Intersecting Quadrics: An Efficient and Exact Implementation

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Abstract

We present the first complete, exact, and efficient C++ implementation for parameterizing the intersection of two implicit quadrics with integer coefficients of arbitrary size. It is based on the near-optimal algorithm recently introduced by Dupont et al. [6] and builds upon Levin's seminal work [11].

Unlike existing implementations, it correctly identifies and parameterizes all the connected components of the intersection in all cases, returning parameterizations with rational functions whenever such parameterizations exist. In addition, the coefficient rings of the parameterizations are either minimal or involve one possibly unneeded square root.

We prove upper bounds on the size of the coefficients of the output parameterizations and compare these bounds to observed values. We give other experimental results and present some examples.

1 Introduction

Computing an explicit representation of the intersection of two general quadrics (i.e., quadratic surfaces) is a fundamental problem in areas such as solid modeling, computational geometry, and computer graphics. The range of applications covers well-known problems like computing arrangements [14, 17], boundary evaluation [16], and convex hull computation [9].

Past work. Until recently, the only known general method for computing a parametric representation of the intersection between two arbitrary quadrics was that of J. Levin [11]. This method is based on an analysis of the pencil generated by the two quadrics, i.e., their set of linear combinations.

Though useful for curve tracing, Levin's method has serious limitations. When the intersection is singular or reducible, a parameterization by rational functions is known to exist, but Levin's pencil method fails to find it and generates a parameterization that involves the square root of some polynomial. In addition, since it introduces algebraic numbers of very high degree (for instance in the computation of eigenvalues and eigenvectors), a correct implementation using exact arithmetic is essentially out of reach. In addition, when a floating point representation of numbers is used, the method may output results that are wrong (geometrically and topologically) and it may even fail to produce any parameterization at all and crash.

Over the years, Levin's seminal work has been extended and refined in several different directions. Wilf and Manor [24] use a classification of quadric intersections by the Segre characteristic (see [2]) to drive the parameterization of the intersection by the pencil method. Recently, Wang, Goldman, and Tu [22] further improved the method making it capable of computing structural information on the intersection and its various connected components and able to produce a parameterization by rational functions when such a parameterization exists. Whether the refined algorithm is numerically robust is open to question.

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Another method of algebraic flavor was introduced by Farouki, Neff, and O'Connor [7] for parameterizing the intersection in degenerate situations. In such cases, using a combination of classical concepts (Segre characteristic) and algebraic tools (factorization of multivariate polynomials), the authors show that explicit information on the morphological type of the intersection curve can be reliably obtained. A notable feature of this method is that it can output an exact parameterization of the intersection in simple cases, when the input quadrics have rational coefficients. No implementation is reported however.

Rather than restricting the type of the intersection, others have sought to restrict the type of the input quadrics, taking advantage of the fact that geometric insights can then help compute the intersection curve [13, 19]. Specialized routines are devised to compute the intersection curve in each particular case. Such geometric approaches are however essentially limited to the class of so-called natural quadrics, i.e., the planes, right cones, circular cylinders, and spheres.

Apart from [6], perhaps the most interesting of the known algorithms for computing an explicit representation of the intersection of two arbitrary quadrics is the method of Wang, Joe, and Goldman [23]. This algebraic method is based on a birational mapping between the intersection curve and a plane cubic curve. The cubic curve is obtained by projection from a point lying on the intersection. Then the classification and parameterization of the intersection are obtained by invoking classical results on plane cubics. The authors claim that their algorithm is the first to produce a complete topological classification of the intersection (singularities, number, and types of connected components, etc.). Numerical robustness issues have however not been studied and the intersection may not be correctly classified. Also, the center of projection is currently computed using Levin's (enhanced) method: with floating point arithmetic, the center of projection will in general not exactly lie on the curve, which is another source of numerical instability.

Contributions. In this paper, we present the first complete, exact, and efficient implementation of an algorithm for parameterizing the intersection of two arbitrary quadrics, given in implicit form, with integer coefficients. (Note that quadrics with rational or finite floating-point coefficients can be trivially converted to integer form.) This implementation is based on the parameterization method described in [6], itself built upon Levin's pencil method.

Precisely, our implementation has the following features:

- it computes an exact parameterization of the intersection of two quadrics with integer coefficients of arbitrary size;
- it places no restriction of any kind on the type of the intersection or the type of the input quadrics;
- it correctly identifies, separates, and parameterizes all the connected components of the intersection and gives all the relevant topological information;
- the parameterization is rational when one exists; otherwise the intersection is a smooth quartic and the parameterization involves the square root of a polynomial;
- the parameterization is either optimal in the degree of the extension of $\mathbb{Z}$ on which its coefficients are defined or, in a small number of well-identified cases, involves one extra possibly unnecessary square root;
- the implementation is carefully designed so that the size of the coefficients is kept small;
- it is fast and efficient, and can routinely compute parameterizations of the intersection of quadrics with input coefficients having ten digits in less than 50 milliseconds on a mainstream PC.

Our code can be downloaded from the LORIA and INRIA web sites[1]. The C++ implementation can also be queried via a web interface.

The paper is organized as follows. After some preliminaries, we recall in Section 3 the main ideas of the parameterization algorithm we introduced in [6]. In Section 4, we prove theoretical bounds on the size of the output coefficients when the intersection is generic and compare those bounds to observed values. A similar work is carried out in Section 5 for singular intersections and the results are used to validate a key design choice we made in our implementation. After describing our implementation (Section 6), we then give experimental results

[1] `http://www.loria.fr`, `http://www.inria.fr`

and performance evaluation in Section 7, both on random and real data. Finally, we show the output produced by our implementation for some examples in Section 8, before concluding.

2 Preliminaries

In what follows, all the matrices considered are 4×4 real matrices, unless otherwise specified. We call a *quadric* associated with a symmetric matrix S the set

$$Q_S = \{\mathbf{x} \in \mathbb{P}^3 \mid \mathbf{x}^T S\mathbf{x} = 0\},$$

where $\mathbb{P}^3 = \mathbb{P}^3(\mathbb{R})$ denotes the real projective space of dimension 3 ($\mathbf{x}^T S\mathbf{x}$ is quadratic and homogeneous in the coordinates of $\mathbf{x}$). In the rest of this paper, points and parameterizations are assumed to live in projective space. Recall that a point of $\mathbb{P}^3$ has four coordinates.

We define the *inertia* of S and Q_S as the pair

$$\sigma_S = (\max(\sigma^+, \sigma^-), \min(\sigma^+, \sigma^-)),$$

where σ^+ (resp. σ^-) is the number of positive (resp. negative) eigenvalues of S. The *rank* of S is the sum $\sigma^+ + \sigma^-$. Recall that Sylvester's Inertia Law asserts that the inertia of S (and thus the rank) is invariant by a real projective transformation [10].

We call *projective cones* (or simply *cones*) the quadrics of rank 3 and *pairs of planes* the quadrics of rank 2. For the benefit of the reader, we recall that, in affine real space, quadrics of inertia $(4, 0)$ are empty, quadrics of inertia $(3, 1)$ are ellipsoids, hyperboloids of two sheets, or elliptic paraboloids, and quadrics of inertia $(2, 2)$ are hyperboloids of one sheet or hyperbolic paraboloids (see [6] for a complete characterization of affine quadrics). Also, quadrics of inertia $(2, 1)$ are cones or cylinders. All the quadric *surfaces* except those of inertia $(3, 1)$ are ruled surfaces, i.e., surfaces that are swept by a one-dimensional family of lines.

Given two matrices S and T, let $R(\lambda, \mu) = \lambda S + \mu T$. The set $\{R(\lambda, \mu) \mid (\lambda, \mu) \in \mathbb{P}^1(\mathbb{R})\}$ is called the *pencil* of matrices generated by S and T. For the sake of simplicity, we sometimes write a member of the pencil $R(\lambda) = \lambda S - T, \lambda \in \mathbb{R} \cup \{\infty\}$. Associated to a pencil of matrices is a pencil of quadrics $\{Q_{R(\lambda,\mu)} \mid (\lambda, \mu) \in \mathbb{P}^1\}$. Recall the classical result that the intersection of two distinct quadrics of a pencil is independent of the choice of the two quadrics.

The equation $\det R(\lambda, \mu) = 0$ is called the *determinantal equation* of the pencil. The *singular* quadrics of the pencil are exactly the quadrics $Q_{R(\lambda,\mu)}$ such that $\det R(\lambda, \mu) = 0$. Note that a quadric of the pencil is singular if and only if it has rank less than or equal to 3.

