Flux approximation on unfitted meshes and application to multiscale hybrid-mixed methods

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FLUX APPROXIMATION ON UNFITTED MESHES AND APPLICATION TO MULTISCALE HYBRID-MIXED METHODS*

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ABSTRACT. The flux variable determines the approximation quality of hybridization-based numerical methods. This work proves that approximating flux variables in discontinuous polynomial spaces from the L^2 orthogonal projection is super-convergent on meshes that are not aligned with jumping coefficient interfaces. The results assume only the local regularity of exact solutions in physical partitions. Based on the proposed flux approximation, we demonstrate that the mixed hybrid multiscale (MHM) finite element method is superconvergent in unfitted meshes, supporting the numerics presented in MHM seminal works.

1. INTRODUCTION

Many numerical algorithms rely on their accuracy in approximating flux variables defined on the skeleton of geometric partitions of physical domains. Finite volume methods, discontinuous finite element methods and hybrid finite element methods are examples of numerical methods of this type, but we also find the fundamental importance of flux recovery in some domain decomposition methods. As a result, there has been growing interest in developing discrete fluxes with optimal convergence properties (see [5], [16], [15], and [14] for instance).

In this work, we are interested in retrieving discrete fluxes associated with the solution u of partial differential equations defined in a domain Ω composed of regions ω where the regularity of the solution is high, although u can only have moderate overall regularity. We assume that the geometric partition $\mathcal{T}_\mathcal{H}$ of Ω used to define the discrete flux is general, with characteristic length $\mathcal{H}$, and composed of polytopal elements whose boundary may not fit on the interfaces of ω. Within such a scenario, we demonstrate that the exact flux λ can be accurately approximated through its L^2 orthogonal projection into a discontinuous polynomial space of degree $\ell \geq 0$ on faces of $\partial\mathcal{T}_\mathcal{H}$ the boundary of $\mathcal{T}_\mathcal{H}$, notably,

$$\inf_{\mu_H \in \Lambda_H} \|\lambda - \mu_H\|_{-1/2,\partial\mathcal{T}_\mathcal{H}} = O(H^{\ell+3/2}),$$

where H is a characteristic length associated to the discretization of $\partial\mathcal{T}_\mathcal{H}$. Here, we understand a flux variable λ as the normal component of vector (tensor) functions $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ restricted to the skeleton of $\mathcal{T}_\mathcal{H}$. We have borrowed the term flux from fluid flows, although

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it may represent other physical quantities (e.g., traction in the elasticity model). This aim is similar to that found in the fictitious domain method [3] or in CutFEM [6], to name a few.

We take advantage of the proposed flux approximation to renew the analysis of the mixed hybrid multiscale method (MHM) on unfitted meshes. The MHM method was originally proposed in [11] and a priori and a posteriori error estimates proposed in [2], and extended to polygonal elements in [4] (see [13] for an abstract framework). It is conceived from the primal hybridization of the original model, and characterizes the exact solution in terms of a global formulation placed on the skeleton of a domain partition and independent local problems. Lagrange multipliers play the role of Neumann boundary conditions for local problems. Such decomposition leads to discretization, decouples global and local problems and gives rise to the MHM method. Regarding the analysis, we note that the original technique used to prove that the MHM method converges is fundamentally based on the accuracy of the approximation flux in a polynomial space $\Lambda_{H,\ell}$ of degree ℓ on the boundary partition (see [7] for a recent alternative proof). Specifically, if u_H denotes the MHM solution, the convergence of u_H toward u in the broken H^1 norm $|\cdot|_{1,\mathcal{T}_\mathcal{H}}$ behaves as follows (c.f. [2]): for $0 \leq q \leq \ell$, we have

$$|u - u_H|_{1,\mathcal{T}_\mathcal{H}} \leq C_\ell \inf_{\mu_H \in \Lambda_{H,\ell}} \|\lambda - \mu_H\|_\Lambda \leq C_\ell H^{q+1} |u|_{H^{q+2}(\mathcal{T}_\mathcal{H})},$$

where $\|\cdot\|_\Lambda$ is a norm on space Λ of trace of functions in $\boldsymbol{H}(\mathrm{div}, \Omega)$ on boundary elements, and C_ℓ is a positive constant depending on ℓ. We note that the above estimate depends on the regularity of the exact solution placed on the geometric partition of Ω rather than the physical partition. Also, the constant depends on the degree of the polynomial, but lacks its precise dependence, which is important for establishing convergence with respect to ℓ.

Therefore, in addition to the proposed discrete fluxes in general unfitted meshes, this work fills the gap in the original numerical analysis of the MHM method, to contribute to

- demonstrate that the MHM method is superconvergent. Specifically, the convergence rate of $|u - u_H|_{1,\mathcal{T}_\mathcal{H}}$ behaves like $O(H^{\ell+3/2})$ when $H \to 0$ (and $\mathcal{H}$ stay fixed), which was numerically anticipated in [12] and [4];
- prove that the MHM method achieves convergence in unfitted meshes assuming local regularity in the physical domain unlike the previous MHM literature;
- explicit the dependence of the constant on the error estimates in the polynomial degree ℓ. We show that the MHM method converges optimally when $\ell \to \infty$. Such a convergence result is also new.

The outline of this article is as follows: We close this section with the functional setting given in an abstract form to be particularized in the next sections. Section 2 includes a description of the Ω partitions, the physical partition and the mesh, followed by the definition of broken spaces and associated norms. In Section 3, we define continuous and discrete fluxed and estimate the error involved. These error estimates are used in Section 4 to revisit the analysis of the MHM method. The conclusions follow in Section 5.

