Sampled-Data Estimator for Nonlinear Systems with Uncertainties and Arbitrarily Fast Rate of Convergence

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Abstract

We study a class of continuous-time nonlinear systems with discrete measurements, model uncertainty, and sensor noise. We provide an estimator of the state for which the observation error enjoys a variant of the exponential input-to-state stability property with respect to the model uncertainty and sensor noise. A valuable novel feature is that the overshoot term in this stability estimate only involves a recent history of uncertainty values. Also, the rate of exponential convergence can be made arbitrarily large by reducing the supremum of the sampling intervals. Our proof uses a recently developed trajectory based approach. We illustrate our work using a model for a pendulum whose suspension point is subjected to an unknown time-varying bounded horizontal oscillation.

Key words: Estimation, nonlinear systems, delay, stability.

1 Introduction

Estimator design plays a central role in current control systems research; see for instance Borri *et al.* (2017), Buccella *et al.* (2014), Cacace *et al.* (2014*a*), Cacace *et al.* (2014*b*), and Parikh *et al.* (2017). One significant criterion for quantifying the performance of an estimator is the sensitivity of the estimation error to measurement or model uncertainty. A standard approach to quantifying sensitivity to uncertainty is input-to-state stability (or ISS), which bounds the norm of the input-to-state stable state by the sum of two terms, namely, a decaying term plus an overshoot term that usually depends on the supremum of the uncertainty from the initial time to the current time. One useful estimator result was Mazenc *et al.* (2015*a*), which provides estimators for continuous-time systems of the form

$$\begin{cases} \dot{x}(t) = Ax(t) + \mu(y(t), u(t)) \\ y(t) = Cx(t) \end{cases} \tag{1}$$

where x is the state, A and C are constant matrices of appropriate dimensions, y is the output, μ is a nonlinear function, and the input u can be a control. The estimators in Mazenc *et al.* (2015*a*) and Mazenc *et al.* (2020*b*) are finite time observers incorporating delays, and their instant of convergence can be chosen by the user as an arbitrarily small positive value. In particular, Mazenc *et al.* (2020*b*) requires delayed measurements of the output. Being fixed time observers, they possess a potential advantage over traditional observers that are only asymptotically converging. However, Mazenc *et al.* (2015*a*) and Mazenc *et al.* (2020*b*) only apply when continuous output measurements $y(t)$ are available. This can pose challenges in settings where only discrete noisy output measurements of the form

$$y(t) = Cx(t_i) + \delta_o(t_i) \text{ for all } t \in [t_i, t_{i+1}) \text{ and } i \geq 0 \tag{2}$$

are available, where the bounded piecewise continuous function δ_o represents measurement noise and t_i is an increasing sequence of nonnegative values. In such a case, Mazenc *et al.* (2015*a*) does not apply.

This motivates this work, which helps overcome the preceding challenge by adapting a technique from Mazenc *et al.* (2015*a*) to cases where the lengths $t_{i+1} - t_i$ of the sampling intervals are not required to be constant and where the output is discrete. Our method borrows key ideas from the work Karafyllis and Kravaris (2009), which redesigns asymptotic observers to accommodate systems with discrete measurements. A key method in Karafyllis and Kravaris (2009) is a dynamic extension that predicts the measured variables between consecutive measurements. Here we show that ideas from Karafyllis and Kravaris (2009) combined with the method from Mazenc *et al.* (2015*a*) yield a state estimate whose most significant

⋆ Special cases of this work have been published in the conference paper Mazenc *et al.* (2020*a*); see Section 1 for a comparison between this work and the conference version.

feature is that it leads to a useful variant of ISS where the overshoot term has a fading effect, meaning, it only depends on a recent history of the uncertainty instead of the supremum of the uncertainty from the initial time to the current time, and where the decaying term also converges with a rate of convergence that can be made arbitrarily large by reducing the supremum of the sampling intervals.

In the sequel, we study systems of the form

$$\begin{cases} \dot{x}(t) = A(t)x(t) + \varphi(Cx(t), t) + \delta_s(t) \\ y(t) = Cx(t_i) + \delta_o(t) \text{ for all } t \in [t_i, t_{i+1}) \text{ and } i \geq 0 \end{cases} \tag{3}$$

where x is valued in $\mathbb{R}^n$ with $n \geq 2$, the output y is valued in $\mathbb{R}$, $C \in \mathbb{R}^{1\times n}$ is a nonzero constant matrix, the matrix valued function $A : \mathbb{R} \to \mathbb{R}^{n\times n}$ is bounded and continuous, φ is a nonlinear function, the sampling times t_i form an increasing sequence with $t_0 = 0$ and $\lim_{i\to+\infty} t_i = +\infty$ and are not required to be evenly spaced, the locally bounded piecewise continuous function δ_s represents model uncertainty (and in particular can model uncertainties in the nonlinearity φ), and δ_o is as above (but see Section 4 where the terms $\varphi(Cx(t), t) + \delta_s(t)$ in (3) are replaced by nonlinearities of the form $\varphi(t, x(t), \delta_s(t))$ under additional conditions involving the sample rates). We design continuous-discrete observers which differ from the ones of Mazenc *et al.* (2015*a*) because ours will not converge in finite time, but our observers converge with a rate of convergence of the form $c_1 \ln(c_2 \sup_i\{t_{i+1} - t_i\})$ for certain constants $c_1 < 0$ and $c_2 > 0$. Hence, the convergence rate can be made arbitrarily large by sampling frequently enough.

Our study of connections between sample rates in the output and convergence rates of the observer is strongly motivated by electrical engineering and other applications. To obtain the sampled measurements $y(t)$ in (3) in applications, one needs sensors, which are sometimes limited by their sample rate. Therefore, to obtain higher sampling rates, it may be necessary to remove old sensors and replace them by new sensors that may offer higher sampling rates. When observers are used to estimate states for use in controls, increasing the sample rates can improve the performance of controls, which may be implemented on a digital signal processor (or DSP). However, a given DSP may have limited computational capabilities and therefore may only be able to process data at a given rate. In addition to the computational capacity of the DSP, the computational capabilities for control implementations can depend on the complexity of the control. Therefore, by quantifying the rate of convergence of our observer in terms of the sample rate, our work provides guidance on when to replace sensors or DSPs by new sensors or new DSPs that allow the faster sample rates that may be called for by our theory to achieve desired convergence rates.

Our results use a trajectory based approach from Ahmed *et al.* (2018), Mazenc and Malisoff (2015), and Mazenc *et al.* (2017). Since the observer formula of Mazenc *et al.* (2015*a*) uses delays, one cannot directly apply Karafyllis and Kravaris (2009) to solve this problem. This paper improves on our conference paper Mazenc *et al.* (2020*a*) by allowing time-varying coefficients A, and model and sensor uncertainty, and including the new extension (in Section 4 below) that allows more general nonlinearities, and by providing a new application to a model of a pendulum whose suspension point is subjected to an unknown time-varying bounded horizontal oscillation and whose friction is time-varying. This contrasts with Mazenc *et al.* (2020*a*), which did not consider uncertainties and required constant A's.

On the other hand, even in the special case where A is constant and the uncertainties δ_s and δ_o are the zero functions, this paper improves the results derived in Mazenc *et al.* (2020*a*), by providing less restrictive conditions on the sample rates; see Section 2.3.3 below. Moreover, although the formula in our main trajectory based lemma shares features with exponential ISS, our trajectory based approach makes it possible to prove our novel variant of the ISS property as indicated above, where the supremum of the uncertainties is only over a recent history of uncertainty values. This sets our work apart from earlier works that did not use the trajectory based approach and so led to a classical ISS estimate that lacks this key fading effect of only depending on recent uncertainty values.

We use the following notation. The dimensions of our Euclidean spaces are arbitrary, unless otherwise noted. The standard Euclidean 2-norm and the corresponding matrix norm are denoted by $|\cdot|$, and I is the identity matrix. For any constant $T > 0$, C_{in} is the set of all continuous functions $\phi : [-T, 0] \to \mathbb{R}^a$. We define $\Xi_t \in C_{\text{in}}$ by

$$\Xi_t(m) = \Xi(t + m) \tag{4}$$

for all choices of Ξ, $m \leq 0$, and $t \geq 0$ such that $t + m$ is in the domain of Ξ. We use $|\cdot|_S$ to denote the essential supremum over a set S, $|\mathcal{M}|_\infty$ is the essential supremum of any function $\mathcal{M}$ over its domain, and Φ_A is the fundamental solution associated with the matrix A in (3) (as defined in (Sontag, 1998, Appendix C.4)).

