An Algebraic Approach Towards the Fine-Grained Complexity of Graph Coloring Problems

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***Abstract*—In this paper we are interested in the fine-grained complexity of determining whether there is an homomorphism from an input graph G to a fixed graph H (the H-coloring problem). The starting point is that these problems can be viewed as constraint satisfaction problems (CSPs), and that (partial) polymorphisms of binary relations are of paramount importance in the study of complexity classes of such CSPs.**

Thus, we first investigate the expressivity of binary symmetric relations E_H and their corresponding (partial) polymorphisms $\mathrm{pPol}(E_H)$. For irreflexive graphs we observe that there is no pair of graphs H and H' such that $\mathrm{pPol}(E_H) \subseteq \mathrm{pPol}(E_{H'})$, unless $E_{H'} = \emptyset$ or $H = H'$. More generally we show the existence of an n-ary relation R whose partial polymorphisms strictly subsume those of H and such that CSP(R) is $\mathcal{NP}$-complete if and only if H contains an odd cycle of length at most n. Motivated by this we also describe the sets of total polymorphisms of every nontrivial clique, every odd cycle, as well as certain cores. We finish the paper with some noteworthy questions for future research.

I. Introduction

This paper aims to improve our understanding of fine-grained complexity of constraint satisfaction problems (CSPs) [9]. In a *constraint satisfaction problem* (CSP), given a set of variables X and a set of constraints of the form $R(\mathbf{x})$ for $\mathbf{x} \in X^k$ and some k-ary relation R, the objective is to assign values from X to a domain V such that every constraint in C is satisfied. This problem is usually denoted by CSP(Γ), with the additional stipulation that every relation occurring in a constraint comes from the set of relations Γ, and it is typically phrased as the decision problem of verifying whether a solution exists.

Specifically, we are interested in the so-called *H-COLORING problem* [7][1]. Recall that a function $f\colon V_G \to V_H$ is said to be a *homomorphism* between the two graphs G and H if it "preserves" the edge relation, that is, if for every edge $(u, v) \in E_G$, we have $(f(u), f(v)) \in E_H$. In that case, we use the notation $f : G \to H$. For each graph H, the homomorphy notion naturally entails the following decision problem, which is equivalent to the problem CSP($\{E_H\}$) (henceforth written CSP(E_H)).

[1] Thoughout this paper, we assume that all graphs are finite, simple (no loops) and undirected. Every graph H is defined by its set V_H of vertices and its set E_H of edges.

H-COLORING PROBLEM. *Given a graph G, decide whether there is a homomorphism $f : G \to H$.*

Clearly, the H-COLORING problem subsumes the well-known k-COLORING problem, $k \geq 1$, that asks for a coloring of the vertices of a graph using at most k colors such that each pair of adjacent vertices are assigned different colors. Indeed, it corresponds to the case where $H = K_k$, the complete graph (clique) of size k.

Hell and Nešetřil [7] showed that the H-COLORING problem is in $\mathcal{P}$ (the class of problems decidable in polynomial time) whenever H is bipartite, and it is $\mathcal{NP}$-complete otherwise. Our goal is to bring some light into the actual exponential-complexity of the H-COLORING problem when H is non-bipartite. On the one hand, there are already some strong upper-bounds results on the fine-grained complexity of k-COLORING for $k \geq 3$. Björklund *et al.* [1] proved that k-COLORING is solvable in time $O^*(2^n)$ (i.e.. $O(2^n \times n^{O(1)})$) where n is the number of vertices of the input graph. Fomin *et al.* [5] also prove that C_{2k+1}-COLORING is solvable in time $O^*(\binom{n}{n/k}) = O^*((\alpha_k)^n)$, with $\alpha_k \underset{k\to\infty}{\longrightarrow} 1$, and improved algorithms are also known when H has bounded *tree-width* or *clique-width* [5], [15]. On the other hand, lower bounds by Fomin *et al.* [4] rule out the existence of a uniform $2^{O(n)}$ time algorithm under the *exponential-time hypothesis* (*i.e.*, that 3-SAT can not be solved in subexponential time).

We notice, however, that there is a lack of general tools for describing fine-grained properties of CSPs, and in particular we lack techniques for comparing $\mathcal{NP}$-hard H-COLORING problems with each other, e.g., via size-preserving reductions. We explore these ideas through an algebraic approach, by investigating algebraic invariants of graphs. For this purpose, viewing H-COLORING as CSP(E_H) is quite useful as it allows us to use the widely studied theory of the complexity of CSP(Γ), since the former is just the special case when $\Gamma = \{R\}$ is a singleton containing a binary, symmetric relation. In particular, it was shown that the fine-grained complexity of CSP(R) only depends on the so-called *partial polymorphisms* of R [9], [3]. Briefly, a *polymorphism* is a higher-arity homomorphism from the relation to the relation itself. Additionally, a polymorphism that is not necessarily everywhere defined is known as a *partial polymorphism*, and we write $\text{Pol}(R)$ (respectively, $\text{pPol}(R)$) for the set of all (partial) polymorphisms of a releation R. If H is a graph, then by $\text{pPol}(H)$, we simply mean the set of partial polymorphisms of the edge relation E_H. It is then known that partial polymorphisms correlate to fine-grained complexity in the sense that if $\text{pPol}(R) \subseteq \text{pPol}(R')$ and if $\text{CSP}(R)$ is solvable in $O^*(c^n)$ time for some $c > 1$ then $\text{CSP}(R')$ is also solvable in $O^*(c^n)$ time [9].

