An Interval Observer for Continuous-Time Persidskii Systems

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***Abstract*—The paper deals with design of interval observers for a class of generalized Persidskii systems. The conditions of stability for the suggested nonlinear interval observer are formulated using linear matrix inequalities. The nonnegativity of this class of models is investigated for nonlinearities satisfying the incremental passivity conditions. The efficiency of the proposed observer is demonstrated on a Lotka-Volterra model.**

I. Introduction

There are many application domains, where estimation or control tasks have to be solved under presence of important nonlinearity and uncertainty of the process models. To give a few popular examples: systems biology and neuroscience, where identification of parameters and modeling are very complicated [1], or collaborative robotics, where presence of multiple agents (including humans) introduces a lot of uncertainty [2]. Moreover, finding a generic control or estimation solution in the nonlinear setting is difficult, and frequently various canonical models are considered, *e.g.*, linear parameter-varying systems, Lur'e models [3], [4], homogeneous dynamics.

Usually, a state observer contains a copy of the process dynamics, then in the presence of uncertain parameters or/and external disturbances, such an estimator design, which has to follow the ideal value of the state, can be realized under restrictive assumptions only [5]. However, an interval observation remains more feasible since this kind of estimators using input-output information calculates only a set of admissible values (interval) for the state at each instant of time [6], [7]. It is worth highlighting that the interval estimation is not a reduction of the original observation problem, in fact it is an improvement since the interval mean value can be utilized as the state pointwise estimate, while the interval bounds provide an evaluation of the observation accuracy for the assumed model uncertainty. There are several methods to design interval/set-membership estimators [8], [9], [10], and this work deals with a design based on the theory of nonnegative systems [11], [12], [13], [14]. In such a way the main difficulty for synthesis consists in ensuring simultaneously the stability and nonnegativity of the interval estimation error dynamics by a proper design of the observer gains.

The objective of this paper is to propose an interval observer for a class of Persidskii systems. This class of nonlinear models was first introduced for stability analysis in [15], where a linear combination of the integrals of the nonlinearities was used as a Lyapunov function. Next, that result was extended in [16] by augmenting the Lyapunov function through a combination of the absolute values of the states. Further, this class of nonlinear systems was studied in the context of diagonal stability [17], [18], sliding mode control [19], [20], [21] and Lur'e systems [3], with applications to opinion dynamics [22], neural networks [23], [24], [25] and others [26]. Its main advantage consists in availability of a canonical form of Lyapunov function. Hence, to analyze stability of the suggested nonlinear observer the Lyapunov function method will be used, whose application results in verification of linear matrix inequalities (LMIs). The conditions of nonnegativity for the Persidskii systems are revisited. Finally, the utility of the developed theory is demonstrated on the problem of interval estimation for a population model supporting mutualistic interactions in the Lotka-Volterra form.

Notation

- $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$, where $\mathbb{R}$ is the set of real numbers. $\mathbb{Z}$ is the set of integers, $\mathbb{Z}_+ = \mathbb{Z} \cap \mathbb{R}_+$.
- $|\cdot|$ denotes the absolute value in $\mathbb{R}$, $\|\cdot\|$ is used for the Euclidean norm on $\mathbb{R}^n$.
- For a measurable function $d : \mathbb{R}_+ \to \mathbb{R}^m$ and $[t_0, t_1) \subset \mathbb{R}_+$ define the norm $\|d\|_{[t_0,t_1)} = \text{ess sup}_{t\in[t_0,t_1)}\|d(t)\|$, then $\|d\|_\infty = \|d\|_{[0,+\infty)}$ and the set of d with the property $\|d\|_\infty < +\infty$ we further denote as $\mathcal{L}^m_\infty$ (the set of essentially bounded measurable functions).
- A continuous function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}$ if $\alpha(0) = 0$ and the function is strictly increasing. The function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and it is increasing to infinity.
- A finite series of integers $1, 2, ..., n$ is denoted by $\overline{1,n}$, and $\{\overline{1,n}\} = \{1, 2, ..., n\}$.
- Denote the identity matrix of dimension $n\times n$ by I_n, the vector of dimension n or the matrix of dimension $n\times m$ with all elements equal 1 by $\mathtt{1}_n$ and $\mathtt{1}_{n\times m}$, respectively. The canonical basis vectors in $\mathbb{R}^n$ are denoted as $\mathfrak{e}_i =$

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$[0 \dots 0\ 1\ \dots 0]^\top$ for $i = \overline{1,n}$, where 1 appears in the i^{th} position.