2 Main result

2.1 Assumptions

Our main assumptions on (3) are as follows:

Assumption 1 *There are constants $\overline{T} > 0$ and $\underline{T} > 0$ such that for all integers $i \geq 0$, the inequalities*

$$\underline{T} \leq t_{i+1} - t_i \leq \overline{T} \tag{5}$$

are satisfied and $t_0 = 0$. □

Assumption 2 *The function φ is continuous. Also, there is a constant $\overline{\varphi} \geq 0$ such that $|\varphi(y_a, t) - \varphi(y_b, t)| \leq \overline{\varphi}|y_a - y_b|$ holds for all real values y_a and y_b and all $t \geq 0$.* □

Assumption 3 *There is a constant $\tau > 0$ such that for each $t \in \mathbb{R}$, the matrix*

$$\Omega(t) = \begin{pmatrix} C \\ C\Phi_A(t-\tau, t) \\ \vdots \\ C\Phi_A(t-(n-1)\tau, t) \end{pmatrix} \tag{6}$$

has an inverse $\Omega^{-1}(t)$ *that satisfies* $\sup_{t\geq 0}|\Omega^{-1}(t)|<\infty$. $\square$

Assumption 3 generalizes the observability condition on (A,C) from the special case of Mazenc *et al.* (2020*a*) where A is constant, and can be checked by computing the determinant of $\Omega(t)$ for $t\in[0,\tau]$ when A has period τ; see Section 2.3 for more on the connection between Assumption 3 and observability, and for sufficient conditions for our assumptions to be satisfied including cases where A is not periodic, and for ways to compute Φ_A to facilitate checking our assumptions and implementing our observer.

We set

$$\Psi(t)=\Omega^{-1}(t) \tag{7}$$

for all $t\in\mathbb{R}$, and

$$\lambda(\tau)=|CA\Psi|_\infty\big(\sqrt{n}+\mathcal{G}(\tau)\sqrt{\tau}\big)+|C|\overline{\varphi}, \tag{8}$$

where

$$\begin{aligned}&\mathcal{G}(\tau)=\sqrt{\sum_{j=1}^{n-1} j\sigma_j^2},\text{ and}\\ &\sigma_j=\sup\{|C\Phi_A(r,s)|:0\leq s-r\leq j\tau, r\geq 0\}\end{aligned} \tag{9}$$

for $j=1,\dots,n-1$. Our last assumption is as follows, which implies that $\lambda(\tau)$ must be positive:

Assumption 4 *The inequality*

$$\overline{T}<\tfrac{1}{\lambda(\tau)} \tag{10}$$

is satisfied. $\square$

Since the right side of (10) is independent of the constants $\underline{T}$ and $\overline{T}$ that we used to bound the intersample times $t_{i+1}-t_i$ in (5), and since we can choose

$$\overline{T}=\sup_{i\geq 0}(t_{i+1}-t_i)\ \text{ and }\ \underline{T}=\inf_{i\geq 0}(t_{i+1}-t_i),$$

it follows that we can always satisfy (5) and (10) by sampling the output faster (and so reducing $\overline{T}$) while maintaining a positive lower bound $\underline{T}$ on the intersample times $t_{i+1}-t_i$. In the next section, our ability to choose $\overline{T}$ as small as we want will play a key role in showing how we can ensure arbitrarily fast convergence of our observer, while ensuring the fading effect in the ISS overshoot that we explained in Section 1.

2.2 Estimator Design

Before providing the formulas for our estimator, we provide an informal summary of its structure and the properties of our estimation algorithm. Our estimator design is composed of an interconnection of (a) a continuous-discrete system whose state is an estimator of the unperturbed output and which is reinitialized at the sample times t_i using measurements $y(t_i)$ of the perturbed output and (b) an estimator variable that is computed from continuous measurements from the interconnected continuous-discrete system. Using this estimator variable, the estimation error has the required features of ensuring arbitrarily fast convergence (by sampling fast enough) and our variant of ISS whose overshoot only depends on a recent history of the uncertainties at each time.

We use the continuous-discrete system:

$$\begin{cases}\dot{\omega}(t)=CA(t)\hat{x}(t)+C\varphi(\omega(t),t)\\ \qquad\text{for all } t\in[t_i,t_{i+1}) \text{ and } i\geq 0\\ \omega(t_i)=y(t_i)\text{ for all } i\geq 0\\ \hat{x}(t)=\Psi(t)\mathcal{U}_1(\omega_t)+\Psi(t)\mathcal{U}_2(\omega_t,t)\end{cases} \tag{11}$$

with $\hat{x}$ valued in $\mathbb{R}^n$, the perturbed measurements $y(t_i)$ from (3), ω valued in $\mathbb{R}$, $\tau>0$ satisfying Assumption 3, and

$$\mathcal{U}_1(\omega_t)=\begin{pmatrix}\omega(t)\\ \omega(t-\tau)\\ \vdots\\ \omega(t-(n-1)\tau)\end{pmatrix}\text{ and} \tag{12}$$

$$\mathcal{U}_2(\omega_t,t)=\begin{pmatrix}0\\ C\Phi_A(t-\tau,t)\Delta H_1(t)\\ \vdots\\ C\Phi_A(t-(n-1)\tau,t)\Delta H_{n-1}(t)\end{pmatrix} \tag{13}$$

and where

$$\Delta H_j(t)=H_j(t)-\Phi_A(t,t-j\tau)H_j(t-j\tau)$$

and the $\mathbb{R}^n$-valued functions H_j solve

$$\dot{H}_j(t)=A(t)H_j(t)+\varphi(\omega(t),t) \tag{14}$$

for $j=1,\dots,n-1$ and any constant initial functions that are defined over $[-(n-1)\tau,0]$. Our $\hat{x}$ formula is reminiscent of the observer of Mazenc *et al.* (2015*a*), and the ω-subsystem of our observer is inspired by the one from Karafyllis and Kravaris (2009). Notice that the predicted output ω only requires the perturbed sampled output measurements $y(t_i)$, and in particular, it does not require derivatives of the predicted output. Also, while the formula for $\mathcal{U}_1$ implies that past values of the predicted output are used, these are readily available for implementations because the dynamics for ω only contain known measured quantities.

While we can apply variation of parameters on the intervals $[t-j\tau,t]$ to check that

$$\begin{aligned}&\Phi_A(t-j\tau,t)\left(H_j(t)-\Phi_A(t,t-j\tau)H_j(t-j\tau)\right)=\\ &\textstyle\int_{t-j\tau}^t\Phi_A(t-j\tau,m)\varphi(\omega(m),m)\mathrm{d}m\end{aligned} \tag{15}$$

holds for all $t\geq(n-1)\tau$ and $j=1,2,\dots,n-1$ and $t\in\mathbb{R}$, we use our expressions (13) and (14) as a way to get formulas for estimators that do not contain any integrations.

In terms of the constant $\mathcal{G}(\tau)$ from (9) and the function

$$\begin{aligned}\mathcal{M}(t)=&|\delta_o|_{[t-\overline{T},t]}+\overline{T}\big(|C|+\\ &+\mathcal{G}(\tau)\sqrt{\tau}|CA\Phi|_\infty\big)|\delta_s|_{[t-\overline{T}-(n-1)\tau,t]}\end{aligned} \tag{16}$$

and the constant

$$\sigma_*=\sup\{|\Phi_A(r,s)|:r\geq 0, s\in[r,r+(n-1)\tau]\}$$

and τ from Assumption 3, our main result is:

Theorem 1 *Let (3) satisfy Assumptions 1-4. Then for all constant initial functions for (3) and (11) and all uncertainty functions (δ_s, δ_o) in (3), and with the choice*

$$c_* = \sqrt{n}|\Psi|_\infty\big(1 + |C|(n-1)\overline{\varphi}\tau\sigma_*\big), \tag{17}$$

we have that

$$\begin{aligned}|\omega(t) - Cx(t)| \le \tfrac{\mathcal{M}(t)}{1-\overline{T}\lambda(\tau)} + \\ \sup_{\ell\in[0,\overline{T}+(n-1)\tau]} |\omega(\ell) - Cx(\ell)| e^{\frac{\ln(\overline{T}\lambda(\tau))}{\overline{T}+(n-1)\tau}(t-\overline{T}-(n-1)\tau)}\end{aligned} \tag{18}$$

for all $t \ge \overline{T} + (n-1)\tau$ and

$$\begin{aligned}|x(t) - \hat{x}(t)| \le \tfrac{\sqrt{n}|\Psi|_\infty|\mathcal{M}|_{[t-(n-1)\tau,t]}}{1-\overline{T}\lambda(\tau)} + \\ c_* \sup_{\ell\in[0,\overline{T}+(n-1)\tau]} |\omega(\ell) - Cx(\ell)| e^{\frac{\ln(\overline{T}\lambda(\tau))}{\overline{T}+(n-1)\tau}(t-\overline{T}-2(n-1)\tau)} \\ +\sqrt{n}|\Psi|_\infty|C|(n-1)\tau\sigma_*|\delta_s|_{[t-(n-1)\tau,t]}\end{aligned} \tag{19}$$

for all $t \ge \overline{T} + 2(n-1)\tau$. □

2.3 Remarks on Theorem 1

Before proving Theorem 1, we provide several remarks concerning the feasibility of checking its assumptions, the implementability of the observer, and the rate of convergence guarantees that are provided by the theorem.

