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***Abstract*— An attitude tracking problem for a full-actuated rigid body in 3-D is studied using a model of rotation dynamics on the Lie group $SO(3)$. A generalized homogeneous control is adopted to achieve tracking of a smooth attitude trajectory in a finite-time. The attitude dynamics on the Lie algebra $\mathfrak{so}(3)$ is utilized to design the homogeneous control, since $SO(3)$ is not a vector space. A switch control is proposed for achieving global finite-time tracking by combining a global asymptotic control with local homogeneous control. The simulations illustrate the performance of the proposed control algorithm.**

I. INTRODUCTION

The rigid body orientation (or attitude) control problem occurs in different applications, such as satellites, quadrotors, airplanes, terrestrial mobile robots etc. In the case of underactuated dynamics, a fast and robust attitude control helps to achieve high performance of a translation control. There are several alternate representations for orientation of a rigid body in the three dimensional space, and, of course, all of them have three degrees of freedom. This means that any representation with more than three parameters must have some constraints. The most common representations are Euler angles (see, e.g. [1], [2]) and quaternions [3]. The Euler angles have singularity in a transformation from the time derivative of Euler angles to angular velocity [4]. The Euler angles are used for a small rotation range. Quaternions can represent the attitude position globally, but with an ambiguity, which may cause an unwinding phenomenon in control design [5]. The representation by a rotation matrix is a global representation with additional orthogonality constraint. The main difficulty is that the set of all rotation matrices is not a vector space but a manifold known as Special Orthogonal group ($SO(3)$).

The geometric control on $SO(3)$ is studied for quadrotors [6], [7] and spacecrafts [8]. According to a general attitude geometric control technique [9], [10], a configuration error function needs to be predefined due to the orthogonal constraint such that the error dynamics is stable. The rotation group is not a vector space, so the conventional control design methods cannot be directly applied. However, the Lie algebra $\mathfrak{so}(3)$ is the tangent space of the associated Lie group $SO(3)$ at the identity element of the group, and it completely captures the local structure of the group [11]. Hence, an alternative approach to geometric control can based on dynamics in the Lie algebra [12], [13], [14] being a vector space.

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The homogeneity is symmetry with respect to the dilation. All linear and many essential nonlinear models of mathematical physics are homogeneous (symmetric) in a generalized sense [15]. Homogeneous control laws appear as solutions to some classical control problems, such as minimum-time feedback control for the chain of integrators [16]. Most of the high-order sliding mode control and estimation algorithms are homogeneous in a generalized sense [17]. Homogeneity allows time constraints in control systems to be fulfilled easily using proper tuning of the so-called homogeneity degree [18]. There are some advantages of homogeneous control compared to linear control:

- finite-time and fixed-time convergence rate;
- robustness with respect to a large class of uncertainties;
- elimination of an unbounded "peaking" effect;
- possibility of design a globally bounded finite-time stabilizing controller.

In this paper, homogeneous control is used to track a smooth attitude trajectory in finite time. For this purpose, the attitude dynamic on Lie algebra is used. An invariant set avoiding ambiguity for the attitude dynamic with homogeneous control is derived. Then, for global finite-time stability, a switch control that combines the local homogeneous attitude control with a global asymptotic control is proposed. In comparison to prior finite-time attitude control methods, homogeneous control is based on the implicit Lyapunov function and requires fewer tuning parameters. The settling time is globally bounded and can be estimated for sufficiently small initial states.

This paper is organized as follows: Section II contains the solution to the paper's problem. Section III provides an overview of homogeneous systems and control, as well as the $SO(3)$ group and its associated Lie algebra. The section IV presents the key results on homogeneous control design using Lie algebraic attitude dynamics. A switched control for global finite-time tracking is proposed. Section V includes a simulation that exhibits the control algorithm's performance.

NOTATION

$\mathbb{R}$ is the set of real numbers, $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$; $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$; $\mathbf{0}$ denotes the zero vector from $\mathbb{R}^n$; $\mathrm{diag}\{\lambda_i\}_{i=1}^n$ is the diagonal matrix with elements λ_i; $P \succ 0 (\prec 0, \succeq 0, \preceq 0)$ for $P \in \mathbb{R}^{n\times n}$ means that the matrix P is symmetric and positive (negative) definite (semidefinite); $C(X,Y)$ denotes the space of continuous functions $X \to Y$, where X, Y are subsets of normed vector spaces; $C^p(X,Y)$ is the space of functions continuously differentiable at least up to the order p; $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ represent the minimal and maximal eigenvalue of a matrix $P = P^\top$; for $P \succeq 0$ the square root of P is a matrix $M = P^{\frac{1}{2}}$ such that $M^2 = P$; $SO(3) \subset \mathbb{R}^{3\times 3}$ is a special orthogonal group (discussed below) and $\mathfrak{so}(3)$ is the corresponding Lie algebra; $(\cdot)^\wedge$ is a map from vector to skew matrix $(\cdot)^\wedge : \mathbb{R}^3 \to \mathfrak{so}(3)$: $\left[[x_1, x_2, x_3]^\top\right]^\wedge \to \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$. And $(\cdot)^\vee : \mathfrak{so}(3) \to \mathbb{R}^3$ is the inverse map of $(\cdot)^\wedge$. $\mathrm{Exp}(\cdot) = \exp[(\cdot)^\wedge]$ is a map from $\mathbb{R}^3$ to $SO(3)$, $\mathrm{Log}(\cdot) = [\log(\cdot)]^\vee$ is a map from $SO(3)$ to $\mathbb{R}^3$.