Thus, describing the inclusion structure between sets of the form $\text{pPol}(H)$ would allow us to relate the fine-grained complexity of H-COLORING problems with each other, but, curiously, we manage to prove that *no* non-trivial inclusions of this form exist, suggesting that partial polymorphisms of graphs are not easy to relate via set inclusion. As a follow-up question we also study inclusions of the form $\text{pPol}(H) \subseteq \text{pPol}(R)$ when R is an arbitrary relation, and manage to give a non-trivial condition based on the length of the shortest odd cycle of H. Concretely, we prove that it is possible to find an n-ary relation R with $\text{pPol}(E_H) \subsetneq \text{pPol}(R)$ where $\text{CSP}(R)$ is $\mathcal{NP}$-complete, if and only if H contains an odd-cycle of length at most n. This result suggests that the size of the smallest odd-cycle is an interesting parameter when regarding the complexity of H-COLORING. As observed above, the smaller $\text{pPol}(E_H)$ is, the harder $\text{CSP}(E_H)$ (and thus H-COLORING) is. In other words, the greater the smallest odd-cycle of H is, the easier the H-COLORING problem is. This fact supports the already known algorithms presented in [5].

Despite this trivial inclusion structure, it could still be of great interest to provide a succinct description of $\text{pPol}(H)$ for some noteworthy choices of non-bipartite, core H. As a first step in this project we concentrate on the total polymorphisms of H, and conclude that *projective graphs* [11] appear to be a reasonable class to target since the total polymorphisms of projective cores are *essentially at most unary*. While we do not manage to give a complete description of projective graphs (which would resolve an open conjecture by Okrasa and Rzążewski [13]) we manage to prove that several well-known families of graphs, e.g., cliques, odd-cycles, and other core graphs, are projective. Importantly, our proofs use the algebraic approach and are significantly simpler than existing proofs, and suggest that the algebraic approach might be a cornerstone in completely describing projective graphs.

This paper is organized as follows. In Section II, we recall the basic notions and preliminary results needed throughout the paper. We investigate the order structure of classes of graph (partial) polymorphisms in Section III where we show the aforementioned main results. In Section IV we focus on projective and core graphs and present several representation results. We conclude with a discussion on the ongoing research and observe a few consequences of our results in the form of open conjectures.

II. Preliminaries

Throughout the paper we use the following notation.

For any $n \in \mathbb{N}$, $[n]$ denotes the set $\{1, \ldots, n\}$. For every set V, $n \geq 1$ and $t = (t_1, \ldots, t_n) \in V^n$, $t[i]$ denotes t_i, and given a relation $R \subseteq [k]^n$ for some $k \geq 1$, we write $\text{ar}(R)$ for its arity n. For all $m \geq 1$ and $i \in [m]$, we write $\pi_i^m : V^m \to V$ for the *projection* on the i-th coordinate (the set V will always be implicit in the context).

For H a graph and $V \subseteq V_H$, we denote by $H[V]$ the graph induced by V on H. We use the symbol $\uplus$ to express the disjoint union of sets, and $+$ to express the disjoint union of graphs.

For a unary function $f : V \to V$, and an m-ary function $g : V^m \to V$ we write $f \circ g$ for their *composition* that is defined by $(f \circ g)(x_1, \ldots, x_m) = f(g((x_1, \ldots, x_m)))$, for every $(x_1, \ldots, x_m) \in V^m$.

Also, for $k \geq 3$, K_k and C_k denote respectively a k-clique and a k-cycle.

A. Graph homomorphisms and cores

For two graphs G and H a function $f : V_G \to V_H$ is a *homomorphism* from G to H if $\forall (u, v) \in E_G, (f(u), f(v)) \in E_H$. In this case, f is also called an H*-coloring* of G, and we denote this fact by $f : G \to H$. The graph G is said to be H*-colorable*, which we denote by $G \to H$, if there exists $f : G \to H$. For a graph H, the H-COLORING problem thus asks whether a given graph G is H-colorable.

Theorem 1 ([7])**.** *H-COLORING is in $\mathcal{P}$ whenever H is bipartite, and it is $\mathcal{NP}$-complete, otherwise.*

A key notion in the proof of Theorem 1 is the notion of a *graph core*: let $core(H)$ be the smallest induced subgraph H' of H such that $H \to H'$. The graph H is said to be a *core* if $H = core(H)$. Note that the core of a graph H is unique up to isomorphism and that the problems H-COLORING and $core(H)$-COLORING are equivalent. Thus, for both classical and fine-grained complexity, it is sufficient to consider $core(H)$-COLORING. Moreover, it is not difficult to verify that cliques and odd-cycles are cores. Notice that a graph H is a core if and only if every H-coloring of H is bijective.