2.3.1 Structure of Observer

Our observer (11)-(14) calls for resetting ω at each time t_i when a new output value becomes available. This implies that at each time t, the future sampling times can be uncertain. Theorem 1 gives a convergence rate to zero of

$$r = c_1 \ln(c_2\overline{T}) \tag{20}$$

of the second right side term in (19), where

$$c_1 = -\tfrac{1}{\overline{T}+(n-1)\tau} < 0 \text{ and } c_2 = \lambda(\tau) > 0. \tag{21}$$

Since r diverges to ∞ as $\overline{T} \to 0$, the continuous time case (where the state is computed exactly in finite time) can be interpreted as a limiting case as $\overline{T} \to 0$. This contrasts with works such as Mazenc *et al.* (2015*b*) on continuous-discrete observer designs that do not provide methods to adjust the sample times to achieve arbitrarily fast convergence.

Our observer formulas also call for computing the Φ_A values in the formulas for Ψ and $\mathcal{U}_2$ in (11) and (13). This can be done using the fact (e.g., (Malisoff, 2020, Section 4.3)) that

$$\Phi_A(t, s) = \alpha_A(t)\beta_A(s) \tag{22}$$

for all real t and s, where α_A and β_A are the unique solutions of the matrix differential equations

$$\dot{\alpha}_A(t) = A(t)\alpha_A(t) \text{ and } \dot{\beta}_A(t) = -\beta_A(t)A(t) \tag{23}$$

that satisfy $\alpha_A(0) = \beta_A(0) = I$. Then we can interconnect our observer design (with $\Phi_A(t - j\tau, t)$ in (13) and Ω replaced by $\alpha_A(t - j\tau)\beta_A(t)$ for $j = 1, \ldots, n-1$) with (23) to compute the required fundamental solution matrices. The observer calls for storing values of the product $\Phi_A(t, s) = \alpha_A(t)\beta_A(s)$ (but not the values of the factors α_A and β_A themselves) on an interval of length $(n-1)\tau$. We found this storage to be feasible in practice, and we encountered no difficulties with divergence of α_A or α_B. However, in practice one could reinitialize the dynamics (23) periodically at times ki for integers $i \ge 0$ and a large enough value $k > 0$.

2.3.2 Discussion of (18)-(19)

Although we can compute an upper bound for $|\omega(t)-Cx(t)|$ over the interval $[0, \overline{T} + (n-1)\tau]$ (which would depend on the initial conditions), we omit this upper bound computation because we believe that it has no significant interest from a practical point of view. This is because $[0, \overline{T} + (n-1)\tau]$ is in a sense arbitrary, since τ and $\overline{T}$ are chosen by the user. Other observer approaches for (3) (such as those of Andrieu and Nadri (2010), Dinh *et al.* (2015), and Mazenc *et al.* (2015*b*)) would either use high gain or do not achieve the arbitrarily fast convergence property from Theorem 1. Also, since the suprema of the uncertainties δ_s and δ_o are only over recent histories of the state, it follows that all of the right side terms of (18)-(19) converge to 0 as $t \to \infty$ if the uncertainties decay to 0. As noted above, the fading effect (where only a recent history of the uncertainties is used to define the overshoot terms on the right sides) contrasts with standard ISS results where the suprema would be over $[0, t]$. Hence, we believe that our novel observer design adds significant value relative to the literature.

2.3.3 Comparison with Mazenc et al. (2020a)

Even in the special case from Mazenc *et al.* (2020*a*) where A is constant, Theorem 1 is less restrictive than the main result from Mazenc *et al.* (2020*a*), because Mazenc *et al.* (2020*a*) required $\overline{T}\lambda(\overline{T}, \tau) < 1$ with the choices

$$\begin{aligned}&\lambda(\overline{T}, \tau) = \\ &\mathcal{F}(\tau)\sqrt{n}\left[1 + |C|\overline{\varphi}e^{|A|(n-1)\tau}(\overline{T} + (n-1)\tau)\right] + |C|\overline{\varphi}\end{aligned} \tag{24}$$

and $\mathcal{F}(\tau) = |CA\Psi|$, where (24) was obtained using the conservative bounds $|CA| \le |C||A|$ and $|e^M| \le e^{|M|}$ for suitable square matrices M; see Section 5.2 below. Therefore, rather than recovering the same performance as Mazenc *et al.* (2020*a*) in the special case where A is constant and where the uncertainties are 0, here we obtain better results in terms of performance in such cases. Theorem 1 can also provide a tighter estimate than traditional ISS estimates, because instead of suprema of (δ_s, δ_o) over an interval from the initial time $\overline{T} + (n-1)\tau$ or $\overline{T} + 2(n-1)\tau$ to the current time t, the suprema are only over a recent history of (δ_s, δ_o). Hence, the conclusions of Theorem 1 are less restrictive than ISS. Moreover, as the measurement step converges to zero, our theorem gives the limiting version

$$\begin{aligned}|x(t) - \hat{x}(t)| \le \tfrac{\sqrt{n}|\Psi|_\infty|\mathcal{M}|_{[t-(n-1)\tau,t]}}{1-\overline{T}\lambda(\tau)} + \\ +\sqrt{n}|\Psi|_\infty|C|(n-1)\tau\sigma_*|\delta_s|_{[t-(n-1)\tau,t]}\end{aligned} \tag{25}$$

of t (19) for all $t \geq \overline{T} + 2(n-1)\tau$. An analogous limiting estimate holds for the output estimation error (18).

2.3.4 Checking Assumption 3

When A is constant, we have $\Phi_A(t,s) = e^{A(t-s)}$ for all real t and s. Therefore, when A is constant, our function Ω from (6) is the constant matrix

$$\overline{\Omega} = \begin{pmatrix} C \\ Ce^{-A\tau} \\ \vdots \\ Ce^{-(n-1)A\tau} \end{pmatrix} \tag{26}$$

which allows us to check Assumption 3 by computing eigenvalues. Also, (6) has period τ when A has period τ. Therefore, when A has period τ, we can check Assumption 3 by checking for the invertibility of Ω on $[0, \tau]$ (since the boundedness from Assumption 3 will then follow because of the continuity of Ω^{-1}). Also, by Mazenc *et al.* (2020*a*), when A is constant and (A, C) is observable, we can satisfy Assumption 3 by choosing $\tau > 0$ such that $(-A^\top, C^\top)$ is τ-sampled controllable, using (Sontag, 1998, Theorem 4 and Lemma 3.4.1) and the fact that the observability of (A, C) implies that $(-A^\top, C^\top)$ is controllable.