- $\mathrm{diag}\{g\} \in \mathbb{D}^n_+$ represents a diagonal matrix of dimension $n \times n$ with a vector $g \in \mathbb{R}^n_+$ on the main diagonal, where $\mathbb{D}^n_+ \subset \mathbb{R}^{n\times n}$ is the set of nonnegative diagonal matrices.
- For a matrix $A \in \mathbb{R}^{n\times n}$ the vector of its eigenvalues is denoted by $\lambda(A)$, its i^{th} row and column by $A^{(i)}$ and $A^{[i]}$, respectively, for $i = \overline{1,n}$; $\|A\|_{\max} = \max_{i,j=\overline{1,n}} |A_{i,j}|$ (the elementwise maximum norm, it is not sub-multiplicative) and $\|A\|_2 = \sqrt{\max_{i=\overline{1,n}} \lambda_i(A^\top A)}$ (the induced matrix norm), the relations $\|A\|_{\max} \le \|A\|_2 \le n\|A\|_{\max}$ are satisfied between these norms.
- The relation $P \prec 0$ ($P \succ 0$) means that a symmetric matrix $P \in \mathbb{R}^{n\times n}$ is negative (positive) definite.

II. Preliminaries

The used standard stability notions and their definitions can be found in [27].

In this paper, it is assumed that if the upper limit of a summation or a sequence is smaller than the lower one, then the corresponding terms (conditions) have to be omitted.

A. Interval arithmetic

For two vectors $x_1, x_2 \in \mathbb{R}^n$ or matrices $A_1, A_2 \in \mathbb{R}^{n\times n}$, the relations $x_1 \le x_2$ and $A_1 \le A_2$ are understood elementwise.

Given a matrix $A \in \mathbb{R}^{m\times n}$, define $A^+ = \max\{0, A\}$, $A^- = A^+ - A$ (similarly for vectors) and denote the matrix of absolute values of all elements by $|A| = A^+ + A^-$.

Lemma 1. [28] *Let $x \in \mathbb{R}^n$ be a vector variable, $\underline{x} \le x \le \overline{x}$ for some $\underline{x}, \overline{x} \in \mathbb{R}^n$. If $A \in \mathbb{R}^{m\times n}$ is a constant matrix, then*

$$A^+\underline{x} - A^-\overline{x} \le Ax \le A^+\overline{x} - A^-\underline{x}. \tag{1}$$

B. Nonnegative systems

A matrix $A \in \mathbb{R}^{n\times n}$ is called *Hurwitz* if all its eigenvalues have negative real parts, it is called *Metzler* if all its elements outside the main diagonal are nonnegative.

Any solution of the linear system

$$\dot{x}(t) = Ax(t) + B\omega(t),\ t \ge 0, \tag{2}$$

with $x(t) \in \mathbb{R}^n$ and a Metzler matrix $A \in \mathbb{R}^{n\times n}$, is elementwise nonnegative for all $t \ge 0$ provided that $x(0) \ge 0$, $\omega : \mathbb{R}_+ \to \mathbb{R}^q_+$ and $B \in \mathbb{R}^{n\times q}_+$ [29], [30]. The dynamical systems are called *nonnegative* if for initial conditions in $\mathbb{R}^n_+$ the nonnegativity of the solutions is guaranteed [30].

C. Persidskii systems

Consider the following class of systems [31], [32]:

$$\dot{x}(t) = A_0x(t) + \sum_{j=1}^{M} A_j f^j(H_jx(t)) + Bu(t) + d(t), \tag{3}$$

$$y(t) = \begin{bmatrix} C_0x(t) \\ C_1f^1(H_1x(t)) \\ \vdots \\ C_Mf^M(H_Mx(t)) \end{bmatrix} + v(t),\ t \ge 0,$$

where $x(t) = [x_1(t) \dots x_n(t)]^\top \in \mathbb{R}^n$ is the state vector, $x(0) \in \mathbb{R}^n$; $u(t) \in \mathbb{R}^m$ is the known input, $u \in \mathcal{L}^m_\infty$; $y(t) \in \mathbb{R}^p$ is the output signal, $p = \sum_{k=0}^{M} p_k$ and $C_g \in \mathbb{R}^{p_g\times k_g}$ for $g = \overline{0,M}$, which is available for measurements and containing the perturbation $v(t) \in \mathbb{R}^p$, $v \in \mathcal{L}^p_\infty$; and $d(t) \in \mathbb{R}^n$ is the external disturbance, $d \in \mathcal{L}^n_\infty$; $f^j : \mathbb{R}^{k_j} \to \mathbb{R}^{k_j}$ with diagonal structure $f^j(s) = [f^j_1(s_1) \dots f^j_{k_j}(s_{k_j})]^\top$, $j = \overline{1,M}$ are the continuous functions ensuring existence and uniqueness of solutions in (3) for $t \ge 0$, the matrices $A_g \in \mathbb{R}^{n\times k_g}$ and $H_g \in \mathbb{R}^{k_g\times n}$ for $g = \overline{0,M}$, where for brevity we use the convention $k_0 = n$ and $H_0 = I_n$ with $f^0(x) = x$.