II. Problem Statement

The equations of rigid body attitude motion can be represented as:

$$\dot{R} = R\omega^\wedge, \quad R(0) = R_0 \tag{1a}$$

$$\dot{\omega} = J^{-1}\left(\omega \times J\omega + M\right), \quad \omega(0) = \omega_0, \tag{1b}$$

where $R \in SO(3)$ is the rotation matrix with respect to the inertial frame, $\omega \in \mathbb{R}^3$ is the angular velocity in body frame, $J \in \mathbb{R}^{3\times 3}$ is the moment of inertia matrix, $M \in \mathbb{R}^3$ is the torque expressed in the body frame. For the rigid body whose dynamics is governed by (1), this problem investigates the following problem.

Problem 1: Given a desired attitude trajectory $R_d \in C^2\left(\mathbb{R}_+, SO(3)\right)$, we need to design a homogeneous (in a certain sense) control law M that ensures a *local uniform finite-time tracking*, i.e., there exists a locally bounded function $T : \Omega_R \times \Omega_\omega \to \mathbb{R}_+$ such that

$$R(t) = R_d(t), \;\; \forall t \geq T(R_0, \omega_0), \forall R_0 \in \Omega_{R_d}, \forall \omega_0 \in \Omega_{\omega_d},$$

where $\Omega_{R_d} \subset SO(3)$ and $\Omega_{\omega_d} \subset \mathbb{R}^3$ are the sets of admissible initial conditions depended on the desired trajectory R_d.

III. Preliminaries

A. Homogeneous system

1) Linear dilations in $\mathbb{R}^n$: By definition, the homogeneity is a dilation symmetry [19], [20], [21], [22], [15]. A dilation [23] is a one-parameter group $\mathbf{d}(s), s \in \mathbb{R}$ of transformations satisfying the limit property $\lim_{s\to s^\infty} \|\mathbf{d}(s)x\| = e^{s^\infty}, s^\infty = \pm\infty, \forall x \neq \mathbf{0}$.

Examples of dilations in $\mathbb{R}^n$ are

- Uniform dilation (L. Euler, 18th century): $\mathbf{d}(s) = e^s I$, where I is the identity matrix $\mathbb{R}^n$;
- Weighted dilation (Zubov 1958) : $\mathbf{d}(s) = \begin{pmatrix} e^{r_1 s} & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & e^{r_n s} \end{pmatrix}$, where $r_i > 0, i = 1, 2, \dots, n$.
- Geometric dilation is a flow generated by unstable C^1 vector field in $\mathbb{R}^n$ (see [20], [24]).

In this paper we deal only with the so-called linear (geometric) dilation in $\mathbb{R}^n$ is defined as follows

$$\mathbf{d}(s) = e^{sG_\mathbf{d}} := \sum_{i=0}^{+\infty} \frac{s^i G_\mathbf{d}^i}{i!}, \quad s \in \mathbb{R}^n, \tag{2}$$

where $G_\mathbf{d} \in \mathbb{R}^{n\times n}$ is an anti-Hurwitz matrix called the generator of the dilation $\mathbf{d}$.

Definition 1: A dilation $\mathbf{d}$ is monotone if $s \to \|\mathbf{d}(s)x\|$ is a monotone increasing function for any $x \neq \mathbf{0}$.

It is worth noting that monotonicity of the dilation may depend of the norm in $\mathbb{R}^n$. Also, any linear dilation in $\mathbb{R}^n$ is monotone [25] provided that the norm in $\mathbb{R}^n$ is defined as follows

$$\|x\|_* = \sqrt{x^\top P x}, \quad PG_\mathbf{d} + G_\mathbf{d}^\top P \succ 0, P \succ 0. \tag{3}$$

2) Canonical Homogeneous Norm: The linear dilation introduces an alternative norm topology in $\mathbb{R}^n$ by means of a homogeneous norm (see, e.g., [26] for an example of a homogeneous norm induced by the weighted dilation).

Definition 2: [25] The functional $\|\cdot\|_\mathbf{d} : \mathbb{R}^n \to [0, +\infty)$ defined as $\|\mathbf{0}\|_\mathbf{d} = 0$ and

$$\|u\|_\mathbf{d} = e^{s_u}, \text{ where } s_u \in \mathbb{R} : \left\|\mathbf{d}\left(-s_u\right)u\right\|_* = 1,$$

is called the canonical homogeneous norm in $\mathbb{R}^n$, where $\mathbf{d}$ is a monotone dilation in $\mathbb{R}^n$, where $\|\cdot\|_*$ is a norm in $\mathbb{R}^n$.