We can also satisfy Assumption 3 when A is not periodic but has the form $A(t) = A_0 + \Delta_A(t)$ where (A_0, C) is observable, when the sup norm of the continuous time varying part Δ_A is small enough. To find a bound on the allowable sup norms of Δ_A, we can use the following three step procedure. First, we can choose τ such that the constant matrix $\bar{\Omega}$ from (26) is invertible when A is replaced by A_0, using the reasoning from the preceding paragraph. Second, for the preceding choice of τ and with the choice $\bar{\Psi} = \bar{\Omega}^{-1}$, we can choose a constant $\bar{\epsilon} > 0$ such that $|M^{-1}| \leq 2|\bar{\Psi}|$ for all matrices M that satisfy $|M - \bar{\Omega}| < \bar{\epsilon}$. Finally, for the preceding choices of τ and $\bar{\epsilon} > 0$, we can use the estimates

$$\left|\Phi_A(t - j\tau, t) - e^{-j\tau A_0}\right| \leq j\tau|\Delta_A|_\infty e^{(|A_0| + |\Delta_A|_\infty)j\tau} \tag{27}$$

for $j = 1, \ldots, n-1$ (e.g., from (Mazenc *et al.*, 2020*b*, Lemma 1)) to compute a constant $\bar{\delta} > 0$ such that $|\Omega(t) - \bar{\Omega}|_\infty < \bar{\epsilon}$ for all Δ_A's that satisfy $|\Delta_A|_\infty < \bar{\delta}$. Then Assumption 3 will hold if $|\Delta_A|_\infty < \bar{\delta}$. Moreover, we can use the estimate

$$|\Phi_A(t,s)| \leq e^{|t-s|(|A_0| + |\Delta_A|_\infty)} \tag{28}$$

from (Mazenc *et al.*, 2020*b*, Lemma 1) for all real t and s to compute upper bounds on the functions σ_j from (9) to check Assumption 4 when A is nonconstant.

3 Proof of Theorem 1

First note that Assumptions 1-2 ensure that the system consisting of (3) and (11) is forward complete. By applying variation of parameters to (3) over any interval $[s, t]$ with $0 \leq s \leq t$, we have

$$\begin{aligned} x(t) \; &= \; \Phi_A(t,s)x(s) \\ &\quad + \textstyle\int_s^t \Phi_A(t,m)[\varphi(Cx(m), m) + \delta_s(m)]\mathrm{d}m. \end{aligned} \tag{29}$$

Thus, we can use the semigroup property of Φ_A to check that for all $p \in \{0, ..., n-1\}$, we have

$$\begin{aligned} &C\Phi_A(t - p\tau, t)x(t) = Cx(t - p\tau) \\ &+ C\int_{t-p\tau}^t \Phi_A(t - p\tau, m)\varphi^\sharp(m)\mathrm{d}m \end{aligned} \tag{30}$$

for all $t \geq p\tau$, where

$$\varphi^\sharp(m) = \varphi(Cx(m), m) + \delta_s(m). \tag{31}$$

Setting

$$\Delta_1(x_t) = \Psi(t) \begin{pmatrix} Cx(t) \\ Cx(t-\tau) \\ \vdots \\ Cx(t-(n-1)\tau) \end{pmatrix} \text{ and} \tag{32}$$

$$\begin{aligned} &\Delta_2(x_t) = \\ &\Psi(t)\begin{pmatrix} 0 \\ C\int_{t-\tau}^t \Phi_A(t-\tau, m)\varphi^\sharp(m)\mathrm{d}m \\ \vdots \\ C\int_{t-(n-1)\tau}^t \Phi_A(t-(n-1)\tau, m)\varphi^\sharp(m)\mathrm{d}m \end{pmatrix}, \end{aligned} \tag{33}$$

we deduce from (30) that

$$x(t) = \Delta_1(x_t) + \Delta_2(x_t) \tag{34}$$

for all $t \geq (n-1)\tau$.

We next use the variables

$$e_\omega(t) = \omega(t) - Cx(t) \text{ and } e_x(t) = \hat{x}(t) - x(t) \tag{35}$$

and we choose $\underline{i}$ to be the integer such that

$$t_{\underline{i}-1} < (n-1)\tau \text{ and } t_{\underline{i}} \geq (n-1)\tau. \tag{36}$$

Using the functions

$$D_{3,i}(x_t, \omega_t) = \omega(t - i\tau) - Cx(t - i\tau) \tag{37}$$

for $i = 0, \ldots, n-1$ and

$$\begin{aligned} &D_{4,i}(x_t, \omega_t) = \\ &C\textstyle\int_{t-i\tau}^t \Phi_A(t - i\tau, m)[\varphi(\omega(m), m) - \varphi^\sharp(m)]\mathrm{d}m, \end{aligned} \tag{38}$$

one can use (15) and (11)-(13) to check that

$$\left\{ \begin{aligned} \dot{e}_\omega(t) \; &= \; CA(t)e_x(t) + C[\varphi(\omega(t), t) \\ &\quad\; -\varphi(Cx(t), t)] \\ &\quad\; -C\delta_s(t) \text{ for all } t \in [t_i, t_{i+1}) \\ e_\omega(t_i) \; &= \; \delta_0(t_i) \\ e_x(t) \; &= \; \Psi(t)\mathcal{U}_1(e_{\omega,t}) + \Psi(t)\begin{pmatrix} 0 \\ D_{4,1}(x_t, \omega_t) \\ \vdots \\ D_{4,n-1}(x_t, \omega_t) \end{pmatrix} \end{aligned} \right. \tag{39}$$

for all $i \in \mathbb{N}$ with $i \geq \underline{i}$, where we use the definition of $\mathcal{U}_1$

from (12) (with ω_t replaced by $e_{\omega,t}$ in the $\mathcal{U}_1$ formula in (12)). For all $i \in \mathbb{N}$ with $i \geq \underline{i}$, this gives

$$\left\{\begin{array}{rcl} \dot{e}_\omega(t) &=& CA(t)\Psi(t)\mathcal{U}_1(e_{\omega,t}) - C\delta_s(t) \\ && +CA(t)\Psi(t)\begin{pmatrix} 0 \\ D_{4,1}(x_t,\omega_t) \\ \vdots \\ D_{4,n-1}(x_t,\omega_t)\end{pmatrix} \\ && +C[\varphi(\omega(t),t) \\ && -\varphi(Cx(t),t)] \text{ if } t \in [t_i, t_{i+1}) \\ e_\omega(t_i) &=& \delta_o(t_i) \end{array}\right. \tag{40}$$

By integrating (40) over the interval $[t_i, t)$, we obtain

$$\begin{array}{rcl} e_\omega(t) &=& C\int_{t_i}^t \left[\varphi(\omega(\ell),\ell) - \varphi(Cx(\ell),\ell) - \delta_s(\ell)\right]\mathrm{d}\ell \\ && +\delta_o(t_i) + \mathcal{U}_3(t,x_t,\omega_t) + \mathcal{U}_4(t,x_t,\omega_t) \end{array} \tag{41}$$

for all $t \geq 0$ and $i \in \mathbb{N}$ such that $t \in [t_i, t_{i+1})$, where

$$\begin{array}{l} \mathcal{U}_3(t,x_t,\omega_t) = \sum_{j=1}^{n}\int_{t_i}^t A_j^\sharp(m)e_\omega(m-(j-1)\tau)\mathrm{d}m \\ \text{and } \mathcal{U}_4(t,x_t,\omega_t) = \\ \sum_{j=2}^{n}\int_{t_i}^t A_j^\sharp(\ell)\int_{\ell-(j-1)\tau}^{\ell} C\Phi_A(\ell-(j-1)\tau,m)\Delta\varphi(m)\mathrm{d}m\mathrm{d}\ell \end{array} \tag{42}$$

where

$$\Delta\varphi(m) = \varphi(\omega(m),m) - \varphi^\sharp(m) \tag{43}$$

and $A_j^\sharp$ is the jth component of the vector valued function $A^\sharp = CA\Psi$ for $1 \leq j \leq n$ when $t \geq \overline{T} + (n-1)\tau$. To use our trajectory based method, we find a suitable upper bound for the right side of (41).