1.1. **Functional setting and norms.** If $\mathcal{U} \subset \Omega$ is a measurable set, $L^2(\mathcal{U})$ is the usual Lebesgue space of square-integrable functions, equipped with its inner product $(\cdot,\cdot)_{\mathcal{U}}$ and the associated norm $\|\cdot\|^2_{0,\mathcal{U}} := (\cdot,\cdot)_{\mathcal{U}}$. We also employ the notation $\boldsymbol{L}^2(\mathcal{U}) := [L^2(\mathcal{U})]^d$, and we keep the same notation for its inner product and norm. For $m \in \mathbb{N}^\star$, $H^m(\mathcal{U})$ is the usual Sobolev space that we equip with the norm

$$\|v\|^2_{H^m(\mathcal{U})} := \sum_{\substack{\boldsymbol{\alpha}\in\mathbb{N}^d\\ |\boldsymbol{\alpha}|\leq m}} \left(\frac{1}{d^2_{\mathcal{U}}}\right)^{m-|\boldsymbol{\alpha}|} \|\partial^{\boldsymbol{\alpha}} v\|^2_{\mathcal{U}}$$

and semi-norm

$$|v|^2_{H^m(\mathcal{U})} := \sum_{\substack{\boldsymbol{\alpha}\in\mathbb{N}^d\\ |\boldsymbol{\alpha}|= m}} \|\partial^{\boldsymbol{\alpha}} v\|^2_{\mathcal{U}}$$

for all $v \in H^m(\mathcal{U})$, where $d_{\mathcal{U}}$ is the diameter of $\mathcal{U}$ and the the partial derivative $\partial^{\boldsymbol{\alpha}}$ is understood in the sense of distributions. We refer the reader to [1] for an in-depth discussion of these spaces. We shall also use the Sobolev space $\boldsymbol{H}(\mathrm{div},\mathcal{U})$ of functions $\boldsymbol{w} \in \boldsymbol{L}^2(\mathcal{U})$ with $\boldsymbol{\nabla}\cdot\boldsymbol{w} \in L^2(\mathcal{U})$, see [10].

We write $H^{1/2}(\partial\mathcal{U})$ for the image of $H^1(\mathcal{U})$ by the trace operator. Its dual, that we denote by $H^{-1/2}(\partial\mathcal{U})$, is the image of $\boldsymbol{H}(\mathrm{div},\mathcal{U})$ by the normal trace operator, and we reserve the notation $\langle\cdot,\cdot\rangle_{\partial\mathcal{U}}$ for the duality pairing between $H^{-1/2}(\partial\mathcal{U})$ and $H^{1/2}(\partial\mathcal{U})$.

If $\mathcal{P}$ is a collection of non-overlapping measurable sets, we introduce for $m \in \mathbb{N}^\star$ the broken Sobolev space

$$H^m(\mathcal{P}) := \left\{ v \in L^2(\Omega) \mid v|_\omega \in H^m(\omega) \text{ for all } \omega \in \mathcal{P} \right\},$$

with its norm and semi-norm

$$\|v\|^2_{H^m(\mathcal{P})} := \sum_{\omega\in\mathcal{P}} \|v\|^2_{m,\omega} \quad \text{and} \quad |v|^2_{H^m(\mathcal{P})} := \sum_{\omega\in\mathcal{P}} |v|^2_{H^m(\omega)} \quad \text{for all } v \in H^m(\mathcal{P}).$$

If $\mathcal{V} \subset \Omega$ is contained in an hyperplane and measurable with respect to the surface measure, we employ the same notations as above for $L^2(\mathcal{V})$ its norm and inner-product, with integration performed with respect to the surface measure. $H^m(\mathcal{V})$ is also defined likewise, with multi-indices running over $\mathbb{N}^{d-1}$. Finally, if $\mathcal{Q}$ is a collection of such disjoint sets $\mathcal{V}$, then $H^m(\mathcal{Q})$ is the associated broken Sobolev space. We will also need the (possibly infinite) Sobolev-Slobodeckij semi-norm

$$|v|^2_{H^{1/2}(\mathcal{V})} := \int_{\mathcal{V}}\int_{\mathcal{V}} \frac{|v(\boldsymbol{x})-v(\boldsymbol{y})|^2}{|\boldsymbol{x}-\boldsymbol{y}|^d} d\boldsymbol{y} d\boldsymbol{x} \qquad \text{for all } v \in L^2(\mathcal{V}),$$

and its piecewise version

$$|v|^2_{H^{1/2}(\mathcal{Q})} := \sum_{\mathcal{V}\in\mathcal{Q}} |v|^2_{H^{1/2}(\mathcal{V})} \qquad \text{for all } v \in L^2(\mathcal{Q}).$$

Let $U \subset \mathbb{R}^d$ be a closed polytopal domain. We denote by h_U the diameter of the smallest ball containing U, and by ρ_U the diameter of the largest ball such that U is star-shaped.

We also denote by $\mathcal{F}_U$ the set of "faces" of U. Then, the shape-regularity parameter of U is the constant

$$\beta_U := \frac{h_U}{\rho_U}.$$

When considering a collection $\mathcal{C} := \{U_1, \dots, U_n\}$ of such sets, we let $\beta_{\mathcal{C}} := \max_{1 \le j \le n} \beta_{U_j}$.

2. Partitions of the Domain

We consider a Lipschitz polytopal domain $\Omega \subset \mathbb{R}^d$, with $d \in \{2, 3\}$, and denote by d_Ω the diameter of Ω. We decompose Ω into two separate and independent partitions. They are detailed next.

2.1. **Physical partition.** We assume that Ω is partitioned into "physical subdomains" $\omega \in \mathcal{P}_\Omega$. We will assume that each ω has a Lipschitz boundary. As a result [19, Theorem 5, Page 181], there exists extension operators $E_\omega : L^2(\omega) \to L^2(\Omega)$ satisfying $(E_\omega v) = v$ for all $v \in L^2(\Omega)$ and such that, for all $m \in \mathbb{N}$, $E_\omega : H^m(\omega) \to H^m(\Omega)$ with

$$\|E_\omega v\|_{m,\Omega} \le C_{\mathrm{E},\omega,m} \|v\|_{m,\omega}$$

for all $v \in H^m(\omega)$ for some constants $C_{\mathrm{E},\omega,m}$, and we set $C_{\mathrm{E},\mathcal{P}_\Omega,m} := \max_{\omega \in \mathcal{P}_\Omega} C_{\mathrm{E},\omega,m}$. This physical partition typically corresponds to regions of space occupied by different materials, each being linked with a constant (or smooth) coefficient in the considered model problem (more in Section 4). Importantly, we may expect the model's solution to be smooth in each of the physical subdomains $\omega \in \mathcal{P}_\Omega$.