For any linear monotone dilation in $\mathbb{R}^n$, the canonical homogeneous norm is continuous on $\mathbb{R}^n$ and locally Lipschitz continuous on $\mathbb{R}^n \backslash \{\mathbf{0}\}$. Moreover, it is differentiable on $\mathbb{R}^n \backslash \{\mathbf{0}\}$ provided that $\|\cdot\|_*$ is given by (3) (see [25]):

$$\frac{\partial \|x\|_\mathbf{d}}{\partial x} = \|x\|_\mathbf{d} \frac{x^\top \mathbf{d}^\top(-\ln\|x\|_\mathbf{d}) P \mathbf{d}(-\ln\|x\|_\mathbf{d})}{x^\top \mathbf{d}^\top(-\ln\|x\|_\mathbf{d}) P G_\mathbf{d} \mathbf{d}(-\ln\|x\|_\mathbf{d}) x}. \tag{4}$$

3) Homogeneous Systems:

Definition 3: A vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ (resp. a function $h : \mathbb{R}^n \to \mathbb{R}$) is said to be $\mathbf{d}$-homogeneous of degree $\mu \in \mathbb{R}$ if

$$f(\mathbf{d}(s)) = e^{\mu s}\mathbf{d}(s) f(x), \quad \forall x \in \mathbb{R}^n, \quad \forall s \in \mathbb{R},$$

$$(\text{resp. } h(\mathbf{d}(s)) = e^{\mu s} h(x), \quad \forall x \in \mathbb{R}^n, \quad \forall s \in \mathbb{R}),$$

where $\mathbf{d}$ is a linear dilation in $\mathbb{R}^n$.

In [25] it is shown that any $\mathbf{d}$-homogeneous system

$$\dot{x} = f(x), \quad t > 0, \quad x(0) = x_0 \in \mathbb{R}^n \tag{5}$$

is diffeomorphic on $\mathbb{R}^n \backslash \{\mathbf{0}\}$ to a standard homogeneous system. This means that many important results known for standard and weighted homogeneous systems hold for linear homogeneous systems as well. The following theorem is the straightforward corollary of Zubov-Rosier Theorem on homogeneous Lyapunov function [19], [27].

Theorem 1: Let f be a continuous $\mathbf{d}$-homogeneous vector field of degree $\mu \in \mathbb{R}$. The system (5) is globally uniformly asymptotically stable if and only if there exists a positive definite $\mathbf{d}$-homogeneous function $V : \mathbb{R}^n \to [0, +\infty)$ such that $V \in C(\mathbb{R}^n) \cap C^1(\mathbb{R}^n \backslash \{\mathbf{0}\})$ and

$$\dot{V}(x) \leq -\rho V^{1+\mu}(x), \quad \forall x \neq 0.$$

The latter theorem immediately implies that asymptotically stable homogeneous system (5) is

- globally uniformly finite-time stable[1] for $\mu < 0$;
- globally uniformly exponentially stable for $\mu = 0$;
- globally uniformly nearly fixed-time stable[2] for $\mu > 0$.

[1] The system (5) is finite-time stable it is Lyapunov stable and $\exists T(x_0) : \|x(t)\| = 0, \forall t \geq T(x_0), \forall x_0 \in \mathbb{R}^n$.

[2] The system (5) is uniformly nearly fixed-time stable it is Lyapunov stable and $\forall r > 0, \exists T_r > 0 : \|x(t)\| < r, \forall t \geq T_r$ independently of $x_0 \in \mathbb{R}^n$.

B. Homogeneous Control for Linear Plants in $\mathbb{R}^n$

Let us consider the linear control system as

$$\dot{x} = Ax + Bu(x), \quad t > 0, \quad x(0) = x_0 \tag{6}$$

where $x(t) \in \mathbb{R}^n$ is the system state, $u : \mathbb{R}^n \to \mathbb{R}^m$ is the feedback control to be designed, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are system matrices.

The system (6) is said to be $\mathbf{d}$-homogeneously stabilizable of degree $\mu \in \mathbb{R}$ if there exists a feedback $u : \mathbb{R}^n \to \mathbb{R}^m$ such that $f(x) = Ax+Bu(x)$ is $\mathbf{d}$-homogeneous of degree μ, and the closed-loop system is globally asymptotically stable. The disturbance-free (i.e., $g = \mathbf{0}$) system (6) is $\mathbf{d}$-homogeneously stabilizable with a degree $\mu \neq 0$ if and only if the pair $\{A, B\}$ is controllable [28].