To this end, note that the Cauchy-Schwarz inequality followed by Hölder's inequality give

$$\begin{array}{rcl} |\mathcal{U}_3(t,x_t,\omega_t)| &\leq& \int_{t_i}^t |A^\sharp(m)||\mathcal{U}_1(e_{\omega,m})|\mathrm{d}m \\ &\leq& \sqrt{\int_{t_i}^t \sum_{j=1}^{n}\left(A_j^\sharp(m)\right)^2\mathrm{d}m} \\ && \times\sqrt{\int_{t_i}^t \sum_{j=1}^{n} e_\omega^2(m-(j-1)\tau)\mathrm{d}m} \\ &\leq& \sqrt{\overline{T}}|CA\Psi|_\infty\sqrt{\sum_{j=0}^{n-1}\int_{t_i-j\tau}^{t-j\tau} e_\omega^2(m)\mathrm{d}m} \\ &\leq& \overline{T}|CA\Psi|_\infty\sqrt{n}|e_\omega|_{[t-\overline{T}-(n-1)\tau,t]} \end{array} \tag{44}$$

and similarly,

$$\begin{array}{l} |\mathcal{U}_4(t,x_t,\omega_t)| \leq \\ \sqrt{\int_{t_i}^t \sum_{i=2}^{n}\left(A_j^\sharp(m)\right)^2\mathrm{d}m} \\ \times\sqrt{\int_{t_i}^t \sum_{j=1}^{n-1}\left(\int_{\ell-j\tau}^{\ell}|C\Phi_A(\ell-j\tau,m)||e_\omega^\sharp(m)|\mathrm{d}m\right)^2\mathrm{d}\ell} \end{array} \tag{45}$$

and so also

$$\begin{array}{l} |\mathcal{U}_4(t,x_t,\omega_t)| \\ \leq \sqrt{\overline{T}}|CA\Psi|_\infty\sqrt{\sum_{j=1}^{n-1}\int_{t_i}^t j\tau\mathrm{d}\ell\sigma_j^2|e_\omega^\sharp|^2_{[t-\overline{T}-(n-1)\tau,t]}} \\ \leq \overline{T}|CA\Psi|_\infty\sqrt{\sum_{j=1}^{n-1} j\tau\sigma_j^2}|e_\omega^\sharp|_{[t-\overline{T}-(n-1)\tau,t]} \\ \leq \overline{T}\sqrt{\tau}|CA\Psi|_\infty\mathcal{G}(\tau)|e_\omega^\sharp|_{[t-\overline{T}-(n-1)\tau,t]} \end{array} \tag{46}$$

if $t \geq \overline{T} + (n-1)\tau$, where we used the notation (9) and

$$e_\omega^\sharp(m) = \overline{\varphi}|e_\omega(m)| + |\delta_s(m)|, \tag{47}$$

and where the first inequality in (46) applied Jensen's inequality to the inside integral in the double integral. By using our bounds (44)-(46) to bound the last two right side terms in (41) and recalling our definitions of $\overline{\varphi}$ and $\overline{T}$, and by defining the intervals $\mathcal{I}(t) = [t - \overline{T} - (n-1)\tau, t]$, we get

$$\begin{array}{rcl} |e_\omega(t)| &\leq& |C|\overline{T}\overline{\varphi}|e_\omega|_{[t-\overline{T},t]} + |\delta_o|_{[t-\overline{T},t]} \\ && +|C|\overline{T}|\delta_s|_{[t-\overline{T},t]} \\ && +\overline{T}|CA\Psi|_\infty\sqrt{n}|e_\omega|_{\mathcal{I}(t)} \\ && +\overline{T}\sqrt{\tau}|CA\Psi|_\infty\mathcal{G}(\tau)|e_\omega^\sharp|_{\mathcal{I}(t)} \\ &\leq& \overline{T}\lambda(\tau)|e_\omega|_{\mathcal{I}(t)} + \mathcal{M}(t) \end{array} \tag{48}$$

for all $t \geq \overline{T} + (n-1)\tau$, where λ is as defined in (8) and $\mathcal{M}$ is defined in (16). Then (10) in Assumption 4 and (Mazenc *et al.*, 2017, Lemma 1) (which we also include in the appendix below) applied to the function $w(t) = |e_\omega(t + \overline{T} + (n-1)\tau)|$ and the constant $T_* = \overline{T} + (n-1)\tau$ ensure that

$$\begin{array}{l} |e_\omega(t)| \leq \\ \sup_{\ell\in[0,\overline{T}+(n-1)\tau]}|e_\omega(\ell)|e^{\frac{\ln(\overline{T}\lambda(\tau))}{\overline{T}+(n-1)\tau}(t-\overline{T}-(n-1)\tau)} + \frac{\mathcal{M}(t)}{1-\overline{T}\lambda(\tau)} \end{array} \tag{49}$$

for all $t \geq \overline{T} + (n-1)\tau$. This concludes the proof of the first conclusion of the theorem. The second conclusion now follows from the formula for e_x in (39).

4 More General Systems

4.1 Statement of Result

Under alternative conditions involving $\overline{T}$, we can generalize the preceding work to cover systems of the form

$$\left\{\begin{array}{rcl} \dot{x}(t) &=& A(t)x(t) + \varphi(t, x(t), \delta_s(t)) \\ y(t) &=& Cx(t_i) + \delta_o(t) \text{ for all } t \in [t_i, t_{i+1}) \text{ and } i \geq 0 \end{array}\right. \tag{50}$$

by replacing Assumptions 2 and 4 by the two assumptions that we give next, where we continue the other notation from above. First, we replace our Assumption 2 by:

Assumption 5 *The function φ is continuous. Also, there is a constant $\overline{\varphi} \geq 0$ such that $|\varphi(t, x_a, d_a) - \varphi(t, x_b, d_b)| \leq \overline{\varphi}(|x_a - x_b| + |d_a - d_b|)$ holds for all x_a, x_b, d_a, and d_b in $\mathbb{R}^n$ and all $t \geq 0$.* □

Fixing a constant $\overline{\varphi}$ that satisfies Assumption 5, and also using the constants

$$\begin{array}{l}\lambda_a = |CA\Psi|_\infty \sqrt{n},\ \lambda_b = \sqrt{\tau}|CA|\Psi|_\infty \mathcal{G}(\tau) + |C|\overline{\varphi},\\ \lambda_c = \sqrt{n}|\Psi|_\infty |C|\tau(n-1)|\Phi_A|_\infty \overline{\varphi},\ \text{and}\ \lambda_d = |\Psi|_\infty \sqrt{n},\end{array} \tag{51}$$

we then replace Assumption 4 by the following:

Assumption 6 *The inequalities*

$$0 < \overline{T}\lambda_a < 1,\ 0 < \lambda_c < 1,\ \text{and}\ 0 < \frac{\overline{T}\lambda_b\lambda_d}{(1-\overline{T}\lambda_a)(1-\lambda_c)} < 1 \tag{52}$$

are satisfied. □

We also replace the continuous-discrete system (11) by

$$\begin{cases} \dot{\omega}(t) = CA(t)\hat{x}(t) + C\varphi(t, \hat{x}(t), 0) \\ \qquad \text{for all } t \in [t_i, t_{i+1}) \text{ and } i \geq 0 \\ \omega(t_i) = y(t_i) \text{ for all } i \geq 0 \\ \hat{x}(t) = \Psi(t)\mathcal{U}_1(\omega_t) + \Psi(t)\mathcal{U}_2(\omega_t, t) \end{cases} \tag{53}$$

and we replace $\varphi(\omega(t), t)$ by $\varphi(t, \hat{x}(t), 0)$ in (14). Then we have this analog of Theorem 1 for (50) and (53):

Theorem 2 *Let (50) satisfy Assumptions 1, 3, 5, and 6. Then we can construct positive constants $\bar{c}_i$ for $i = 1, 2, 3$ such that with the choices*

$$\mathcal{D}(t) = \max\{|\omega(t) - Cx(t)|, |x(t) - \hat{x}(t)|\} \tag{54}$$

and $T_* = \overline{T} + (n-1)\tau$, and for all constant initial functions for (50) and (53) and all measurable essentially bounded functions (δ_s, δ_o) in (50), we have

$$\mathcal{D}(t) \leq \bar{c}_1 e^{-\bar{c}_2 t}|\mathcal{D}|_{[0,T_*]} + \bar{c}_3(|\delta_s|_{[0,t]} + |\delta_o|_{[0,t]}) \tag{55}$$

for all $t \geq 0$. □

Remark 1 It is tempting to surmise that Theorem 2 is less restrictive than Theorem 1, because the system (50) includes (3) as a special case. However, that is not the case, because as we illustrate in Section 5.2, Assumption 6 can lead to a smaller (and therefore more conservative) allowable upper bound $\overline{T}$ on the sample rate as compared with Assumption 4 when the nonlinearity φ only depends on the t and the output y and when the uncertainty δ_s is additive. Therefore, Assumption 6 is the price to pay to allow the more general nonlinearities and uncertainties in (50). For simplicity, we assume in this section that δ_s has the same dimension as the state. However, the results remain true if δ_s and the state x have different dimensions. Also, we will see in the proof of Theorem 2 that the constants $\bar{c}_i$ in Theorem 2 can be constructed using small gain arguments, e.g., those from the proof of (Liu *et al.*, 2020, Theorem 2.6). □