For two graphs G and H, we define their *cross product* $G \times H$ as the graph with $V_{G \times H} = V_G \times V_H$ and

$$E_{G \times H} = \{((u_1, v_1), (u_2, v_2)) \mid (u_1, u_2), \in E_G, (v_1, v_2) \in E_H\}.$$

Clearly, for graphs A,B and C, we have that $(V_A \times V_B) \times V_C$ and $V_A \times (V_B \times V_C)$ are in bijection and thus, up to isomorphism, the cross product is associative. Hence, for each $m \geq 1$, we can define $H^m = \underbrace{H \times \ldots \times H}_{m \text{ times}}$.

B. Polymorphisms, pp/qfpp-definitions

Even though the previous definitions apply only to graphs, *i.e.*, binary symmetric and irreflexive relations, we will need to introduce the following notions for relations R of arbitrary arity.

Definition 2. *Let V be a finite set, $n, m \geq 1$ be integers, and let $R \subseteq V^n$ be an n-ary relation on V. A partial function $f : dom(f) \to V$, with $dom(f) \subseteq V^m$, is said to be a* partial polymorphism of R *if for every $n \times m$ matrix $A = (A_{i,j}) \in V^{n\times m}$ such that for every $j \in [m]$, the j-th column $A_{*,j} \in R$ and for every $i \in [n]$, the i-th row $A_{i,*} \in dom(f)$, the column $(f(A_{1,*}), \ldots, f(A_{n,*}))^\top \in R$. In the case when $dom(f) = V^m$, f is a said to be a* total polymorphism *(or just a* polymorphism*) of R. We denote the sets of total and partial polymorphisms of R by* $\mathrm{Pol}(R)$ *and* $\mathrm{pPol}(R)$*, respectively.*

Every (partial) function over a set V is a (partial) polymorphism of both the empty relation (denoted by $\emptyset$) and the equality relation $\mathrm{EQ}_V = \{(x, x) \mid x \in V\}$ over V (or simply EQ when the domain is clear from the context).

For a graph H, we sometimes use $\mathrm{pPol}(H)$ and $\mathrm{Pol}(H)$ instead of $\mathrm{pPol}(E_H)$ and $\mathrm{Pol}(E_H)$, where E_H is viewed as a binary relation over the domain V_H. Note that $\mathrm{Pol}(H)$ is exactly the set of H-colorings of H^m for $m \geq 1$, and that $\mathrm{pPol}(H)$ is exactly the set of H-colorings of the induced subgraphs of H^m for $m \in \mathbb{N}$.

Definition 3. *Let R be a relation over a finite domain V. An n-ary relation R' over V is said to have a* primitive positive-definition *(pp-definition) w.r.t. R if there exists $m, m', n' \in \mathbb{N}$ such that*

$$R'(x_1, \ldots, x_n) \equiv \exists x_{n+1}, \ldots, x_{n+n'}\colon R(\mathbf{x}_1) \wedge \ldots \wedge R(\mathbf{x}_m) \wedge EQ(\mathbf{y}_1) \wedge \ldots \wedge EQ(\mathbf{y}_{m'}) \quad (1)$$

where each $\mathbf{x}_i$ is an $\mathrm{ar}(R)$-ary tuple and each $\mathbf{y}_i$ is an binary tuple of variables from $x_1, \ldots, x_n$, $x_{n+1}, \ldots, x_{n+n'}$. Each term of the form $R(\mathbf{x}_i)$ or $EQ(\mathbf{y}_j)$ is called an atom *or a* constraint *of the pp-definition* (1)*.*

In addition, if $n' = 0$, then in (1) is called a *quantifier-free primitive positive-definition* (qfpp-definition) of R'. Let $\langle R \rangle_{\nexists}$ and $\langle R \rangle$ be the sets of qfpp-definable and of pp-definable, respectively, relations over R.

Theorem 4 ([14])**.** *Let R and R' be two relations over the same finite domain. Then (1) $R' \in \langle R \rangle_{\nexists}$ if and only if $\mathrm{pPol}(R) \subseteq \mathrm{pPol}(R')$ and (2) $R' \in \langle R \rangle$ if and only if $\mathrm{Pol}(R) \subseteq \mathrm{Pol}(R')$.*

C. CSPs and polymorphisms

We now recall the link between the complexity of CSPs and the algebraic tools described in the previous section (recall that the H-COLORING problem is the same problem as $\mathrm{CSP}(E_H)$).

Theorem 5 ([8])**.** *Let R and R' be two relations over the same finite domain where $\mathrm{Pol}(R) \subseteq \mathrm{Pol}(R')$. Then $\mathrm{CSP}(R')$ is polynomial-time many-one reducible to $\mathrm{CSP}(R)$.*

Let R be a relation over a finite domain V. Define: $\mathsf{T}(R) = \inf\{c > 1 : \mathrm{CSP}(R)$ is solvable in time $O^*(c^n)\}$, where n is the number of variables in $\mathrm{CSP}(R)$, (with the notation $O^*(v_n) = O(v_n \times n^{O(1)})$ for all $(v_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$).