The model (3) belongs to the class of *Persidskii system* [16], [17] under the following passivity (or sector) condition imposed on f^j: for any $j = \overline{1,M}$ and $i = \overline{1,k_j}$:

$$sf^j_i(s) > 0 \quad \forall s \in \mathbb{R} \setminus \{0\}.$$

In the property above it is stated that all nonlinearities belong to a sector and may take zero values at zero only. If $H_1 = I_n$ and $A_r = 0$ for all $r = \overline{2,M}$, then we recover the system studied by Persidskii in [16]. In the case of $M = 1$, (3) belongs also to the class of Lur'e systems widely investigated in the absolute stability theory [4]. In this work we will need a more strong *incremental passivity* (or monotonicity) property ensured by the following assumption:

Assumption 1. *For any $j = \overline{1,M}$ and $i = \overline{1,k_j}$:*

$$(s_1 - s_2)\left(f^j_i(s_1) - f^j_i(s_2)\right) > 0 \quad \forall s_1, s_2 \in \mathbb{R},\ s_1 \ne s_2.$$

Taking $s_2 = 0$, and since $f^j_i(0) = 0$, Assumption 1 implies the sector condition, while additionally asking certain monotonicity of the nonlinearities (it is also called incremental passivity of f^j_i).

After a proper re-indexing and decomposition of f^j, there exists $\nu \in \{\overline{0,M}\}$ such that for all $z = \overline{1,\nu}$ and $i = \overline{1,k_z}$:

$$\lim_{s\to\pm\infty} f^z_i(s) = \pm\infty;$$

and due to Assumption 1 for all $j = \overline{1,M}$ and $i = \overline{1,k_j}$:

$$\lim_{s\to\pm\infty} \int_0^s f^j_i(\sigma)d\sigma = +\infty.$$

Thus, some of the nonlinearities are radially unbounded, and $\nu = 0$ corresponds to the case when all nonlinearities are bounded (at least for negative or positive argument).

III. Problem statement

Consider the system (3). The following assumptions will be used in this work.

Assumption 2. *Let $x(0) \in [\underline{x}_0, \overline{x}_0]$ for known $\underline{x}_0, \overline{x}_0 \in \mathbb{R}^n$.*

Assumption 3. *There exists known signals $\underline{d}, \overline{d} \in \mathcal{L}_\infty^n$ and a constant $V > 0$ such that $\underline{d}(t) \le d(t) \le \overline{d}(t)$ and $-V\mathtt{1}_p \le v(t) \le V\mathtt{1}_p$ for all $t \ge 0$.*

Assumptions 2 and 3 represent standard hypotheses for the interval estimation [6], [7].

The objective of this work is to design an *interval observer* for the system (3), which is a dynamical system that utilizes the properties stated in assumptions 1–3 (*i.e.*, takes the information on the initial conditions $[\underline{x}_0, \overline{x}_0]$, the admissible bounds on the values of the exogenous input $[\underline{d}(t), \overline{d}(t)]$, the upper bound on the measurement noise V), the information about the system matrices A_k, H_k, B and C_k for $k = \overline{0, M}$, and generates interval estimates $\underline{x}(t), \overline{x}(t) \in \mathbb{R}^n$ such that $\overline{x} - \underline{x} \in \mathcal{L}_\infty^n$ and

$$\underline{x}(t) \le x(t) \le \overline{x}(t) \quad \forall t \ge 0. \tag{4}$$

Further, we design an interval observer for the system (3) under an additional assumption given in Section V.

IV. Nonnegativity in Persidskii systems

In the simplest disturbance-free case of (3) with $M = 1$ and $H_1 = I_n$ (the conventional form of Persidskii systems):

$$\dot{x}(t) = A_0 x(t) + A_1 f^1(x(t)) + Bu(t),$$

the nonnegativity conditions can be similar to the ones for (2) [16], [33]: the matrices A_0 and A_1 are Metzler and $Bu(t) \in \mathbb{R}^m_+$ for all $t \ge 0$. However, for $H_1 \ne I_n$ the conditions become more sophisticated:

Lemma 2. *For any $x(0) \in \mathbb{R}^n_+$, the special case of* (3)*,*

$$\dot{x}(t) = A_1 f^1(H_1 x(t)),$$

satisfying Assumption 1, has $x(t) \ge 0$ for all $t \ge 0$, provided that for all $i = \overline{1, k_1}$ one of the following properties is verified:

i) $H_1^{(i)} \ge 0$ and $A_1^{[i]} \ge 0$,

ii) $H_1^{(i)} \le 0$ and $A_1^{[i]} \le 0$.

