Asymptotic expansion of the solution of Maxwell's equations in polygonal plane domains

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N° ????

Juillet 2005

Thème NUM

Rapport de recherche

N 0249-6399

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Thème NUM — Systèmes numériques
Projet CALVI

Rapport de recherche n° ???? — Juillet 2005 — 26 pages

Abstract: This paper is mainly concerned with the structure of the singular and regular parts of the solution of time-harmonic Maxwell's equations in polygonal plane domains. The asymptotic behaviour of the solution near corner points of the domain is studied by means of discrete Fourier transformation. A detailed functional analysis of the solution shows that the boundary value problem does not belong locally to H^2 when the boundary of the domain has non-acute angles, and explicit formulas for the singularity functions and their corresponding coefficients are given.

Key-words: Maxwell's equations, singularities, Fourier method

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[†] This work was partly done while the author was visiting INRIA-Lorraine (Project: CALVI), Nancy. The visit was jointly sponsored by INRIA and the Ministry of Higher Education (MINESUP), Yaounde, Cameroon, under the programme "Mobilité Acadèmique". The author gratefully acknowledges the support. The author would also like to thank staff of INRIA for their hospitality.

Unité de recherche INRIA Lorraine
LORIA, Technopôle de Nancy-Brabois, Campus scientifique,

Développement asymptotique de la solution des équations de Maxwell dans des domaines polygonaux

Résumé : Cet article étudie la structure des singularités de la solution des équations de Maxwell harmoniques dans des polygones plans. Le comportement asymptotique de la solution au voisinage des coins est étudié à l'aide de séries de Fourier. Une analyse détaillée du problème aux limites montre que la solution n'est pas localement H^2 lorsque la frontière du domaine présente des angles obtus ou rentrants, On présente des formules explicites pour les fonctions singulières et les coefficients de singularité.

Mots-clés : Équations de Maxwell, singularités, méthode de Fourier.

1 Introduction

In this paper, we consider the electric field $\mathbf{E} = (E_1, E_2)$ of time-harmonic Maxwell's equations in a bounded and simply connected polygonal subset Ω of $\mathbb{R}^2$ with boundary $\Gamma := \partial\Omega$, representing an isotropic and homogeneous medium and with perfect condutor boundary condition, i.e.

$$\begin{cases} \mathbf{curl}\,\mathrm{curl}\,\mathbf{E} - \alpha^2\mathbf{E} = \mathbf{f} & \text{in} \quad \Omega \\ \mathrm{div}\,\mathbf{E} = 0 & \text{in} \quad \Omega \\ \mathbf{E} \wedge \mathbf{n} = E_1 n_2 - E_2 n_1 = 0 & \text{on} \quad \Gamma \end{cases} \tag{1.1}$$

where $\alpha^2 \neq 0$ is some given complex number, $\mathbf{n} = (n_1, n_2)$ denotes the unit outward normal on Γ, and the given datum $\mathbf{f} = (f_1, f_2)$ satisfies the condition

$$\mathrm{div}\,\mathbf{f} = 0 \quad \text{in} \quad \Omega \tag{1.2}$$

We have used the notations curl and **curl** to distinguish between the scalar and vectorial meanings of the *curl* operator defined as follows:

$$\begin{aligned} \mathrm{curl}\,\mathbf{E} &= \frac{\partial E_2}{\partial x_1} - \frac{\partial E_1}{\partial x_2} \\ \mathbf{curl}\,\omega &= \Big(\frac{\partial \omega}{\partial x_2}, -\frac{\partial \omega}{\partial x_1}\Big) \end{aligned}$$

We observe that the operator $\mathbf{E} \mapsto \mathbf{curl}\,\mathrm{curl}\,\mathbf{E} - \alpha^2\mathbf{E}$ is not elliptic and the usual H^1-conforming nodal finite element method (cf. [6, 7]) is not appropriate for the approximation of the weak solution of (1.1). In this case non standard finite element methods have to be employed (cf. [4, 14, 30]).

However, since $-\triangle\mathbf{E} = \mathbf{curl}\,\mathrm{curl}\,\mathbf{E} - \mathbf{grad}\,\mathrm{div}\,\mathbf{E}$, the solution of (1.1) can be found by solving the following equivalent system of elliptic boundary value problem (the so-called *regularized Maxwell's equations*) (cf. [5, 19, 23]).

$$\begin{cases} -\triangle\mathbf{E} - \alpha^2\mathbf{E} = \mathbf{f} & \text{in} \quad \Omega \\ \mathbf{E} \wedge \mathbf{n} = 0 & \text{on} \quad \Gamma \\ \mathrm{div}\,\mathbf{E} = 0 & \text{on} \quad \Gamma \end{cases} \tag{1.3}$$

We will consider subsequently problem (1.3) rather than (1.1).

For the associated weak formulation of problem (1.3) we introduce the Hilbert spaces

$$\begin{aligned} H_0(\mathrm{curl},\Omega) &:= \Big\{\mathbf{v}\in L_2(\Omega)^2:\ \mathrm{curl}\,\mathbf{v}\in L_2(\Omega) \text{ and } \mathbf{v}\wedge\mathbf{n}=0 \text{ on } \Gamma\Big\} \\ H_0(\mathrm{curl},\mathrm{div},\Omega) &:= \Big\{\mathbf{v}\in H_0(\mathrm{curl},\Omega):\mathrm{div}\,\mathbf{v}\in L_2(\Omega)\Big\} \\ H_N(\Omega) &:= \Big\{\mathbf{v}\in H^1(\Omega)^2:\mathbf{v}\wedge\mathbf{n}=0 \text{ on } \Gamma\Big\} \end{aligned} \tag{1.4}$$

equipped with the norms

$$\begin{aligned} \|\mathbf{v}\|_{H_0(\mathrm{curl},\Omega)} &:= \Big(\|\mathbf{v}\|^2_{L_2(\Omega)^2}+\|\ \mathrm{curl}\,\mathbf{v}\|^2_{L_2(\Omega)}\Big)^{1/2} \\ \|\mathbf{v}\|_{H_0(\mathrm{curl},\mathrm{div},\Omega)} &:= \Big(\|\mathbf{v}\|^2_{L_2(\Omega)^2}+\|\ \mathrm{curl}\,\mathbf{v}\|^2_{L_2(\Omega)}+\|\mathrm{div}\,\mathbf{v}\|^2_{L_2(\Omega)}\Big)^{1/2} \\ \|\mathbf{v}\|_{H_N(\Omega)} &:= \|\mathbf{v}\|_{H^1(\Omega)^2} \end{aligned} \tag{1.5}$$

The weak solution of (1.3) is obtained by solving the variational problem: Find $\mathbf{u}\in H_0(\mathrm{curl},\mathrm{div},\Omega)$ such that

$$a(\mathbf{u},\mathbf{v}) := (\mathrm{curl}\,\mathbf{u},\mathrm{curl}\,\mathbf{v})+(\mathrm{div}\,\mathbf{u},\mathrm{div}\,\mathbf{v})-\alpha^2(\mathbf{u},\mathbf{v}) = (\mathbf{f},\mathbf{v}) =: h(\mathbf{v}), \forall\mathbf{v}\in H_0(\mathrm{curl},\mathrm{div},\Omega) \tag{1.6}$$

where $(\cdot,\cdot)$ denotes the usual scalar product in $L_2(\Omega)$. If $\alpha^2\neq 0$ is not an eigenvalue of the Dirichlet-Laplace operator in Ω, then the variational problem (1.6) has for any datum $\mathbf{f}\in L_2(\Omega)^2$ a unique solution $\mathbf{u}\in H_0(\mathrm{curl},\mathrm{div},\Omega)$ (cf. [5, 10, 11, 23]).

It is a well known fact (cf. [13, 16, 22]), that the spaces $H_0(\mathrm{curl},\mathrm{div},\Omega)$ and $H_N(\Omega)$ coincide if the domain Ω is convex. In this case, the solution $\mathbf{u}$ of (1.6) can be sought in $H_N(\Omega)$ (cf. [9]) and approximated by means of the usual nodal H^1-conforming finite element method. However, if the domain Ω is non-convex, then the space $H_N(\Omega)$ is a proper subspace of the space $H_0(\mathrm{curl},\mathrm{div},\Omega)$, and the solution $\mathbf{u}$ of (1.6) in this case can not be approximated by means of the usual nodal H^1-conforming finite element method, even with local mesh refinements. This fact has motivated the analysis of the regularity and the singularities of the solution of Maxwell's equations in non-convex domains and the construction of special adaptive nodal H^1-conforming finite elements for its approximation (cf. [1, 2, 3, 5, 19, 23, 27]). In all these papers, it is shown by means of some abstract arguments on the decomposition of the space $H_0(\mathrm{curl},\mathrm{div},\Omega)$, that the solution $\mathbf{u}$ of (1.6) can be split in the form $\mathbf{u}=\mathbf{u}^{re}+\mathbf{u}^{si}$ with $\mathbf{u}^{re}\in H^1(\Omega)^2$ and where the singular part $\mathbf{u}^{si}$ is the gradient of the singular function of the Dirichlet problem for the Laplace operator. Relying on this H^1-regularity results, *Assous et al.* [1, 2], *Bonnet-Ben Dhia et al.* [5], *Hazard & Lohrengel* [19] proposed for the approximation of the solution $\mathbf{u}$ of (1.6) the so-called *Singular field method* which introduces into the finite element subspaces the well known singular functions. However, since the rate of convergence of this adaptive method depends on the smoothness of the regular part $\mathbf{u}^{re}$ of the splitting, the rate of convergence in this case is only of the order $O(h^s)$ $(0 < s < 1)$ (cf. [19, 23]). This shows that the splitting of the solution $\mathbf{u}$ and the accompany adaptive solution method are not optimal from the point of view of numerical analysis.

In accordance with several papers, see for example [12, 17, 18, 21, 24, 25, 26, 28], it is natural to suppose that for $\mathbf{f} \in L_2(\Omega)^2$, the solution $\mathbf{u} \in H_0(\mathrm{curl}, \mathrm{div}, \Omega)$ of the second order elliptic boundary value problem (1.6) is H^2-regular if the physical domain Ω is sufficiently smooth. Moreover, if the solution entails singularities due to boundary irregularities, then one can split the solution $\mathbf{u}$ in the form $\mathbf{u} = \mathbf{u}^{re} + \mathbf{u}^{si}$ with $\mathbf{u}^{re} \in H^2(\Omega)^2$ and with a well defined singular part $\mathbf{u}^{si}$.

Further regularity results for Maxwell's equations can be found in [8, 10]. In [8], $H^{1/2}$-regularity is proved for general Lipschitz domains and in [10], optimal regularity results are derived by means of the classical Mellin transformation (cf. [21]), namely, H^{s+1}-regularity and the corresponding singularity functions for a right hand side from the space H^{s-1}. The method of the Mellin transformation does not yield explicit formulas for the coefficients of the singularity functions.