Theorem 6 ([9])**.** *Let R and R' be relations over a finite domain V. If $\mathrm{pPol}(R) \subseteq \mathrm{pPol}(R')$, then $\mathsf{T}(R') \leq \mathsf{T}(R)$.*

These two theorems motivate our study of polymorphisms of graphs: since $\mathrm{CSP}(E_H)$ is the same problem as H-COLORING, key information about the fine-grained complexity of H-COLORING is contained in the set $\mathrm{pPol}(E_H)$.

III. The inclusion structure of partial polymorphisms of graphs

In this section we study the inclusion structure of sets of the form $\mathrm{pPol}(H)$ when H is a graph with $V_H = V$ for a fixed, finite set V. In other words, we are interested in describing the set

$$\mathcal{H} = \{\mathrm{pPol}(H) \mid H \text{ is a graph over } V\}$$

partially ordered by set inclusion. Here, one may observe that the requirement that $V_H = V_{H'} = V$ is not an actual restriction. Indeed, if $V_{H'} \subsetneq V$, then we can easily obtain a graph over V simply by adding isolated vertices, with no impact on the set of partial polymorphisms.

A. Trivial inclusion structure

Our starting point is to establish $\mathrm{pPol}(H) \subseteq \mathrm{pPol}(H')$ when H, H' are non-bipartite graphs, since it implies that (1) H-COLORING and H'-COLORING are both $\mathcal{NP}$-complete, and (2) $\mathsf{T}(H') \leq \mathsf{T}(H)$, i.e., that H'-COLORING is not strictly harder than H-COLORING.

Inclusions of this kind e.g. raise the question whether there exist, for every fixed finite domain V, an $\mathcal{NP}$-hard H-coloring problem which is (1) maximally easy, or (2) maximally hard[2].

As we will soon prove, the set $\mathcal{H}$ does not admit any non-trivial inclusions, in the sense that $\mathrm{pPol}(H) \subseteq \mathrm{pPol}(H')$ implies that either $H = H'$ or $E_{H'} = \emptyset$, for *all* $\mathrm{pPol}(H), \mathrm{pPol}(H') \in \mathcal{H}$.

Theorem 7. *Let H and H' be two graphs with the same finite domain $V_H = V_{H'} = V$. Then $\mathrm{pPol}(H) \subseteq \mathrm{pPol}(H')$ if and only if $H = H'$ or $E_{H'} = \emptyset$.*

Proof. To prove sufficiency, assume that $H = H'$ or that H' has no edges. Then $\mathrm{pPol}(H) = \mathrm{pPol}(H')$ or $\mathrm{pPol}(H) \subseteq \mathrm{pPol}(H')$ since in the latter case $\mathrm{pPol}(H')$ contains every partial function.

To prove necessity, assume that $\mathrm{pPol}(H) \subseteq \mathrm{pPol}(H')$. Then, by Theorem 4, E_H qfpp-defines $E_{H'}$. However, the only possible atoms using E_H and two variables x and y are: (1) $E_H(x, x)$ and $E_H(y, y)$, which cannot appear by irreflexivity, unless $E_{H'} = \emptyset$ and (2) $E_H(x, y)$ and $E_H(y, x)$, which are equivalent by symmetry. Also, if the qfpp-definition would contain an equality constraint $\mathrm{EQ}(x, y)$, then $E_{H'}$ would not be irreflexive, unless $E_{H'} = \emptyset$. Hence, any qfpp-definition of $E_{H'}$ either (1) contains $E_H(x, x)$, $E_H(y, y)$ or $\mathrm{EQ}(x, y)$, meaning that $E_{H'} = \emptyset$, or (2) only contains $E_H(x, y)$ or $E_H(y, x)$, meaning that $H = H'$. $\square$

[2]Here, "maximally" refers to the function T.

B. Higher-arity inclusions

As proven in Theorem 7, the expressivity of binary irreflexive symmetric relations is rather limited, in the sense that $\mathcal{H}$ does not admit *any* non-trivial inclusions. It is thus natural to ask whether anything at all can be said concerning inclusions of the form $\text{pPol}(H) \subseteq \text{pPol}(R)$ when R is an arbitrary relation. In particular, under which conditions does there exist an n-ary R such that $\text{pPol}(H) \subsetneq \text{pPol}(R)$, given that H-COLORING and CSP(R) are both $\mathcal{NP}$-complete? We give a remarkably sharp classification and, assuming that $\mathcal{P} \neq \mathcal{NP}$, we prove that an n-ary relation R with the stated properties exists if and only if H contains an odd-cycle of length $\leq n$.

The following definition and lemma are particularly usefull when establishing our classification.