3 Algorithm description

In this section, we give a brief presentation of the basic ideas underpinning our near-optimal parameterization method [6].

From now on, S and T are two symmetric 4×4 matrices with entries in $\mathbb{Z}$. By abuse of language, we will often talk about (and manipulate) objects with rational coefficients, with the understanding that, in projective space, such coefficients can trivially be converted to integers.

3.1 Near-optimal parameterization algorithm

Let $\{Q_{R(\lambda,\mu)} \mid (\lambda, \mu) \in \mathbb{P}^1\}$, with $R(\lambda, \mu) = \lambda S + \mu T$, be a pencil of quadrics. The main idea of existing methods for parameterizing the intersection of two quadrics based on an analysis of their pencil (Levin's and derivatives) is as follows: find a quadric Q_R of some particularly simple form in the pencil generated by Q_S and Q_T (assume $Q_R \neq Q_S$), parameterize this quadric, plug the parameterization $\mathbf{X}$ of Q_R in the equation of Q_S, solve the resulting equation $\mathbf{X}^T S\mathbf{X} = 0$, and plug the result in $\mathbf{X}$, finally giving the parameterization of the intersection.

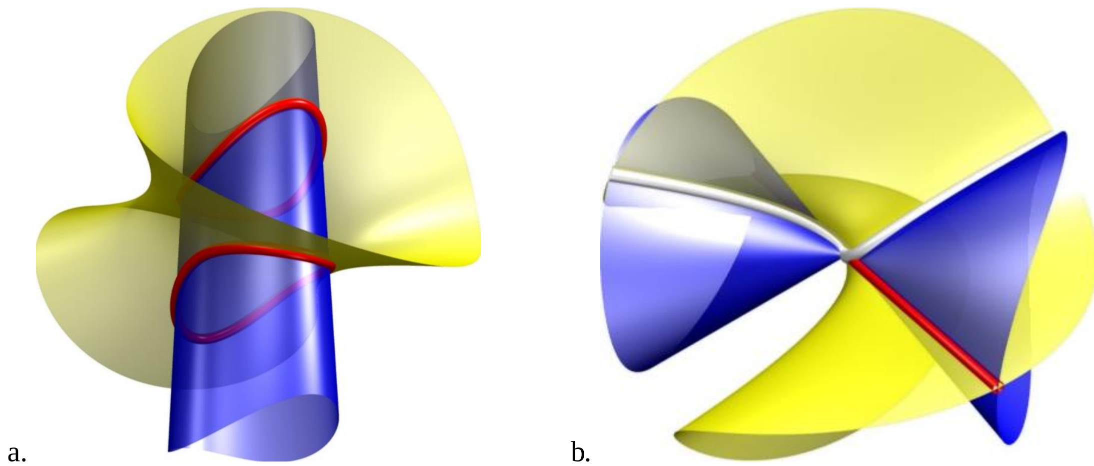

Figure 1: Examples of intersections (rendered with Surf [20]). a. Smooth quartic, with two affinely finite components. b. Cubic and tangent line.

The key to making this procedure work in practice is to find a quadric Q_R that is ruled and thus admits a parameterization that is linear in one of its parameters so that the equation $\mathbf{X}^T S \mathbf{X} = 0$ has degree 2. Levin's main result was to prove that a pencil of quadrics always contains at least one *"simple" ruled quadric* [11]. Furthermore, Levin showed how to compute such a quadric by first finding the zeros of the determinant of the upper left 3×3 submatrix of $R(\lambda, \mu)$, a cubic equation. Since cubic equations have generically no rational root (by Hilbert's Irreducibility Theorem), Levin's algorithm introduces non-rational numbers at an early stage and, in practice, floating-point arithmetic has to be used, resulting in numerical robustness problems.

The principal contribution of [6] was to show that, by a careful choice of the intermediate quadric Q_R, the appearance of algebraic numbers can be kept to a minimum. One major result is encapsulated in Theorem 3 of [6]: except when the intersection is reduced to two real points, the pencil contains at least one ruled quadric whose coefficients are rational and such a quadric can be easily computed. In addition, thanks to new worst-case optimal (in the number of square roots) parameterizations of ruled projective quadrics, we can always find such a rational ruled quadric Q_R with a parameterization involving only one square root.

Some of the basic ingredients used in our algorithm to infer information about the intersection are the Segre classification of pencils and its refinement over the reals (the Canonical Form Theorem for pairs of real symmetric matrices – see [21]), a projective setting, ad hoc projective transformations to compute the canonical form of a projective quadric, and Sylvester's Inertia Law [10].

The basic principles underlying the design of our implementation are as follows:

- compute structural information on the intersection and its various real components as early as possible;
- use the structural information gathered to drive the parameterization process and make the right choices so that the output is optimal or near-optimal from the point of view of the degree of the extension of $\mathbb{Z}$ on which its coefficients are defined.

In our implementation we were interested not just in optimizing the number of square roots in the output but also in minimizing the size of the output coefficients. For this reason, the basic philosophy is to use as intermediate ruled quadric Q_R a quadric with rational coefficients of the smallest rank that we can easily find, the rationale being, for instance, that the parameterization of a cone involves coefficients of smaller asymptotic size than the coefficients of the parameterization of a quadric of inertia $(2, 2)$. There are essentially two cases: (i) Q_R has rank 4; (ii) Q_R has rank 3 or less.

Case (i): Q_R has rank 4

The main case where Q_R has rank 4 is when the intersection is a smooth quartic (Figure 1.a). In this situation, the quartic determinantal equation $\det R(\lambda) = 0$ has no multiple root. It could well be that at least one of its simple roots is rational and that a Q_R with rank less than 4 could have been used, but checking this via the Rational Root Theorem can be very time consuming[2]. Since generically a degree-four equation has no rational root, we prefer instead to isolate the real zeros of the determinantal equation using an implementation of Uspensky's algorithm [15]. We then take (at most two) rational test points λ_i outside the isolating intervals in the areas where $\det R(\lambda) > 0$. If one of the quadrics $R(\lambda_i)$ has inertia $(4, 0)$, the intersection is empty (it is a complex smooth quartic), a consequence of Finsler's Theorem (see [6]). Otherwise, we proceed.

We now have a quadric $R_0 = R(\lambda_0)$ of inertia $(2, 2)$ and a range of values $I = [a, b]$ such that $\lambda_0 \in I$ and $\det R(\lambda) > 0$ for all $\lambda \in I$. In the worst case, the parameterization of Q_R involves two square roots [6]. We can improve this situation as follows. First, compute a point $\mathbf{p}_0$ on Q_{R_0}. Approximate this point by a point $\mathbf{p}$ with integer coordinates (recall that $\mathbf{p}$ is a projective point). Find the quadric $Q_R = Q_{R(\lambda_1)}$ through $\mathbf{p}$. If $\mathbf{p}$ is close enough to $\mathbf{p}_0$, then $\lambda_1 \in I$ and $\det R > 0$. We thus have a quadric of inertia $(2, 2)$ containing a point in $\mathbb{P}^3(\mathbb{Z})$: such a quadric can be parameterized with at most one square root [6].

Plugging the parameterization $\mathbf{X}((u, v), (s, t))$ of Q_R, with $(u, v), (s, t) \in \mathbb{P}^1$, in the equation of any other quadric of the pencil gives a bihomogeneous equation that has degree two in (u, v) and two in (s, t). Solving this equation for (s, t) in terms of (u, v) and replugging in the parameterization of Q_R gives a parameterization of the smooth quartic:

$$\mathbf{X}(u, v) = \mathbf{X}_1(u, v) \pm \mathbf{X}_2(u, v)\sqrt{\Delta(u, v)},$$

where $\mathbf{X}_1(u, v)$ (resp. $\mathbf{X}_2(u, v)$) is a vector of homogeneous polynomials of degree 3 (resp. 1) and $\Delta(u, v)$ is a homogeneous polynomial of degree 4. Δ and the polynomials of $\mathbf{X}_1$ and $\mathbf{X}_2$ have coefficients in $\mathbb{Z}(\sqrt{\det R})$.

If $\det R$ is a square, then all of these polynomials have rational coefficients and the parameterization is *optimal* in terms of the degree of the extension of $\mathbb{Z}$ on which it is defined. If $\det R$ is not a square, then we can only conclude that the parameterization is *near-optimal*: it might well be that there exists another quadric $Q_{R'}$ of inertia $(2, 2)$ in the pencil, containing a rational point, such that $\det R'$ is a square, implying that $\sqrt{\det R}$ could have been avoided in the output (see Section 8.2 for an example). Finding such a quadric however implies, in general, finding a rational point on a hyperelliptic curve (see [6]), a problem known to be very hard.