2.2. **Geometrical partitions.** The domain is partitioned into a computational mesh $\mathcal{T}_\mathcal{H}$ characterized by a size $\mathcal{H} > 0$. This partition is made of polytopal regions K, and we collect the element boundaries ∂K in the set $\partial \mathcal{T}_\mathcal{H}$. We denote by $\mathcal{F}_\mathcal{H}$ the faces of partition $\mathcal{T}_\mathcal{H}$, and for $K \in \mathcal{T}_\mathcal{H}$, $\mathcal{F}_K$ is the set of faces of K. For the sake of simplicity, we assume that for two distinct regions $K_\pm \in \mathcal{T}_\mathcal{H}$, that when intersection $\partial K_+ \cap \partial K_-$ is non-empty, it is either a full face, a full edge, or a single vertex of both regions. We highlight that we *do not* assume any conformity between the partition $\mathcal{T}_\mathcal{H}$ with the physical partition $\mathcal{P}_\Omega$.

We denote by $C_{\mathrm{qu}}(\mathcal{T}_\mathcal{H})$ the quasi-uniformity constant of $\mathcal{T}_\mathcal{H}$, i.e., the smallest real number such that

$$\mathcal{H} \le C_{\mathrm{qu}}(\mathcal{T}_\mathcal{H}) \mathcal{H}_K \qquad \forall K \in \mathcal{T}_\mathcal{H}.$$

Then, for each $K \in \mathcal{T}_\mathcal{H}$, there exists a constant $C_{\mathrm{tr},K}$ solely depending on β_K and $C_{\mathrm{qu}}(\mathcal{T}_\mathcal{H})$ such that

$$\|v\|_{\mathcal{F}_K}^2 \le C_{\mathrm{tr},K}^2 \left(\mathcal{H}^{-1} \|v\|_K^2 + \mathcal{H} \|\boldsymbol{\nabla} v\|_K^2 \right) \tag{2.1}$$

and

$$|v|_{H^{1/2}(\mathcal{F}_K)} \le C_{\mathrm{tr},K} \|\boldsymbol{\nabla} v\|_K \tag{2.2}$$

for all $v \in H^1(K)$, see, e.g., [9, Lemmas 6.1 and 6.4]. We write $C_{\mathrm{tr},\mathcal{T}_\mathcal{H}} := \max_{K \in \mathcal{T}_\mathcal{H}} C_{\mathrm{tr},K}$, which only depends on $\beta_{\mathcal{T}_\mathcal{H}}$ and $C_{\mathrm{qu}}(\mathcal{T}_\mathcal{H})$. Notice that because $C_{\mathrm{qu}}(\mathcal{T}_\mathcal{H})$ enters our analysis, our results are essentially relevant on quasi-uniform meshes where the ratio between the maximal and minimal element diameter is not large.

We further introduce another level of geometrical discretization. Namely, each face $F \in \mathcal{F}_{\mathcal{H}}$ is partitioned into a mesh $\mathcal{M}_H^F$ with elements D and characteristic length H. For $0 \leq q \leq \ell+1$, and $\ell \geq 0$, the orthogonal projector $\pi_{D,\ell} : L^2(D) \to \mathbb{P}_\ell(D)$ satisfies

$$\|\xi - \pi_{D,\ell}\xi\|_D \leq \left(\frac{C_{\mathrm{P},D,q} H_D}{\ell+1}\right)^{q+1} |\xi|_{H^{q+1}(D)} \qquad \text{for all } \xi \in H^{q+1}(D) \tag{2.3}$$

where $C_{\mathrm{P},D,q}$ only solely depends on β_D and q and, if $d=3$, on the ratio $h_D/\min_{e \in \mathcal{F}_D} h_e$, see e.g. [8, Lemma 4.2 and Remark 2]. In addition, using Banach space interpolation theory (see, e.g. [20, Chapters 22, 34 and 36]), we can combine the case $q=0$ and $q=1$ of (2.3) to show that

$$\|\xi - \pi_{D,\ell}\xi\|_D \leq \left(\frac{C_{\mathrm{P},D,\mathrm{h}} H_D}{\ell+1}\right)^{1/2} |\xi|_{H^{1/2}(D)} \quad \text{for all } \xi \in H^{1/2}(D), \tag{2.4}$$

for some constant $C_{\mathrm{P},D,\mathrm{h}}$ only depending on β_D. We denote by $\mathcal{M}_H := \cup_{F \in \mathcal{F}_{\mathcal{H}}} \mathcal{M}_H^F$ the global skeletal mesh, and we set

$$C_{\mathrm{P},\mathcal{M}_H,q} := \max_{F \in \mathcal{F}_{\mathcal{H}}} \max_{D \in \mathcal{M}_H^F} C_{\mathrm{P},D,q}, \qquad C_{\mathrm{P},\mathcal{M}_H,\mathrm{h}} := \max_{F \in \mathcal{F}_{\mathcal{H}}} \max_{D \in \mathcal{M}_H^F} C_{\mathrm{P},D,\mathrm{h}}.$$

We also set $\mathcal{M}_H^{\partial K} := \cup_{F \in \mathcal{F}_K} \mathcal{M}_H^F$ for each $K \in \mathcal{T}_{\mathcal{H}}$ and $\mathcal{M}_H^\omega := \{D \in \mathcal{M}_H \mid D \subset \omega\}$.

Remark 2.1 (Constraint on skeleton meshes)**.** *In contrast to $\mathcal{T}_{\mathcal{H}}$, we assume that the partition $\mathcal{M}_H$ fits the physical partition $\mathcal{P}_\Omega$. It means that every element $D \in \mathcal{M}_H$ entirely belongs to a single physical subdomain $\omega \in \mathcal{P}_\Omega$.*

3. Flux interpolation

This section presents our first set of results, where we construct an interpolation operator for flux variables and estimate errors.