Theorem 2: [28] If the linear equation

$$AG_0 + BY_0 = G_0A + A, \quad G_0B = \mathbf{0}. \tag{7}$$

has a solution $G_0 \in \mathbb{R}^{n\times n}$ and $Y_0 \in \mathbb{R}^{m\times n}$ such that G_0-I_n is invertible, then for any $\mu \neq 0$ such that $G_{\mathbf{d}} = I_n + \mu G_0$ is anti-Hurwitz, the *disturbance-free* system (6) can always be homogeneously stabilized by the following control

$$u(x) = K_0x + \|x\|_{\mathbf{d}}^{1+\mu} K\mathbf{d}\left(-\ln \|x\|_{\mathbf{d}}\right)x, \quad K = YX^{-1} \tag{8}$$

with any $K_0 = Y_0(G_0 - I_n)^{-1}$, $X \in \mathbb{R}^{n\times n}, Y \in \mathbb{R}^{m\times n}$ by solving the following algebraic system

$$\begin{cases} XA_0^\top + A_0X + Y^\top B^\top + BY + \rho\left(XG_{\mathbf{d}}^\top + G_{\mathbf{d}}X\right) = \mathbf{0} \\ XG_{\mathbf{d}}^\top + G_{\mathbf{d}}X \succ 0, \quad X \succ 0 \end{cases} \tag{9}$$

where the canonical homogeneous norm $\|\cdot\|_{\mathbf{d}}$ is induced by the norm $\|x\|_* = \sqrt{x^\top X^{-1}x}$. Then the perturbed system (6) is

- globally uniformly finite-time stable for $\mu < 0$;
- globally uniformly exponentially stable for $\mu = 0$;
- globally nearly fixed-time stable for $\mu > 0$.

The proof of this theorem is based on the use of the canonical homogeneous norm as a Lyapunov function of the closed-loop system (6), (8) with $g = \mathbf{0}$:

$$\frac{d}{dt}\|x(t)\|_{\mathbf{d}} = -\rho\|x(t)\|_{\mathbf{d}}^{1+\mu}. \tag{10}$$

The formula (10) implies that the canonical homogeneous norm is a Lyapunov function for the closed-loop system. The feasibility of the algebraic system is proven in [29]. The existence of an appropriate solution for (7) is studied in [28]. It can be shown that for any solution of (7) the matrix $G_0 - I$ is invertible. Moreover, for a sufficiently small $|\mu|$ we have $G_{\mathbf{d}} = I + \mu G_0$ is anti-Hurwitz.

C. Special orthogonal group $SO(3)$ and Lie algebra $\mathfrak{so}(3)$

In this subsection, a brief summary about $SO(3)$ group and $\mathfrak{so}(3)$ algebra is given (see, e.g., [11],[30], [7] for more details). The $SO(3)$ group can be represented as follows:

$$SO(3) = \left\{R \in \mathbb{R}^{3\times 3} \mid RR^T = I, \det R = I\right\} \tag{11}$$

The Lie algebra $\mathfrak{so}(3)$ consists of 3×3 skew-matrices with the Lie bracket given by the commutator

$$\mathfrak{so}(3) = \left\{X \in \mathbb{R}^{3\times 3} | X^\top = -X\right\}. \tag{12}$$

Since $X = x^\wedge$ for $x \in \mathbb{R}^3$, then $\mathfrak{so}(3)$ is isomorphic to $\mathbb{R}^3$. Notice that for $x \in \mathbb{R}^3$ and $R \in SO(3)$ one holds

$$Rx^\wedge R^\top = (Rx)^\wedge. \tag{13}$$

The exponential map $\exp(\cdot)$ is a surjective map, which maps the elements of Lie algebra $\mathfrak{so}(3)$ to elements of $SO(3)$. For $\phi \in \mathbb{R}^3$ we have $\phi^\wedge \in \mathfrak{so}(3)$ and the exponential map is given by

$$\exp\left(\phi^\wedge\right) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\phi^\wedge\right)^n \tag{14}$$

For the $\mathfrak{so}(3)$ group parameterized by $\boldsymbol{\phi} \in \mathbb{R}^3$, the left Jacobian of the group $SO(3)$ is

$$J_\ell(\phi) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!}\left(\phi^\wedge\right)^n = \int_0^1 \left(e^{\phi^\wedge}\right)^\alpha d\alpha. \tag{15}$$

It relates the time derivatives of R and ϕ as follows [11]

$$\dot{R}R^\top = \left(J_\ell(\phi)\dot{\phi}\right)^\wedge \tag{16}$$

Notice that the inverse of J_ℓ is given by

$$J_\ell(\phi)^{-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!}\left(\phi^\wedge\right)^n \tag{17}$$

where B_n is Bernoulli number and $B_1 = -\frac{1}{2}$. Moreover, for

$$\left\{\phi \in \mathbb{R}^3 \mid \phi = \varphi q,\, q^\top q = 1,\, |\varphi| = \|\phi\|,\, q \in \mathbb{R}^3, \varphi \in \mathbb{R}\right\}$$

the following representations of J_ℓ and J_ℓ^{-1}

$$J_\ell(\varphi, q) = \frac{\sin\varphi}{\varphi}I + \left(1 - \frac{\sin\varphi}{\varphi}\right)qq^T + \frac{1-\cos\varphi}{\varphi}q^\wedge \tag{18}$$

$$J_\ell^{-1}(\varphi, q) = \frac{\varphi}{2}\cot\frac{\varphi}{2}I + \left(1 - \frac{\varphi}{2}\cot\frac{\varphi}{2}\right)qq^T - \frac{\varphi}{2}q^\wedge \tag{19}$$

hold as well.