Case (ii): Q_R has rank strictly less than 4

Though not generic, the situation where Q_R has rank strictly less than 4 happens quite often in practice since it covers in particular all the types of intersection corresponding, in the Segre characterization, to the determinantal equation having a single multiple root λ_0. Indeed, in that case, the multiple root is both real (otherwise its complex conjugate would also be a multiple root of $\det R(\lambda) = 0$) and rational (otherwise its algebraic conjugate would also be a multiple root of $\det R(\lambda)$). So the associated quadric $Q_R = Q_{R(\lambda_0)}$ has rational coefficients and has rank 3 or less.

The general philosophy for parameterizing the intersection is to parameterize Q_R, plug the parameterization in any other quadric of the pencil, and solve the resulting equation in the parameters. There are however many situations in which this procedure can be simplified by the fact that we can find a rational point on Q_R outside its singular locus and thus parameterize Q_R rationally, and that we know enough information on the intersection to greatly simplify the solving and factorization of the equation in the parameters.

Let us illustrate this on the example of an intersection consisting of a cubic and a line that are tangent (Figure 1.b). The determinantal equation in this case has a quadruple root corresponding to a cone Q_R of inertia $(2, 1)$. By the above argument, Q_R has rational coefficients. So the vertex $\mathbf{c}$ of Q_R has rational coordinates. $\mathbf{c}$ is the point of tangency of the cubic and the line of the intersection. Assume $Q_R \neq Q_S$. The line of the intersection is necessarily rational (otherwise its conjugate would be in the intersection). This line can be found by intersecting the cone Q_R with the plane tangent to Q_S at $\mathbf{c}$. Picking any point $\mathbf{p}$ with rational coordinates on this line other than $\mathbf{c}$ gives a non-singular rational point on the cone. A projective cone having a rational point $\mathbf{p}$ other than its singular locus can be rationally parameterized. Plugging this parameterization in Q_S gives an equation in the parameters

[2] If however one of the initial quadrics has rank 3, then it should be used to parameterize the intersection. Doing so results in a parameterization having the same algebraic complexity in the worst case, but of smaller coefficient size.

of the cone which factors into two terms of total degree 1 and 3. Each factor can then be solved rationally for one parameter in terms of the other. The linear factor yields the line of the intersection and the cubic factor yields the cubic.

4 Height of output coefficients: smooth quartics

In this section and the next, we prove theoretical bounds on the height of the coefficients of the parameterizations computed by our intersection software. We start by defining the notions of height and asymptotic height.

4.1 Definition of height

In what follows, we bound the asymptotic height of the coefficients of the parameterization of the intersection of two quadrics S and T with respect to the size of the coefficients of S and T. The height of such a coefficient is roughly its logarithm with base the maximum of the coefficients of S and T (in absolute value); if such a coefficient has a polynomial expression in terms of the coefficients of S and T, its asymptotic height is the (total) degree of this polynomial. However, a precise definition of the height of these coefficients needs care for various reasons. First, we want to compare, and thus define, *observed heights* (the heights computed for specific values of the input) and *theoretical asymptotic heights*.

We face the following problem for computing theoretical asymptotic heights of the coefficients of the parameterizations. Despite being, ultimately, only functions of the input S and T, these coefficients, in the smooth quartic case, are functions of not just S and T but also of an intermediate rational point $\mathbf{p}$ which depends implicitly (and not explicitly) on S, T. Since obtaining a bound on the height of $\mathbf{p}$ is very hard, we chose to express the asymptotic height of the parameterization as a function of the height of $\mathbf{p}$. As it turns out, the height of $\mathbf{p}$ can, in practice, be neglected, so it is not really a problem (see the discussion at the end of Section 4.2).

In what follows, the *size* of an integer e is $\log_{10}|e|$ (assuming $|e| > 1$). The *size* of an algebraic number $e_1 + \sqrt{\delta}\, e_2$, where e_1, e_2, δ are integers and any two factors of δ are relatively prime, is the maximum of the sizes of e_1, e_2, and δ. The *size* of a vector or matrix, with at most a constant number of entries, is the maximum size of the entries.

The *height* of an entity E (an integer, a vector, or a matrix) with respect to another entity x (also an integer, a vector, or a matrix) is the size of e over the size of x (assuming that the sizes of e and x are nonzero); note that if E and x are integers, the height is also equal to $\log_{|x|}|E|$. The *asymptotic height* of a function $f(x)$ with respect to an integer x is the limit of the height of $f(x)$ with respect to x when x tends to infinity. If a function f depends on a set X of variables, the *asymptotic height* of $f(X)$ with respect to X is the sum of the asymptotic heights of f with respect to each of the variables of X. For instance, if f is a polynomial in a constant number of variables, the asymptotic height of f with respect to these variables is the (total) degree of f. Finally, if $F(X)$ is matrix of functions depending on a set of variables X, the *asymptotic height* of $F(X)$ with respect to X is the maximum of the asymptotic heights of the entries of the matrix.

We mostly consider in the sequel heights and asymptotic heights with respect to S and T (that is with respect to the set of coefficients of S and T). *Heights* and *asymptotic heights* are thus considered with respect to S and T unless specified otherwise.

4.2 Height of the parameterization in the smooth quartic case

We consider now the case of a smooth quartic. This case is important because it is the generic intersection situation (given two random quadrics, a non-empty intersection is a smooth quartic with probability 1) and because it is also the worst case from the point of view of the height of the coefficients involved.

Let Q_R be the quadric of inertia $(2, 2)$ used to parameterize the intersection and $\mathbf{p}$ a point of $\mathbb{P}^3(\mathbb{Z})$ on Q_R, as described in Section 3.1.

Proposition 1. *The parameterization of a smooth quartic*

$$\mathbf{X}(u,v) = \mathbf{X}_1(u,v) \pm \mathbf{X}_2(u,v)\sqrt{\Delta(u,v)}$$

is such that

- $\mathbf{X}_1$ *has asymptotic height at most* $27 + 36h_{\mathbf{p}}$,
- $\mathbf{X}_2$ *has asymptotic height at most* $8 + 11h_{\mathbf{p}}$,
- $\Delta(u,v)$ *has asymptotic height at most* $38 + 50h_{\mathbf{p}}$,

where $h_{\mathbf{p}}$ *is the asymptotic height of* $\mathbf{p}$.

Proof. We first show how the parameterization of Q_R is computed and then bound the height of its coefficients.

Let P be a projective transformation sending the point $\mathbf{p}_0 = (1,0,0,0)^T$ to the point $\mathbf{p}$. Let Y denote the quadric obtained from R through the projective transformation P: $Y = P^T R\, P$. It follows from Sylvester's Inertia Law [10] that Y has the same inertia as R, i.e. $(2,2)$. Moreover, the point $\mathbf{p}_0$ belongs to Q_Y since $P\,\mathbf{p}_0 = \mathbf{p}$.

Let $\mathbf{x}$ denote the vector $(x_1,x_2,x_3,x_4)^T$. Let L be 1/2 times the differential of quadric Q_Y at $\mathbf{p}_0$ (one can trivially show that L is the first row of Y) and let i be such that $Y_{1,i} \neq 0$ (such an i necessarily exists). We compute the polynomial division of $Q_Y = \mathbf{x}^T Y \mathbf{x}$ by $L\mathbf{x}$ with respect to the variable x_i. The result of the division is

$$Y_{1,i}^2\,(\mathbf{x}^T Y \mathbf{x}) = (L\mathbf{x})\,(L'\mathbf{x}) + A, \tag{1}$$

where the ξ-th coordinate of L' is equal to $L'_\xi = -Y_{i,i}\,Y_{1,\xi} + 2\,Y_{1,i}\,Y_{i,\xi}$ for $\xi = 1,\ldots,4$ and

$$A = c_j\,x_j^2 + c_k\,x_k^2 + 2\,c_{jk}\,x_j\,x_k$$

where j and k are equal to the two values in $\{2,3,4\}$ distinct from i, and c_j, c_k, and c_{jk} are coefficients defined as follows:

$$\begin{aligned} c_\xi &= Y_{\xi,\xi}\,Y_{i,1}^2 + Y_{i,i}\,Y_{\xi,1}^2 - 2\,Y_{\xi,1}\,Y_{i,1}\,Y_{i,\xi}, \quad \xi \in \{j,k\},\\ c_{jk} &= Y_{j,k}\,Y_{i,1}^2 + Y_{j,1}\,Y_{k,1}\,Y_{i,i} - (Y_{j,1}\,Y_{k,i} + Y_{k,1}\,Y_{j,i})\,Y_{i,1}. \end{aligned}$$