Definition 8. *Let $n, m \geq 1$ be integers, H be a graph, R be a relation of arity n over V_H, and let $M = (M_{i,j})$ be a $n \times m$ matrix of elements of V_H. We say that M is an R-*wall *for H if:*

1) $\forall j \in [m],\ (M_{1,j}, ..., M_{n,j})^\top \in R$, and
2) $\forall (i,i') \in [n]^2,\ \exists j \in [m],\ (M_{i,j}, M_{i',j})^\top \notin E_H$.

In the following lemma, we say that, for a relation R, that CSP(R) is *trivial* if every instance of CSP(R) is satisfiable. Clearly, if CSP(R) is trivial, it is not $\mathcal{NP}$-complete.

Lemma 9. *Let H a graph and let R be an n-ary relation over V_H. Suppose that* $\text{pPol}(H) \subseteq \text{pPol}(R)$ *and that there exists an R-wall M for H. Then, CSP(R) is trivial.*

Proof. Using property 2) of Definition 8, it is easy to check that any partial function f whose domain is the set of rows of M is in $\text{pPol}(H)$. In particular, f can be chosen to be of constant value $a \in V_H$. Then, from $\text{pPol}(H) \subseteq \text{pPol}(R)$ it follows that $f \in \text{pPol}(R)$. Combining this with property 1) of Definition 8, we conclude that $(a, \ldots, a)^\top \in R$. Since the valuation sending all variables to a satisfies any instance of CSP(R), the proof is now complete. ☐

We now propose a construction of an R-wall for a graph H without odd-cycles of length at most $n := \text{ar}(R)$, and such that $\text{pPol}(H) \subsetneq \text{pPol}(R)$.

Lemma 10. *Let H be a graph without odd-cycles of length $\leq n$, and let $R \neq \emptyset$ be an n-ary relation such that* $\text{pPol}(H) \subseteq \text{pPol}(R)$. *If $\forall (x_1, \ldots, x_n) \in (V_H)^n, R(x_1, \ldots, x_n) \implies E_H(x_1, x_2)$, then R qfpp-defines E_H.*

Proof. Suppose that $R(x_1, \ldots, x_n) \implies E_H(x_1, x_2)$, and let $(a_1, \ldots, a_n) \in R \neq \emptyset$. Since H has no odd-cycle of size $\leq n$, $\{a_1, \ldots a_n\}$ induces a bipartite graph in H: there exists a partition $A \uplus B$ of $\{a_1, \ldots, a_n\}$ such that $E_{H[\{a_1,\ldots,a_n\}]} \subseteq (A \times B) \uplus (B \times A)$. For $(x, y) \in E_H$, define $f\colon \{a_1, \ldots, a_n\} \to V_H$ by $f(a_i) = x$, if $a_i \in A$, and $f(a_i) = y$, if $a_i \in B$. Since $(x, y) \in E_H$, we have that $f \in \text{pPol}(H)$, and since $\text{pPol}(H) \subseteq \text{pPol}(R)$, we also have that $f \in \text{pPol}(R)$. As $(a_1, \ldots, a_n) \in R$, it follows $(f(a_i))_{1 \leq i \leq n} \in R$.

This proves that $E_H(x, y) \implies R(\mathbf{x}_{A,B}(x, y))$, where $\mathbf{x}_{A,B}(x, y)[i] = x$ if $a_i \in A$ and $\mathbf{x}_{A,B}(x, y)[i] = y$ if $a_i \in B$. Reversely, $R(x_1, \ldots, x_n) \implies E_H(x_1, x_2)$ and $(a_1, a_2) \in E_H$ causes $R(\mathbf{x}_{A,B}(x, y)) \implies E_H(x, y)$. Hence, $E_H(x, y) \equiv R(\mathbf{x}_{A,B}(x, y))$, and R qfpp-defines E_H. ☐

Lemma 11. *Let $n \geq 1$, H be a graph without odd-cycles of length $\leq n$, and let $R \neq \emptyset$ be an n-ary relation such that* $\text{pPol}(H) \subsetneq \text{pPol}(R)$. *Then, for all $(i, i') \in [n]^2$ with $i < i'$, there is $(x_1^{(i,i')}, \ldots, x_n^{(i,i')})^\top \in R$ with $(x_i^{(i,i')}, x_{i'}^{(i,i')})^\top \notin E_H$.*

Proof. We show only the existence for $i = 1$ and $i' = 2$; the other cases can be proven similarly. For the sake of a contradiction, suppose that $\forall (x_1, \ldots, x_n) \in (V_H)^n, (x_1, \ldots, x_n) \in R \implies (x_1, x_2) \in E_H$. By Lemma 10 we have $E_H \in \langle R \rangle_{\not\exists}$, and by Theorem 4, $\text{pPol}(R) \subseteq \text{pPol}(E_H)$. This contradicts our hypothesis that $\text{pPol}(H) \subsetneq \text{pPol}(R)$. ☐

This leads to the following corollary whose proof provides a simple construction of an R-wall for graph H in the conditions of Lemma 11.

