s < p$, then the system is not left invertible (equivalently, the inherent delay is infinite), for any choice of the parameters.

Concerning the inherent delay when $s = p$, instead, no conclusion can be driven about its value for some particular parameters: the inherent delay can be smaller or larger than the generic one, as we will show in an example in Section 5. $\square$

A final remark is about the computational complexity. As discussed in Dion et al. (2003), Section 10, the computations of s and α_i's involved in the statements of Prop. 6 and Thm. 7 can be cast as optimization problems for which there is a rich literature of polynomial-time algorithms (maximum flow, and minimum-cost flow, respectively).

5. EXAMPLE

In this section, we give an example of structured system, to illustrate the definitions and results given in Sect. 4, in particular Theorem 7 and Remark 8.

Consider the structured system with the following matrices:

$$A = \begin{bmatrix} 0 & a_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{43} & 0 & 0 \\ 0 & 0 & a_{53} & a_{54} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & b_{12} \\ b_{21} & 0 \\ 0 & b_{32} \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} c_{11} & 0 & c_{13} & 0 & 0 \\ 0 & 0 & 0 & c_{24} & c_{25} \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The corresponding digraph $\mathcal{G}$ is depicted in Figure 1.

We want to apply Theorem 7 to find the generic inherent delay of this structured system. In this example, the size of the input is $p = 2$.

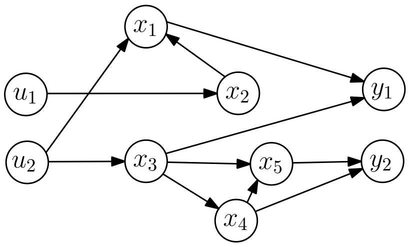

Fig. 1. Digraph $\mathcal{G}$ of the structured system in Section 5.

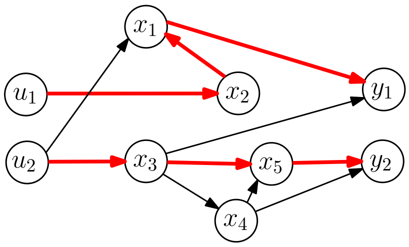

Fig. 2. Red edges represent an U-Y linking of size 2 with minimum number of X-vertices, formed by the two paths (u_1, x_2), (x_2, x_1), (x_1, y_1), and (u_2, x_3), (x_3, x_5), (x_5, y_2).

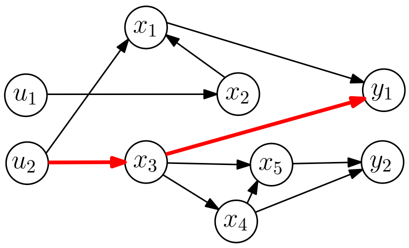

Fig. 3. Red edges represent an U-Y linking of size 1 with minimum number of X-vertices: the path (u_2, x_3), (x_3, y_1).

We can see in Figure 2 that in $\mathcal{G}$ there exists an U-Y linking of size two, i.e., there are two vertex-disjoints paths, each originated in a vertex in U and ending in a vertex in Y. Clearly, no U-Y linking can have a size larger than the size of U, and hence the maximum size of a U-Y linking in this example is $s = p = 2$. The linking in Figure 2 has four X-vertices (x_1, x_2, x_3 and x_5) and this is the smallest number of X-vertices among all U-Y linkings of size two. Indeed, the only other U-Y linkings of size two have the same first path, and the second path which is either (u_2, x_3), (x_3, x_4), (x_4, y_2), which has the same number of X-vertices, or (u_2, x_3), (x_3, x_4), (x_4, x_5), (x_5, y_2), which has more X-vertices. Hence, in this case $\alpha_2 = 4$.

We then compute $\alpha_1 = 1$ by looking at Figure 3, which shows the U-Y linking of size 1 (i.e., the path from some u in U to some y in Y) with the smallest number of X-vertices.

With $s = p = 2$, $\alpha_p = \alpha_2 = 4$ and $\alpha_{p-1} = \alpha_1 = 1$, by Theorem 7 this structured system has generic delay $\alpha_p - \alpha_{p-1} = 3$.