Moreover, if $H_1^{(i)} = \kappa \mathfrak{e}_r^\top$ for some $\kappa \in \mathbb{R}$, $i \in \{\overline{1, k_1}\}$ and $r \in \{\overline{1, n}\}$, then $(A_1)_{r,i}$ can be arbitrary.

All proofs are skipped due to space limitations.

The conditions given in Lemma 2 generalize the previous ones: it is easy to check that if $H_1 = I_n$, then A_1 can be chosen Metzler according to Lemma 2.

Since the proposed nonnegativity conditions depend on the pair of matrices A_1 and H_1, then further we will say that

$$(A_1, H_1) \in \mathcal{N}$$

if the restrictions formulated in Lemma 2 are verified (hence, the property $(A_0, H_0) = (A_0, I_n) \in \mathcal{N}$ implies that A_0 is Metzler).

Corollary 1. *Under Assumption 1 the system* (3) *is nonnegative if $(A_k, H_k) \in \mathcal{N}$ for all $k = \overline{0, M}$ and $Bu(t) + d(t) \ge 0$ for all $t \ge 0$.*

Similarly can be formulated conditions for nonnegativity of discrepancy between two solutions of (3) with different initial conditions:

Corollary 2. *Under Assumption 1, consider a copy of* (3)*:*

$$\dot{\xi}(t) = A_0 \xi(t) + \sum_{j=1}^{M} A_j f^j(H_j \xi(t)) + Bu(t) + d(t),$$

with the same inputs u, d and $\xi(0) \in \mathbb{R}^n$ verifying $x(0) \ge \xi(0)$, then $x(t) \ge \xi(t)$ for all $t \ge 0$ provided that $(A_k, H_k) \in \mathcal{N}$ for all $k = \overline{0, M}$.

The conditions of corollaries 1 and 2 will be used later in the design of interval observer.

Remark 1. As in the case of (2), if $(A_k, H_k) \notin \mathcal{N}$, then a transformation of coordinates can be considered. Indeed, the transformation $z = Sx$ with an invertible matrix $S \in \mathbb{R}^{n \times n}$ keeps the Persidskii form of (3) in the new coordinates and [13] can be used to derive S. However, to find such a S providing Metzler property to the matrix $\tilde{A}_0 = SA_0S^{-1}$ and simultaneously $(A_j, H_j) \in \mathcal{N}$ for all $j = \overline{1, M}$ can be difficult. If all matrices A_j are sufficiently close to each other, then the result of [34] can be adapted to derive S.

However, if such a transformation S cannot be found, then interval inclusion of the nonlinearities can be utilized for design of interval observers. Such an inclusion is well developed in the theory of the bounding decomposition functions for mixed monotone mappings [35], which in our case has a simple realization:

Lemma 3. *Let $s, \underline{s}, \overline{s} \in \mathbb{R}^{k_1}$ and $f^1 : \mathbb{R}^{k_1} \to \mathbb{R}^{k_1}$ verify Assumption 1, then*

$$\underline{s} \le s \le \overline{s} \Leftrightarrow f^1(\underline{s}) \le f^1(s) \le f^1(\overline{s}).$$

If $\underline{x} \le x \le \overline{x}$ for some $x, \underline{x}, \overline{x} \in \mathbb{R}^n$, $H_1 \in \mathbb{R}^{k_1 \times n}$ and $f^1 : \mathbb{R}^{k_1} \to \mathbb{R}^{k_1}$ verifies Assumption 1, then $\underline{s} = H_1^+ \underline{x} - H_1^- \overline{x} \le s = H_1 x \le H_1^+ \overline{x} - H_1^- \underline{x} = \overline{s}$ due to (1), and using Lemma 3 we obtain:

$$f^1(H_1^+ \underline{x} - H_1^- \overline{x}) \le f^1(H_1 x) \le f^1(H_1^+ \overline{x} - H_1^- \underline{x}). \tag{5}$$

V. Interval observer design

The system (3) can be equivalently presented as follows:

$$\begin{aligned} \dot{x}(t) \;=\; & (A_0 - L_0 C_0) x(t) + \sum_{j=1}^{M} (A_j - L_j C_j) f^j(H_j x(t)) \\ & + L\,(y(t) - v(t)) + Bu(t) + d(t), \end{aligned}$$

where $L = [L_0 \; L_1 \ldots L_M] \in \mathbb{R}^{n \times p}$, with $L_k \in \mathbb{R}^{n \times p_k}$ for $k = \overline{0, M}$, is an observer gain to be properly chosen next. As

usual [6], [7], such a gain L has to ensure *nonnegativity* and *stability* of the dynamics of the interval estimation errors. Let us focus first on the former property, then following corollaries 1 and 2 we may ask for existence of L providing $(A_k - L_k C_k, H_k) \in \mathcal{N}$ for all $k = \overline{0, M}$, which is however a restrictive requirement. For brevity of presentation we will not introduce a transformation of coordinates discussed in Remark 1, which can relax the conservatism in some cases (synthesis of the transformation matrix S is not an easy task). There is another approach presented in [36] that mixes the advantages of a (partial) transformation of coordinates with simplicity of formulation in the original variables, whose utilization requires the following additional conditions:

Assumption 4. *There exist $\ell \in \{\overline{0, M}\}$, $L \in \mathbb{R}^{n\times p}$ and $N \in \mathbb{R}^{n\times p_0}$ such that $(D_r, H_r) \in \mathcal{N}$ for all $r = \overline{0, \ell}$, where we denote $D_k = TA_k - L_k C_k$ for all $k = \overline{0, M}$ and $T = I_n - NC_0$.*

Obviously, if $\ell = M$ and $T = I_n$ or $N = 0$, then we have just the initial case without any transformation of coordinates. Note that in Remark 1 the transformation S acts on A_j with $j = \overline{1, M}$ in a similar way yielding SA_j for the dynamics of the new variables. The variable ℓ determines the number of terms in (3) that can be well processed by L and N, the choice $\ell \geq 0$ (that the linear part $D_0 = TA_0 - L_0C_0$ is always Metzler) is not obligatory and accepted to streamline the presentation.

Under Assumption 4 the interval observer for (3) can be chosen as follows:

$$
\begin{aligned}
&\underline{x}(t) = \underline{z}(t) + Ny_0 - |N|V,\ \overline{x}(t) = \overline{z}(t) + Ny_0 + |N|V,\\
&\dot{\underline{z}}(t) = \sum_{r=0}^{\ell} \left(D_r f^r(H_r \underline{x}(t)) - \underline{\kappa}_r(\underline{x}(t))\right) + TBu(t) + Ly(t)\\
&+ \sum_{k=\ell+1}^{M} D_k^+ f^k(H_k^+ \underline{x}(t) - H_k^- \overline{x}(t)) - D_k^- f^k(H_k^+ \overline{x}(t) - H_k^- \underline{x}(t))\\
&\qquad -|L|V + T^+ \underline{d}(t) - T^- \overline{d}(t),\\
&\dot{\overline{z}}(t) = \sum_{r=0}^{\ell} \left(D_r f^r(H_r \overline{x}(t)) + \overline{\kappa}_r(\overline{x}(t))\right) + TBu(t) + Ly(t)\\
&+ \sum_{k=\ell+1}^{M} D_k^+ f^k(H_k^+ \overline{x}(t) - H_k^- \underline{x}(t)) - D_k^- f^k(H_k^+ \underline{x}(t) - H_k^- \overline{x}(t))\\
&\qquad +|L|V + T^+ \overline{d}(t) - T^- \underline{d}(t),\\
&\underline{z}(0) = T^+ \underline{x}_0 - T^- \overline{x}_0,\ \overline{z}(0) = T^+ \overline{x}_0 - T^- \underline{x}_0,
\end{aligned}
\tag{6}
$$

where $y_0 \in \mathbb{R}^{p_0}$ represents the first p_0 (linear in the state) measured outputs, $\underline{\kappa}_r, \overline{\kappa}_r : \mathbb{R}^n \to \mathbb{R}^n_+$ are derived using the following rules: if $H_r^{(i)} = \varkappa \mathfrak{e}_g^\top$ for some $\varkappa \in \mathbb{R}$, $i \in \{\overline{1, k_r}\}$ and $g \in \{\overline{1, n}\}$ with $(D_r)_{g,i}\varkappa < 0$, then

$$
\begin{aligned}
\underline{\kappa}_{r,g}(\underline{x}(t)) &= |(D_r)_{g,i}| \left(f_i^r(H_r \underline{x}(t) + 2H_r^+ |N|V) - f_i^r(H_r \underline{x}(t))\right),\\
\overline{\kappa}_{r,g}(\overline{x}(t)) &= |(D_r)_{g,i}| \left(f_i^r(H_r \overline{x}(t)) - f_i^r(H_r \overline{x}(t) - 2H_r^+ |N|V)\right),
\end{aligned}
$$

and the elements of $\underline{\kappa}_r, \overline{\kappa}_r$ are zeros otherwise. The desired interval inclusion of the state is realized by (6) under the introduced hypotheses:

Lemma 4. *Let assumptions 1–4 be satisfied for the system* (3)*, then the interval observer* (6) *ensures the property* (4)*.*