Corollary 12. *Let $n \geq 1$, H be a graph without an odd-cycle of length $\leq n$, and let $R \neq \emptyset$ be an n-ary relation such that* $\text{pPol}(H) \subsetneq \text{pPol}(R)$. *Then there is an R-wall for H.*

Proof. Using the notation of Lemma 11, we can take the $n \times \frac{n(n-1)}{2}$ matrix M, whose $\frac{n(n-1)}{2}$ columns are the $(x_1^{(i,i')}, \ldots, x_n^{(i,i')})^\top$, for each $(i, i') \in [n]^2$ with $i < i'$. ☐

In the following theorem we need to assume that $\mathcal{P} \neq \mathcal{NP}$ simply because if $\mathcal{P} = \mathcal{NP}$, then the relation $R = \emptyset$ trivially satisfies the stated conditions since CSP(R) would be $\mathcal{NP}$-complete. Furthermore, we only consider non-bipartite graphs since otherwise H-COLORING is in $\mathcal{P}$.

Theorem 13. *Assume $\mathcal{P} \neq \mathcal{NP}$. Let H be a non-bipartite graph, and k be the length of the smallest odd-cycle of H. Then there exists an n-ary relation R with* $\text{pPol}(H) \subsetneq \text{pPol}(R)$ *and for which* CSP(R) *is $\mathcal{NP}$-complete if and only if $k \leq n$.*

Proof. We sketch the most important ideas.

Suppose first that $k > n$. In this case, H does not have an odd-cycle of length $\leq n$. Again for the sake of a contradiction, suppose that such a relation R exists. Note that $R \neq \emptyset$ since CSP(R) is $\mathcal{NP}$-complete. Using Corollary 12, there exists an R-wall for H. Then, by Lemma 9, CSP(R) is trivial. This contradicts the fact that CSP(R) is $\mathcal{NP}$-complete, and thus such a relation R does not exist.

Suppose now that $k \leq n$. Define $R(x_1, \ldots, x_n) \equiv E_H(x_1, x_2) \wedge E_H(x_2, x_3) \wedge \ldots \wedge E_H(x_{k-1}, x_k) \wedge E_H(x_k, x_1)$. Since k is the length of the smallest odd-cycle of H, it follows that $R = \{(x_1, \ldots, x_n) \mid (x_1, \ldots, x_k) \text{ forms a } k\text{-cycle}\}$ (the variables $x_{k+1}, \ldots, x_n$ are inessential). Note that this is not true when k is even.

We then proceed as follows. Since E_H qfpp-defines R, we have that $\text{pPol}(H) \subseteq \text{pPol}(R)$, by Theorem 4. Also, the inclusion is strict since, for any edge (x, y) of H, the function $f : \{x, y\} \to V_H$ that maps both x and y to any $a \in V_H$, belongs to $\text{pPol}(R) \setminus \text{pPol}(H)$. Indeed, $f \in \text{pPol}(R)$ because $\{x, y\}^n \cap R = \emptyset$, since it is impossible to form an odd-cycle with only x and y.

To prove that CSP(R) is $\mathcal{NP}$-complete, consider $C_k(H)$, the subgraph of H, with $V_{C_k(H)} = V_H$, and where each edge of H that does not belong to a cycle of length k has been removed. Note that as H contains a k-cycle, $C_k(H)$ also contains a k-cycle. Hence, CSP($E_{C_k(H)}$), which is the same problem as the $C_k(H)$-COLORING problem, is $\mathcal{NP}$-hard since $C_k(H)$ is not bipartite (by Theorem 1).

It is easy to see that R pp-defines $E_{C_k(H)}(x_1, x_2) \equiv \exists x_3, \ldots, x_n, R(x_1, x_2, x_3 \ldots, x_n)$. From Theorem 4 and 5, it then follows that CSP(R) is $\mathcal{NP}$-hard. In addition, it is in $\mathcal{NP}$, thus showing its $\mathcal{NP}$-completeness. □

IV. Projective and core graphs

In this section we study the inclusion structure of sets of total polymorphisms. We are particularly interrested in graphs H with small sets of polymorphisms since, intuitively, they correspond to the hardest H-COLORING problems. This motivates the following definitions.

An m-ary function f is said to be *essentially at most unary* if it is of the form $f = f' \circ \pi_i^m$ for some $i \in [m]$ and some unary function f'. Larose [10] says that a graph H is *projective* if every *idempotent polymorphism* (i.e., $f(x, \ldots, x) = x$ for every $x \in V_H$) is a projection. Okrasa and Rzążewski [13] showed that the polymorphisms of a core graph H are all essentially at most unary if and only if H is projective. Since it is sufficient to study cores in the context of H-COLORING, determining whether H is projective is particularly interesting.

In this section we use the algebraic approach for proving that a given graph is a projective core, that is, both projective and a core. As we will see, this enables simpler proofs than those of [10], and suggests the possibility of completely characterizing projective cores.

Using the following theorem, our proofs of projectivity can be seen as reductions from cliques.

Theorem 14 ([2],[12])**.** *For $k \geq 3$, K_k is projective.*

Let $\mathfrak{S}_k$ be the set of permutations over $[k]$. It then follows that $\text{Pol}(K_k) = \{\sigma \circ \pi_i^m \mid \sigma \in \mathfrak{S}_k, m \geq 1, i \in [m]\}$. Corollary 15 below implies that the graphs we will consider in this subsection are projective cores.