The principal objectives of this paper are the following: Firstly, to present a constructive and direct method that will lead to obtaining all singularities of the solution of Maxwell's equations in bounded plane domains with corners and their corresponding coefficients. Secondly, give precise conditions on the domain Ω that will guarantee H^2-regularity of the solution $\mathbf{u} \in H_0(\mathrm{curl}, \mathrm{div}, \Omega)$ of problem (1.6).

The main results of this paper presented in Theorem 2.2 can be summarised as follows: Let the domain Ω be a polygonal domain with N vertices K_j of angle ω_j. Let (r_j, θ_j) and $\eta(r_j)$ be, respectively, the local polar coordinates and a truncation function associated with the j^{th}-vertex $(j = 1, 2, \cdots, N)$. Then, for each $\mathbf{f} \in L_2(\Omega)^2$ the unique solution $\mathbf{u} \in H_0(\mathrm{curl}, \mathrm{div}, \Omega)$ of the variational problem (1.6) is provided with the following additional properties:

If $0 < \omega_j < 2\pi$ for $j \in \{1, 2, \cdots, N\}$, then there exist unique real numbers $\gamma_{j,i}$, $(i = 1, 2, 3)$ such that for $\lambda_{j,i} := \frac{i\pi}{\omega_j}$ $(i = 1, 2, 3)$, $\mathbf{u}$ can be split in the form $\mathbf{u} = \mathbf{w} + \mathbf{s}$, where

$$\begin{aligned} u_1(x_1,x_2) &= w_1(x_1,x_2)+s_r(r,\theta), \quad u_2(x_1,x_2)=w_2(x_1,x_2)+s_\theta(r,\theta) \\ s_r(r,\theta) &:= \sum_{j=1}^{N}\sum_{i=1}^{3}\sum_{0<\lambda_{j,i}<2} \eta_j(r_j)\gamma_{j,i} r_j^{\lambda_{j,i}-1}\sin\lambda_{j,i}\theta_j \\ s_\theta(r,\theta) &:= \sum_{j=1}^{N}\sum_{i=1}^{3}\sum_{0<\lambda_{j,i}<2} \eta_j(r_j)\gamma_{j,i} r_j^{\lambda_{j,i}-1}\cos\lambda_{j,i}\theta_j \\ \mathbf{w}\in H^2(\Omega)^2 \text{ and } & \sum_{j=1}^{N}\sum_{i=1}^{3}\sum_{0<\lambda_{j,i}<2} |\gamma_{j,i}| + \|\mathbf{w}\|_{H^2(\Omega)^2} \le C\|\mathbf{f}\|_{L_2(\Omega)^2} \end{aligned}$$

2 Singularity functions for Maxwell's equations

2.1 Maxwell's equations in a sector

It is well known that the leading singularities of the solution of boundary value problems is generated by the principal parts of the differential operator. Hence, for the regularity analysis of the solution $\mathbf{u} \in H_0(\mathrm{curl}, \mathrm{div}, \Omega)$ of the boundary value problem (1.6), it suffices to consider the Maxwell equation:
Find $\mathbf{u} \in H_0(\mathrm{curl}, \mathrm{div}, \Omega)$ such that

$$a(\mathbf{u},\mathbf{v}) := (\mathrm{curl}\,\mathbf{u}, \mathrm{curl}\,\mathbf{v}) + (\mathrm{div}\,\mathbf{u}, \mathrm{div}\,\mathbf{v}) = (\mathbf{f},\mathbf{v}) =: h(\mathbf{v}), \forall \mathbf{v} \in H_0(\mathrm{curl}, \mathrm{div}, \Omega) \tag{2.1}$$

The sesquilinear form $a(\cdot,\cdot)$ from (2.1) is coercive on $H_0(\mathrm{curl}, \mathrm{div}, \Omega)$, that is, there is a constant $C > 0$ such that

$$a(\mathbf{v},\mathbf{v}) \ge C\big(\|\mathbf{v}\|^2_{L_2(\Omega)^2} + \|\ \mathrm{curl}\,\mathbf{v}\|^2_{L_2(\Omega)} + \|\mathrm{div}\,\mathbf{u}\|^2_{L_2(\Omega)}\big), \forall \mathbf{u} \in H_0(\ \mathrm{curl}, \mathrm{div}, \Omega) \tag{2.2}$$

see e.g. [10]. Hence, there is a unique solution $\mathbf{u} \in H_0(\mathrm{curl}, \mathrm{div}, \Omega)$ of the variational problem (2.1) and we proceed to analyse its regularity.

Since the regularity of solutions of elliptic boundary value problems is a local property, we consider some corner point $K \in \Gamma$, with interior angle $\omega \in (0, 2\pi)$ with $\omega \neq \pi$, and assume for convenience that K coincides with the origin of the Cartesian coordinate system (x_1, x_2). In the (x_1, x_2)-plane we introduce local polar coordinates (r, θ) with respect to the corner at $K = (0, 0)$ viz.

$$x_1 = -r\cos(\theta+\theta_1), \quad x_2 = -r\sin(\theta+\theta_1) \tag{2.3}$$

and define some circular sector G_o in Ω with radius R_0 and angle ω by (cf. Fig. 1)

$$\bar{G}_o := \{(x_1, x_2):\ 0 \le r \le R_0,\ 0 \le \theta \le \omega\}, \quad G_o := \bar{G}_o \setminus \partial G_0 \tag{2.4}$$

so that $\bar{G}_o$ contains only one corner point of Ω.

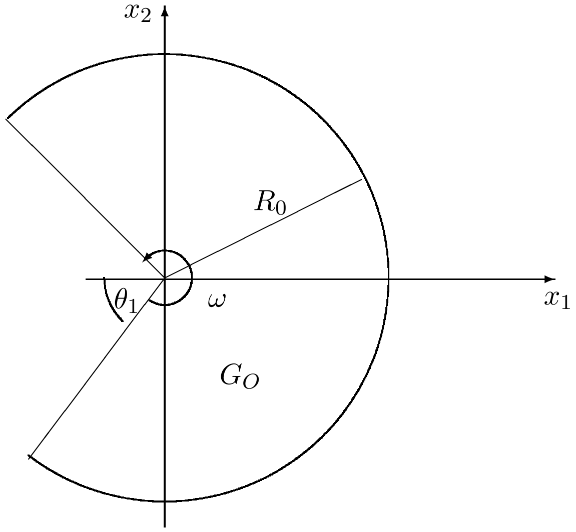

Fig. 1

For regularity analysis of the solution $\mathbf{u}$ of (2.1) near the corner point $K = (0, 0)$, we introduce a smooth truncation function $\eta \in C^\infty[0, \infty)$ which depends only on the distance r to the corner K by

$$\eta(r) := \begin{cases} 1 & \text{for} \quad 0 \le r \le R_0/3 \\ \\ 0 & \text{for} \quad r \ge 2R_0/3 \end{cases} \quad , 0 \le \eta \le 1 \tag{2.5}$$

with r and R_0 from (2.4). Obviously, the function $\mathbf{u}_\eta := \eta\mathbf{u}$ describes the solution $\mathbf{u}$ of (2.1) in the neighborhood G_0 of the corner at $K = (0, 0)$. In fact, it can be shown (cf. [17, 18, 20, 31]) that $\mathbf{u}_\eta$ is the unique solution of a boundary value problem of the form:
Find $\mathbf{u}_\eta \in H_0(\text{curl}, \text{div}, G_o)$ such that

$$a(\mathbf{u}_\eta, \mathbf{v}) := (\text{curl}\,\mathbf{u}_\eta,\ \text{curl}\,\mathbf{v}) + (\text{div}\,\mathbf{u}_\eta, \text{div}\,\mathbf{v}) = (\mathbf{f}_\eta, \mathbf{v}) =: h\eta(\mathbf{v}), \forall \mathbf{v} \in H_0(\text{curl}, \text{div}, G_o) \tag{2.6}$$

where the new right hand side $\mathbf{f}_\eta$ now depends on the functions η, $\mathbf{f}$, $\mathbf{u}$ and the first partial derivatives of $\mathbf{u}$, and satisfies

$$\|\mathbf{f}_\eta\|_{L_2(G_o)^2} \leq C\|\mathbf{f}\|_{L_2(\Omega)^2} \tag{2.7}$$

Remark 2.1. *The global regularity of the solution* $\mathbf{u}$ *of the variational problem* (*2.1*) *is determined by the regularity of the solution* $\mathbf{u}_\eta$ *of* (*2.6*) *and by the regularity of the part* $(1-\eta)\mathbf{u}$ *of* $\mathbf{u}$ *away from the corner* K*. But it is well known (cf.* [*12, 15, 17, 18, 21*]*), that for elliptic boundary value problems,* $\mathbf{f} \in L_2(\Omega)^2$ *implies* $\mathbf{u} \in H^2(\Omega \cap N)^2$ *for any compact subset* N *of* Ω *that does not contain corner points of* Ω*. Thus, it is sufficient to analyse only the regularity of the solution* $\mathbf{u}_\eta$ *of* (*2.6*)*.*

For simplicity we omit in the sequel the subscript η. It will be clear from the context if we are referring to the solution of (2.1) or (2.6).

It is easy to see that the variational problems (2.6) is the generalized formulation of the boundary value problem:

$$\begin{cases} -\triangle \mathbf{u} = \mathbf{f} & \text{in} \quad G_o \\ \mathbf{u} \wedge \mathbf{n} = 0 & \text{on} \quad \partial G_o \\ \operatorname{div} \mathbf{u} = 0 & \text{on} \quad \partial G_o \end{cases} \tag{2.8}$$

2.2 The solution of Maxwell's equations in a sector

In this subsection we determine the exact expression for the solution $\mathbf{u}$ of the boundary value problem (2.8).