We assume in the following that $c_j \neq 0$ (if $c_j = 0$ but $c_k \neq 0$, we exchange the roles of j and k; otherwise the analysis is different but similar and we omit it here). For clarity we denote in the following

$$c = c_j \quad \text{and} \quad r = Y_{1,i}.$$

We consider the projective transformation M such that, in the new projective frame, the quadric Q_Y has equation (up to a factor)

$$\mathbf{x}'^T M^T Y M \mathbf{x}' = 4\,x'_1\,x'_2 + {x'_3}^2 - c\,{x'_4}^2.$$

In accordance with Equation (1) we choose $x'_1 = L\mathbf{x}$, $x'_2 = L'\mathbf{x}$. We apply Gauss' decomposition of quadratic forms into sum of squares to A and set $x'_3 = c\,x_j + c_{jk}\,x_k$ and $x'_4 = x_k$. Precisely, we define M such that its adjoint has its first row equal to L, its second row equal to L', and the last two rows equal to zero except for the entry $(3,j)$ equal to c, the entry $(3,k)$ equal to c_{jk}, and the entry $(4,k)$ equal to 1.

Straightforward computations show that the four columns of M can be simplified by the factors $r\,c$, r, $2\,r$, and $2\,r^2$, respectively. We then get

$$\mathbf{x}^T M^T Y M \mathbf{x} = r^2 c\,(4x_1x_2 + x_3^2 - \det(Y)\,x_4^2). \tag{2}$$

If i,j,k are equal to $2,3,4$ respectively, M is equal to

$$M = \begin{pmatrix} Y_{2,2} & -c & Y_{2,2}\,Y_{1,3} - r\,Y_{2,3} & M_{1,4}\\ -2\,r & 0 & -r\,Y_{1,3} & M_{2,4}\\ 0 & 0 & r^2 & M_{3,4}\\ 0 & 0 & 0 & r\,c \end{pmatrix},$$

$$
\begin{aligned}
M_{1,4} &= r\,(Y_{1,4}\,(Y_{2,2}\,Y_{3,3} - Y_{2,3}^2) + Y_{3,4}(r\,Y_{2,3} - Y_{2,2}\,Y_{1,3}) + Y_{2,4}(Y_{1,3}\,Y_{2,3} - r\,Y_{3,3})),\\
M_{2,4} &= r\,(Y_{1,4}\,(Y_{1,3}\,Y_{2,3} - r\,Y_{3,3}) + Y_{1,3}\,(r\,Y_{3,4} - Y_{1,3}\,Y_{2,4})),\\
M_{3,4} &= r\,(-r^2\,Y_{3,4} - Y_{2,2}\,Y_{1,3}Y_{1,4} + r\,(Y_{1,3}\,Y_{2,4} + Y_{1,4}\,Y_{2,3})).
\end{aligned}
$$

We can easily parameterize the quadric of Equation (2) and the parameterization of the original Q_R is, with $\delta = \det(Y)$ and (u, v) and (s, t) in $\mathbb{P}^1(\mathbb{R})$,

$$
PM\,\left(ut\sqrt{\delta},\quad sv\sqrt{\delta},\quad (us - tv)\sqrt{\delta},\quad us + tv\right)^T. \tag{3}
$$

We now bound the asymptotic height of this parameterization with respect to S, T and $\mathbf{p}$. For simplicity, asymptotic heights are referred to as *heights* until the end of the proof. First note that the matrix Y is equal to $P^T RP$, where R is the matrix $\lambda_1 S + \mu_1 T$ of the pencil such that $(\lambda_1, \mu_1) \in \mathbb{P}^1$ is solution of

$$
\mathbf{p}^T(\lambda_1 S + \mu_1 T)\mathbf{p} = 0. \tag{4}
$$

So $(\lambda_1, \mu_1) = (-\mathbf{p}^T T\mathbf{p}, \mathbf{p}^T S\mathbf{p})$ has height $1 + 2h_{\mathbf{p}}$ and $R = \lambda_1 S + \mu_1 T$ has height $2 + 2h_{\mathbf{p}}$. Since $P\mathbf{p}_0 = \mathbf{p}$, the first column of P has height $h_{\mathbf{p}}$ and the rest of P has height 0. We can now deduce the heights of the entries of $Y = P^T RP$. Note first that $Y_{1,1}$ is zero because $\mathbf{p}_0$ belongs to Q_Y. A straightforward computation thus gives that the first line and column of Y have height $2 + 3h_{\mathbf{p}}$ and the other entries have height $2 + 2h_{\mathbf{p}}$. Note that it follows that $\delta = \det Y$ has height $8 + 10h_{\mathbf{p}}$ and that, when δ is a square, $\sqrt{\delta}$ has height $4 + 5h_{\mathbf{p}}$.

It directly follows from the heights of the coefficients of Y and P that the heights of the four columns of PM are, respectively,

$$
2 + 3h_{\mathbf{p}},\quad 6 + 9h_{\mathbf{p}},\quad 4 + 6h_{\mathbf{p}},\quad \text{and}\quad 8 + 11h_{\mathbf{p}}.
$$

The worst case for the height of the coefficients of the parameterization of Q_R happens when $\sqrt{\delta}$ is a square, because the height of these coefficients is at least the height of PM which is larger than the height of δ. We can thus assume for the rest of the proof that $\sqrt{\delta}$ is a square. It then follows from (3) that the coordinates of the parameterization of Q_R are polynomials of the form

$$
\rho_1\,ut + \rho_2\,sv + \rho_3\,us + \rho_4\,tv. \tag{5}
$$

The height of ρ_1 is the sum of the heights of the first column of PM and of $\sqrt{\delta}$. Similarly, we get that the heights of $\rho_1, \ldots, \rho_4$ are

$$
h_{\rho_1} = 6 + 8h_{\mathbf{p}},\quad h_{\rho_2} = 10 + 14h_{\mathbf{p}},\quad \text{and}\quad h_{\rho_3} = h_{\rho_4} = 8 + 11h_{\mathbf{p}}.
$$

When substituting the parameterization of Q_R into the equation of one of the initial quadrics (say Q_S), we obtain an equation which can be written as

$$
a\,s^2 + b\,st + c\,t^2 = 0, \tag{6}
$$

where a, b, and c depend on (u, v) and whose heights are

$$
h_a = 1 + 2\max(h_{\rho_2},\ h_{\rho_3}) = 21 + 28h_{\mathbf{p}},
$$

$$
h_b = 1 + \max(h_{\rho_2}, h_{\rho_3}) + \max(h_{\rho_1}, h_{\rho_4}) = 19 + 25h_{\mathbf{p}},
$$

$$
h_c = 1 + 2\max(h_{\rho_1},\ h_{\rho_4}) = 17 + 22h_{\mathbf{p}}.
$$

When substituting the solution $(s = 2c,\ t = -b \pm \sqrt{b^2 - 4ac})$ into each coordinate, of the form (5), of the parameterization (3) we obtain a parameterization of the smooth quartic in which each coordinate has the form

$$
\chi_1(u, v) \pm \chi_2(u, v)\sqrt{\Delta(u, v)}.
$$

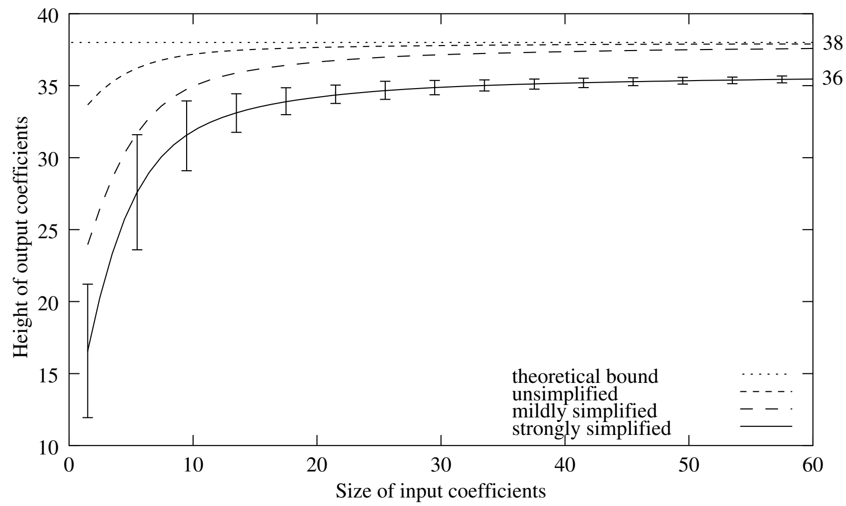

Figure 2: Evolution of the height of $\Delta(u,v)$ (smooth quartic case) as a function of the size of the input, with the standard deviation displayed on the simplified plot.