Lemma 1: If $\phi \in C^1(\mathbb{R}, \mathbb{R}^3)$ then the time derivative $\frac{d}{dt}J_\ell^{-1}(\phi(t))$ exists for $t : \|\phi(t)\| \in [0, 2\pi)$ and

$$\phi(t) = 0 \quad \Rightarrow \quad \frac{d}{dt}J_\ell^{-1}(\phi(t)) = \left(\dot{\phi}(t)\right)^\wedge.$$

Proof: On the one hand, considering the time derivative of J_ℓ^{-1} in the form (19) we derive

$$\frac{d}{dt}J_\ell^{-1} = a_1I - a_1qq^\top + a_2\dot{q}q^\top + a_2q\dot{q}^\top - \frac{\dot{\phi}}{2}q^\wedge - \frac{\phi}{2}\dot{q}^\wedge$$

where $a_1 = \frac{d}{dt}\left(\frac{\phi}{2}\cot\frac{\phi}{2}\right) = \frac{\dot{\phi}}{2}\cot\frac{\phi}{2} - \frac{\phi\dot{\phi}}{4}\frac{1}{\sin^2\frac{\phi}{2}}$, $a_2 = 1 - \frac{\phi}{2}\cot\frac{\phi}{2}$. Hence, the inverse of left Jacobian is differentiable on $\phi \in \mathbb{R} \setminus \{2n\pi\}_{n\in\mathbb{Z}}$.

On the other hand, calculating the time derivative of J_ℓ^{-1} using (17) we obtain

$$\frac{d}{dt}J_\ell^{-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!}\frac{d}{dt}\left(\phi^\wedge\right)^n \tag{20}$$

Hence, for $\phi = \mathbf{0}$ we derive $\frac{d}{dt}J_\ell^{-1} = \dot{\phi}^\wedge$. Therefore, the inverse of left Jacobian is differentiable on $\varphi \in \mathbb{R} \setminus \{2n\pi\}_{n\in\mathbb{Z}\setminus\{0\}}$. ∎

IV. Homogeneous Finite-time Tracking Control

A. Attitude dynamics on Lie algebra

Homogeneity (dilation symmetry) is defined on vector spaces (see above). Since the $SO(3)$ group is not a vector space, so the conventional homogeneity-based control design is not applicable to the system (1a). However, since the Lie algebra $\mathfrak{so}(3)$ is isomorphic to $\mathbb{R}^3$, then using the exponential map from $\mathfrak{so}(3)$ to $SO(3)$, the attitude kinematics can be represented in Lie algebra [11]. Aiming to design a homogeneous controller on the Lie algebra, a rotation error on $\mathfrak{so}(3)$ is introduced as follows:

$$\begin{aligned} R = e^{\theta_e^\wedge} R_d \Rightarrow R_e := RR_d^\top = e^{\theta_e^\wedge} = \mathrm{Exp}(\theta_e) \\ \Rightarrow \theta_e = \mathrm{Log}(RR_d^\top) \end{aligned} \tag{21}$$

where $\theta_e \in \mathbb{R}^3$ and $\mathrm{Exp}, \mathrm{Log}$ are defined above in Notation. Combining the kinematics (16) with the attitude dynamics (1b), the attitude error dynamics on vector space can be derived for $\|\theta_e\| < \pi$.

Lemma 2: With a smooth attitude trajectory $R_d \in C^2(\mathbb{R}, SO(3))$, the error dynamics of (21) for $\|\theta_e\| \in [0, \pi)$, can be represented as follows:

$$\dot{\xi} = A\xi + Bu \tag{22}$$

where $\xi = \begin{bmatrix} \theta_e \\ \dot{\theta}_1 \end{bmatrix}$, $A = \begin{bmatrix} \mathbf{0} & I_3 \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad B = \begin{bmatrix} \mathbf{0} \\ I_3 \end{bmatrix}$,

$$\theta_1 = J_l^{-1}(\theta_e) R_e \omega_e,$$

$$u = \frac{d\left(J_l^{-1}\right)}{dt} R_e \omega_e + J_l^{-1} R_e \omega_e \times \omega_e + J_l^{-1} R_e \dot{\omega}_e,$$

$J_l(\theta_e) = \int_0^1 \left(e^{\theta_e^\wedge}\right)^\alpha d\alpha$, $\omega_e = R_d(\omega - \omega_d)$, and

$$\omega_d = \left(R_d^\top \dot{R}_d\right)^\vee$$

is the desired angular velocity in body frame.