The one-to-one mapping (2.3) transforms the circular sector G_o into the rectangle

$$\tilde{G}_o := \{(r,\theta) : 0 < r < R_0,\ 0 < \theta < \omega\}$$

in the (r,θ)-coordinate system. Consequently, for each function $u(x_1,x_2)$, $(x_1,x_2) \in G_o$, some function $\tilde{u}(r,\theta)$ on $\tilde{G}_o$ is uniquely defined by

$$u(r,\theta) = u(-r\cos\theta, -r\sin\theta) \tag{2.9}$$

and for each vector field $\mathbf{u} = (u_1(x_1,x_2), u_2(x_1,x_2))$, $(x_1,x_2) \in G_o$, some vector field $\tilde{\mathbf{u}} = (u_r(r,\theta), u_\theta(r,\theta))$, $(r,\theta) \in \tilde{G}_o$ is defined uniquely by

$$u_r = -u_1\cos\theta - u_2\sin\theta, \quad u_\theta = u_1\sin\theta - u_2\cos\theta \tag{2.10}$$

Accordingly, the boundary value problem (2.8) can be written in terms of the local polar coordinates r, θ in the form:

$$\begin{aligned}
-\frac{\partial^2 u_r}{\partial r^2} - \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} + \frac{1}{r^2}u_r = f_r \text{ in } \tilde{G}_o \\
-\frac{\partial^2 u_\theta}{\partial r^2} - \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} + \frac{1}{r^2}u_\theta = f_\theta \text{ in } \tilde{G}_o
\end{aligned} \tag{2.11}$$

and with boundary conditions

$$\begin{aligned}
&\frac{\partial u_r}{\partial r} + \frac{1}{r}u_r + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = 0, \quad u_r = 0 \quad \text{if} \quad \theta = 0 \\
&\frac{\partial u_r}{\partial r} + \frac{1}{r}u_r + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = 0 \quad u_r = 0 \quad \text{if} \quad \theta = \omega \\
&|u_r(r,\theta)|_{r=0} < \infty, \quad |u_\theta(r,\theta)|_{r=0} < \infty \\
&|u_r(r,\theta)|_{r=R_0} = |u_\theta(r,\theta)|_{r=R_0} = 0
\end{aligned} \tag{2.12}$$

In (2.11), (2.12) the derivatives are to be interpreted in the sense of distribution. The boundary value problem (2.11), (2.12) can be solved using the Fourier method of separation of variables. We assume the solution $\tilde{\mathbf{u}} = (u_r, u_\theta)$ and the function $\tilde{\mathbf{f}} = (f_r, f_\theta)$ of the right hand side have the series form

$$\begin{aligned}
u_r(r,\theta) &= \sum_{k=1}^{\infty} u_{rk}(r) \sin \lambda_k \theta, \quad u_\theta(r,\theta) = \sum_{k=1}^{\infty} u_{\theta k}(r) \cos \lambda_k \theta \\
f_r(r,\theta) &= \sum_{k=1}^{\infty} f_{rk}(r) \sin \lambda_k \theta, \quad f_\theta(r,\theta) = \sum_{k=1}^{\infty} f_{\theta k}(r) \cos \lambda_k \theta
\end{aligned} \tag{2.13}$$

where $\lambda_k := \frac{k\pi}{\omega}$ ($\omega \neq \pi$, i.e. $\lambda_k \neq 1, \forall k \in \mathbb{N}$) and where the Fourier coefficients $\{(u_{rk}, u_{\theta k})\}$ are to be determined.

We assume temporarily that the series in (2.13) converge uniformly and substitute (2.13) into (2.11), (2.12). Differentiating term by term and comparing coefficients, we get with respect to $0 \leq r \leq R_0$ and for each $k \in \mathbb{N} := \{1, 2, 3, \cdots\}$ the two-point boundary value problem

$$\begin{aligned}
-u''_{rk} - \frac{1}{r}u'_{rk} + \frac{1+\lambda_k^2}{r^2}u_{rk} - \frac{2\lambda_k}{r^2}u_{\theta k} = f_{rk} \\
-u''_{\theta k} - \frac{1}{r}u'_{\theta k} + \frac{1+\lambda_k^2}{r^2}u_{\theta k} - \frac{2\lambda_k}{r^2}u_{rk} = f_{\theta k}
\end{aligned} \tag{2.14}$$

with boundary conditions

$$|u_{rk}(r)|_{r=0} < \infty, \quad |u_{\theta k}(r)|_{r=0} < \infty, \quad u_{rk}(R_0) = u_{\theta k}(R_0) = 0 \tag{2.15}$$

We observe that (2.14) is a system of linear ordinary differential equations with variable coefficients. However, the substitution $r = e^{\xi}$ transforms the system (2.14) into a system with constant coefficients with respect to ξ and which can readily be solved. In fact, one can verify by a straightforward computation that the solution

$\mathbf{u}_k = (u_{rk}, u_{\theta k})$ of the boundary value problem (2.14), (2.15) ($k \in \mathbb{N}$) is given by

$$\begin{aligned} u_{rk}(r) &= -\frac{r^{\lambda_k-1}}{4(\lambda_k-1)R_0^{2\lambda_k-2}} \int_0^{R_0} \Big(f_{rk}(\tau)+f_{\theta k}(\tau)\Big)\tau^{\lambda_k}d\tau \\ &+ \frac{r^{\lambda_k+1}}{4(\lambda_k+1)R_0^{2\lambda_k+2}} \int_0^{R_0} \Big(-f_{rk}(\tau)+f_{\theta k}(\tau)\Big)\tau^{\lambda_k+2}d\tau \\ &+ \frac{r^{-(\lambda_k+1)}}{4(\lambda_k+1)} \int_0^{r} \Big(f_{rk}(\tau)-f_{\theta k}(\tau)\Big)\tau^{\lambda_k+2}d\tau \\ &+ \frac{r^{1-\lambda_k}}{4(\lambda_k-1)} \int_0^{r} \Big(f_{rk}(\tau)+f_{\theta k}(\tau)\Big)\tau^{\lambda_k}d\tau \\ &+ \frac{r^{\lambda_k-1}}{4(\lambda_k-1)} \int_r^{R_0} \Big(f_{rk}(\tau)+f_{\theta k}(\tau)\Big)\tau^{2-\lambda_k}d\tau \\ &- \frac{r^{\lambda_k+1}}{4(\lambda_k+1)} \int_r^{R_0} \Big(-f_{rk}(\tau)+f_{\theta k}(\tau)\Big)\tau^{-\lambda_k}d\tau \\ &=: \gamma_k r^{\lambda_k-1} + \delta_k r^{\lambda_k+1} + F_k(r) + G_k(r) + H_k(r) - I_k(r) \end{aligned} \tag{2.16}$$

$$\begin{aligned} u_{\theta k}(r) &= -\frac{r^{\lambda_k-1}}{4(\lambda_k-1)R_0^{2\lambda_k-2}} \int_0^{R_0} \Big(f_{rk}(\tau)+f_{\theta k}(\tau)\Big)\tau^{\lambda_k}d\tau \\ &- \frac{r^{\lambda_k+1}}{4(\lambda_k+1)R_0^{2\lambda_k+2}} \int_0^{R_0} \Big(-f_{rk}(\tau)+f_{\theta k}(\tau)\Big)\tau^{\lambda_k+2}d\tau \\ &- \frac{r^{-(\lambda_k+1)}}{4(\lambda_k+1)} \int_0^{r} \Big(f_{rk}(\tau)-f_{\theta k}(\tau)\Big)\tau^{\lambda_k+2}d\tau \\ &+ \frac{r^{1-\lambda_k}}{4(\lambda_k-1)} \int_0^{r} \Big(f_{rk}(\tau)+f_{\theta k}(\tau)\Big)\tau^{\lambda_k}d\tau \\ &+ \frac{r^{\lambda_k-1}}{4(\lambda_k-1)} \int_r^{R_0} \Big(f_{rk}(\tau)+f_{\theta k}(\tau)\Big)\tau^{2-\lambda_k}d\tau \\ &+ \frac{r^{\lambda_k+1}}{4(\lambda_k+1)} \int_r^{R_0} \Big(-f_{rk}(\tau)+f_{\theta k}(\tau)\Big)\tau^{-\lambda_k}d\tau \\ &=: \gamma_k r^{\lambda_k-1} - \delta_k r^{\lambda_k+1} - F_k(r) + G_k(r) + H_k(r) + I_k(r) \end{aligned} \tag{2.17}$$

>From the above analysis, we conclude that the solution $\tilde{\mathbf{u}} = (u_r(r,\theta), u_\theta(r,\theta))$ of the Maxwell equations (2.11), (2.12) has, for suitable real numbers λ_k, the repre-

sentation

$$u_r(r,\theta)=\sum_{k=1}^{\infty} v_{rk}(r)\sin\lambda_k\theta, \quad u_\theta(r,\theta)=\sum_{k=1}^{\infty} v_{rk}(r)\cos\lambda_k\theta \tag{2.18}$$

where the Fourier coefficients u_{rk} and $u_{\theta k}$ ($k\in\mathbb{N}$) are given by (2.16).

2.3 Regularities and singularities of solution of Maxwell's equations

Of course, the boundedness of the integrals in (2.16) and (2.17) and the regularity of the solution $\tilde{\mathbf{u}}$ of problem (2.11), (2.12) depend on the value of the parameter λ_k ($k\in\mathbb{N}$). In this subsection, we investigate the influence of the parameter λ_k on the regularity of the solution $\tilde{\mathbf{u}}$ of problem (2.11), (2.12) and deduce from it the global regularity of the solution of time-harmonic Maxwell's equations (1.6).

The transformations (2.9) and (2.10) define mappings $H^l(G_o)\to x^l_{1/2}(\tilde{G}_o)$ ($l=0,1,2$), $H^1(G_o)^2\to V(\tilde{G}_o)$ and $H^2(G_o)^2\to W(\tilde{G}_o)$ with (cf. [20, 29, 31])

$$\begin{aligned}
X^0_{1/2}(\tilde{G}_o) &:= \Big\{\tilde{v}=\tilde{v}(r,\theta): \int_{\tilde{G}_o}|\tilde{v}|^2 r\,dr\,d\theta<\infty\Big\}\\
X^1_{1/2}(\tilde{G}_o) &:= \Big\{\tilde{v}\in X^0_{1/2}(\tilde{G}_o): \frac{\partial\tilde{v}}{\partial r}, \frac{1}{r}\frac{\partial\tilde{v}}{\partial\theta}\in X^0_{1/2}(\tilde{G}_o)\Big\}\\
X^2_{1/2}(\tilde{G}_o) &:= \Big\{\tilde{v}\in X^1_{1/2}(\tilde{G}_o): \frac{\partial^2\tilde{v}}{\partial r^2}, \frac{1}{r}\frac{\partial^2\tilde{v}}{\partial\theta\partial r}-\frac{1}{r^2}\frac{\partial\tilde{v}}{\partial\theta}, \frac{1}{r^2}\frac{\partial^2\tilde{v}}{\partial\theta^2}+\frac{1}{r}\frac{\partial\tilde{v}}{\partial r}\in X^0_{1/2}(\tilde{G}_o)\Big\}\\
V(\tilde{G}_o) &:= \Big\{\tilde{\mathbf{v}}=(v_r,v_\theta)^T\in X^0_{1/2}(\tilde{G}_o)^2: \frac{\partial v_r}{\partial r}, \frac{\partial v_\theta}{\partial r},\\
&\qquad \frac{1}{r}\frac{\partial v_r}{\partial\theta}-\frac{1}{r}v_\theta, \frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{1}{r}v_r\in X^0_{1/2}(\tilde{G}_o)\Big\} \qquad (2.19)\\
W(\tilde{G}_o) &:= \Big\{\tilde{\mathbf{v}}=(v_r,v_\theta)^T\in V(\tilde{G}_o): \frac{\partial^2 v_r}{\partial r^2}, \frac{1}{r}\frac{\partial^2 v_r}{\partial r\partial\theta}-\frac{1}{r^2}\frac{\partial v_r}{\partial\theta}-\frac{1}{r}\frac{\partial v_\theta}{\partial r}+\frac{1}{r^2}v_\theta,\\
&\qquad \frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2}+\frac{1}{r}\frac{\partial v_r}{\partial r}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta}-\frac{1}{r^2}v_r, \frac{\partial^2 v_\theta}{\partial r^2}, \frac{1}{r}\frac{\partial^2 v_\theta}{\partial r\partial\theta}-\frac{1}{r^2}\frac{\partial v_\theta}{\partial\theta}+\frac{1}{r}\frac{\partial v_r}{\partial r}-\frac{1}{r^2}v_r,\\
&\qquad \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2}+\frac{1}{r}\frac{\partial v_\theta}{\partial r}+\frac{2}{r^2}\frac{\partial v_r}{\partial\theta}-\frac{1}{r^2}v_\theta\in X^0_{1/2}(\tilde{G}_o)\Big\}
\end{aligned}$$