The height of the coefficients of χ_1, χ_2, and Δ are

$$h_{\chi_1} = \max(h_{\rho_1} + h_b,\ h_{\rho_2} + h_c,\ h_{\rho_3} + h_c,\ h_{\rho_4} + h_b) = 27 + 36h_{\mathbf{p}},$$

$$h_{\chi_2} = \max(h_{\rho_1},\ h_{\rho_4}) = 8 + 11h_{\mathbf{p}},$$

$$\Delta = \max(2h_b,\ h_a + h_c) = 38 + 50h_{\mathbf{p}}.$$

which concludes the proof. □

Figure 2 shows how the observed height of the coefficients of $\Delta(u,v)$ evolves as a function of the input size s for the three variants of our implementation discussed in Section 6. For each value of s in a set of samples between 0 and 60, we have generated random quadrics with coefficients in the range $[-10^s, 10^s]$, computed the height of the coefficients of the parameterization of the smooth quartic and averaged the results.

The plots of Figure 2 show that the observed height of the coefficients of $\Delta(u,v)$ converges to 38 when no gcd computation is performed for simplifying the output parameterization. Since the asymptotic height of $\Delta(u,v)$ is at most 38 plus 50 times the height of $\mathbf{p}$, this suggests that the asymptotic height of $\mathbf{p}$ is zero. Indeed, we have observed experimentally that the coordinates of $\mathbf{p}$ are integers between -2 and 2 most of the time. Out of thousands of runs we have encountered no example where the size of $\mathbf{p}$ had a significant impact on the height of the coefficients of the parameterization.

Backing this observation by theoretical results is hard, if not out of reach. Let $R = R(\lambda_1, \mu_1)$ be the quadric through $\mathbf{p}$. By Eq. (4), the size of the rational point $\mathbf{p}$ is intimately related to the height of (λ_1, μ_1). It is intuitively clear that if the size of the interval on which (λ_1, μ_1) is taken is small, then the size of $\mathbf{p}$ will increase. It thus seems natural to look for results on the distance between roots of integer polynomials. Various upper and lower bounds are known as a function of the degree of the polynomial and the height of its coefficients (see, e.g., [3]), and pathological examples exhibiting root distances almost matching those bounds can be constructed. However, nothing is known about the average distance between the roots of a polynomial whose coefficients are uniformly distributed between $-h$ and h for some fixed integer h (personal communication with Y. Bugeaud and M. Mignotte).

Figure 2 also shows that the observed height of the coefficients of $\Delta(u,v)$ converges to 36 when gcd computations are performed. We ran experiments with inputs of size up to 10,000 and observed the same limit of 36 on the height of the coefficients when gcd computations are performed. We do not have any explanation as to why the bound of 38 is not reached in that case.

real type of intersection	height of parameterization	inertia of Q_R used
smooth quartic	$38 + 50h_{\mathbf{p}}$	$(2,2)$
nodal quartic	22	$(2,1)$ without rational point
cuspidal quartic	38*	$(2,1)$ with rational point
cubic and secant line	22 (cubic), 9 (line)	$(2,1)$ with rational point**
cubic and tangent line	20 (cubic), 11 (line)	$(2,1)$ with rational point
two tangent conics	$20 + \frac{1}{6}$	$(1,1)$
double conic	$13 + \frac{2}{3}$	$(1,0)$
conic and two lines crossing	$17 + \frac{1}{2}$ (conic) and 9 (lines)	$(1,1)$
two skew lines and a double line	9 (lines) and 4 (double line)	$(1,1)$
two double lines	12	$(1,0)$

Table 1: Asymptotic heights of parameterizations in major cases, when the determinantal equation has a unique multiple root. In the singular cases, these values should be compared to the bound of 27 for each component if a quadric of inertia $(2,2)$ had been used, keeping in mind that the result could also contain an unnecessary square root. Note: $(*)$ Since 38 is larger than 27, it might seem that using a quadric Q_R of inertia $(2,1)$ in the cuspidal quartic case is a bad idea and that a quadric of inertia $(2,2)$ would have given better results. This is in no way the case: since the intersection curve is irreducible, the equation in the parameters using a quadric of inertia $(2,2)$ would also have been irreducible, therefore producing a parameterization involving the square root of some polynomial. $(**)$ We can easily find a rational point on Q_R here only when the intersection points between the cubic and the line are rational. Otherwise, we need to use a quadric Q_R of inertia $(2,2)$.

5 Height of output coefficients: singular intersections

In this section, we analyze two different types of situations to validate a key design choice we made, which is to take the quadric with rational coefficients of lowest possible rank to parameterize the intersection. We first consider the case when the pencil contains a rational cone and then when it contains a rational pair of planes. In both cases, we illustrate the fact that better results are obtained than when using a quadric of inertia $(2,2)$ as intermediate quadric.

Table 1 summarizes the asymptotic heights of the parameterizations in many cases of interest.

5.1 Preliminaries

Let Q_R be a singular quadric corresponding to a rational root $(\lambda_0, \mu_0) \in \mathbb{P}^1(\mathbb{Z})$ of multiplicity $d \geqslant 1$ of the determinantal equation $\det(\lambda S + \mu T) = 0$. Here, we further assume that (λ_0, μ_0) is a representative of the root in $\mathbb{Z}^2$ such that $\gcd(\lambda_0, \mu_0) = 1$. We also assume that Q_R has rank r (recall that $3 \geqslant r \geqslant 4 - d$).

Lemma 2. *The asymptotic height of (λ_0, μ_0) is at most $\frac{4}{d}$, and the asymptotic height of $R = \lambda_0 S + \mu_0 T$ is at most $1 + \frac{4}{d}$.*

Proof. We have that

$$\det(\lambda S + \mu T) = C(\mu_0\lambda - \lambda_0\mu)^d(\alpha_0\lambda^{n-d} + \cdots + \alpha_{n-d}\mu^{n-d}).$$

Since the coefficients of $\det(\lambda S + \mu T)$ are integers, we can assume that the α_i are integers and $C \in \mathbb{Q}$. We can also assume that the gcd of all the α_i is one. Recall that an integer polynomial is called *primitive* if the gcd of all its coefficients is one. Since the product of two primitive polynomials is primitive, by Gauss's Lemma (see [5, §4.1.2]), C is an integer (equal to the gcd of the coefficients of $\det(\lambda S + \mu T)$). Therefore, since the coefficient $C\mu_0^d\alpha_0 = \det S$ of λ^4 has asymptotic height 4, μ_0 has asymptotic height at most $\frac{4}{d}$, and similarly for λ_0. It directly follows that $R = \lambda_0 S + \mu_0 T$ has asymptotic height at most $1 + \frac{4}{d}$. ☐

Lemma 3. *The singular set of Q_R contains a basis of points of asymptotic height at most $r\left(1 + \frac{4}{d}\right)$.*

Proof. Assume first that R has rank 3, i.e., Q_R has a singular point. Finding this singular point amounts to finding a point $\mathbf{c} \in \mathbb{P}^3(\mathbb{Z})$ in the kernel of R, i.e., such that $R\mathbf{c} = 0$. Since R has rank 3, at least one of its 3×3 minors is non-zero. Assume that the upper left 3×3 minor has this property. We decompose R such that R_u is the upper left 3×3 matrix of R and $\mathbf{r}_4$ is the first three coordinates of the last column of R, and $\mathbf{c}$ such that $\mathbf{c}_u$ is the first three coordinates and c_4 is the last. Then $\mathbf{c}$ is found by solving

$$R_u \mathbf{c}_u = -c_4 \mathbf{r}_4.$$

A solution is thus $\mathbf{c} = (-R_u^* \mathbf{r}_4, \det R_u)$, where R_u^* is the adjoint of R_u. The asymptotic heights of R_u^*, $\mathbf{r}_4$, and $\det R_u$ are the asymptotic height of R times 2, 1, and 3, respectively. The asymptotic height of $\mathbf{c}$ is thus 3 times the asymptotic height of R. Hence, $\mathbf{c}$ has asymptotic height at most $3\left(1 + \frac{4}{d}\right)$.