Proof: Using the formula (16), we derive

$$\dot{R}_e (R_e)^{-1} = \left(J_l(\theta_e)\dot{\theta}_e\right)^\wedge. \tag{23}$$

On the other hand, we have

$$\dot{R}_e = \dot{R}R_d^\top + R\dot{R}_d^\top = R\omega^\wedge R_d^\top - R\omega_d^\wedge R_d^\top =$$

$$R(\omega - \omega_d)^\wedge R_d^\top = R_e R_d (\omega - \omega_d)^\wedge R_d^\top = R_e \omega_e^\wedge,$$

where the identity (13) is utilized on the last step.

Therefore, we have

$$\dot{\theta}_e = \theta_1 = J_\ell^{-1}(\theta_e) R_e \omega_e \tag{24}$$

According to Lemma 1, θ_1 is differentiable on $\|\theta_e\| \in [0, \pi)$, so calculating the derivative of θ_1 and using the identity $\omega_e \times \omega_e = \omega_e^\wedge \omega_e$, we complete the proof. ■

Notice that to compute $\dot{\omega}$ the time derivative of ω_d is required. Since the desired rotation trajectory is assumed to be smooth, then

$$\dot{\omega}_d = \frac{d}{dt}\left(R_d^\top \dot{R}_d\right)^\vee = \left(R_d^\top \ddot{R}_d - (\omega_d^\wedge)^2\right)^\vee \in C(\mathbb{R}_+, \mathbb{R}^3).$$

B. Homegeneous attitude control

Since A in (22) is nilpotent and the pair (A, B) is controllable, then a homogeneous controller can be designed using Theorem 2 with $K_0 = 0$.

Theorem 3: [Local homogeneous finite-time tracking] Let

$$u = \|\xi\|_{\mathbf{d}}^{1+\mu} K \mathbf{d}(-\ln\|\xi\|_{\mathbf{d}})\xi, \quad \xi = \begin{pmatrix} \theta_e \\ \theta_1 \end{pmatrix} \in \mathbb{R}^6 \tag{25}$$

be a $\mathbf{d}$-homogeneous controller designed by Theorem 2 for the linear control system (22). If $\xi(0) \in \Omega = \left\{\xi \in \mathbb{R}^6 : \xi^\top P_r \xi < 1\right\}$, where

$$P_r = \mathbf{d}(-\ln r) X^{-1} \mathbf{d}(-\ln r), \quad r = \left(\frac{\pi^2}{\lambda_{\max}(X)}\right)^{\frac{1}{g_{11}}}$$

and g_{11} is the first element of the generator $G_{\mathbf{d}} = \{g_{i,j}\}$, then the controller

$$\begin{aligned} M =& J\left[R_d^\top \tilde{u} - \omega_d \times (\omega - \omega_d) + \dot{\omega}_d\right] - \omega \times J\omega \\ \tilde{u} =& R_e^\top J_\ell \left(u - \tfrac{dJ_\ell^{-1}}{dt} R_e \omega_e\right) \end{aligned} \tag{26}$$

guarantees that the attitude error dynamics (22) is:

- uniformly finite-time stable for $\mu < 0$:
$$\xi(t) = \mathbf{0}, \quad \forall t \geq \|\xi(0)\|_{\mathbf{d}}^{-\mu}/(-\mu\rho);$$
- uniformly exponentially stable for $\mu = 0$:
$$\|\xi(t)\|_{\mathbf{d}} \leq e^{-\rho t}\|\xi(0)\|_{\mathbf{d}}, \quad \forall t \geq 0;$$
- uniformly nearly fixed-time stable for $\mu > 0$:
$$\|\xi(t)\|_{\mathbf{d}} \leq r, \quad \forall t \geq \frac{1}{\mu r^\mu}.$$

By Theorem 2, such a system (22) with the controller (25) is globally uniformly finite-time stable. However, in our case the model (22) is valid only locally $\|\theta_e\| < \pi$. Since the canonical homogeneous norm is the Lyapunov function of the closed-loop system (see Theorem 2), then the set $\Omega = \{\xi : \|\xi\|_{\mathbf{d}} \leq r\}$ is its invariant set. If $r > 0$ is such that $\Omega \subset \{\xi : \|\theta_e\| < \pi\}$, then Ω defines the set of admissible initial conditions such that the corresponding trajectories of the closed-loop system never leave the domain. It has been shown in [15] that $\|\xi\|_{\mathbf{d}} \leq r$ is equivalent to

$$\xi^\top \mathbf{d}(-\ln r) X^{-1} \mathbf{d}(-\ln r)\xi \leq 1$$

Since the equivalent representation of the inequality $\|\theta_e\| < \pi$ is

$$\xi^\top \begin{bmatrix} \frac{1}{\pi^2} I_3 & 0 \\ 0 & 0 \end{bmatrix} \xi < 1$$

then the matrix inequality

$$\begin{bmatrix} \frac{I}{\pi^2} & 0 \\ 0 & 0 \end{bmatrix} < \mathbf{d}(-\ln r) X^{-1} \mathbf{d}(-\ln r)$$

guarantees the inclusion $\Omega \subset \{\xi : \|\theta_e\| < \pi\}$.