The norms in these spaces are derived from the corresponding norms of Sobolev spaces in Cartesian plane coordinates. We have

$$
\begin{aligned}
\|\tilde v\|_{X^0_{1/2}(\tilde G_o)} &:= \Big\{\int_{\tilde G_o} |\tilde v|^2 r dr d\theta\Big\}^{1/2}, \quad \|\tilde v\|_{X^1_{1/2}(\tilde G_o)} := \{\|\tilde v\|^2_{X^0_{1/2}(\tilde G_o)} + |\tilde v|^2_{X^1_{1/2}(\tilde G_o)}\}^{1/2} \\
|\tilde v|_{X^1_{1/2}(\tilde G_o)} &:= \Big\{\int_{\tilde G_o} \Big(\Big|\frac{\partial \tilde v}{\partial r}\Big|^2 + \Big|\frac{1}{r}\frac{\partial \tilde v}{\partial \theta}\Big|^2\Big) r dr d\theta\Big\}^{1/2} \qquad (2.20) \\
\|\tilde v\|_{X^2_{1/2}(\tilde G_o)} &:= \{\|\tilde v\|^2_{X^1_{1/2}(\tilde G_o)} + |\tilde v|^2_{X^2_{1/2}(\tilde G_o)}\}^{1/2} \\
|\tilde v|_{X^2_{1/2}(\tilde G_o)} &:= \Big\{\int_{\tilde G_o} \Big(\Big|\frac{\partial^2 \tilde v}{\partial r^2}\Big|^2 + 2\Big|\frac{1}{r}\frac{\partial^2 \tilde v}{\partial\theta\partial r} - \frac{1}{r^2}\frac{\partial \tilde v}{\partial\theta}\Big|^2 + \Big|\frac{1}{r^2}\frac{\partial^2 \tilde v}{\partial \theta^2} + \frac{1}{r}\frac{\partial \tilde v}{\partial r}\Big|^2\Big) r dr d\theta\Big\}^{1/2} \\
\|\tilde{\mathbf v}\|_{V(\tilde G_o)} &:= \Big\{\int_{\tilde G_o} \Big(|v_r|^2 + |v_\theta|^2 + \Big|\frac{\partial v_r}{\partial r}\Big|^2 + \Big|\frac{\partial v_\theta}{\partial r}\Big|^2 + \Big|\frac{1}{r}\frac{\partial v_r}{\partial \theta} - \frac{1}{r}v_\theta\Big|^2 \\
&\quad + \Big|\frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{1}{r}v_r\Big|^2\Big) r dr d\theta\Big\}^{1/2} \qquad (2.21) \\
|\tilde{\mathbf v}|_{W(\tilde G_o)} &:= \Big\{\int_{\tilde G_o} \Big(\Big|\frac{\partial^2 v_r}{\partial r^2}\Big|^2 + 2\Big|\frac{1}{r}\frac{\partial^2 v_r}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{1}{r}\frac{\partial v_\theta}{\partial r} + \frac{1}{r^2}v_\theta\Big|^2 + \Big|\frac{\partial^2 v_\theta}{\partial r^2}\Big|^2 \\
&\quad + \Big|\frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta} - \frac{1}{r^2}v_r\Big|^2 + 2\Big|\frac{1}{r}\frac{\partial^2 v_\theta}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial v_\theta}{\partial\theta} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{1}{r^2}v_r\Big|^2 \\
&\quad + \Big|\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} + \frac{2}{r^2}\frac{\partial v_r}{\partial\theta} - \frac{1}{r^2}v_\theta\Big|^2\Big) r dr d\theta\Big\}^{1/2}
\end{aligned}
$$

The systems of trigonometric functions $\{\sin(\lambda_k\theta)\}_{k\in\mathbb N}$ and $\{\cos(\lambda_k\theta)\}_{k\in\mathbb N}$ with $\lambda_k > 0$ (real) and $0 < \theta < \omega$, are both orthogonal and complete in $L_2(0,\omega)$. Thus, functions $\tilde v = \tilde v(r,\theta) \in X^0_{1/2}(\tilde G_o)$ can be represented by converging Fourier series with respect to the above systems and with Fourier coefficients $v_k(r)$, $0 < r < R_0$.

Lemma 2.1. *For $v_r, v_\theta \in X^0_{1/2}(\tilde G_o)$ and $\lambda_k > 0$, there are Fourier coefficients $v_{rk}(r), v_{\theta k}(r)$ $(k \in \mathbb N)$ defined by*

$$v_{rk}(r) := \frac{2}{\omega}\int_0^\omega v_r(r,\theta)\sin\lambda_k\theta d\theta, \quad v_{\theta k}(r) := \frac{2}{\omega}\int_0^\omega v_\theta(r,\theta)\cos\lambda_k\theta d\theta, \qquad (2.22)$$

that satisfy for almost any $0 < r < R_0$ the relations (use the abbreviation $\xi = r, \theta$)

$$v_r(r,\theta) = \sum_{k=1}^{\infty} v_{rk}(r)\sin\lambda_k\theta, \quad v_\theta(r,\theta) = \sum_{k=1}^{\infty} v_{\theta k}(r)\cos\lambda_k\theta, \tag{2.23}$$

$$\|v_\xi\|^2_{X^0_{1/2}(\tilde G_o)} = \frac{\omega}{2}\sum_{k=1}^{\infty}\|v_{\xi k}\|^2_{L_{1/2}(0,R_0)} < \infty, \tag{2.24}$$

where $L_{1/2}(0,R_0) := \{w = w(r) : \int_0^{R_0} |w|^2 r dr < \infty\}$.
For $v_r, v_\theta \in X^1_{1/2}(\tilde G_o)$*, relation (2.24) holds and*

$$|v_\xi|^2_{X^1_{1/2}(\tilde G_o)} = \frac{\omega}{2}\sum_{k=1}^{\infty}\Big\{\Big\|\frac{\partial v_{\xi k}}{\partial r}\Big\|^2_{L_{1/2}(0,R_0)} + \lambda_k^2\Big\|\frac{v_{\xi k}}{r}\Big\|^2_{L_{1/2}(0,R_0)}\Big\} < \infty$$

$$\|v_\xi\|^2_{X^1_{1/2}(\tilde G_o)} = \|v_\xi\|^2_{X^0_{1/2}(\tilde G_o)} + |v_\xi|^2_{X^1_{1/2}(\tilde G_o)} \tag{2.25}$$

For $v_r, v_\theta \in X^2_{1/2}(\tilde G_o)$*, relations (2.24) and (2.25) hold and additionally (use the abbreviation* $X := L_{1/2}(0,R_0)$*)*

$$|v_\xi|^2_{X^2_{1/2}(\tilde G_o)} = \frac{\omega}{2}\sum_{k=1}^{\infty}\Big\{\Big\|\frac{\partial^2 v_{\xi k}}{\partial r^2}\Big\|^2_X + 2\lambda_k^2\Big\|\frac{1}{r}\frac{\partial v_{\xi k}}{\partial r} - \frac{v_{\xi k}}{r^2}\Big\|^2_X + \Big\|\frac{1}{r}\frac{\partial v_{\xi k}}{\partial r} - \frac{\lambda_k^2}{r^2}v_{\xi k}\Big\|^2_X\Big\} < \infty$$

$$\|v_\xi\|^2_{X^2_{1/2}(\tilde G_o)} = \|v_\xi\|^2_{X^1_{1/2}(\tilde G_o)} + |v_\xi|^2_{X^2_{1/2}(\tilde G_o)} \tag{2.26}$$

For $\tilde{\mathbf v} = (v_r, v_\theta)^T \in V(\tilde G_o)$*, the identity*

$$\|\tilde{\mathbf v}\|^2_{V(\tilde G_o)} = \frac{\omega}{2}\sum_{k=1}^{\infty}\Big\{\|\tilde{\mathbf v}_k\|^2_{X^2} + \Big\|\frac{\partial\tilde{\mathbf v}_k}{\partial r}\Big\|^2_{X^2} + \Big\|\frac{\lambda_k}{r}v_{rk} - \frac{1}{r}v_{\theta k}\Big\|^2_X + \Big\|\frac{1}{r}v_{rk} - \frac{\lambda_k}{r}v_{\theta k}\Big\|^2_X\Big\} < \infty \tag{2.27}$$

holds; additionally, for $\tilde{\mathbf v} = (v_r, v_\theta)^T \in W(\tilde G_o)$,

$$\begin{aligned} |\tilde{\mathbf v}|^2_{W(\tilde G_o)} = \frac{\omega}{2}\sum_{k=1}^{\infty}\Big\{&\Big\|\frac{\partial^2\tilde{\mathbf v}_k}{\partial r^2}\Big\|^2_{X^2} + 2\Big\|\frac{\lambda_k}{r}\frac{\partial v_{rk}}{\partial r} - \frac{\lambda_k}{r^2}v_{rk} - \frac{1}{r}\frac{\partial v_{\theta k}}{\partial r} + \frac{1}{r^2}v_{\theta k}\Big\|^2_X \\ &+ \Big\|\frac{1}{r}\frac{\partial v_{rk}}{\partial r} - \frac{\lambda_k^2}{r^2}v_{rk} + \frac{2\lambda_k}{r^2}v_{\theta k} - \frac{1}{r^2}v_{rk}\Big\|^2_X \\ &+ 2\Big\|\frac{\lambda_k}{r^2}v_{\theta k} - \frac{\lambda_k}{r}\frac{\partial v_{\theta k}}{\partial r} + \frac{1}{r}\frac{\partial v_{rk}}{\partial r} - \frac{1}{r^2}v_{rk}\Big\|^2_X \\ &+ \Big\|\frac{1}{r}\frac{\partial v_{\theta k}}{\partial r} - \frac{\lambda_k^2}{r^2}v_{\theta k} + \frac{2\lambda_k}{r^2}v_{rk} - \frac{1}{r^2}v_{\theta k}\Big\|^2_X\Big\} < \infty \end{aligned} \tag{2.28}$$

$$\|\tilde{\mathbf v}\|^2_{W(\tilde G_o)} = \|\tilde{\mathbf v}\|^2_{V(\tilde G_o)} + |\tilde{\mathbf v}|^2_{W(\tilde G_o)}$$

Proof. The proof of this lemma is similar to the proof of Lemma 3.2 in [20] and so we omit it.■

The following relationship between the spaces $X^1_{1/2}(G_o)$ and $V(G_o)$, and $X^2_{1/2}(G_o)$ and $W(G_o)$, respectively, will be useful subsequently.