4.2 Proof of Theorem 2

We indicate the changes needed in the proof of Theorem 1 to prove Theorem 2. In equations (29) and the formula (30) for $\varphi^\sharp$, we replace $\varphi(Cx(m), m) + \delta_s(m)$ by $\varphi(m, x(m), \delta_s(m))$. Then, in the formula (38) for D_4, we replace $\varphi(\omega(m), m)$ by $\varphi(m, \hat{x}(m), 0)$. Now each $D_{4,i}$ only depends on x_t and $\hat{x}_t$, so we can write it as $D_{4,i}(x_t, \hat{x}_t)$. Then we replace the terms $C[\varphi(\omega(t), t) - \varphi(Cx(t), t)] - C\delta_s(t)$ in (39) and (40) by $C[\varphi(m, \hat{x}(m), 0) - \varphi(m, x(m), \delta_s(m))]$. Also, we replace the integrand $\varphi(\omega(\ell), \ell) - \varphi(Cx(\ell), \ell) - \delta_s(\ell)$ in the formula for $e_\omega(t)$ in (41) by $\varphi(m, \hat{x}(m), 0) - \varphi(m, x(m), \delta_s(m))$, and we replace $\varphi(\omega(m), m)$ in the formula (43) for $\Delta\varphi(m)$ by $\varphi(m, \hat{x}(m), 0)$. Then $\mathcal{U}_4$ is a function of $(t, x_t, \hat{x}_t)$, so we write it as $\mathcal{U}_4(t, x_t, \hat{x}_t)$. Then, we replace $e_\omega^\sharp$ by $e_x^\sharp$ in the upper bounds for $|\mathcal{U}_4(t, x_t, \hat{x}_t)|$ in (45)-(46), where $e_x^\sharp(m) = \overline{\varphi}\big(|e_x(m)| + |\delta_s(m)|\big)$. We also use the function

$$\begin{array}{l}\mathcal{M}_*(t) = \\ |\delta_o|_{[0,t]} + \overline{\varphi}\overline{T}\big(|C| + \sqrt{\tau}|CA\Psi|_\infty \mathcal{G}(\tau)\big)|\delta_s|_{[0,t]}.\end{array} \tag{56}$$

Then, since (52) gives $\overline{T}\lambda_a \in (0,1)$, we can use the reasoning of the last part of the proof of Theorem 1 to get

$$\begin{array}{rl}|e_\omega(t)| \leq & |C|\overline{T}\overline{\varphi}(|e_x|_{[t-\overline{T},t]} + |\delta_s|_{[t-\overline{T},t]}) \\ & +\overline{T}|CA\Psi|_\infty\sqrt{n}|e_\omega|_{\mathcal{I}(t)} + |\delta_o|_{[t-\overline{T},t]} \\ & +\overline{T}\sqrt{\tau}|CA\Psi|_\infty\mathcal{G}(\tau)|e_x^\sharp|_{[0,t]} \\ \leq & \overline{T}\lambda_a|e_\omega|_{\mathcal{I}(t)} + \overline{T}\lambda_b|e_x|_{[0,t]} + \mathcal{M}_*(t)\end{array} \tag{57}$$

for all $t \geq T_*$ and therefore also

$$\begin{array}{l}|e_\omega(t)| \leq \\ |e_\omega|_{[0,T_*]}e^{\frac{\ln(\overline{T}\lambda_a)}{T_*}(t-T_*)} + \frac{\overline{T}\lambda_b|e_x|_{[0,t]}}{1-\overline{T}\lambda_a} + \frac{\mathcal{M}_*(t)}{1-\overline{T}\lambda_a}\end{array} \tag{58}$$

for all $t \geq 0$, by applying (Mazenc *et al.*, 2017, Lemma 1), where we also used the fact that the first term on the right side of (58) upper bounds $|e_\omega(t)|$ for all $t \in [0, T_*]$, in order to ensure that (58) holds for all $t \geq 0$. Also, with the preceding change in D_4, we can combine the last equation in (39) with our definitions of λ_c and λ_d from (51), to obtain

$$|e_x(t)| \leq \lambda_d|e_\omega|_{[0,t]} + \lambda_c|e_x|_{\mathcal{I}(t)} + \lambda_c|\delta_s|_{[0,t]} \tag{59}$$

for all $t \geq 0$. Recalling from Assumption 6 that $\lambda_c \in (0,1)$, we can now again apply (Mazenc *et al.*, 2017, Lemma 1), this time to (59) (in a similar way to the approach that we used to obtain (58)) to conclude that

$$|e_x(t)| \leq |e_x|_{[0,T_*]}e^{\frac{\ln(\lambda_c)t}{T_*}} + \frac{\lambda_d|e_\omega|_{[0,t]}}{1-\lambda_c} + \frac{\lambda_c|\delta_s|_{[0,t]}}{1-\lambda_c} \tag{60}$$

for all $t \geq 0$. By (58) and (60) and the negativeness of $\ln(\overline{T}\lambda_a)/T_*$ and $\ln(\lambda_c)/T_*$, the conclusion now follows from the last inequality in Assumption 6 and small gain arguments, e.g., the proof of (Liu *et al.*, 2020, Theorem 2.6).

5 Illustrations

5.1 Pendulum with Horizontal Acceleration

Consider a pendulum of mass M and length L whose suspension point is subjected to an unknown time varying piecewise continuous bounded horizontal acceleration $h(t)$ and having a time varying friction $k(t)$ and a torque input T_c. As noted in (Khalil, 2002, p.627), this produces

$$ML\ddot{\theta} + Mg\sin(\theta) + k(t)L\dot{\theta} = \frac{T_c}{L} + Mh(t)\cos(t), \tag{61}$$

where $g = 9.81$ is the gravitational constant, and where we assume that sampled perturbed measurements $y(t_i) = c_o\theta(t_i) + \delta_o(t_i)$ are available at sample times t_i that satisfy our Assumption 1 for some constants $\underline{T} > 0$ and $\overline{T} > 0$, where $c_o > 0$ is a known constant whose effect we will study in what follows. Then (61) and its output measurements can be written in the form (3) with the choices $n = 2$,

$$A(t) = \begin{pmatrix} 0 & 1 \\ 0 & -\frac{k(t)}{M} \end{pmatrix}, \; x = \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix}, \; C = \begin{pmatrix} c_o \\ 0 \end{pmatrix}^\top,$$
$$\varphi(\ell, t) = \begin{pmatrix} 0 \\ -\frac{g}{L}\sin\left(\frac{\ell}{c_o}\right) + \frac{1}{ML^2}T_c(t) \end{pmatrix}, \text{ and} \tag{62}$$
$$\delta_s(t) = \begin{pmatrix} 1 \\ \frac{1}{L}h(t)\cos(\theta(t)) \end{pmatrix}.$$

We assume that $L = 1$, and that $k(t) = 0.1 + \Delta_k(t)$ where the function $\Delta_k : [0, \infty) \to [0, 0.1]$ can represent the increase in friction over time when Δ_k is increasing. This differs from the case in Khalil (2002) which did not allow sampled outputs, uncertainties, or time varying friction. We choose $\overline{\varphi} = g/c_0$. We next compute bounds on the allowable supremum $\overline{T}$ under which Theorem 1 applies.