Indeed, we have a local homogeneous attitude control and the invariant set Ω has been derived. The attitude control can be extended to a global one by combining it with a global asymptotic control. A global asymptotic control is implemented outside of invariant set Ω, and once the states are in the set Ω, the control is switched to a homogeneous one.

Notice the fact that there is no continuous time-invariant feedback control that globally asymptotically stabilizes the attitude dynamics [4]. If $\|\mathrm{Log} RR_d^\top\| = \pi$, then θ_e is not uniquely defined, i.e., there is ambiguity of the map from $SO(3)$ to $\mathfrak{so}(3)$: $\theta_e = \pi q$ or $\theta_e = -\pi q$, where $q \in \mathbb{R}^3$: $R_e q = q, \|q\| = 1$. As in [31], a jump mechanism is introduced when $\|\mathrm{Log} RR_d^\top\| = \pi$ and $\frac{d\|\mathrm{Log} RR_d^\top\|}{dt} > 0$ to globally represent the attitude dynamic in Lie algebra coordinates:

$$\theta_e^+(t) = -\theta_e^-(t), \quad \dot{\theta}_e^+(t) = \dot{\theta}_e^-(t), \tag{27}$$

the superscripts $-$ and $+$ to refer to values before and after the jump. Using a global asymptotic controller proposed in [31], a global (locally homogeneous) switched control can be designed easily.

Corollary 1 (Global finite-time attitude control): Let the control be defined as follows

$$M = \begin{cases} M^\star & \text{if} \quad \xi \notin \Omega & (28a) \\ \tilde{M} & \text{if} \quad \xi \in \Omega & (28b) \end{cases}$$

$$\begin{aligned} M^\star &= J\left[R_d^\top u^\star - \omega_d \times (\omega - \omega_d) + \dot{\omega}_d\right] - \omega \times J\omega \\ u^\star &= R_e^\top J_\ell \left(-K_p\theta_e - K_d\dot{\theta}_e - \frac{dJ_\ell^{-1}}{dt} R_e\omega_e\right) \end{aligned} \tag{29}$$

where $\tilde{M}$ is a local homogeneous control designed by Theorem 3, K_p and K_d are two positive definite matrices. Then the closed-loop system (22),(28) is globally uniformly asymptotically stable and it is finite-time stable for $\mu < 0$.

The proof of stability of the system with the global asymptotic control is based on the Lyapunov function:

$$V = \frac{1}{2}\theta_e^\top (K_p + \gamma K_d)\,\theta_e + \frac{1}{2}\dot{\theta}_e^\top \dot{\theta}_e + \gamma \dot{\theta}_e^\top \theta_e \tag{30}$$

where $0 < \gamma < \min\left\{\sqrt{\lambda_{\min}(K_p)}, \lambda_{\min}(K_d)\right\}$. It can be shown that V is positive definite and $\dot{V}$ is negative definite excluding the jump point. For the jump, it satisfies $V^+(t) < V^-(t)$ across any jump. Thus, V is strictly decreasing along every trajectory of the system, then the system is globally asymptotically stable (more details see [31]). The global asymptotic control $M^\star$ ensure the states will enter the invariance set Ω, then the homogeneous control will guarantee the states converge to the origin in finite-time.

Such a switched control (28) would guarantee a global finite-time stabilization, but it does not allow a global settling time estimate. The exact settling time estimate $T(x_0) = \|\xi(0)\|_{\mathbf{d}}^{-\mu}/(-\mu\rho)$ is valid only for $\xi(0) \in \Omega$.

V. Simulation Results

The simulation is created in Simulink with solver ODE 4 and the time step: 10^{-4} for

$$J = \begin{bmatrix} 1.0 \times 10^{-2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 8.2 \times 10^{-3} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 1.48 \times 10^{-2} \end{bmatrix} \mathrm{kg\,m^2}$$

The initial states are: $R(0) = I_3$, $\omega(0) = [50, 0, 0]^\top \mathrm{rad/s}$, $\dot{\omega}(0) = [0, 0, 0]^\top \mathrm{rad/s^2}$.