Note that if $T = I_n$ (or $N = 0$), then $\underline{\kappa}_r = \overline{\kappa}_r = 0$ for all $r = \overline{0, \ell}$. To demonstrate that (6) is indeed an interval observer for (3) it is left to prove the boundedness of $\overline{x}(t) - \underline{x}(t)$. To this end denote

$$
\begin{aligned}
&\mathcal{D}_r = \begin{bmatrix} D_r & 0 \\ 0 & D_r \end{bmatrix},\ \mathcal{H}_r = \begin{bmatrix} H_r & 0 \\ 0 & H_r \end{bmatrix},\ r = \overline{0, \ell};\\
&\mathcal{D}_k = \begin{bmatrix} D_k^+ & -D_k^- \\ -D_k^- & D_k^+ \end{bmatrix},\ \mathcal{H}_k = \begin{bmatrix} H_k^+ & -H_k^- \\ -H_k^- & H_k^+ \end{bmatrix},\ k = \overline{\ell+1, M};\\
&\Gamma = \begin{bmatrix} I_n & -I_n \end{bmatrix}.
\end{aligned}
$$

Theorem 1. *Let assumptions 1–4 be satisfied for the system* (3)*, the functions $\underline{\kappa}_r, \overline{\kappa}_r$ be globally bounded for $r = \overline{0, \ell}$ and $v_0(t) \in \mathbb{R}^{p_0}$ (representing the first p_0 elements of the output perturbation v) be continuously differentiable or $N = 0$, then* (6) *is an interval observer provided that there exist $P_1^\top = P_1 \in \mathbb{R}^{2n\times 2n}$, $P_2^\top = P_2 \in \mathbb{R}^{n\times n}$, $\Lambda^j \in \mathbb{D}_+^{2k_j}$ for $j = \overline{1, M}$, $\Phi^\top = \Phi \in \mathbb{R}^{2n\times 2n}$, $\Psi \in \mathbb{R}^{2n\times 2n}$, $\Omega_g \in \mathbb{R}^{2n\times 2k_g}$, $\Upsilon_{g,g} \in \mathbb{D}_+^{2k_g}$ ($\tilde{\Upsilon}_{g,g} = \mathcal{H}_g^\top \Upsilon_{g,g} \mathcal{H}_g$) for $g = \overline{0, M}$, $\Upsilon_{z,s} \in \mathbb{D}_+^{2n}$ ($\tilde{\Upsilon}_{z,s} = \mathcal{H}_z \Upsilon_{z,s} \mathcal{H}_s^\top$) for $z = \overline{0, M-1}$ and $s = \overline{z+1, M}$ such that the following LMIs are satisfied:*

$$
\begin{aligned}
&P_1 \succeq 0,\ P_2 \succ 0,\ \Phi \succ 0,\ Q \preceq 0,\\
&\rho \left(P + 2\sum_{j=1}^{M} \mathcal{H}_j^\top \Lambda^j \mathcal{H}_j\right) \preceq \sum_{g=0}^{\nu} \tilde{\Upsilon}_{g,g} + 2\sum_{z=0}^{\nu-1} \sum_{s=z+1}^{\nu} \tilde{\Upsilon}_{z,s},
\end{aligned}
$$

for some $\rho > 0$, where $P = P_1 + \Gamma^\top P_2 \Gamma$ and

$$
Q = \begin{bmatrix}
-\Psi^\top - \Psi & P + \Psi^\top \mathcal{D}_0 - \Omega_0 & \cdots & P + \Psi^\top \mathcal{D}_M - \Omega_M & \Psi^\top \\
P + \mathcal{D}_0^\top \Psi - \Omega_0^\top & \Omega_0^\top \mathcal{D}_0 + \mathcal{D}_0^\top \Omega_0 + \tilde{\Upsilon}_{0,0} & \cdots & \Omega_0^\top \mathcal{D}_M + \mathcal{D}_0^\top \Omega_M + \tilde{\Upsilon}_{0,M} & \Omega_0^\top \\
\Lambda^1 \mathcal{H}_1 + \mathcal{D}_1^\top \Psi - \Omega_1^\top & \mathcal{D}_1^\top \Omega_0 + \Omega_1^\top \mathcal{D}_0 + \tilde{\Upsilon}_{0,1} & \cdots & \mathcal{D}_1^\top \Omega_M + \Omega_1^\top \mathcal{D}_M + \tilde{\Upsilon}_{1,M} & \Omega_1^\top \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\Lambda^M \mathcal{H}_M + \mathcal{D}_M^\top \Psi - \Omega_M^\top & \mathcal{D}_M^\top \Omega_0 + \Omega_M^\top \mathcal{D}_0 + \tilde{\Upsilon}_{0,M} & \cdots & \mathcal{D}_M^\top \Omega_M + \Omega_M^\top \mathcal{D}_M + \tilde{\Upsilon}_{M,M} & \Omega_M^\top \\
\Psi & \Omega_0 & \cdots & \Omega_M & -\Phi
\end{bmatrix}.
$$