The extension to general rank r is similar: Q_R contains in this case a linear space of dimension $3-r$ of singular points. One can extract a non-singular submatrix of R of size r and points in the kernel of R have asymptotic height r with respect to the coefficients of the matrix. The result follows. □

5.2 When Q_R is a cone

5.2.1 Parameterization of a cone

Assume now that Q_R is a real cone with vertex $\mathbf{c}$ containing a rational point $\mathbf{p} \neq \mathbf{c}$. We want to find a rational parameterization of Q_R. First, we apply to R a projective transformation P sending the point $(0,0,0,1)^T$ to $\mathbf{c}$ and the point $(0,0,1,0)^T$ to $\mathbf{p}$. We are left with the problem of parameterizing the cone $Q_{P^T RP}$ with apex $(0,0,0,1)^T$ and going through the point $(0,0,1,0)^T$. Such a cone has equation

$$a_1 x^2 + a_2 xy + a_3 y^2 + a_4 yz + a_5 xz = 0. \tag{7}$$

A parameterization of this cone is given by

$$\mathbf{X}'(u,v,s) = \begin{pmatrix} a_5 & 0 & a_4 & 0 \\ 0 & a_4 & a_5 & 0 \\ -a_1 & -a_3 & -a_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u^2 \\ v^2 \\ uv \\ s \end{pmatrix}, \qquad (u,v,s) \in \mathbb{P}^{\star 2}(\mathbb{R}). \tag{8}$$

Here, $\mathbb{P}^{\star 2}(\mathbb{R})$ is the real quasi-projective space defined as the quotient of $\mathbb{R}^3 \setminus \{0,0,0\}$ by the equivalence relation $\sim$ where $(x,y,z) \sim (x',y',z')$ if and only if there exists $\lambda \in \mathbb{R} \setminus \{0\}$ such that $(x,y,z) = (\lambda x', \lambda y', \lambda^2 z')$. Lifting the parameterization to the original space by multiplying by matrix P, we have a parameterization of Q_R.

Let h_R (resp. $h_\mathbf{p}, h_\mathbf{c}$) denote the asymptotic height of R (resp. of $\mathbf{p}, \mathbf{c}$). From the above, we can deduce the following.

Lemma 4. *The parameterization $\mathbf{X}(u,v,s)$ of Q_R is such that:*

- *the asymptotic height of the coefficients of u^2, v^2, uv is $h_R + h_\mathbf{p}$;*
- *the asymptotic height of the coefficients of s is $h_\mathbf{c}$.*

Proof. The matrix P has its third column set to $\mathbf{p}$ and its fourth column set to $\mathbf{c}$. We complete it so that it indeed represents a real projective transformation (i.e., its columns form a basis of $\mathbb{P}^3$). So the first two columns have height 0 in R, $\mathbf{p}$, and $\mathbf{c}$. Computing $P^T RP$, we see that the height of a_1, a_2, and a_3 is the height of R and the asymptotic height of a_4 and a_5 is the sum of the asymptotic heights of R and $\mathbf{p}$. From this, we see that the asymptotic height of the coefficients of u^2, v^2, uv in $\mathbf{X}(u,v,s) = P\mathbf{X}'(u,v,s)$ is the sum of the asymptotic heights of R and $\mathbf{p}$; also the height of the coefficients of s is the height of $\mathbf{c}$. □

5.2.2 Cubic and tangent line

We now consider the case of an intersection consisting of a cubic and a tangent line. In this case, we can parameterize the intersection using an intermediate rational quadric Q_R of inertia either $(2,2)$ or $(2,1)$: the pencil contains an instance of both types of quadrics.

We prove the following theoretical bounds on the asymptotic height of the coefficients of the parameterizations of the cubic and the line.

Proposition 5. *When a quadric Q_R of inertia $(2,2)$ is used to parameterize the intersection, the parameterizations of the cubic and the line have asymptotic height at most 27 plus 36 times the asymptotic height of the point $\mathbf{p} \in Q_R$ used for parameterizing Q_R.*

Proof. The bounds found in the proof of Proposition 1 apply here, and in particular, the bounds $h_{\rho_1}, \ldots, h_{\rho_4}$, h_a, h_b, and h_c on the heights of the coefficients of Equations (5) and (6). Equation (6) factors here into two terms, one of degree 0 and the other of degree 2 in, say, (u, v), and both linear in, say, (s, t); Equation (6) can thus be written as

$$(\alpha s + \beta t)(\alpha' s + \beta' t) = a\, s^2 + b\, st + c\, t^2 = 0,$$

where α, β are constants and α', β' are polynomials in (u, v). Since $\alpha\beta' + \beta\alpha' = b$, α and the coefficients of α' have asymptotic height at most h_b. Similarly, $\beta\beta' = c$ thus β and the coefficients of β' have asymptotic height at most h_c. Substituting the solutions $(s = \beta,\ t = -\alpha)$ and $(s = \beta',\ t = -\alpha')$ into the parameterization (3), we get parameterizations of the cubic and the line whose coefficients have asymptotic height at most

$$h_c + \max(h_{\rho_2},\ h_{\rho_3}) = h_b + \max(h_{\rho_1},\ h_{\rho_4}) = 27 + 36 h_{\mathbf{p}}$$

where $h_{\mathbf{p}}$ is asymptotic height of $\mathbf{p}$. □

Proposition 6. *When a quadric Q_R of inertia $(2,1)$ is used to parameterize the intersection, then asymptotically the parameterization of the line has height at most 11, and the parameterization of the cubic has height at most 20.*

Proof. We follow the algorithm outline given in Section 3.1 to determine the asymptotic height of the output.

Here, the determinantal equation has a quadruple root (λ_0, μ_0) corresponding to a quadric Q_R of inertia $(2,1)$. The asymptotic height h_R of $R = \lambda_0 S + \mu_0 T$ is at most 2, by Lemma 2. The asymptotic height $h_{\mathbf{c}}$ of the singular point $\mathbf{c}$ of Q_R is at most 6, by applying Lemma 3 with $d = 4$ and $r = 3$.

Since the line of the intersection is the (double) intersection of Q_R and the tangent plane to Q_S at $\mathbf{c}$, any point $\mathbf{p}$ on this line satisfies

$$R\mathbf{p} = S\mathbf{c}. \tag{9}$$

(Observe that if $\mathbf{p}$ is a solution, any $a_1\mathbf{p} + a_2\mathbf{c}$ is also solution.) The right-hand side $S\mathbf{c}$ of (9) has asymptotic height at most $6 + 1 = 7$. As in the proof of Lemma 3, one can assume that $\det R_u \neq 0$ and there is a unique point $\mathbf{p}$ having zero as last coordinate. Point $\mathbf{p}$ satisfies $\mathbf{p}_u = R_u^*(S\mathbf{c})_u$ and thus, its asymptotic height $h_{\mathbf{p}}$ is at most $4 + 7 = 11$. Overall, the coefficients of the line $(\mathbf{c}, \mathbf{p})$ have height 11.