3.1. **Continuous and discrete fluxes.** We consider that the continuous flux variable belongs to the space

$$\Lambda(\partial\mathcal{T}_{\mathcal{H}}) := \left\{ \mu \in \prod_{K \in \mathcal{T}_{\mathcal{H}}} H^{-1/2}(\partial K) \;\middle|\; \begin{array}{l} \exists \boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div},\Omega); \\ \boldsymbol{\sigma}\cdot\boldsymbol{n}_K = \mu|_{\partial K} \ \forall K \in \mathcal{T}_{\mathcal{H}} \end{array} \right\}.$$

If $\mu \in \Lambda(\partial\mathcal{T}_{\mathcal{H}})$ and $v \in H^1(\mathcal{T}_{\mathcal{H}})$, we define the pairing

$$\langle \mu, v\rangle_{\partial\mathcal{T}_{\mathcal{H}}} = \sum_{K \in \mathcal{T}_{\mathcal{H}}} \langle \mu, v\rangle_{\partial K}.$$

For $\mu \in \Lambda(\mathcal{T}_{\mathcal{H}})$, we define the (semi) norm

$$\|\mu\|_\Lambda = \sup_{\substack{v \in \widetilde{H}^1(\mathcal{T}_{\mathcal{H}}) \\ \|\boldsymbol{\nabla}\tilde{v}\|_{\mathcal{T}_{\mathcal{H}}}=1}} \langle \mu, \tilde{v}\rangle_{\partial\mathcal{T}_{\mathcal{H}}}, \tag{3.1}$$

where $\widetilde{H}^1(\mathcal{T}_{\mathcal{H}})$ stands for the space of functions in $H^1(\mathcal{T}_{\mathcal{H}})$ with zero mean value in each $K \in \mathcal{T}_{\mathcal{H}}$. Notice that $\|\cdot\|_\Lambda$ becomes a norm when restricted to the subspace $\mathcal{N}(\partial\mathcal{T}_{\mathcal{H}}) \subset \Lambda(\partial\mathcal{T}_{\mathcal{H}})$ defined by

$$\mathcal{N}(\partial\mathcal{T}_{\mathcal{H}}) := \{\mu \in \Lambda(\partial\mathcal{T}_{\mathcal{H}}) \mid \langle \mu, v_0\rangle_{\partial\mathcal{T}_{\mathcal{H}}} = 0 \quad \text{for all } v_0 \in \mathbb{P}_0(\mathcal{T}_{\mathcal{H}})\}\,. \tag{3.2}$$

For a given integer $\ell \in \mathbb{N}$, we introduce an interpolation operator that is well defined for all $\mu \in \Lambda \cap L^2(\partial\mathcal{T}_{\mathcal{H}})$ by setting

$$(\pi_{H,\ell}\mu)|_D = \pi_{D,\ell}\mu \qquad \text{for all } D \in \mathcal{M}_H$$

and it follows that the discrete flux $\pi_{H,\ell}\mu \in \Lambda(\mathcal{T}_{\mathcal{H}}) \cap L^2(\partial\mathcal{T}_{\mathcal{H}})$.

3.2. **Error estimates.** We start with a duality result that is similar to [3].

Lemma 3.1 (Duality). *For all $\mu \in \Lambda(\partial\mathcal{T}_{\mathcal{H}}) \cap L^2(\partial\mathcal{T}_{\mathcal{H}})$, we have*

$$\|\mu - \pi_{H,\ell}\mu\|_\Lambda \le C_{\mathrm{tr},\mathcal{T}_{\mathcal{H}}} \left(\frac{C_{\mathrm{P},\mathcal{M}_H,\mathrm{h}} H}{\ell+1}\right)^{1/2} \|\mu - \pi_{H,\ell}\mu\|_{\partial\mathcal{T}_{\mathcal{H}}}. \tag{3.3}$$

Proof. Let $v \in H^1(\mathcal{T}_{\mathcal{H}})$. For all $K \in \mathcal{T}_{\mathcal{H}}$, we have

$$\langle \mu - \pi_{H,\ell}\mu, v\rangle_{\partial K} = (\mu - \pi_{H,\ell}\mu, v)_{\partial K} = \sum_{D \in \mathcal{M}_H^{\partial K}} (\mu - \pi_{D,\ell}\mu, v)_D.$$

Recalling that $\pi^D_{H,\ell}\mu$ is the $L^2(D)$ projection of μ onto $\mathbb{P}_\ell(D)$, and using (2.4), we have

$$\begin{aligned}(\mu - \pi_{D,\ell}\mu, v)_D = (\mu - \pi_{D,\ell}\mu, v - \pi_{D,\ell}v)_D &\le \|\mu - \pi_{D,\ell}\mu\|_D \|v - \pi_{D,\ell}v\|_D \\ &\le \left(\frac{C_{\mathrm{P},D,\mathrm{h}} H_D}{\ell+1}\right)^{1/2} \|\mu - \pi_{D,\ell}\mu\|_D |v|_{H^{1/2}(D)},\end{aligned}$$

and therefore, involving (2.2), we obtain

$$\begin{aligned}\langle \mu - \pi_{H,\ell}\mu, v\rangle_{\partial K} &\le \left(\frac{C_{\mathrm{P},\mathcal{M}_H,\mathrm{h}} H}{\ell+1}\right)^{1/2} \|\mu - \pi_{H,\ell}\mu\|_{\mathcal{F}_K} |v|_{H^{1/2}(\mathcal{F}_K)} \\ &\le C_{\mathrm{tr},\mathcal{T}_{\mathcal{H}}} \left(\frac{C_{\mathrm{P},\mathcal{M}_H,\mathrm{h}} H}{\ell+1}\right)^{1/2} \|\mu - \pi_{H,\ell}\mu\|_{\mathcal{F}_K} \|\boldsymbol{\nabla} v\|_K.\end{aligned}$$