We can also find the inherent delay by using the results from Sect. 2, involving matrices $M_0, \ldots, M_5$, in order to explore some cases with non-generic parameters (parameters in some proper subvariety, for which the results are different from the generic ones). Simple computations give

$$M_5 = \left[\begin{array}{cc|cc|cc|cc|cc|cc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\beta & 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \gamma & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & \beta & 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \delta & 0 & \gamma & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & \beta & 0 & 0 & \alpha & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \delta & 0 & \gamma & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & \beta & 0 & 0 & \alpha & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \delta & 0 & \gamma & 0 & 0 & 0 & 0
\end{array}\right],$$

with $\alpha = c_{11}b_{12} + c_{13}b_{32}$, $\beta = c_{11}a_{12}b_{21}$, $\gamma = c_{24}a_{43}b_{32} + c_{25}a_{53}b_{32}$ and $\delta = c_{25}a_{54}a_{43}b_{32}$.

Matrices M_i for $i = 0, 1, 2, 3, 4$ are the square submatrices of M_4 obtained by restricting the attention to first $2(i+1)$ rows and $2(i+1)$ columns.

It is then easy to compute the rank of such matrices, from which the inherent delay can be obtained by finding the smallest L such that $\operatorname{rank} M_L - \operatorname{rank} M_{L-1} = 2$. Different results will be obtained, for different parameters.

Generically, α, β, γ and δ are non-zero, and hence $\operatorname{rank} M_0 = 0$, $\operatorname{rank} M_1 = 1$, $\operatorname{rank} M_2 = 2$, $\operatorname{rank} M_3 = 4$, $\operatorname{rank} M_4 = 6$ and $\operatorname{rank} M_5 = 8$. This confirms that the inherent delay is generically equal to 3, as obtained above by applying Theorem 7.

We can now look at some particular parameters, for which the inherent delay is different from the generic one.

First, for some parameters, we have $\alpha = 0$ and non-zero β, γ, δ, so that $\operatorname{rank} M_0 = 0$, $\operatorname{rank} M_1 = 0$, $\operatorname{rank} M_2 = 2$, $\operatorname{rank} M_3 = 4$, $\operatorname{rank} M_4 = 6$ and $\operatorname{rank} M_5 = 8$. Hence, for such parameters the inherent delay is equal to 2, which is smaller than the generic one.

For some other parameters, we have $\gamma = 0$ and non-zero α, β, δ, so that $\operatorname{rank} M_0 = 0$, $\operatorname{rank} M_1 = 1$, $\operatorname{rank} M_2 = 2$, $\operatorname{rank} M_3 = 3$, $\operatorname{rank} M_4 = 5$ and $\operatorname{rank} M_5 = 7$, which gives an inherent delay of 4, larger than the generic one.

There are also some parameters for which the system is not left invertible (the inherent delay is infinite), for example when $\gamma = \delta = 0$ and α, β are non-zero, which gives $\operatorname{rank} M_0 = 0$, $\operatorname{rank} M_1 = 1$, $\operatorname{rank} M_2 = 2$, $\operatorname{rank} M_3 = 3$, $\operatorname{rank} M_4 = 4$ and $\operatorname{rank} M_5 = 5$. Recall that $n = 5$ and that a system is left-invertible if and only if $\operatorname{rank} M_n - \operatorname{rank} M_{n-1} = p$, which fails in this case.

6. CONCLUSION

In this paper, we have studied discrete-time linear time-invariant structured systems with an unknown input, and their delay-L invertibility. Exploiting results from linear systems theory and from structured systems theory, we have given a graphical condition that characterizes the generic inherent delay, namely the non-negative integer L_{in} such that the structured system is generically delay-L left invertible if and only if $L \geq L_{\text{in}}$ (or infinite if no such finite integer exists). We have also presented an example that illustrates how some particular parameters can exist such that the inherent delay is smaller than the generic one, and some others such that the inherent delay is larger than the generic one.

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