Corollary 15. *Let H be a graph on $[k]$ with $k \geq 3$. Then E_H pp-defines the relation $NEQ_k = \{(x, x') \in V_H \mid x \neq x'\}$ if and only if H is a projective core.*

Proof. First observe that $\text{NEQ}_k = E_{K_k}$. From Theorem 4 and using the definitions of cores and of projective graphs, we thus have that the following assertions are equivalent:

1) $\text{NEQ}_k \in \langle E_H \rangle$;
2) $\text{Pol}(H) \subseteq \text{Pol}(K_k)$;
3) all polymorphisms of H are essentially at most unary, and all unary polymorphisms of H are bijective;
4) H is a projective core. □

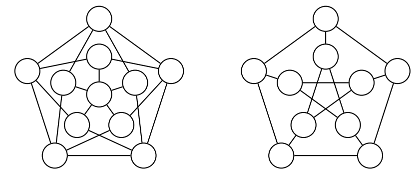

Figure 1. The Grötzsch graph (left) and the Petersen graph (right)

Following exact steps in the proof of Corollary 15, we can obtain the following result.

Corollary 16. *Let G and H be two graphs on the same set of vertices, with G projective (respectively, a core), and such that E_H that pp-defines E_G. Then H is also projective (respectively, a core).*

Pp-definitions thus explains the property of being projective (respectively, a core). We hope that this viewpoint helps to discover new classes of projective graphs. For example, Corollary 15 enables a much simpler proof of the following theorem by Larose [10].

Theorem 17 ([10],[11])**.** *Let $k \geq 3$ be an odd integer. The k-cycle C_k is a projective core.*

Proof. We claim that

$$\text{NEQ}_k(x, x') \equiv \exists x_2, \ldots, x_{k-2} \colon E_{C_k}(x, x_2) \wedge E_{C_k}(x_2, x_3) \wedge \ldots \wedge E_{C_k}(x_{k-3}, x_{k-2}) \wedge E_{C_k}(x_{k-2}, x').$$

To see this, note that for any two vertices x and x' in C_k, $x \neq x'$ if and only if there exists an odd-path from x to x' of size $< k$ (since k is odd). In other words, $x \neq x'$ if and only if there exists a $(k-2)$-path from x to x' (by going through the same edge as many times as necessary, $k-2$ being odd). By Corollary 15, it then follows that C_k is a projective core. □

There are also other examples of cores that are projective, other than k-cliques for $k \geq 3$ and k-cycles. For instance, Okrasa and Rzążewski [13] proved that the so-called *Grötzsch graph* (see Figure 1) is a projective core.

Theorem 18. *The Grötzsch graph is a projective core.*[3]

Proof. We provide an alternative proof using our algebraic framework. Let E_G be the set of edges of the Grötzsch graph. Note that the Grötzsch graph has 11 vertices. We can see that E_G pp-defines NEQ_{11}:

$$\text{NEQ}_{11}(x, x') \equiv \exists x_2, x_3 \colon E_G(x, x_2) \wedge E_G(x_2, x_3) \wedge E_G(x_3, x').$$

[3]We acknowledge Mario Valencia-Pabon for pointing out that the same result and proof applies to the Petersen graph.

From Corollary 15 it follows that the Grötzsch graph is a projective core. □

Complements $\overline{C_k}$ of odd-cycles of length $k \geq 5$ are also projective cores. Since $\overline{C_5} = C_5$ has already been studied, we take a look at $\overline{C_{2p+1}}$, for $p \geq 3$. The following result is an immediate corollary of Larose [10], but we give a algebraic proof using Corollary 15.

Theorem 19. *$\overline{C_{2p+1}}$ is a projective core for $p \geq 3$.*

Proof. It is not difficult to see that $\mathrm{NEQ}_{2p+1}(x_1, x_4) \equiv \exists x_2, x_3, w_1, \ldots, w_{p-2} : R_1 \wedge R_2 \wedge R_3$, where

1) $R_1 = \bigwedge_{i \in [3]} E_{\overline{C_{2p+1}}}(x_i, x_{i+1})$,
2) $R_2 = \bigwedge_{i \in [4], j \in [p-2]} E_{\overline{C_{2p+1}}}(x_i, w_j)$, and
3) $R_3 = \bigwedge_{(j,j') \in [p-2]^2, j<j'} E_{\overline{C_{2p+1}}}(w_j, w_{j'})$.