By summation, we see that

$$\langle \mu - \pi_{H,\ell}\mu, v\rangle_{\partial\mathcal{T}_{\mathcal{H}}} \le C_{\mathrm{tr},\mathcal{T}_{\mathcal{H}}} \left(\frac{C_{\mathrm{P},\mathcal{M}_H,q} H}{\ell+1}\right)^{1/2} \|\mu - \pi_{H,\ell}\mu\|_{\partial\mathcal{T}_{\mathcal{H}}} \|\boldsymbol{\nabla} v\|_{\mathcal{T}_{\mathcal{H}}}$$

for all $v \in H^1(\mathcal{T}_{\mathcal{H}})$, and (3.3) follows by definition (3.1) of $\|\cdot\|_\Lambda$ and noting $\mu - \pi_{H,\ell}\mu \in \mathcal{N}(\partial\mathcal{T}_{\mathcal{H}})$. $\square$

As a direct consequence of Lemma 3.1 and (2.3), we have the following result.

Corollary 3.2 (Approximation). *Let $0 \le q \le \ell+1$. Assuming $\mu \in \Lambda(\partial\mathcal{T}_{\mathcal{H}}) \cap H^{q+1}(\partial\mathcal{T}_{\mathcal{H}})$, it holds*

$$\|\mu - \pi_{H,\ell}\mu\|_{\Lambda} \le C_{\mathrm{tr},\mathcal{T}_{\mathcal{H}}} \left(\frac{C_{\mathrm{A},\mathcal{M}_H,q} H}{\ell+1}\right)^{q+3/2} |\mu|_{H^{q+1}(\mathcal{M}_H)}, \tag{3.4}$$

where $C_{\mathrm{A},\mathcal{M}_H,q} := \max(C_{\mathrm{P},\mathcal{M}_H,\mathrm{h}}, C_{\mathrm{P},\mathcal{M}_H,q})$.

In practical application, the variable $\lambda \in \Lambda(\partial\mathcal{T}_{\mathcal{H}})$ to be approximated is related to the "flux" of the solution u to the model problem under consideration. For instance, $\lambda|_{\partial K} = \boldsymbol{\nabla} u \cdot \boldsymbol{n}_K$ for the Laplace operator. This motivates the main result of this section.

Theorem 3.3 (Interpolation error estimate). *Let $0 \le q \le \ell+1$ and $\mu \in \Lambda(\partial\mathcal{T}_{\mathcal{H}}) \cap H^{q+1}(\partial\mathcal{T}_{\mathcal{H}})$. Assume that there exists $u \in H^{q+3}(\mathcal{P}_\Omega)$ such that*

$$|\mu|_{H^{q+1}(\mathcal{M}_H)} \le |u|_{H^{q+2}(\mathcal{M}_H)}.$$

Then, we have

$$\begin{aligned} &\|\mu - \pi_{H,\ell}\mu\|_{\Lambda} \le \\ &\qquad C_{\mathrm{E},\mathcal{P}_\Omega,q+3} C_{\mathrm{tr},\mathcal{T}_{\mathcal{H}}} \left(\frac{C_{\mathrm{A},\mathcal{M}_H,q} H}{\ell+1}\right)^{q+3/2} \left(\mathcal{H}^{-1/2}\|u\|_{H^{q+2}(\mathcal{P}_\Omega)} + \mathcal{H}^{1/2}\|u\|_{H^{q+3}(\mathcal{P}_\Omega)}\right). \end{aligned} \tag{3.5}$$

Proof. In view of (3.4), it is sufficient to establish that

$$|u|_{H^{q+2}(\mathcal{M}_H)} \le C_{\mathrm{E},\mathcal{P}_\Omega,q+3} C_{\mathrm{tr},\mathcal{T}_{\mathcal{H}}} \left(\mathcal{H}^{-1/2}\|u\|_{H^{q+2}(\mathcal{P}_\Omega)} + \mathcal{H}^{1/2}\|u\|_{H^{q+3}(\mathcal{P}_\Omega)}\right).$$

First, because the mesh $\mathcal{M}_H$ fits the physical partition $\mathcal{P}_\Omega$, we have

$$\begin{aligned} |u|^2_{H^{q+2}(\mathcal{M}_H)} = \sum_{D\in\mathcal{M}_H} |u|^2_{H^{q+2}(D)} &\le \sum_{\omega\in\mathcal{P}_\Omega} \sum_{D\in\mathcal{M}_H^\omega} |u|^2_{H^{q+2}(D)} \\ &= \sum_{\omega\in\mathcal{P}_\Omega} \sum_{D\in\mathcal{M}_H^\omega} \|E_\omega u\|^2_{H^{q+2}(D)} \le \sum_{\omega\in\mathcal{P}_\Omega} \|E_\omega u\|^2_{H^{q+2}(\partial\mathcal{T}_{\mathcal{H}})}. \end{aligned}$$

On the other hand, for $K \in \mathcal{T}_{\mathcal{H}}$, we can apply (2.1) to $\partial^\alpha(E_\omega u)$ for $|\alpha| \le q+2$. It follows that

$$\|E_\omega u\|^2_{H^{q+2}(\partial K)} \le C^2_{\mathrm{tr},K} \left(\mathcal{H}^{-1}\|E_\omega u\|^2_{H^{q+2}(K)} + \mathcal{H}\|E_\omega u\|^2_{H^{q+3}(K)}\right),$$

and therefore

$$\begin{aligned} \|E_\omega u\|^2_{H^{q+2}(\partial\mathcal{T}_{\mathcal{H}})} &\le C^2_{\mathrm{tr},\mathcal{T}_{\mathcal{H}}} \left(\mathcal{H}^{-1}\|E_\omega u\|^2_{H^{q+2}(\Omega)} + \mathcal{H}\|E_\omega u\|^2_{H^{q+3}(\Omega)}\right) \\ &\le C^2_{\mathrm{E},\mathcal{P}_\Omega,q+3} C^2_{\mathrm{tr},\mathcal{T}_{\mathcal{H}}} \left(\mathcal{H}^{-1}\|u\|^2_{H^{q+2}(\omega)} + \mathcal{H}\|u\|^2_{H^{q+3}(\omega)}\right), \end{aligned}$$

for all $\omega \in \mathcal{P}_\Omega$. By summation over $\omega \in \mathcal{P}_\Omega$, it follows that

$$|u|^2_{H^{q+2}(\mathcal{M}_H)} \le C^2_{\mathrm{E},\mathcal{P}_\Omega,q+3} C^2_{\mathrm{tr},\mathcal{T}_{\mathcal{H}}} \left(\mathcal{H}^{-1}\|u\|^2_{H^{q+2}(\mathcal{P}_\Omega)} + \mathcal{H}\|u\|^2_{H^{q+3}(\mathcal{P}_\Omega)}\right).$$