To this end, notice that in terms of our notation from Theorem 1, our choices (62) give

$$\Phi_A(t, s) = \begin{pmatrix} 1 & \int_s^t e^{-\int_s^r k(\ell)/M \mathrm{d}\ell} \mathrm{d}r \\ 0 & e^{-\int_s^t k(\ell)/M \mathrm{d}\ell} \end{pmatrix} \tag{63}$$

so with the choice (6), the matrix $\Psi = \Omega^{-1}$ has the form

$$\Psi(t) = \begin{pmatrix} \int_t^{t-\tau} e^{-\int_t^r k(\ell)/M \mathrm{d}\ell} \mathrm{d}r & 0 \\ -1 & 1 \end{pmatrix} \frac{1}{c_o \int_t^{t-\tau} e^{-\int_t^r k(\ell)/M \mathrm{d}\ell} \mathrm{d}r} \tag{64}$$

and so is bounded for any choice of $\tau > 0$. For simplicity, we choose $\tau = 1$ in the rest of this subsection. Then

$$CA(t)\Psi(t) = (-1 \;\; 1) \frac{1}{\int_t^{t-1} e^{-\int_t^r k(\ell)/M \mathrm{d}\ell} \mathrm{d}r} \tag{65}$$

so our lower bound of 0.1 on $k(t)$ gives

$$|CA\Psi|_\infty \leq \tfrac{\sqrt{2}}{10M}(e^{0.1/M} - 1)^{-1}. \tag{66}$$

Also, since

$$\mathcal{G}(1) = c_o \sup \left\{ \sqrt{1 + \left(\int_s^r e^{-\int_s^q k(\ell)/M \mathrm{d}\ell} \mathrm{d}q \right)^2} : \right. \tag{67}$$
$$\left. s \geq r \geq 0, s - r \leq 1 \right\},$$

our upper bound of 0.2 on k gives the upper bound

$$\mathcal{G}(1) \leq c_o \sqrt{1 + e^{0.04/M^2}}. \tag{68}$$

Since $c_o > 0$ can be taken to be arbitrarily small, the assumptions of Theorem 1 will then be satisfied if $\overline{T}$ satisfies $\sqrt{2\overline{T}}|CA\Psi|_\infty + \overline{T}g < 1$. which holds if

$$\overline{T}\left(\tfrac{1}{5M}\left(e^{0.1/M} - 1\right)^{-1} + 9.81 \right) < 1. \tag{69}$$

When $M = 1$, this gives the approximate upper bound of 0.08538768 on the allowable $\overline{T}$ values, and a limiting upper bound of $\overline{T} < 1/9.81 = 0.101936799$ as $M \to +\infty$.

5.2 Comparison with Mazenc et al. *(2020*a*)*

Theorem 1 provides less restrictive conditions on $\overline{T}$ than Mazenc *et al.* (2020*a*), even when A is constant and there are no uncertainties. To illustrate this point, we revisit the example from Mazenc *et al.* (2020*a*), which is the particular case where $n = 2$, $C = (1, 0)$ and

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \text{ and } \varphi(y, t) = \begin{pmatrix} \epsilon\sin(y) + u^2(t) \\ -u(t) \end{pmatrix} \tag{70}$$

for any constant $\epsilon > 0$, where u represents a control input. Then (A, C) is observable, so Assumptions 1-3 can be satisfied with $\overline{\varphi} = \epsilon$; see Section 2.3.4. By noting that

$$e^{-Am} = \begin{pmatrix} \cos(m) & -\sin(m) \\ \sin(m) & \cos(m) \end{pmatrix} \tag{71}$$

for all $m \in \mathbb{R}$ and using our notation from Section 2.3, we get

$$\overline{\Omega} = \begin{pmatrix} C \\ Ce^{-A\tau} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \cos(\tau) & -\sin(\tau) \end{pmatrix} \tag{72}$$

and for any $\tau \in \left(0, \frac{\pi}{2}\right]$, we have

$$\overline{\Psi} = \begin{pmatrix} 1 & 0 \\ \frac{\cos(\tau)}{\sin(\tau)} & -\frac{1}{\sin(\tau)} \end{pmatrix}. \tag{73}$$

Using our formula (24) and $\tau = \pi/2$, this gives

$$\overline{\Psi} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{74}$$
$$\text{and } \lambda\left(\overline{T}, \tfrac{\pi}{2}\right) = \sqrt{2}\left[1 + \epsilon e^{\frac{\pi}{2}}\left(\overline{T} + \tfrac{\pi}{2}\right)\right] + \epsilon.$$

Choosing $\epsilon = \frac{1}{6}$ as in Mazenc *et al.* (2020*a*), we get

$$\lambda\left(\overline{T}, \frac{\pi}{2}\right) = \sqrt{2}\left[1 + \frac{e^{\frac{\pi}{2}}}{6}\left(\overline{T} + \frac{\pi}{2}\right)\right] + \frac{1}{6}, \tag{75}$$

and then the requirement $\overline{T}\lambda(\overline{T}, \pi/2) < 1$ from Mazenc *et al.* (2020*a*) produces the requirement

$$\overline{T} \leq 0.27242. \tag{76}$$

Also, the supremum of the set of all $\overline{T}$'s that satisfy the requirements (52) of Theorem 2 would be 0.28. By contrast, if we use Theorem 1 with the preceding choices of the model parameters, then we get the requirement

$$\overline{T} < \frac{1}{\sqrt{2} + \sqrt{\pi/2} + 1/6} = 0.352834 \tag{77}$$

which is a 29% increase in the upper bound as compared with (76). Hence, in this special case, our result is significantly less conservative than Mazenc *et al.* (2020*a*).

On the other hand, Theorem 1 also allows uncertainties, which were not allowed in Mazenc *et al.* (2020*a*). We next

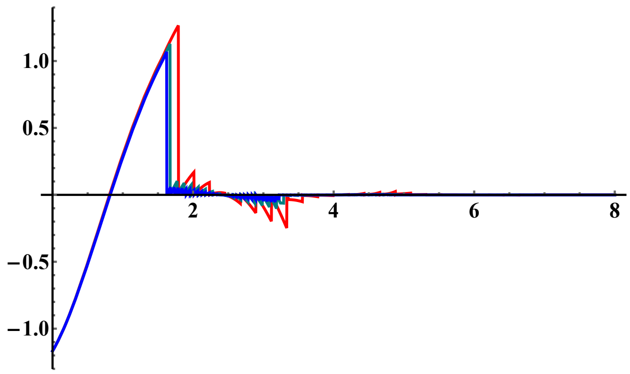

Fig. 1. Tracking Error $\hat{x}_2(t) - x_2(t)$ Converging to 0 over Time with J Values 1 (Red), 0.5 (Green), and 0.25 (Blue) with $\delta_0 = 0$ and $\delta_s = 0$ after 4 Second Transient Period

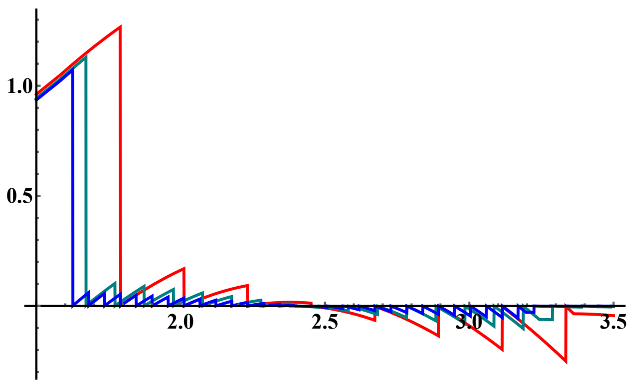

Fig. 2. Tracking Error $\hat{x}_2(t) - x_2(t)$ over Time with J Values 1 (Red), 0.5 (Green), and 0.25 (Blue) with $\delta_0 = 0$ and $\delta_s = 0$ Zoomed in to Transient Period

present some Mathematica simulations, which illustrate the effects of the uncertainties on the performance of the observers from Theorem 1 when applied with the choices (70). We use the nonuniformly spaced sampling instants $t_i = \big(0.9 + 0.1\sin(i\pi/2)\big)iJ\overline{T}$ whose sampling intervals alternate between lengths $0.8J\overline{T}$, $0.9J\overline{T}$, and $J\overline{T}$ (but similar reasoning applies for other nonuniform sampling) with $\overline{T}$ chosen as the upper bound in (76) and the input $u(t) = \sin(t)$ in (70), with the choice $J = 1$, then with $J = 1/2$, and finally with $J = 0.25$. Also, we used the initial conditions $x_1(0) = x_2(0) = 1$ and $\omega(0) = 0$.

We plot the resulting observation error values of $\hat{x}_2(t) - x_2(t)$ for the three different choices of the sampling rate, in Figures 1-6. Since the figures show rapid convergence of the errors to 0, with faster convergence as j decreases, when the uncertainty is 0, and an ISS like property for nonzero values of the uncertainty δ_0 or δ_s, they help illustrate Theorem 1 for the special case of (70).