We can now compute the asymptotic height of the parameterization $\mathbf{X}(u, v, s)$ of Q_R as defined in Section 5.2.1. By Lemma 4, the asymptotic height $h_{u,v}$ of coefficients of u^2, v^2, uv in $\mathbf{X}(u, v, s)$ is $h_R + h_{\mathbf{p}}$, and the asymptotic height h_s of the coefficient of s is $h_{\mathbf{c}}$. Plugging $\mathbf{X}(u, v, s)$ in the equation of any other quadric of the pencil gives an equation in the parameters of the form

$$as^2 + b(u, v)s + c(u, v) = 0, \tag{10}$$

where $b(u, v)$ and $c(u, v)$ have asymptotic heights respectively equal to

$$1 + h_{u,v} + h_s = 1 + h_R + h_{\mathbf{p}} + h_{\mathbf{c}}, \quad \text{and} \quad 1 + 2h_{u,v} = 1 + 2(h_R + h_{\mathbf{p}}).$$

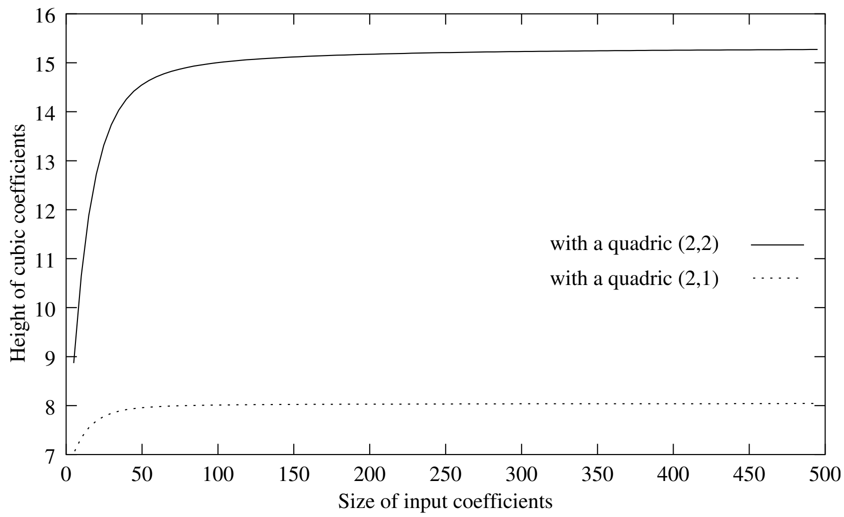

Figure 3: Observed height of the parameterization of the cubic in the cubic and tangent line case.

Observe that $a = 0$ since the singularity of the cone, which is a point of the intersection, is reached at $(u, v) = (0, 0)$ and at this point $s \neq 0$ necessarily (because $\mathbf{X}(u, v, s)$ is a faithful parameterization of the cone). We also know that (10) has a linear factor corresponding to the line of the intersection. By construction (see (8)), this line $(\mathbf{c}, \mathbf{p})$ is represented in parameter space by the line $a_5\, u + a_4\, v = 0$, where a_4 and a_5 have asymptotic height $h_R + h_{\mathbf{p}}$ (see the proof of Lemma 4). So, after factoring out the linear term, (10) can be rewritten as

$$b'(u, v)s + c'(u, v) = 0. \tag{11}$$

The asymptotic height $h_{b'}$ of $b'(u, v)$ is $1 + h_{\mathbf{c}}$, the difference of the asymptotic heights of $b(u, v)$ and of the linear factor. Similarly, the asymptotic height $h_{c'}$ of $c'(u, v)$ is $1 + h_R + h_{\mathbf{p}}$, the difference of the asymptotic heights of $c(u, v)$ and of the linear factor. We plug the solution of (11) in s into the parameterization $\mathbf{X}(u, v, s)$ of Q_R. Multiplying by $b'(u, v)$ to clear the denominators, we get a parameterization of the cubic of asymptotic height

$$\max(h_{u,v} + h_{b'}, h_s + h_{c'}) = 1 + h_R + h_{\mathbf{p}} + h_{\mathbf{c}} \leqslant 1 + 2 + 11 + 6 = 20.$$

□

The difference in the asymptotic heights of the parameterizations underscored in the above two propositions is vindicated by some experiments we made. Figure 3 shows the observed heights of the coefficients of the parameterization of the cubic when a quadric Q_R of inertia $(2, 2)$ or $(2, 1)$ is used. The plots clearly show that the coefficients of the cubic are smaller when a cone is used to parameterize the intersection. The fact that the observed heights are, in the limit, so different from the theoretical bounds (8 instead of 20 when a cone is used) is most likely a consequence of the way the random quadrics are generated: it does not reflect a truly random distribution in the space of quadrics with integer coefficients of given size intersecting in a cubic and a tangent line, as explained in Section 6.3.

Figure 4 further reinforces our choice of using a cone: the parameterizations have not only smaller coefficients, they are also faster to compute.

5.3 When Q_R is a pair of planes

5.3.1 Parameterization of a pair of planes

We now suppose that the singular quadric Q_R corresponding to a root of multiplicity d of the determinantal equation is a pair of planes (i.e., has inertia $(1, 1)$). Let $\mathbf{p}_1$ and $\mathbf{p}_2$ two distinct points on the singular line of Q_R. Let P be a projective transformation matrix sending the point $(0, 0, 1, 0)^T$ to $\mathbf{p}_1$ and the point $(0, 0, 0, 1)^T$ to $\mathbf{p}_2$. We are

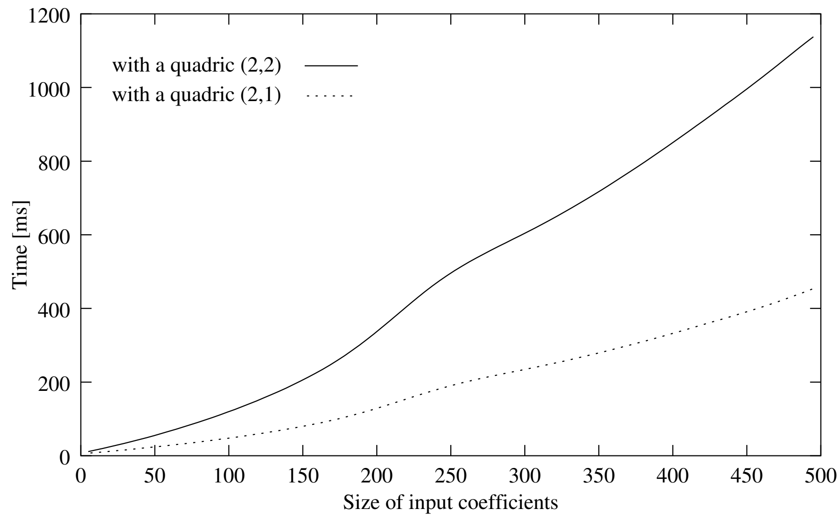

Figure 4: Computation time for the cubic and tangent line case.

left with the problem of parameterizing the pair of planes Q_{P^TRP} whose singular line contains $(0,0,1,0)^T$ and $(0,0,0,1)^T$. Such a pair of planes has equation

$$a_1x^2 + 2a_2xy + a_3y^2 = 0,$$

and it can be parameterized by $M_{\pm}(u,v,s)^T$ with

$$M_{\pm} = \begin{pmatrix} -a_2 \pm \sqrt{\delta} & 0 & 0 \\ a_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \delta = a_2^2 - a_1a_3, \quad (u,v,s) \in \mathbb{P}^2.$$

Lifting this parameterization to the original space by multiplying by matrix P, we obtain a parameterization of Q_R.

Let h_R (resp. $h_{\mathbf{p}_1}, h_{\mathbf{p}_2}$) denote the asymptotic height of R (resp. of $\mathbf{p}_1, \mathbf{p}_2$). From the above, we deduce the following.

Lemma 7. *The asymptotic height of the coefficients a_i in $M_{\pm}$ is h_R. Furthermore, if δ is a square, the parameterization $\mathbf{X}_{\pm}(u,v,s)$ is such that:*

- *the asymptotic height of the coefficients of u is h_R;*
- *the asymptotic heights of the coefficients of v and s are $h_{\mathbf{p}_1}$ and $h_{\mathbf{p}_2}$, respectively.*

Proof. In the parameterization of the pair of planes, the first two columns of P can be completed with 0 and 1 so that it is a non-singular matrix. A straightforward computation then gives that the height of a_1, a_2, and a_3 is the height of R. Hence, the coefficient of u in $\mathbf{X}_{\pm}(u,v,s)$ has same asymptotic height as R, and the coefficients of v and s have the same heights as $\mathbf{p}_1$ and $\mathbf{p}_2$, respectively. □

5.3.2 Two tangent conics

We now consider the case of two tangent conics. This time, we have three possibilities for Q_R: inertia $(2,2)$, $(2,1)$, or $(1,1)$.

Proposition 8. *When the intersection consists of two tangent conics, the parameterization of each of the conics is as follows:*

- *when Q_R has inertia $(1,1)$, the parameterization has asymptotic height at most $20+\frac{1}{6}$;*
- *when Q_R has inertia $(2,1)$, the parameterization has asymptotic height at most $30+\frac{5}{6}$;*
- *when Q_R has inertia $(2,2)$, the parameterization has asymptotic height at most 27 plus 36 times the asymptotic height of the point on Q_R used for parameterizing Q_R; moreover the coefficients may contain an unnecessary square root.*

Proof. The determinantal equation in this case has a real rational triple root corresponding to a pair of planes and a real rational simple root corresponding to a real cone. The pencil also contains quadrics of inertia $(2,2)$. The rational point of tangency $\mathbf{p}$ of the two conics is the point of intersection of the singular line of the pair of planes with any other quadric of the pencil.