$\square$

We close this section with an important property of our interpolation operator (c.f. [4])

Proposition 1 (Mass conservation)**.** *Assume that* $\lambda \in \Lambda(\partial\mathcal{T}_\mathcal{H}) \cap L^2(\partial\mathcal{T}_\mathcal{H})$*. Then, we have*

$$\langle \pi_{\ell,H}\lambda, v_0\rangle_{\partial\mathcal{T}_\mathcal{H}} = \langle \lambda, v_0\rangle_{\partial\mathcal{T}_\mathcal{H}} \quad \forall v_0 \in \mathbb{P}_0(\mathcal{T}_\mathcal{H}). \tag{3.6}$$

Proof. Since λ (and $\pi_{\ell,H}\lambda$) belongs to $L^2(\partial\mathcal{T}_\mathcal{H})$ by assumption, we can regroup the duality pairings into face-by-face L^2 products, leading to

$$\langle \lambda - \pi_{H,\ell}\lambda, v_0\rangle_{\partial\mathcal{T}_\mathcal{H}} = \sum_{F\in\mathcal{F}_\mathcal{H}} (\lambda - \pi_{H,\ell}\lambda,\, [\![v_0]\!])_F = \sum_{F\in\mathcal{F}_\mathcal{H}} \sum_{D\in\mathcal{M}_H^F} (\lambda - \pi_{D,\ell}\lambda,\, [\![v_0]\!])_D = 0$$

since $[\![v_0]\!] \in \mathbb{P}_0(D)$ for all $D \in \mathcal{M}_H$, and $\pi_{D,\ell}$ is the orthogonal projection onto $\mathbb{P}_\ell(D) \supset \mathbb{P}_0(D)$. $\square$

4. The MHM method for the Poisson problem

In this section, we revisit the convergence analysis of the MHM method using the interpolation operator introduced in Section 3. This analysis improves over the existing works [2]. In particular, we obtain better constants and optimal rates in H. In addition, we are able to establish ℓ-convergence when the mesh is fixed the polynomial degree is increased, which is new in the MHM context.

4.1. **Model problem.** Throughout this section, we fix $f \in L^2(\Omega)$ and focus on the model problem of finding $u \in H^1_0(\Omega)$ such that

$$(\boldsymbol{A}\boldsymbol{\nabla} u, \boldsymbol{\nabla} v)_\Omega = (f, v)_\Omega \qquad \text{for all } v \in H^1_0(\Omega), \tag{4.1}$$

where, for a.e. $\boldsymbol{x}$ in Ω, $\boldsymbol{A}(\boldsymbol{x})$ is a symmetric matrix. We assume that $\boldsymbol{A}$ is measurable and that there exists two constants $0 < a_{\min} \le a_{\max} < +\infty$ such that

$$a_{\min} \le \min_{\substack{\boldsymbol{\xi}\in\mathbb{R}^d\\ |\boldsymbol{\xi}|=1}} \boldsymbol{A}(\boldsymbol{x})\boldsymbol{\xi}\cdot\boldsymbol{\xi} \qquad a_{\min} \le \max_{\substack{\boldsymbol{\xi}\in\mathbb{R}^d\\ |\boldsymbol{\xi}|=1}} \boldsymbol{A}(\boldsymbol{x})\boldsymbol{\xi}\cdot\boldsymbol{\xi} \le a_{\max}$$

for a.e. $\boldsymbol{x}$ in Ω. For the sake of simplicity, we introduce the weighted norm

$$\|\boldsymbol{v}\|^2_{\boldsymbol{A},\mathcal{T}_\mathcal{H}} := \sum_{K\in\mathcal{T}_\mathcal{H}} \int_K \boldsymbol{A}\boldsymbol{v}\cdot\boldsymbol{v} \qquad \text{for all } \boldsymbol{v} \in \boldsymbol{L}^2(\mathcal{T}_\mathcal{H}).$$

4.2. **MHM formulation.** Let us introduce

$$\mathbb{P}_0^\perp(\mathcal{T}_\mathcal{H}) := \left\{ v \in H^1(\mathcal{T}_\mathcal{H}) \mid (v, v_0)_{\mathcal{T}_\mathcal{H}} = 0 \ \text{ for all } v_0 \in \mathbb{P}_0(\mathcal{T}_\mathcal{H}) \right\}.$$

Owing to Poincaré inequality, it is easily seen that the application $\|\boldsymbol{\nabla}\cdot\|_{\boldsymbol{A},\mathcal{T}_\mathcal{H}}$ is a norm over $\mathbb{P}_0^\perp(\mathcal{T}_\mathcal{H})$. Then, we define the mappings $T : \Lambda(\partial\mathcal{T}_\mathcal{H}) \to \mathbb{P}_0^\perp(\mathcal{T}_\mathcal{H})$ and $\widehat{T} : L^2(\Omega) \to \mathbb{P}_0^\perp(\mathcal{T}_\mathcal{H})$ by requiring that

$$(\boldsymbol{A}\boldsymbol{\nabla} T(\mu), \boldsymbol{\nabla} v)_{\mathcal{T}_\mathcal{H}} = \langle \mu, v\rangle_{\partial\mathcal{T}_\mathcal{H}}, \quad (\boldsymbol{A}\boldsymbol{\nabla}\widehat{T}(g), \boldsymbol{\nabla} v)_{\mathcal{T}_\mathcal{H}} = (g, v)_{\mathcal{T}_\mathcal{H}} \quad \text{for all } v \in \mathbb{P}_0^\perp(\mathcal{T}_\mathcal{H}), \tag{4.2}$$

with $\mu \in \Lambda(\partial\mathcal{T}_\mathcal{H})$ and $g \in L^2(\Omega)$.