Lemma 2.2. *Let the vector-valued function* $\tilde{\mathbf{v}} = (v_r, v_\theta)^T \in X^0_{1/2}(\tilde{G}_o)^2$ *be defined as in (2.23).*

(i) *If* $v_r, v_\theta \in X^1_{1/2}(\tilde{G}_o)$ *and* $\lambda_k > 1$ *then there exists a constant* $C > 0$ *independent of* λ_k *such that*

$$\tilde{\mathbf{v}} = (v_r, v_\theta)^T \in V(\tilde{G}_o) \quad \text{and} \quad \|\tilde{\mathbf{v}}\|_{V(\tilde{G}_o)} \le C\|\tilde{\mathbf{v}}\|_{X^1_{1/2}(\tilde{G}_o)^2} \tag{2.29}$$

(ii) *If* $v_r, v_\theta \in X^2_{1/2}(\tilde{G}_o)$ *and* $\lambda_k > 2$ *then there exists a constant* $C > 0$ *independent of* λ_k *such that*

$$\tilde{\mathbf{v}} = (v_r, v_\theta)^T \in W(\tilde{G}_o) \quad \text{and} \quad \|\tilde{\mathbf{v}}\|_{W(\tilde{G}_o)} \le C\|\tilde{\mathbf{v}}\|_{X^2_{1/2}(\tilde{G}_o)^2} \tag{2.30}$$

Proof. (i) From relations (2.27), (2.25), and the fact that $\lambda_k > 1$ follows immediately the estimate

$$\|\tilde{\mathbf{v}}\|^2_{V(\tilde{G}_o)} \le \omega \sum_{k=1}^{\infty} \Big\{ \|\tilde{\mathbf{v}}_k\|^2_{X^2} + \Big\|\frac{\partial \tilde{\mathbf{v}}_k}{\partial r}\Big\|^2_{X^2} + \lambda_k^2 \Big\|\frac{\tilde{\mathbf{v}}_k}{r}\Big\|^2_{X^2} \Big\} = 2\|\tilde{\mathbf{v}}\|^2_{X^1_{1/2}(\tilde{G}_o)^2}. \tag{2.31}$$

(ii) First we note that for $\lambda_k > 2$ the inequality

$$\lambda_k^2 \le \frac{16}{9}\Big(\lambda_k - \frac{1}{\lambda_k}\Big)^2 \tag{2.32}$$

holds. Using the relation

$$\Big(\lambda_k - \frac{1}{\lambda_k}\Big)\frac{v_{\xi k}}{r^2} = \frac{1}{\lambda_k}\Big(\frac{1}{r}\frac{\partial v_{\xi k}}{\partial r} - \frac{1}{r^2}v_{\xi k}\Big) - \frac{1}{\lambda_k}\Big(\frac{1}{r}\frac{\partial v_{\xi k}}{\partial r} - \frac{\lambda_k^2}{r^2}v_{\xi k}\Big) \quad \text{for} \quad \xi = r, \theta$$

one deduces the estimates

$$\begin{aligned} \Big(\lambda_k - \frac{1}{\lambda_k}\Big)^2\Big\|\frac{v_{\theta k}}{r^2}\Big\|^2_X &\le \frac{2}{\lambda_k^2}\Big\|\frac{1}{r}\frac{\partial v_{\theta k}}{\partial r} - \frac{1}{r^2}v_{\theta k}\Big\|^2_X + \frac{2}{\lambda_k^2}\Big\|\frac{1}{r}\frac{\partial v_{\theta k}}{\partial r} - \frac{\lambda_k^2}{r^2}v_{\theta k}\Big\|^2_X \\ \Big(\lambda_k - \frac{1}{\lambda_k}\Big)^2\Big\|\frac{v_{rk}}{r^2}\Big\|^2_X &\le \frac{2}{\lambda_k^2}\Big\|\frac{1}{r}\frac{\partial v_{rk}}{\partial r} - \frac{1}{r^2}v_{rk}\Big\|^2_X + \frac{2}{\lambda_k^2}\Big\|\frac{1}{r}\frac{\partial v_{rk}}{\partial r} - \frac{\lambda_k^2}{r^2}v_{rk}\Big\|^2_X \end{aligned} \tag{2.33}$$

The application of triangle inequality and relations (2.32) and (2.33) gives

$$\begin{aligned}
&\Big\|\frac{1}{r}\frac{\partial v_{rk}}{\partial r}-\frac{\lambda_k^2}{r^2}v_{rk}+\frac{2\lambda_k}{r^2}v_{\theta k}-\frac{1}{r^2}v_{rk}\Big\|_X^2+\Big\|\frac{1}{r}\frac{\partial v_{\theta k}}{\partial r}-\frac{\lambda_k^2}{r^2}v_{\theta k}+\frac{2\lambda_k}{r^2}v_{rk}-\frac{1}{r^2}v_{\theta k}\Big\|_X^2\\
&\quad\le C\Big(\Big\|\frac{1}{r}\frac{\partial v_{rk}}{\partial r}-\frac{\lambda_k^2}{r^2}v_{rk}\Big\|_X^2+\Big\|\frac{1}{r}\frac{\partial v_{\theta k}}{\partial r}-\frac{\lambda_k^2}{r^2}v_{\theta k}\Big\|_X^2+\lambda_k^2\Big\|\frac{v_{rk}}{r^2}\Big\|_X^2+\lambda_k^2\Big\|\frac{v_{\theta k}}{r^2}\Big\|_X^2\Big)\\
&\quad\le C\Big(2\lambda_k^2\Big\|\frac{1}{r}\frac{\partial v_{\theta k}}{\partial r}-\frac{1}{r^2}v_{\theta k}\Big\|_X^2+\Big\|\frac{1}{r}\frac{\partial v_{\theta k}}{\partial r}-\frac{\lambda_k^2}{r^2}v_{\theta k}\Big\|_X^2\\
&\quad+2\lambda_k^2\Big\|\frac{1}{r}\frac{\partial v_{rk}}{\partial r}-\frac{1}{r^2}v_{rk}\Big\|_X^2+\Big\|\frac{1}{r}\frac{\partial v_{rk}}{\partial r}-\frac{\lambda_k^2}{r^2}v_{rk}\Big\|_X^2\Big)
\end{aligned}\tag{2.34}$$

Again, we use the triangle inequality to get the estimate

$$\begin{aligned}
&2\Big\|\frac{\lambda_k}{r}\frac{\partial v_{rk}}{\partial r}-\frac{\lambda_k}{r^2}v_{rk}-\frac{1}{r}\frac{\partial v_{\theta k}}{\partial r}+\frac{1}{r^2}v_{\theta k}\Big\|_X^2+2\Big\|\frac{\lambda_k}{r^2}v_{\theta k}-\frac{\lambda_k}{r}\frac{\partial v_{\theta k}}{\partial r}+\frac{1}{r}\frac{\partial v_{rk}}{\partial r}-\frac{1}{r^2}v_{rk}\Big\|_X^2\\
&\quad\le C\Big(2\lambda_k^2\Big\|\frac{1}{r}\frac{\partial v_{\theta k}}{\partial r}-\frac{1}{r^2}v_{\theta k}\Big\|_X^2+2\lambda_k^2\Big\|\frac{1}{r}\frac{\partial v_{rk}}{\partial r}-\frac{1}{r^2}v_{rk}\Big\|_X^2\Big)
\end{aligned}\tag{2.35}$$

Finally, assertion (2.30) follows from the definition of the norms $\|\tilde{\mathbf{v}}\|_{W(\tilde{G}_o)}$ and $\|\tilde{\mathbf{v}}\|_{X^2_{1/2}(\tilde{G}_o)^2}$ (cf. (2.28) and (2.26)), and relations (2.31)-(2.35). ■

We are now in a position to prove precise regularity assumptions for the solution of Maxwell's equations in a sector.

Lemma 2.3. *Suppose* $\mathbf{f}=(f_1(x_1,x_2),f_2(x_1,x_2))\in L_2(G_o)^2$, *and* $\lambda_k:=\frac{k\pi}{\omega}>1$, *(*$k\in\mathbb{N}$*) and* $\omega\neq\pi$ *from (2.4). Then, the weak solution* $\mathbf{u}=(u_1(x_1,x_2),u_2(x_1,x_2))\in H_0(curl,div,G_o)$ *of the Maxwell equations (2.6) has the property*

$$\mathbf{u}\in H_N(G_o)\quad \textit{and}\quad \|\mathbf{u}\|_{H_N(G_o)}\le C\|\mathbf{f}\|_{L_2(G_o)^2}\tag{2.36}$$

Proof. The solution $\mathbf{u}$ of problem (2.6) in local polar coordinates (r,θ) is given by relation (2.18). We show that for $\lambda_k>1$, the estimate $\|\tilde{\mathbf{u}}\|_{X^1_{1/2}(\tilde{G}_o)^2}\le C\|\tilde{\mathbf{f}}\|_{X^0_{1/2}(\tilde{G}_o)^2}$ holds and use Lemma 2.2 to conclude.
Subsequently, $C>0$ will simply denote a generic constant, which has different values at different points. Using the expression (2.18) for u_r and applying, respectively, Beppo Levi's and Fubini's theorems, we get the estimate

$$\|u_r\|^2_{X^1_{1/2}(\tilde{G}_o)}\le C\sum_{k=1}^{\infty}\int_0^{R_0}\Big\{|u_{rk}|^2+\Big|\frac{\partial u_{rk}}{\partial r}\Big|^2+\Big|\frac{u_{rk}}{r}\Big|^2\Big\}rdr\tag{2.37}$$

>From (2.16), u_{rk} has the representation

$$u_{rk}(r) = \gamma_k r^{\lambda_k - 1} + \delta_k r^{\lambda_k + 1} + F_k(r) + G_k(r) + H_k(r) - I_k(r) \tag{2.38}$$

The terms on the right hand side of (2.38) can be bounded as follows.