The proof is based on introduction of a common state vector $X = [\underline{x}^\top \ \overline{x}^\top]^\top$, whose dynamics can be presented as an extended Persdiskii system:

$$
\begin{aligned}
&\dot{X}(t) = \mathcal{D}_0 X(t) + \sum_{j=1}^{M} \mathcal{D}_j F^j(\mathcal{H}_j X(t)) + \delta(t),\\
&Y(t) = \Gamma X(t),
\end{aligned}
\tag{7}
$$

$$
F^k(s) = \begin{bmatrix} f^k(s) \\ f^k(s) \end{bmatrix},\ k = \overline{0, M},
$$

where $Y(t) \in \mathbb{R}^n$ is an auxiliary output whose boundedness we would like to analyze in the presence of the input

$$
\delta(t) = \begin{bmatrix} \underline{\delta}(t) \\ \overline{\delta}(t) \end{bmatrix},
$$

$$\underline{\delta}(t) = TBu(t) + Ly(t) + N\dot{y}_0(t) - \sum_{r=0}^{\ell} \underline{\kappa}_r(\underline{x}(t)) - |L|V + T^+\underline{d}(t) - T^-\overline{d}(t),$$

$$\overline{\delta}(t) = TBu(t) + Ly(t) + N\dot{y}_0(t) + \sum_{r=0}^{\ell} \overline{\kappa}_r(\overline{x}(t)) + |L|V + T^+\overline{d}(t) - T^-\underline{d}(t).$$

Following [32], [37], the input-to-output stability property [38] of the system (7) is established by using a Lyapunov function:

$$W(X) = X^\top PX + 2\sum_{j=1}^{M}\sum_{i=1}^{2k_j} \Lambda_i^j \int_0^{\mathcal{H}_j^{(i)}X} F_i^j(s)ds,$$

where $P = P_1 + \Gamma^\top P_2 \Gamma$ and $\Lambda^j = \text{diag}[\Lambda_1^j \dots \Lambda_{2k_j}^j]$.

A necessary condition for feasibility of LMIs given in the formulation of Theorem 1 is that all matrices on the main diagonal of Q are nonnegative definite, *i.e.*, $\Omega_g^\top \mathcal{D}_g + \mathcal{D}_g^\top \Omega_g \preceq 0$ for $g = \overline{0, M}$ with some $\Omega_g \in \mathbb{R}^{2n\times 2k_g}$, which can be easily checked. The result of this theorem also allows the asymptotic gain, for the estimation accuracy $Y = \overline{x} - \underline{x}$ from the uncertainty collected in δ, be evaluated.

VI. Interval estimation for a Lotka-Volterra model

Consider the problem of interval state estimation for a generalized Lotka–Volterra model with a support of mutualistic interactions [39], [40]:

$$\dot{x}(t) = \text{diag}\{x(t)\}\left(d(t) + A_1 x(t) + A_2\varphi(x(t))\right),\ t \geq 0,$$

where $x(t) \in \mathbb{R}^n_+$ is the vector of the populations of n biological species, $x(0) \in \mathbb{R}^n_+$; $d(t) \in \mathbb{R}^n$ corresponds to the intrinsic time-varying birth or death rates of the species, $A_1 \in \mathbb{R}^{n\times n}$ is the community matrix, and $A_2 \in \mathbb{R}^{n\times n}$ is the mutualistic interaction strength between the species; $\varphi(x(t)) = [\varphi_1(x_1)\dots\varphi_n(x_n)]^\top = [\frac{x_1}{r_1+x_1}\dots\frac{x_n}{r_n+x_n}]^\top$ is the vector of Michaelis-Menten functions with $r = [r_1 \dots r_n]^\top \in \mathbb{R}^n_+$ being the half-saturation constants. We assume that the coupling matrices A_1 and A_2 are known, but the rates in d are known with an interval error, *i.e.*, there are $\underline{d}(t), \overline{d}(t) \in \mathcal{L}^n_\infty$ such that

$$\underline{d}(t) \leq d(t) \leq \overline{d}(t), \quad \forall t \geq 0.$$

Without loosing generality assume that the levels of the first $\eta > 0$ populations can be measured:

$$\tilde{y}(t) = Cx(t),\ C = [I_\eta\ 0].$$

Lotka–Volterra model is a highly nonlinear dynamical system, which is not in the form of (3), but it can be brought to it by a transformation of coordinates $z = \ln(x)$ (where logarithm is applied elementwise) with

$$\dot{z}(t) = A_1 f^1(z(t)) + A_2 f^2(z(t)) + u + d(t),$$
$$y(t) = [y_0(t)^\top\ y_1(t)^\top\ y_2(t)^\top]^\top,$$
$$y_0(t) = Cz(t),\ y_1(t) = Cf^1(z(t)),\ y_2(t) = Cf^2(z(t)),$$