Then, the continuous MHM formulation consists of finding $(\lambda, u_0) \in \Lambda(\partial\mathcal{T}_\mathcal{H}) \times \mathbb{P}_0(\mathcal{T}_\mathcal{H})$ such that

$$\left\{ \begin{array}{rcll} \langle \mu, T(\lambda)\rangle_{\partial\mathcal{T}_\mathcal{H}} + \langle \mu, u_0\rangle_{\partial\mathcal{T}_\mathcal{H}} & = & \langle \mu, \widehat{T}(f)\rangle_{\partial\mathcal{T}_\mathcal{H}} & \text{for all } \mu \in \Lambda(\partial\mathcal{T}_\mathcal{H}), \\ \langle \lambda, v_0\rangle_{\partial\mathcal{T}_\mathcal{H}} & = & (f, v_0)_{\mathcal{T}_\mathcal{H}} & \text{for all } v_0 \in \mathbb{P}_0(\mathcal{T}_\mathcal{H}). \end{array} \right. \tag{4.3}$$

It is shown in [2] (see also [18]), that actually

$$\lambda|_{\partial K} = \boldsymbol{\nabla} u \cdot \boldsymbol{n}_K|_{\partial K} \qquad \text{and} \qquad u_0|_K = \frac{1}{|K|}\int_K u\, d\boldsymbol{x},$$

for all $K \in \mathcal{T}_{\mathcal{H}}$, and that

$$u = u_0 + T(\lambda) + \widehat{T}(f). \tag{4.4}$$

Let $\Lambda_{H,\ell}(\partial\mathcal{T}_{\mathcal{H}})$ be the finite dimensional subspace of $\Lambda(\partial\mathcal{T}_{\mathcal{H}})$ defined by

$$\Lambda_{H,\ell}(\partial\mathcal{T}_{\mathcal{H}}) := \left\{\mu \in \Lambda(\partial\mathcal{T}_{\mathcal{H}}) \mid \mu|_D \in \mathbb{P}_\ell(D), \quad \text{for all } D \in \mathcal{M}_H^F \text{ and } F \in \mathcal{F}_{\mathcal{H}}\right\}.$$

The discrete formulation then consists of finding $(\lambda_H, u_{0,H}) \in \Lambda_{H,\ell}(\partial\mathcal{T}_{\mathcal{H}}) \times \mathbb{P}_0(\mathcal{T}_{\mathcal{H}})$ such that

$$\left\{\begin{array}{rcll} \langle \mu_H, T(\lambda_H)\rangle_{\partial\mathcal{T}_{\mathcal{H}}} + \langle \mu_H, u_{0,H}\rangle_{\partial\mathcal{T}_{\mathcal{H}}} & = & \langle \mu_H, \widehat{T}(f)\rangle_{\partial\mathcal{T}_{\mathcal{H}}} & \text{for all } \mu_H \in \Lambda_{H,\ell}(\partial\mathcal{T}_{\mathcal{H}}) \\ \langle \lambda_H, v_0\rangle_{\partial\mathcal{T}_{\mathcal{H}}} & = & (f, v_0)_{\mathcal{T}_{\mathcal{H}}} & \text{for all } v_0 \in \mathbb{P}_0(\mathcal{T}_{\mathcal{H}}), \end{array}\right. \tag{4.5}$$

and we set

$$u_H := u_{0,H} + T(\lambda_H) + \widehat{T}(f). \tag{4.6}$$

4.3. **Convergence analysis.** We start with a quasi-optimality result. Because the MHM formulation is a saddle point problem, Galerkin orthogonality cannot be immediately employed, and a compatibility condition is required. The approximation result follows the proof in [17][Lemma 7], but now with optimal constants.

Lemma 4.1 (Best approximation). *We have*

$$\|\boldsymbol{\nabla} T(\lambda - \lambda_H)\|_{\boldsymbol{A},\mathcal{T}_{\mathcal{H}}} = \min_{\substack{\mu_H \in \Lambda_{H,\ell} \\ \langle \lambda - \mu_H, v_0\rangle_{\partial\mathcal{T}_{\mathcal{H}}} = 0\ \forall v_0 \in \mathbb{P}_0(\mathcal{T}_{\mathcal{H}})}} \|\boldsymbol{\nabla} T(\lambda - \mu_H)\|_{\boldsymbol{A},\mathcal{T}_{\mathcal{H}}}. \tag{4.7}$$

In addition, if $\lambda \in L^2(\partial\mathcal{T}_{\mathcal{H}})$, then

$$\|\boldsymbol{\nabla} T(\lambda - \lambda_H)\|_{\boldsymbol{A},\mathcal{T}_{\mathcal{H}}} \le \|\boldsymbol{\nabla} T(\lambda - \pi_{H,\ell}\lambda)\|_{\boldsymbol{A},\mathcal{T}_{\mathcal{H}}}. \tag{4.8}$$

Proof. Consider $\mu_H \in \Lambda_{H,\ell}$ with $\langle \lambda - \mu_H, v_0\rangle_{\partial\mathcal{T}_{\mathcal{H}}} = 0$ for all $v_0 \in \Lambda_H$. We have

$$\begin{aligned} \|\boldsymbol{\nabla} T(\lambda - \lambda_H)\|^2_{\boldsymbol{A},\mathcal{T}_{\mathcal{H}}} &= (\boldsymbol{A}\boldsymbol{\nabla} T(\lambda - \lambda_H), \boldsymbol{\nabla} T(\lambda - \lambda_H))_{\mathcal{T}_{\mathcal{H}}} \\ &= \langle \lambda - \lambda_H, T(\lambda - \lambda_H)\rangle_{\partial\mathcal{T}_{\mathcal{H}}}. \end{aligned} \tag{4.9}$$