$$\int_0^{R_0} \Big\{ \Big|\gamma_k r^{\lambda_k - 1}\Big|^2 + \Big|\frac{\partial(\gamma_k r^{\lambda_k - 1})}{\partial r}\Big|^2 + \Big|\frac{\gamma_k r^{\lambda_k - 1}}{r}\Big|^2 \Big\} r dr \le C(\gamma_k)^2 \tag{2.39}$$

since $\lambda_k > 1$. The constant $(\gamma_k)^2$ can be bounded with the help of Cauchy-Schwarz inequality and the inequality $(a+b)^2 \le 2(a^2+b^2)$ as follows.

$$\begin{aligned} (\gamma_k)^2 &\le C\Big(\int_0^{R_0} \big(f_{rk}(\tau) + f_{\theta k}(\tau)\big)\tau^{\lambda_k} d\tau\Big)^2 \\ &\le C\int_0^{R_0} \big|f_{rk}(\tau) + f_{\theta k}(\tau)\big|^2 \tau d\tau \int_0^{R_0} \big|r^{\lambda_k - 1/2}\big|^2 d\tau \le C\|\tilde{\mathbf{f}}\|_{X^2}^2 \end{aligned} \tag{2.40}$$

$$\int_0^{R_0} \Big\{ \Big|\delta_k r^{\lambda_k + 1}\Big|^2 + \Big|\frac{\partial(\delta_k r^{\lambda_k + 1})}{\partial r}\Big|^2 + \Big|\frac{\delta_k r^{\lambda_k + 1}}{r}\Big|^2 \Big\} r dr \le C(\delta_k)^2 \tag{2.41}$$

For the constant $(\delta_k)^2$ we get by applying Cauchy-Schwartz inequality the bound

$$\begin{aligned} (\delta_k)^2 &\le C\Big(\int_0^{R_0} \big(-f_{rk}(\tau) + f_{\theta k}(\tau)\big)\tau^{\lambda_k + 2} d\tau\Big)^2 \\ &\le C\int_0^{R_0} \big|-f_{rk}(\tau) + f_{\theta k}(\tau)\big|^2 \tau d\tau \int_0^{R_0} \big|r^{\lambda_k + 3/2}\big|^2 d\tau \le C\|\tilde{\mathbf{f}}\|_{X^2}^2 \end{aligned} \tag{2.42}$$

Application of Cauchy-Schwarz inequality and integration by parts formula yield the estimates

$$\begin{aligned} \int_0^{R_0} |F_k(r)|^2 r dr &\le C\int_0^{R_0} \Big| r^{-(\lambda_k+1)} \int_0^r \big(f_{rk}(\tau) - f_{\theta k}(\tau)\big)\tau^{\lambda_k+2} d\tau \Big|^2 r dr \\ &\le C\int_0^{R_0} \big|r^{-(\lambda_k+1)}\big|^2 \Big|\int_0^r \big(f_{rk}(\tau) - f_{\theta k}(\tau)\big)\tau^{\lambda_k+2} d\tau\Big|^2 r dr \\ &\le C\int_0^{R_0} \big|r^{-(\lambda_k+1)}\big|^2 \int_0^r \big|f_{rk}(\tau) - f_{\theta k}(\tau)\big|^2 \tau d\tau \int_0^r \big|\tau^{\lambda_k+3/2}\big|^2 d\tau r dr \\ &\le C\int_0^{R_0} r^3 \int_0^r \big|f_{rk}(\tau) - f_{\theta k}(\tau)\big|^2 \tau d\tau dr \\ &\le C\Big\{ r^4 \int_0^r \big|f_{rk}(\tau) - f_{\theta k}(\tau)\big|^2 \tau d\tau \Big|_0^{R_0} + \int_0^{R_0} r^4 \big|f_{rk}(r) - f_{\theta k}(r)\big|^2 r dr \Big\} \\ &\le C\int_0^{R_0} \big(|f_{rk}(r)|^2 + |f_{\theta k}(r)|^2\big) r dr \le C\|\tilde{\mathbf{f}}\|_{X^2}^2 \end{aligned} \tag{2.43}$$

$$\begin{aligned}
\int_0^{R_0}\Big|\frac{\partial F_k(r)}{\partial r}\Big|^2 r dr &\le C\int_0^{R_0}\Big\{\big|r^{-(\lambda_k+2)}\big|^2\Big|\int_0^r\big(f_{rk}(\tau)-f_{\theta k}(\tau)\big)\tau^{\lambda_k+2}d\tau\Big|^2+r^2\big|f_{rk}(r)-f_{\theta k}(r)\big|^2\Big\}rdr\\
&\le C\Big\{\int_0^{R_0}\big|r^{-(\lambda_k+2)}\big|^2\int_0^r\big|f_{rk}(\tau)-f_{\theta k}(\tau)\big|^2\tau d\tau\int_0^r\big|\tau^{\lambda_k-3/2}\big|^2 d\tau\ rdr\\
&\qquad+\int_0^{R_0}\big|f_{rk}(r)-f_{\theta k}(r)\big|^2 r^3 dr\Big\}\\
&\le C\Big\{\int_0^{R_0} r\int_0^r\big|f_{rk}(\tau)-f_{\theta k}(\tau)\big|^2\tau d\tau+\int_0^{R_0}\big|f_{rk}(r)-f_{\theta k}(r)\big|^2 r^3 dr\Big\} \qquad (2.44)\\
&\le C\Big\{r^2\int_0^r\big|f_{rk}(\tau)-f_{\theta k}(\tau)\big|^2\tau d\tau\Big|_0^{R_0}+\int_0^{R_0}\big|f_{rk}(r)-f_{\theta k}(r)\big|^2 r^3 dr\Big\}\le C\|\tilde{\mathbf{f}}\|^2_{X^2}
\end{aligned}$$

$$\begin{aligned}
\int_0^{R_0}\Big|\frac{F_k(r)}{r}\Big|^2 rdr &\le C\int_0^{R_0}\big|r^{-(\lambda_k+2)}\big|^2\Big|\int_0^r\big(f_{rk}(\tau)-f_{\theta k}(\tau)\big)\tau^{\lambda_k+2}d\tau\Big|^2 rdr\\
&\le C\int_0^{R_0}\big|r^{-(\lambda_k+2)}\big|^2\int_0^r\big|f_{rk}(\tau)-f_{\theta k}(\tau)\big|^2\tau d\tau\int_0^r\big|\tau^{\lambda_k+3/2}\big|^2 d\tau rdr\\
&\le C\int_0^{R_0} r\int_0^r\big|f_{rk}-f_{\theta k}\big|^2\tau d\tau dr \qquad (2.45)\\
&\le C\Big\{r^2\int_0^r\big|f_{rk}(\tau)-f_{\theta k}(\tau)\big|^2\tau d\tau\Big|_0^{R_0}+\int_0^{R_0} r^3\big|f_{rk}(r)-f_{\theta k}(r)\big|^2 dr\Big\}\le C\|\tilde{\mathbf{f}}\|^2_{X^2}
\end{aligned}$$

Proceeding as above one easily verifies the estimates

$$\begin{aligned}
\int_0^{R_0}\Big\{|G_k(r)|^2+\Big|\frac{\partial G_k(r)}{\partial r}\Big|^2+\Big|\frac{G_k(r)}{r}\Big|^2\Big\}rdr &\le C\int_0^{R_0}\{|f_{rk}(r)|^2+|f_{\theta k}(r)|^2\}rdr\\
\int_0^{R_0}\Big\{|H_k(r)|^2+\Big|\frac{\partial H_k(r)}{\partial r}\Big|^2+\Big|\frac{H_k(r)}{r}\Big|^2\Big\}rdr &\le C\int_0^{R_0}\{|f_{rk}(r)|^2+|f_{\theta k}(r)|^2\}rdr\\
\int_0^{R_0}\Big\{|I_k(r)|^2+\Big|\frac{\partial I_k(r)}{\partial r}\Big|^2+\Big|\frac{I_k(r)}{r}\Big|^2\Big\}rdr &\le C\int_0^{R_0}\{|f_{rk}(r)|^2+|f_{\theta k}(r)|^2\}rdr
\end{aligned} \qquad (2.46)$$

Assertion (2.36) follows by combining (2.37) – (2.46) and taking into account completeness relations of the type (2.24) and Lemma 2.2 (i). ■

Lemma 2.4. *Suppose* $\mathbf{f}=(f_1(x_1,x_2),f_2(x_1,x_2))\in L_2(G_o)^2$, *and* $\lambda_k:=\frac{k\pi}{\omega}>2$, $(k\in\mathbb{N})$ *and* $\omega\neq\pi$ *from (2.4). Then, the solution* $\mathbf{u}\in H_0(curl,div,G_o)$ *of equation (2.6) has the property*

$$\mathbf{u}\in H^2(G_o)^2 \quad \textit{and} \quad \|\mathbf{u}\|_{H^2(G_o)^2}\le C\|\mathbf{f}\|_{L_2(G_o)^2} \qquad (2.47)$$

Proof. Owing to relation (2.26) and Lemma 2.3, we need only prove the estimate

$$|\tilde{\mathbf{u}}|_{X^2_{1/2}(\tilde{G}_o)^2} \le C\|\tilde{\mathbf{f}}\|_{X^0_{1/2}(\tilde{G}_o)^2}$$

We use for the component u_r expression (2.18) and apply, respectively, Beppo Levi's and Fubini's theorems, to get the estimate

$$|u_r|^2_{X^2_{1/2}(\tilde{G}_o)} \le C\sum_{k=1}^{\infty}\int_0^{R_0}\Big\{\Big|\frac{\partial^2 u_{rk}}{\partial r^2}\Big|^2 + 2\lambda_k^2\Big|\frac{1}{r}\frac{\partial u_{rk}}{\partial r} - \frac{u_{rk}}{r^2}\Big|^2 + \Big|\frac{1}{r}\frac{\partial u_{rk}}{\partial r} - \frac{\lambda_k^2}{r^2}u_{rk}\Big|^2\Big\}rdr \tag{2.48}$$