where

$$f^1(z) = e^z - 1_n,\ f^2(z) = \varphi(e^z) - c,\ u = A_1 1_n + A_2 c,$$
$$c = [c_1 \dots c_n]^\top,\ c_i = \frac{1}{1+r_i},\ i = \overline{1,n},$$

and Assumption 1 is satisfied for such a choice of f^1 and f^2, the variable $u \in \mathbb{R}^n$ represents the known input. The values of the output $y(t)$ can be calculated from the measured signal $\tilde{y}(t)$ (*i.e.*, $\tilde{y}(t) = Ce^{z(t)}$, then $\ln\tilde{y}(t) = Cz(t) = y_0(t)$ and so on). By assuming that $x(0) \in [\underline{x}_0, \overline{x}_0]$ for some known $\underline{x}_0, \overline{x}_0 \in \mathbb{R}^n_+$, we get $z(0) \in [\ln(\underline{x}_0), \ln(\overline{x}_0)] = [\underline{z}_0, \overline{z}_0]$ and assumptions 2 and 3 are also verified with $V = 0$. To guarantee the fulfillment of Assumption 4 we need to select the observer design gains L and N for given A_1 and A_2 with $A_0 = 0$, which we assume to be successfully done with $\ell = 2$. Hence, the interval observer takes the form (6), and if $N = 0$ in the original coordinates:

$$\dot{\underline{x}}(t) = \text{diag}\{\underline{x}(t)\}[-L_0 C\ln(\underline{x}(t)) + (A_1 - L_1 C)\underline{x}(t) + (A_2 - L_2 C)\varphi(\underline{x}(t)) + L_0\ln(\tilde{y}(t)) + L_1\tilde{y}(t) + L_2\varphi(\tilde{y}(t)) + \underline{d}(t)],$$
$$\dot{\overline{x}}(t) = \text{diag}\{\overline{x}(t)\}[-L_0 C\ln(\overline{x}(t)) + (A_1 - L_1 C)\overline{x}(t) + (A_2 - L_2 C)\varphi(\overline{x}(t)) + L_0\ln(\tilde{y}(t)) + L_1\tilde{y}(t) + L_2\varphi(\tilde{y}(t)) + \overline{d}(t)],$$

where it is worth to highlight the presence of nonlinear output injection terms.

Example. In Fig. 1, the results of simulation are shown (solid lines correspond to the trajectories of the system, dot and dash curves represent the lower and upper interval estimates, respectively) for

$$n = 3,\ \eta = 1,$$
$$A_1 = \begin{bmatrix} -2.68 & 0.50 & 0.16 \\ -0.23 & -2.80 & 0.75 \\ 0.56 & 0.42 & -2.63 \end{bmatrix}, A_2 = \begin{bmatrix} -1.98 & 0.62 & 0.72 \\ -0.05 & -1.46 & 0.09 \\ 0.24 & 0.65 & -1.12 \end{bmatrix},$$
$$r = [1.05\ 1.81\ 1.52]^\top,\ d(t) = \sin(t) + [0.13\ 0.15\ 0.36]^\top,$$

with the bounds

$$\underline{x}_0 = [0.62\ 0.19\ 0.48]^\top,\ \overline{x}_0 = [0.76\ 0.23\ 0.59]^\top,$$
$$\underline{d}(t) = \sin(t) + [0.07\ 0.08\ 0.27]^\top,\ \overline{d}(t) = \sin(t) + [0.20\ 0.21\ 0.45]^\top,$$

and the observer gains:

$$L_0 = [2\ 0\ 0]^\top,\ L_1 = [-0.01\ -0.29\ -0.17]^\top,$$
$$L_2 = [-0.92\ -0.06\ -0.58]^\top.$$

For these values all assumptions are satisfied. As we can see the proposed interval observer generates very reasonable and bounded estimates for a highly nonlinear uncertain system.

VII. Conclusion

The paper addressed the issue of design of interval observers for a class of generalized Persidskii systems. The conditions of nonnegativity in this class of models were established. A possible structure of the interval observer was given, together with respective stability conditions. All conditions are formulated in terms of LMIs. To illustrate the

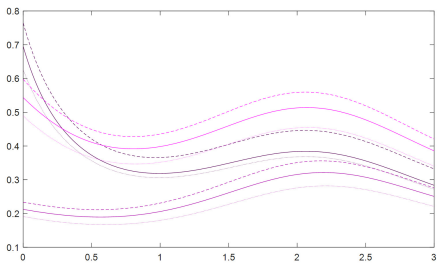


Figure 1. The results of interval estimation by (6) for a Lotka-Volterra model: $x(t), \underline{x}(t), \overline{x}(t)$ versus time t

efficiency of the proposed interval observer, it was applied to a Lotka-Volterra model. Development of these results to interval prediction and model predictive control design as in [41], [42] can be considered as a direction for future research.

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