Then, using the first equations of (4.3) and (4.5), we observe that

$$\langle \mu_H, T(\lambda - \lambda_H)\rangle_{\partial\mathcal{T}_{\mathcal{H}}} = 0,$$

so that, from (4.9), it holds

$$\|\boldsymbol{\nabla} T(\lambda - \lambda_H)\|^2_{\boldsymbol{A},\mathcal{T}_{\mathcal{H}}} = \langle \lambda - \mu_H, T(\lambda - \lambda_H)\rangle_{\partial\mathcal{T}_{\mathcal{H}}} = (\boldsymbol{A}\boldsymbol{\nabla} T(\lambda - \mu_H), \boldsymbol{\nabla} T(\lambda - \lambda_H))_{\mathcal{T}_{\mathcal{H}}}.$$

Next, from the Cauchy-Schwartz inequality, we get

$$\|\boldsymbol{\nabla} T(\lambda - \lambda_H)\|_{\boldsymbol{A},\mathcal{T}_{\mathcal{H}}} \le \|\boldsymbol{\nabla} T(\lambda - \mu_H)\|_{\boldsymbol{A},\mathcal{T}_{\mathcal{H}}},$$

and (4.7) follows. Then, (4.8) follows from (3.6). $\square$

We are now ready to establish the main result of this section.

Theorem 4.2 (Error estimate). *Let $0 \leq q \leq \ell + 1$, and $\ell \geq 0$, and assume that $u \in H^{q+3}(\mathcal{P}_\Omega)$. Then, we have*

$$\|\boldsymbol{\nabla}(u - u_H)\|_{\mathcal{T}_\mathcal{H}} \leq \mathscr{C}_{\mathcal{P}_\Omega,\mathcal{T}_\mathcal{H},q}\sqrt{\frac{a_{\max}}{a_{\min}}}\left(\frac{C_{\mathrm{A},\mathcal{M}_H,q}H}{\ell+1}\right)^{q+3/2}\left(\mathcal{H}^{-1/2}\|u\|_{H^{q+2}(\mathcal{P}_\Omega)} + \mathcal{H}^{1/2}\|u\|_{H^{q+3}(\mathcal{P}_\Omega)}\right). \tag{4.10}$$

Proof. We have

$$\|\boldsymbol{\nabla} T(\lambda - \lambda_H)\|_{\mathcal{T}_\mathcal{H}} \leq a_{\min}^{-1/2}\|\boldsymbol{\nabla} T(\lambda - \lambda_H)\|_{\boldsymbol{A},\mathcal{T}_\mathcal{H}} \leq a_{\min}^{-1/2}\|\boldsymbol{\nabla} T(\lambda - \pi_{H,\ell}\lambda)\|_{\boldsymbol{A},\mathcal{T}_\mathcal{H}},$$

and

$$\begin{aligned}\|\boldsymbol{\nabla} T(\lambda - \pi_{H,\ell}\lambda)\|_{\boldsymbol{A},\mathcal{T}_\mathcal{H}}^2 &= \langle \lambda - \pi_{H,\ell}\lambda, T(\lambda - \pi_{H,\ell}\lambda)\rangle_{\partial\mathcal{T}_\mathcal{H}} \\ &\leq \|\lambda - \pi_{H,\ell}\lambda\|_{\star,\partial\mathcal{T}_\mathcal{H}}\|\boldsymbol{\nabla} T(\lambda - \pi_{H,\ell}\lambda)\|_{\mathcal{T}_\mathcal{H}} \\ &\leq a_{\max}^{1/2}\|\lambda - \pi_{H,\ell}\lambda\|_{\star,\partial\mathcal{T}_\mathcal{H}}\|\boldsymbol{\nabla} T(\lambda - \pi_{H,\ell}\lambda)\|_{\boldsymbol{A},\mathcal{T}_\mathcal{H}},\end{aligned}$$

so that

$$\|\boldsymbol{\nabla} T(\lambda - \lambda_H)\|_{\mathcal{T}_\mathcal{H}} \leq \sqrt{\frac{a_{\max}}{a_{\min}}}\|\lambda - \pi_{H,\ell}\lambda\|_{\star,\partial\mathcal{T}_\mathcal{H}}.$$

Hence, (4.10) follows from (3.5). □

Remark 4.3 (Super-convergence). *Under local regularity assumptions for the exact solution in the physical partition of Ω, the error estimate in Theorem 4.2 indicates that the MHM method achieves superconvergence when the skeleton diameter of the mesh H tends to zero with an additional $O(H^{1/2})$ convergence rate. Furthermore, the estimate (4.10) establishes that the MHM method provides optimal convergent solutions with respect to the degree of polynomial interpolation ℓ on the faces. These are novel results that are supported by numerical evidence presented in previous works. We also recover the (optimal) convergence classically found when $\mathcal{H}$, the diameter of the $\mathcal{T}_\mathcal{H}$ partition, vanishes. This is demonstrated by assuming the exact solution is locally regular on the physical partition, which is also new.*

5. Conclusion

We proposed a strategy to approximate fluxes in partitions not aligned with physical interfaces. Under local regularity assumptions, the theoretical result showed that the approach provides approximate super-converged fluxes driven by exact solution regularity in physical regions. In other words, we replace the standard and unrealistic assumption about the regularity of the exact solution in each mesh element with the regularity of the solution in physical regions. In addition, we highlight the dependence of the constant on the polynomial degree used to approximate the exact fluxes.

We leverage these findings to improve the convergence results for the MHM method applied to the Poisson problem. We mainly prove that the MHM method is super-convergent on non-aligned meshes, assuming exact solution regularity only in the physical partition. Such mathematical analysis supports the numerical evidence originally anticipated in [12]. It is worth mentioning that the results can be easily extended to MHM methods applied to other operators such as the linear elasticity model or the reactive-advective-diffusive equation, for example. Furthermore, the discrete flux can be exploited in other flux-based numerical methods or domain decomposition techniques and inherit from its properties.

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