The set-point tracking problem is considered: $R_d = \exp([\pi, 0, 0]^\top)^\wedge$, $\omega_d = [0, 0, 0]^\top \mathrm{rad/s}$, $\dot{\omega}_d = [0, 0, 0]^\top \mathrm{rad/s^2}$. The global asymptotic controller parameters is selected as : $K_p = K_d = 5I_3$. The homogeneous controller parameters are defined by using MATLAB package Yalmip with solver SEDUMI for $\mu = -0.5$ and $\rho = 5$,

$$G_{\mathbf{d}} = \begin{bmatrix} 1.5I_3 & \mathbf{0} \\ \mathbf{0} & I_3 \end{bmatrix}, X = \begin{bmatrix} 0.0093I_3 & -0.1065I_3 \\ -0.1065I_3 & 1.6297I_3 \end{bmatrix},$$

$$Y = \begin{bmatrix} -0.298I_3 & -8.6483I_3 \end{bmatrix},$$

$$K = \begin{bmatrix} -374.0222I_3 & -29.7578I_3 \end{bmatrix}.$$

The simulation results are depicted in Fig. 1 and Fig. 2 in logarithmic scale. In Fig 3, there are two discontinuous points of the control input M, the first induced by a jump in the system state and the second by the controller switching. Specifically, the switch instant is $t = 0.452s$ (see, Figs. 1 and 2). The convergence becomes faster after switching $t > 0.452s$, and eventually both the attitude error and the angular velocity error reach zero, theoretically, in a finite time. In practice, a convergence to a zone can only be guaranteed due to numerical issues and noise/perturbations. According to the simulation, the proposed switched control can track a smooth trajectory even with a large initial error, and the homogeneous control ensures local finite-time stability.

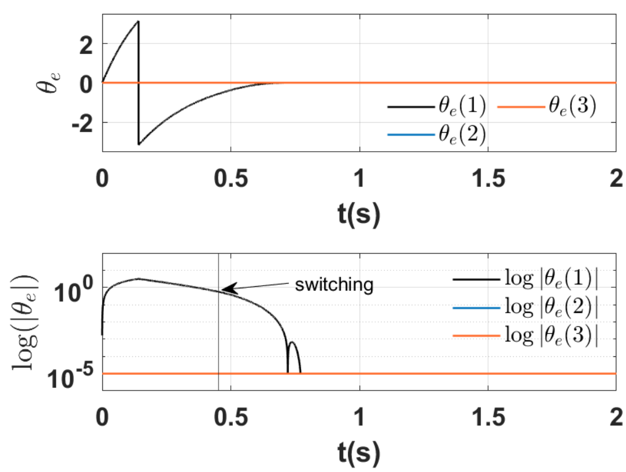

Fig. 1. Rotation error θ_e

VI. Conclusion

In this paper, a homogeneous finite-time controller is designed for attitude tracking under the assumption that initial state is sufficiently close to the desired trajectory. This assumption is not conservative in practice, since usually the desired (planed) trajectory simply starts at the initial state of the rigid body. The stability of the error dynamics is analyzed by using the model on the Lie algebra $\mathfrak{so}(3)$. To achieve finite-time tracking for all initial condition, the presented locally homogeneous controller can be combined

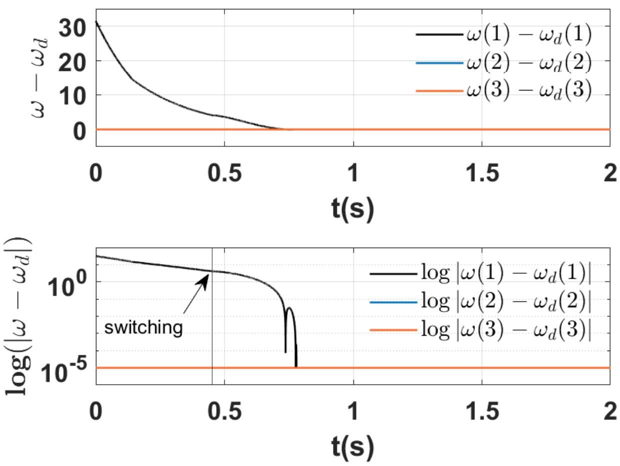

Fig. 2. Angular velocity error $\omega_d - \omega$

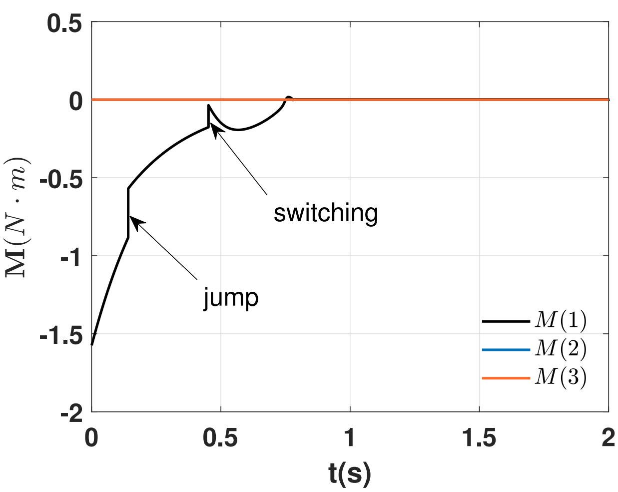

Fig. 3. Control input M

with a globally asymptotically stabilizing algorithm [31]. The simulation results demonstrate that the switch control can track a desired trajectory globally and in a finite time.

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