6 Conclusions

This paper extended the observer design method from Mazenc *et al.* (2015*a*) to allow discrete measurements, model uncertainty, time varying coefficients, and sensor noise. We used a novel trajectory based approach that ensured arbitrarily fast convergence of the observation error to zero, as long as the sampling in the output is frequent enough. Our trajectory based contractivity condition was used in place of standard Lyapunov function methods. We illustrated the value of our method in a pendulum example whose novel features included a time varying horizontal acceleration and a time varying friction. When

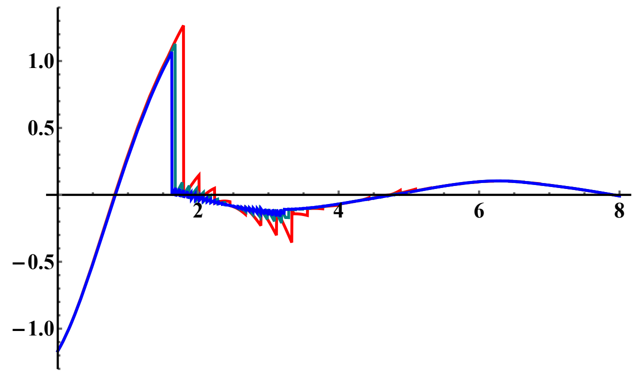

Fig. 3. Tracking Error $\hat{x}_2(t) - x_2(t)$ over Time with J Values 1 (Red), 0.5 (Green), and 0.25 (Blue) with $\delta_0(t) = 0.1\sin(t)$ and $\delta_s = 0$ Converging to Oscillation after 4 Second Transient Period.

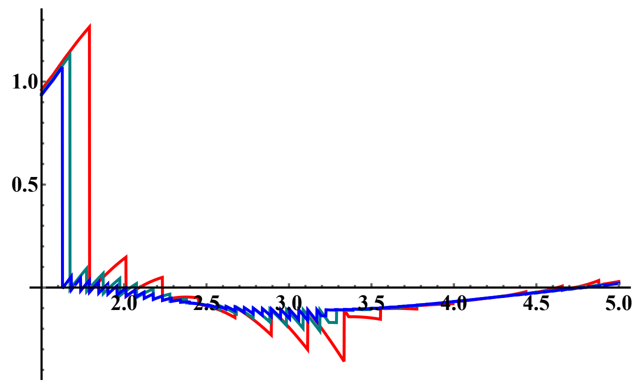

Fig. 4. Tracking Error $\hat{x}_2(t) - x_2(t)$ over Time with J Values 1 (Red), 0.5 (Green), and 0.25 (Blue) with $\delta_0(t) = 0.1\sin(t)$ and $\delta_s = 0$ Zoomed into Transient Period.

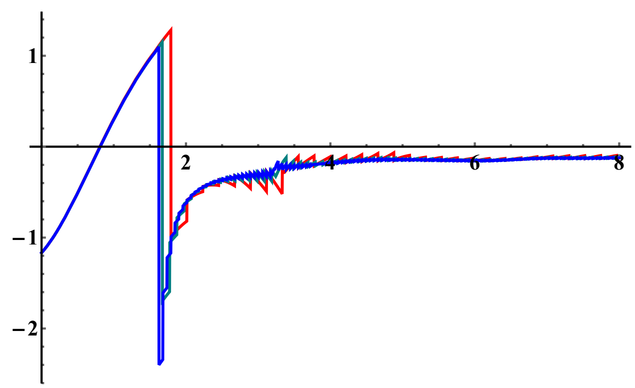

Fig. 5. Tracking Error $\hat{x}_2(t) - x_2(t)$ over Time with J Values 1 (Red), 0.5 (Green), and 0.25 (Blue) with $\delta_0(t) = 5[1 - 0.98(20t/(1+20t))]$ and $\delta_s = 0$ Converging to an Undershoot Level of -0.18 after 5 Seconds.

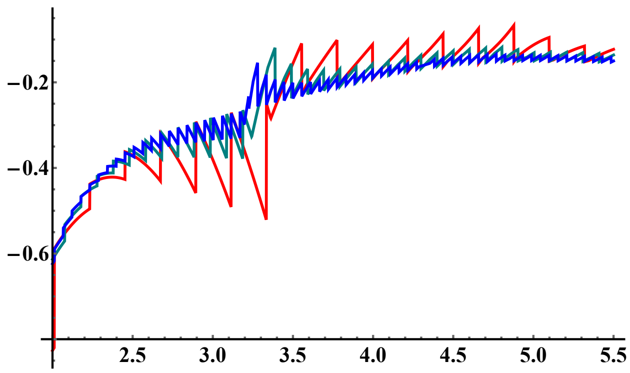

Fig. 6. Tracking Error $\hat{x}_2(t) - x_2(t)$ over Time with J Values 1 (Red), 0.5 (Green), and 0.25 (Blue) with $\delta_0(t) = 5[1 - 0.98(20t/(1 + 20t))]$ and $\delta_s = 0$ Zoomed to Transient Period.

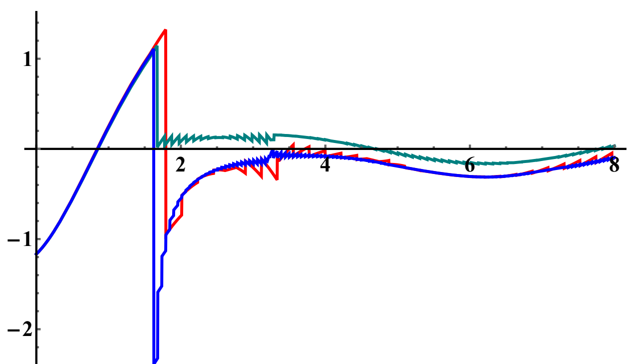

Fig. 7. Tracking Error $\hat{x}_2(t) - x_2(t)$ over Time with J Values 1 (Red), 0.5 (Green), and 0.25 (Blue) with $\delta_0(t) = 0$ and $\delta_s(t) = 0.1(\sin(t), \cos(t))$ Converging to an Oscillation after 5 Seconds.

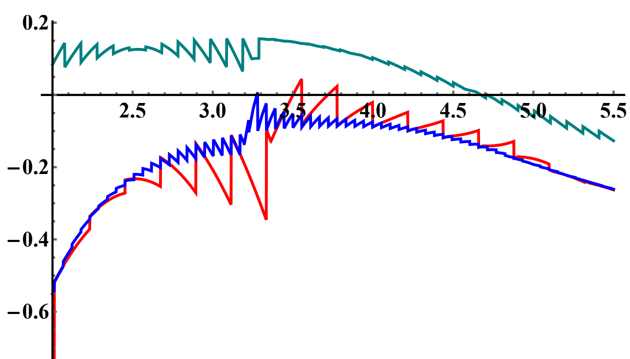

Fig. 8. Tracking Error $\hat{x}_2(t) - x_2(t)$ over Time with J Values 1 (Red), 0.5 (Green), and 0.25 (Blue) with $\delta_0(t) = 0$ and $\delta_s(t) = 0.1(\sin(t), \cos(t))$ Zoomed to Transient Period.

model and measurement uncertainties are present, we provided a variant of input-to-state stability that quantifies the effects of the uncertainties in an overshoot term in our bound on the estimation error.

Our work was motivated by the increasing speeds of available sensors and DSPs, which call for studies of trade-offs between sampling rates in outputs measurements and convergence rates of observers, such as this work. Another motivation was the usefulness of observers in state feedback design, which would entail replacing state values in the stabilizing feedback controls by the estimated values $\hat{x}(t)$ from our estimator construction (11)-(13); see, e.g., Mazenc *et al.* (2020*b*), where such a replacement was done using finite time observers that required continuous unperturbed output measurements (instead of only the discrete perturbed output measurements as required here). We hope to develop analogs for systems governed by hyperbolic PDEs or reaction-diffusion equations, e.g., for the dynamics in Selivanov and Fridman (2019); Katz *et al.* (2021)

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Appendix: Statement of (Mazenc *et al.*, 2017, Lemma 1)

We provide a statement of (Mazenc *et al.*, 2017, Lemma 1), which we used at the end of our proof of Theorem 1 above:

Lemma A.1 Let $T_* > 0$ be a constant. Let $w : [-T_*, \infty) \to [0, \infty)$ be a piecewise continuous locally bounded function and $d : [0, \infty) \to [0, \infty)$ be piecewise continuous. Assume that there exists a constant $\rho \in (0, 1)$ such that

$$w(t) \le \rho |w|_{[t-T_*,t]} + d(t) \tag{A.1}$$

holds for all $t \ge 0$. Then the inequality

$$w(t) \le |w|_{[-T_*,0]} e^{\frac{\ln(\rho)}{T_*} t} + \frac{1}{1-\rho} |d|_{[0,t]} \tag{A.2}$$

holds for all $t \ge 0$.

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