The result then follows from Corollary 15. □

V. Conclusion and Future Research

A. Concluding remarks

In this paper, we have investigated the inclusion structure of the sets of partial polymorphisms of graphs, and proved that for all pairs of graphs H,H' on the same set of vertices, $\mathrm{pPol}(H) \subseteq \mathrm{pPol}(H')$ implies that $H = H'$ or $E_{H'} = \emptyset$. Since this inclusion structure is trivial, it is natural to generalize the question and investigate inclusions of the form $\mathrm{pPol}(H) \subsetneq \mathrm{pPol}(R)$, where H is a graph, but where R is an arbitrary relation. We deemed the case when $\mathrm{CSP}(R)$ was $\mathcal{NP}$-complete to be of particular interest since the problem $\mathrm{CSP}(R)$ then bounds the complexity of H-COLORING from below, in a non-trivial way. We then identified a condition depending on the length of the shortest odd cycle in H, and proved that there exists a such an n-ary relation R if and only if H does not have an odd cycle of length $\leq n$. In an attempt to better understand the algebraic invariants of graphs, we then proceeded by studying total polymorphisms of graphs, with a particular focus on projective graphs, where we used the algebraic approach to obtain simplified and uniform proofs.

B. Future research

Okrasa and Rzążewski [13] observed that a graph H that can be expressed as a disjoint union of two non-empty graphs H_1 and H_2 is not projective, since it admits the binary polymorphism f defined by $f|_{V_{H_1} \times V_H} = (\pi_1^2)|_{V_{H_1} \times V_H}$ and $f|_{V_{H_2} \times V_H} = (\pi_2^2)|_{V_{H_2} \times V_H}$. The same holds for the *cross-product* of non-trivial graphs $H = H_1 \times H_2$ (in which case the graph is said to be decomposable), with the binary polymorphism $f((x, y), (x', y')) \mapsto (x, y')$. Okrasa and Rzążewski also noticed the existence of disconnected cores, such as $G + K_3$ (indecomposable cores are much more difficult to study), where G is the Grötzsch graph from Figure 1. These observations resulted in the following conjecture.

Conjecture 1 ([13]). *Let H be a connected non-trivial core on at least 3 vertices. Then H is projective if and only if it is indecomposable.*

It would be interesting to see if our algebraic approach can settle this conjecture. In fact, by Corollary 15, Conjecture 1 is equivalent to:

Conjecture 2. *Let H be a connected core on $k \geq 3$ verices. Then, H is indecomposable if and only if $\mathrm{NEQ}_k \in \langle E_H \rangle$.*

To advance our understanding of the fine-grained complexity of H-COLORING, it would also be interesting to settle the following question.

Question 20. *Let H be a projective core. Describe* $\mathrm{pPol}(H)$.

For instance, is it possible to relate $\mathrm{pPol}(H)$ with the *treewidth* of H? More generally, are there structural properties of classes of (partial) polymorphisms that translate into bounded width classes of graphs [6]? These questions constitute topics that we are currently investigating.

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References

[1] A. Björklund, T. Husfeldt, and M. Koivisto. Set partitioning via inclusion-exclusion. *SIAM J. Computing*, 39(2):546–563, 2009.

[2] M. Bodirsky. Graph homomorphisms and universal algebra course notes. *TU Dresden*, 2015.

[3] M. Couceiro, L. Haddad, and V. Lagerkvist. Fine-grained complexity of constraint satisfaction problems through partial polymorphisms: A survey. In *2019 IEEE 49th ISMVL 2019*, pages 170–175. IEEE, 2019.

[4] F. V. Fomin, A. Golovnev, A. S. Kulikov, and I. Mihajlin. Lower bounds for the graph homomorphism problem. In *Proceedings of ICALP 2015*, volume 9134 of *Lecture Notes in Computer Science*, pages 481–493. Springer, 2015.

[5] F. V. Fomin, P. Heggernes, and D. Kratsch. Exact algorithms for graph homomorphisms. *Theory of Computing Systems*, 41(2):381–393, 2007.

[6] S. Guillemot and D. Marx. Finding small patterns in permutations in linear time. In *Proceedings of SODA 2014*, pages 82–101. SIAM, 2014.

[7] P. Hell and J. Nešetřil. On the complexity of h-coloring. *Journal of Combinatorial Theory, Series B*, 48(1):92–110, 1990.

[8] P. Jeavons. On the algebraic structure of combinatorial problems. *Theoretical Computer Science*, 200:185–204, 1998.

[9] P. Jonsson, V. Lagerkvist, G. Nordh, and B. Zanuttini. Strong partial clones and the time complexity of SAT problems. *J. Computer and System Sciences*, 84:52 – 78, 2017.

[10] B. Larose. Families of strongly projective graphs. *Discussiones Mathematicae Graph Theory*, 22(2):271–292, 2002.

[11] B. Larose and C. Tardif. Strongly rigid graphs and projectivity. *J. Multiple-Valued Logic*, 7:339–361, 2001.

[12] T. Łuczak and J. Nešetřil. Note on projective graphs. *J. Graph Theory*, 47(2):81–86, 2004.

[13] K. Okrasa and P. Rzążewski. Fine-grained complexity of the graph homomorphism problem for bounded-treewidth graphs. *SIAM J. Comput.*, 50(2):487–508, 2021.

[14] R Pöschel. Galois connections for operations. *Galois Connections and Applications*, 565:231, 2013.

[15] M. Wahlström. New plain-exponential time classes for graph homomorphism. *Theory of Computing Systems*, 49(2):273–282, 2011.