For the sake of brevity, we prove only the estimate

$$\int_0^{R_0}\Big|\frac{\partial^2 u_{rk}}{\partial r^2}\Big|^2 rdr \le C\int_0^{R_0}\{|f_{rk}(r)|^2 + |f_{\theta k}(r)|^2\}rdr$$

since the other terms can be estimated by analogy. We have

$$\begin{aligned}
\frac{\partial^2 u_{rk}}{\partial r^2} \quad = \quad & -\frac{\lambda_k+2}{4R_0^{2\lambda_k-2}}r^{\lambda_k-3}\int_0^{R_0}\Big(f_{rk}(\tau)+f_{\theta k}(\tau)\Big)\tau^{\lambda_k}d\tau \\
& + \frac{\lambda_k}{4R_0^{2\lambda_k+2}}r^{\lambda_k-1}\int_0^{R_0}\Big(-f_{rk}(\tau)+f_{\theta k}(\tau)\Big)\tau^{\lambda_k+2}d\tau \\
& + \frac{\lambda_k+2}{4}r^{-(\lambda_k+3)}\int_0^{r}\Big(f_{rk}(\tau)-f_{\theta k}(\tau)\Big)\tau^{\lambda_k+2}d\tau \\
& + \frac{\lambda_k}{4}r^{-(\lambda_k+1)}\int_0^{r}\Big(f_{rk}(\tau)+f_{\theta k}(\tau)\Big)\tau^{\lambda_k}d\tau \\
& + \frac{\lambda_k-2}{4}r^{\lambda_k-3}\int_r^{R_0}\Big(f_{rk}(\tau)+f_{\theta k}(\tau)\Big)\tau^{2-\lambda_k}d\tau \\
& - \frac{\lambda_k}{4}r^{\lambda_k-1}\int_r^{R_0}\Big(-f_{rk}(\tau)+f_{\theta k}(\tau)\Big)\tau^{-\lambda_k}d\tau - f_{rk} \\
=: \quad & \tilde{\gamma}_k r^{\lambda_k-3} + \tilde{\delta}_k r^{\lambda_k-1} + \tilde{F}_k(r) + \tilde{G}_k(r) + \tilde{H}_k(r) + \tilde{I}_k(r) - f_{rk}
\end{aligned} \tag{2.49}$$

We can now estimate the terms on the right hand side of relation (2.49). By means of Cauchy-Schwarz inequality, integration by parts formula and some simple estimates we get

$$\begin{aligned}
\int_0^{R_0} \Big|\tilde{\gamma}_k r^{\lambda_k - 3}\Big|^2 r dr &\leq C \int_0^{R_0} \big|f_{rk}(\tau) + f_{\theta k}(\tau)\big|^2 \tau d\tau \int_0^{R_0} \big|r^{\lambda_k - 1/2}\big|^2 d\tau \leq C\|\tilde{\mathbf{f}}\|^2_{X^2} \\
\int_0^{R_0} \Big|\tilde{\delta}_k r^{\lambda_k - 1}\Big|^2 r dr &\leq C \int_0^{R_0} \big|-f_{rk}(\tau) + f_{\theta k}(\tau)\big|^2 \tau d\tau \int_0^{R_0} \big|r^{\lambda_k + 3/2}\big|^2 d\tau \leq C\|\tilde{\mathbf{f}}\|^2_{X^2} \\
\int_0^{R_0} |\tilde{F}_k(r)|^2 r dr &\leq C \int_0^{R_0} \big|r^{-(\lambda_k+3)}\big|^2 \int_0^r \big|f_{rk}(\tau) - f_{\theta k}(\tau)\big|^2 d\tau \int_0^r \big|\tau^{\lambda_k+2}\big|^2 d\tau r dr \\
&\leq C \int_0^{R_0} \int_0^r \big|f_{rk}(\tau) - f_{\theta k}(\tau)\big|^2 \tau d\tau dr \\
&\leq C\Big\{ r \int_0^r \big|f_{rk}(\tau) - f_{\theta k}(\tau)\big|^2 d\tau \Big|_0^{R_0} + \int_0^{R_0} r \big|f_{rk}(r) - f_{\theta k}(r)\big|^2 dr \Big\} \\
&\leq C \int_0^{R_0} \big(|f_{rk}(r)|^2 + |f_{\theta k}(r)|^2\big) r dr \leq C\|\tilde{\mathbf{f}}\|^2_{X^2} \qquad (2.50)
\end{aligned}$$

Similar considerations lead to the inequality

$$\int_0^{R_0} \Big\{ |\tilde{G}_k(r)|^2 + |\tilde{H}_k(r)|^2 + |\tilde{I}_k(r)|^2 + |f_k(r)|^2 \Big\} r dr \quad \leq \quad C\|\tilde{\mathbf{f}}\|^2_{L_{1/2}(0,R_0)^2} \qquad (2.51)$$

Assertion (2.47) is finally proved by taking into consideration relations of the type (2.48) – (2.51) and Lemma 2.2 (ii). ■

Remark 2.2. *The results of Lemmas (2.3) and (2.4) show that if* $\omega \in (0, 2\pi)$ *is a nontrivial angle (i.e.* $\omega \neq \pi$*) of the circular sector* G_o *(cf. (2.4)) and* $\lambda_k := \frac{k\pi}{\omega}$ $(k \in \mathbb{N})$*, then the following statements are true for the solution* $\mathbf{u} \in H_0(curl, div, G_o)$ *of the variational problem (2.6).*

(i) *For* $\lambda_k < 1$ $(k \in \mathbb{N})$*, i.e.* $\omega < \pi$*,* $\mathbf{u} \in H^1(G_o)^2$*, i.e.* $\tilde{\mathbf{u}} \in V(\tilde{G}_o)$*.*

(ii) *For* $\lambda_k < 2$ $(k \in \mathbb{N})$*, i.e.* $\omega < \frac{\pi}{2}$*,* $\mathbf{u} \in H^2(G_o)^2$*, i.e.* $\tilde{\mathbf{u}} \in W(\tilde{G}_o)$*.*

(iii) *We see from the relation (cf. (2.49))*

$$\int_0^{R_0} \Big|\tilde{\gamma}_k r^{\lambda_k - 3}\Big|^2 r dr \leq |\tilde{\gamma}|^2 \int_0^{R_0} r^{2\lambda_k - 5} dr \qquad (2.52)$$

that if $\lambda_k = 2$ *(this will be the case if* $\omega = \pi/2$*), then* $\mathbf{u} \notin H^2(G_0)^2$*, since the integral in (2.52) converges only if* $2\lambda_k - 5 > -1$*, i.e.* $\lambda_k > 2$*.*

Theorem 2.1. *Let* $\mathbf{f} = (f_1(x_1,x_2), f_2(x_1,x_2)) \in L_2(G_o)^2$, *i.e.* $\tilde{\mathbf{f}} = (f_r(r,\theta), f_\theta(r,\theta))^T \in X^0_{1/2}(\tilde{G}_o)^2$. *Further, let* $\omega \in (0, 2\pi)$ *be a nontrivial angle of the circular sector* G_o *from (2.4) and* $\lambda_k := \frac{k\pi}{\omega}$ $(k \in \mathbb{N})$. *Then, the solution* $\mathbf{u} = (u_1(x_1,x_2), u_2(x_1,x_2)) \in H_0(curl, div, G_o)$ *of the variational problem (2.6) has the following additional properties.*

(i) *For* $0 < \omega \le \pi$,

$$\mathbf{u} \in H_N(G_o)^2, \quad \|\mathbf{u}\|_{H_N(G_o)^2} \le C\|\mathbf{f}\|_{L_2(G_o)^2} \tag{2.53}$$

(ii) *For* $\pi < \omega < 2\pi$, *there exists a unique real number* γ_1 *such that* $\mathbf{u}$ *can be split in the form* $\mathbf{u} = \mathbf{w} + \mathbf{s}$, *where*

$$\begin{aligned} &u_1(x_1,x_2) = w_1(x_1,x_2) + s_r(r,\theta), \quad s_r(r,\theta) := \gamma_1 r^{\lambda_1 - 1}\sin\lambda_1\theta \\ &u_2(x_1,x_2) = w_2(x_1,x_2) + s_\theta(r,\theta), \quad s_\theta(r,\theta) := \gamma_1 r^{\lambda_1 - 1}\cos\lambda_1\theta \\ &\mathbf{w} \in H^1(G_o)^2 \quad \text{and} \quad |\gamma_1| + \|\mathbf{w}\|_{H^1(G_o)^2} \le C\|\mathbf{f}\|_{L_2(G_o)^2} \end{aligned} \tag{2.54}$$

(iii) *For* $0 < \omega < \frac{\pi}{2}$,

$$\mathbf{u} \in H^2(G_o)^2, \quad \|\mathbf{u}\|_{H^2(G_o)^2} \le C\|\mathbf{f}\|_{L_2(G_o)^2} \tag{2.55}$$

(iv) *For* $\frac{\pi}{2} \le \omega < \pi$, *there exists a unique real number* γ_1 *such that* $\mathbf{u}$ *can be split in the form* $\mathbf{u} = \mathbf{w} + \mathbf{s}$, *where*

$$\begin{aligned} &u_1(x_1,x_2) = w_1(x_1,x_2) + s_r(r,\theta), \quad s_r(r,\theta) := \gamma_1 r^{\lambda_1 - 1}\sin\lambda_1\theta \\ &u_2(x_1,x_2) = w_2(x_1,x_2) + s_\theta(r,\theta), \quad s_\theta(r,\theta) := \gamma_1 r^{\lambda_1 - 1}\cos\lambda_1\theta \\ &\mathbf{w} \in H^2(G_o)^2 \quad \text{and} \quad |\gamma_1| + \|\mathbf{w}\|_{H^2(G_o)^2} \le C\|\mathbf{f}\|_{L_2(G_o)^2} \end{aligned} \tag{2.56}$$

(v) *For* $\pi < \omega < \frac{3\pi}{2}$, *there exist unique real numbers* γ_1 *and* γ_2 *such that* $\mathbf{u}$ *can be split in the form* $\mathbf{u} = \mathbf{w} + \mathbf{s}$, *where*

$$\begin{aligned} &u_1(x_1,x_2) = w_1(x_1,x_2) + s_r(r,\theta), \quad s_r(r,\theta) := \gamma_1 r^{\lambda_1 - 1}\sin\lambda_1\theta + \gamma_2 r^{\lambda_2 - 1}\sin\lambda_2\theta \\ &u_2(x_1,x_2) = w_2(x_1,x_2) + s_\theta(r,\theta), \quad s_\theta(r,\theta) := \gamma_1 r^{\lambda_1 - 1}\cos\lambda_1\theta + \gamma_2 r^{\lambda_2 - 1}\cos\lambda_2\theta \\ &\mathbf{w} \in H^2(G_o)^2 \quad \text{and} \quad |\gamma_1| + |\gamma_2| + \|\mathbf{w}\|_{H^2(G_o)^2} \le C\|\mathbf{f}\|_{L_2(G_o)^2} \end{aligned} \tag{2.57}$$

(vi) *For* $\frac{3\pi}{2} \leq \omega < 2\pi$*, there exist unique real numbers* γ_1*,* γ_2 *and* γ_3 *such that* $\mathbf{u}$ *can be split in the form* $\mathbf{v} = \mathbf{w} + \mathbf{s}$*, where*

$$\begin{aligned}
&u_1(x_1,x_2) = w_1(x_1,x_2) + s_r(r,\theta), \quad u_2(x_1,x_2) = w_2(x_1,x_2) + s_\theta(r,\theta)\\
&s_r(r,\theta) := \gamma_1 r^{\lambda_1-1}\sin\lambda_1\theta + \gamma_2 r^{\lambda_2-1}\sin\lambda_2\theta + \gamma_3 r^{\lambda_2-1}\sin\lambda_3\theta\\
&s_\theta(r,\theta) := \gamma_1 r^{\lambda_1-1}\cos\lambda_1\theta + \gamma_2 r^{\lambda_2-1}\cos\lambda_2\theta + \gamma_3 r^{\lambda_3-1}\cos\lambda_3\theta \qquad (2.58)\\
&\mathbf{w} \in H^2(G_o)^2 \quad \textit{and} \quad |\gamma_1| + |\gamma_2| + |\gamma_3| + \|\mathbf{w}\|_{H^2(G_o)^2} \leq C\|\mathbf{f}\|_{L_2(G_o)^2}
\end{aligned}$$

Proof. Assertions (2.53) and (2.55) are immediate consequences of Lemmas 2.3 and 2.4.
By (2.18), $\mathbf{u} = \tilde{\mathbf{u}} = (u_r(r,\theta), u_\theta(r,\theta))$ can be written in the form

$$\begin{aligned}
u_r(r,\theta) &= u_{r1}(r)\sin\lambda_1\theta + u_{r2}(r)\sin\lambda_2\theta + u_{r3}(r)\sin\lambda_3\theta + \sum_{k=4}^{\infty} u_{rk}(r)\sin\lambda_k\theta\\
&=: u_{r1}(r)\sin\lambda_1\theta + u_{r2}(r)\sin\lambda_2\theta + u_{r3}(r)\sin\lambda_3\theta + \Psi_r(r,\theta) \qquad (2.59)\\
u_\theta(r,\theta) &= u_{\theta 1}(r)\cos\lambda_1\theta + u_{\theta 2}(r)\cos\lambda_2\theta + u_{\theta 3}(r)\cos\lambda_3\theta + \sum_{k=4}^{\infty} u_{\theta k}(r)\cos\lambda_k\theta\\
&:= u_{\theta 1}(r)\cos\lambda_1\theta + u_{\theta 2}(r)\cos\lambda_2\theta + u_{\theta 3}(r)\cos\lambda_3\theta + \Psi_\theta(r,\theta)
\end{aligned}$$

Since $\lambda_k = \frac{k\pi}{\omega} > 2$ for $k \geq 4$ and for all $0 < \omega < 2\pi$, Lemma 2.4 implies that the terms Ψ_r and Ψ_θ in (2.59) belong to $X^2_{1/2}(\tilde{G}_o)$ and satisfy the inequality

$$\|\Psi_r\|_{X^2_{1/2}(\tilde{G}_o)} + \|\Psi_\theta\|_{X^2_{1/2}(\tilde{G}_o)} \leq C\|\tilde{\mathbf{f}}\|_{X^0_{1/2}(\tilde{G}_o)^2}$$

Assertions (2.54), (2.56), (2.57), and (2.58) follow in a straight forward way from relations (2.18), (2.16), Lemmas 2.3 and 2.4. ■

$>$From the above analysis, we can now state the regularity and singularity properties of the solution $\mathbf{u} \in H_0(\mathrm{curl}, \mathrm{div}, \Omega)$ of the time-harmonic Maxwell equations (1.6).

Let Ω be a bounded and simply connected polygonal open subset of $\mathbb{R}^2$ with boundary Γ. We denote by K_j the vertices of Γ and by ω_j the measure of the interior angle at K_j, where the index j ranges from 1 to some integer $N > 0$ and $\omega_j \in (0, 2\pi)$, $\omega_j \neq \pi$ for $j = 1, 2, \cdots, N$. Further, we denote by (r_j, θ_j) $(0 \leq r_j \leq R_j, \quad 0 \leq \theta_j \leq \omega_j)$ local polar coordinates with respect to the vertex K_j (cf. (2.3)), and for every j we introduce a truncation function η_j according to (2.5) in such a way that the supports of the η_j's do not intersect each other.

Theorem 2.2. *For each* $\mathbf{f} \in L_2(\Omega)^2$ *the unique solution* $\mathbf{u} \in H_0(curl, div, \Omega)$ *of the variational problem (1.6) is provided with the following additional properties.*

(i) *If* $0 < \omega_j \leq \pi$ *for* $j \in \{1, \cdots, N\}$, *then*

$$\mathbf{u} \in H_N(\Omega)^2, \quad \|\mathbf{u}\|_{H_N(\Omega)^2} \leq C\|\mathbf{f}\|_{L_2(\Omega)^2}$$

(ii) *If* $0 < \omega_j < 2\pi$ *for* $j \in \{1, \cdots, N\}$, *then there exist unique real numbers* $\gamma_{j,1}$ *such that for* $\lambda_{j,1} := \frac{\pi}{\omega_j}$, $\mathbf{u}$ *can be split in the form* $\mathbf{u} = \mathbf{w} + \mathbf{s}$, *where*

$$u_1(x_1, x_2) = w_1(x_1, x_2) + s_r(r, \theta), \quad s_r(r, \theta) := \sum_j \sum_{0<\lambda_{j,1}<1} \eta_j(r_j)\gamma_{j,1} r_j^{\lambda_{j,1}-1} \sin \lambda_{j,1}\theta_j$$

$$u_2(x_1, x_2) = w_2(x_1, x_2) + s_\theta(r, \theta), \quad s_\theta(r, \theta) := \sum_j \sum_{0<\lambda_{j,1}<1} \eta_j(r_j)\gamma_{j,1} r_j^{\lambda_{j,1}-1} \cos \lambda_{j,1}\theta_j$$

$$\mathbf{w} \in H^1(\Omega)^2 \quad \textit{and} \quad \sum_j \sum_{0<\lambda_{j,1}<1} |\gamma_{j,1}| + \|\mathbf{w}\|_{H^1(\Omega)^2} \leq C\|\mathbf{f}\|_{L_2(\Omega)^2}$$

(iii) *If* $0 < \omega_j < \frac{\pi}{2}$ *for* $j \in \{1, \cdots, N\}$, *then*

$$\mathbf{u} \in H^2(\Omega)^2, \quad \|\mathbf{u}\|_{H^2(\Omega)^2} \leq C\|\mathbf{f}\|_{L_2(\Omega)^2}$$

(iv) *If* $0 < \omega_j < \pi$ *for* $j \in \{1, \cdots, N\}$, *then there exist unique real numbers* $\gamma_{j,1}$ *such that for* $\lambda_{j,1} := \frac{\pi}{\omega_j}$, $\mathbf{u}$ *can be split in the form* $\mathbf{u} = \mathbf{w} + \mathbf{s}$, *where*

$$u_1(x_1, x_2) = w_1(x_1, x_2) + s_r(r, \theta), \quad s_r(r, \theta) := \sum_j \sum_{0<\lambda_{j,1}<2} \eta_j(r_j)\gamma_{j,1} r_j^{\lambda_{j,1}-1} \sin \lambda_{j,1}\theta_j$$

$$u_2(x_1, x_2) = w_2(x_1, x_2) + s_\theta(r, \theta), \quad s_\theta(r, \theta) := \sum_j \sum_{0<\lambda_{j,1}<2} \eta_j(r_j)\gamma_{j,1} r_j^{\lambda_{j,1}-1} \cos \lambda_{j,1}\theta_j$$

$$\mathbf{w} \in H^2(\Omega)^2 \quad \textit{and} \quad \sum_j \sum_{0<\lambda_{j,1}<2} |\gamma_{j,1}| + \|\mathbf{w}\|_{H^2(\Omega)^2} \leq C\|\mathbf{f}\|_{L_2(\Omega)^2}$$

(v) *If* $0 < \omega_j < \frac{3\pi}{2}$ *for* $j \in \{1, 2, \cdots, N\}$, *then there exist unique real numbers* $\gamma_{j,1}$ *and* $\gamma_{j,2}$ *such that for* $\lambda_{j,1} := \frac{\pi}{\omega_j}$ *and for* $\lambda_{j,2} := \frac{2\pi}{\omega_j}$, $\mathbf{u}$ *can be split in the form*

$\mathbf{u} = \mathbf{w} + \mathbf{s}$, *where*

$$\begin{aligned}
u_1(x_1, x_2) &= w_1(x_1, x_2) + s_r(r, \theta), \quad u_2(x_1, x_2) = w_2(x_1, x_2) + s_\theta(r, \theta) \\
s_r(r, \theta) &:= \sum_j \sum_{i=1}^{2} \sum_{0<\lambda_{j,i}<2} \eta_j(r_j)\gamma_{j,i} r_j^{\lambda_{j,i}-1} \sin \lambda_{j,i}\theta_j \\
s_\theta(r, \theta) &:= \sum_j \sum_{i=1}^{2} \sum_{0<\lambda_{j,i}<2} \eta_j(r_j)\gamma_{j,i} r_j^{\lambda_{j,i}-1} \cos \lambda_{j,i}\theta_j \\
\mathbf{w} \in H^2(\Omega)^2 \text{ and } &\sum_j \sum_{i=1}^{2} \sum_{0<\lambda_{j,1}<2} |\gamma_{j,i}| + \|\mathbf{w}\|_{H^2(\Omega)^2} \leq C\|\mathbf{f}\|_{L_2(\Omega)^2}
\end{aligned}$$

(vi) *If* $0 < \omega_j < 2\pi$ *for* $j \in \{1, 2, \cdots, N\}$, *then there exist unique real numbers* $\gamma_{j,i}$, $(i = 1, 2, 3)$ *such that for* $\lambda_{j,i} := \frac{i\pi}{\omega_j}$ $(i = 1, 2, 3)$, $\mathbf{u}$ *can be split in the form* $\mathbf{u} = \mathbf{w} + \mathbf{s}$, *where*

$$\begin{aligned}
u_1(x_1, x_2) &= w_1(x_1, x_2) + s_r(r, \theta), \quad u_2(x_1, x_2) = w_2(x_1, x_2) + s_\theta(r, \theta) \\
s_r(r, \theta) &:= \sum_{j=1}^{N} \sum_{i=1}^{3} \sum_{0<\lambda_{j,i}<2} \eta_j(r_j)\gamma_{j,i} r_j^{\lambda_{j,i}-1} \sin \lambda_{j,i}\theta_j \\
s_\theta(r, \theta) &:= \sum_{j=1}^{N} \sum_{i=1}^{3} \sum_{0<\lambda_{j,i}<2} \eta_j(r_j)\gamma_{j,i} r_j^{\lambda_{j,i}-1} \cos \lambda_{j,i}\theta_j \\
\mathbf{w} \in H^2(\Omega)^2 \text{ and } &\sum_{j=1}^{N} \sum_{i=1}^{3} \sum_{0<\lambda_{j,i}<2} |\gamma_{j,i}| + \|\mathbf{w}\|_{H^2(\Omega)^2} \leq C\|\mathbf{f}\|_{L_2(\Omega)^2}
\end{aligned}$$

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ISSN 0249-6399