Let us first bound the asymptotic height $h_{\mathbf{p}}$ of point $\mathbf{p}$. Let $\mathbf{c}_1, \mathbf{c}_2$ be a basis for the singular set of the pair of planes of the pencil. By Lemma 3, with $d=3$ and $r=2$, $\mathbf{c}_1$ and $\mathbf{c}_2$ have asymptotic height $h_{\mathbf{c}_i}$ at most $\frac{14}{3}$. $\mathbf{p}$ is the point of tangency of the line spanned by $\mathbf{c}_1$ and $\mathbf{c}_2$ with any quadric of the pencil other than the pair of planes. Let $\mathbf{p}=\alpha_0\mathbf{c}_1+\beta_0\mathbf{c}_2$, where $(\alpha_0,\beta_0)\in\mathbb{P}^1$. Then (α_0,β_0) is the double root of the equation

$$(\alpha_0\mathbf{c}_1+\beta_0\mathbf{c}_2)^T S(\alpha_0\mathbf{c}_1+\beta_0\mathbf{c}_2)=0.$$

By Lemma 2, the asymptotic height of (α_0,β_0) is at most $h_{\mathbf{c}_i}+\frac{1}{2}$. Thus, $h_{\mathbf{p}}\leqslant 2h_{\mathbf{c}_i}+\frac{1}{2}\leqslant 2\frac{14}{3}+\frac{1}{2}=\frac{59}{6}$.

Q_R has inertia $(1,1)$. We consider the case where Q_R is the pair of planes of the pencil. We compute a parameterization $\mathbf{X}_\pm(u,v,s)=PM_\pm\,(u,v,s)^T$ of each of the planes of Q_R by sending $(0,0,1,0)^T$ to $\mathbf{c}_1$ and $(0,0,0,1)^T$ to $\mathbf{p}$ as in Section 5.3.1. Plugging each of the $\mathbf{X}_+(u,v,s)$ and $\mathbf{X}_-(u,v,s)$ in the equation of Q_S gives a degree-two homogeneous equation in u, v, and s (i.e., $\mathbf{X}_\pm^T(u,v,s)S\mathbf{X}_\pm(u,v,s)$). This projective conic contains the point $(0,0,1)^T$ since $PM_\pm(0,0,1)^T=\mathbf{p}$ by definition of P and $M_\pm$. Such a conic has equation

$$\mathbf{X}_\pm^T(u,v,s)S\mathbf{X}_\pm(u,v,s)=b_1u^2+b_2uv+b_3v^2+b_4vs+b_5us=0 \tag{12}$$

which can be parameterized, similarly as for (7), by

$$\mathbf{X}'(u',v',s')=\begin{pmatrix} b_5 & 0 & b_4\\ 0 & b_4 & b_5\\ -b_1 & -b_3 & -b_2\end{pmatrix}\begin{pmatrix} u'^2\\ v'^2\\ u'v'\end{pmatrix},\qquad (u',v')\in\mathbb{P}^1(\mathbb{R}).$$

Plugging $\mathbf{X}'(u',v',s')$ into the parameterization of Q_R gives $PM_\pm\mathbf{X}'(u',v',s')$, the parameterizations of the two conics of intersection.

We now compute the asymptotic height of the parameterizations $PM_\pm\mathbf{X}'(u',v',s')$. We assume first that δ in $M_\pm$ is a square. Let h_{b_i}, denote the asymptotic height of b_i, and h_a the asymptotic height of $\{a_1,a_2,a_3\}$ in $M_\pm$. The asymptotic height of the three coordinates of $\mathbf{X}'(u',v',s')$ are, respectively,

$$\max(h_{b_4},h_{b_5}),\quad \max(h_{b_4},h_{b_5}),\quad\text{and}\quad \max(h_{b_1},h_{b_2},h_{b_3}).$$

Thus, the asymptotic height of each of the coordinates of $M_\pm\mathbf{X}'(u',v',s')$ are, respectively,

$$h_a+\max(h_{b_4},h_{b_5}),\quad h_a+\max(h_{b_4},h_{b_5}),\quad \max(h_{b_4},h_{b_5}),\quad\text{and}\quad \max(h_{b_1},h_{b_2},h_{b_3}).$$

The third and fourth columns of P are $\mathbf{c}_1$ and $\mathbf{p}$, and P can be completed with 0 and 1 so that it is a non-singular matrix. Thus, the asymptotic height of $PM_\pm\mathbf{X}'(u',v',s')$ is the maximum of

$$h_a+\max(h_{b_4},h_{b_5}),\quad h_{\mathbf{c}_i}+\max(h_{b_4},h_{b_5}),\quad\text{and}\quad h_{\mathbf{p}}+\max(h_{b_1},h_{b_2},h_{b_3}).$$

Now, the asymptotic height of each b_i is one plus the sum of the asymptotic heights of two of the coefficients of u, v, and s in $\mathbf{X}_\pm(u,v,s)$ (by Equation (12)). Lemma 7 yields

$$h_{b_1}=1+2h_R,\quad h_{b_2}=1+h_R+h_{\mathbf{c}_i},\quad h_{b_3}=1+2h_{\mathbf{c}_i},\quad h_{b_4}=1+h_{\mathbf{c}_i}+h_{\mathbf{p}},\quad h_{b_5}=1+h_R+h_{\mathbf{p}}.$$

Since $h_R \leqslant 1 + \frac{4}{3} = \frac{7}{3}$ by Lemma 2, $h_a \leqslant \frac{7}{3}$ by Lemma 7, $h_{\mathbf{c}_i} \leqslant \frac{14}{3}$, and $h_{\mathbf{p}} \leqslant \frac{59}{6}$, we get $h_{b_1} \leqslant \frac{17}{3}$, $h_{b_2} \leqslant \frac{24}{3}$, $h_{b_3} \leqslant \frac{31}{3}$, $h_{b_4} \leqslant \frac{31}{2}$, and $h_{b_5} \leqslant \frac{79}{6}$. Hence, if δ is a square, the asymptotic height of the parameterization $PM_{\pm}\mathbf{X}'(u', v', s')$ of the two conics of intersection is at most

$$\max\left(\frac{7}{3} + \frac{31}{2}, \quad \frac{14}{3} + \frac{31}{2}, \quad \frac{59}{6} + \frac{31}{3}\right) = \frac{121}{6} = 20 + \frac{1}{6}.$$

Finally, since this bound is larger than the asymptotic height of δ (which is $2h_a \leqslant \frac{14}{3}$), the asymptotic height of $PM_{\pm}\mathbf{X}'(u', v', s')$ can only be less than or equal to $20 + \frac{1}{6}$, even if δ is not a square.

Q_R has inertia $(2, 1)$. Let now Q_R be the cone of the pencil with apex $\mathbf{c}$. By Lemma 4, we have a rational parameterization $\mathbf{X}(u, v, s)$ of Q_R whose coefficients in u^2, v^2, uv have asymptotic height $h_R + h_{\mathbf{p}}$ and whose coefficient in s has asymptotic height $h_{\mathbf{c}}$. Plugging this parameterization into the equation of any other quadric of the pencil gives an equation in the parameters of the form

$$as^2 + b(u, v)s + c(u, v) = 0, \tag{13}$$

where the asymptotic heights of $a, b(u, v)$, and $c(u, v)$ are, respectively,

$$1 + 2h_{\mathbf{c}}, \quad 1 + h_{\mathbf{c}} + h_R + h_{\mathbf{p}}, \quad \text{and} \quad 1 + 2(h_R + h_{\mathbf{p}}).$$

We know (13) factors in two quadratic factors corresponding to the two conics. Also, by construction (see (8)), the ruling of Q_R on which $\mathbf{p}$ lies is represented in parameter space by the line $a_5\, u + a_4\, v = 0$, where a_4, a_5 are as in Section 5.2.1. As in the proof of Lemma 4, the asymptotic height of a_4 and a_5 is $h_R + h_{\mathbf{p}}$. Point $\mathbf{p}$ must be on each conic on intersection, and $\mathbf{p}$ corresponds in parameter space to (u, v, s) such that $s = a_5\, u + a_4\, v = 0$. So (13) rewrites

$$(\alpha_1 s + (a_5 u + a_4 v)\beta_1(u, v))\,(\alpha_2 s + (a_5 u + a_4 v)\beta_2(u, v)) = 0,$$

where β_1 and β_2 are linear in u, v (possibly defined over an extension of $\mathbb{Z}$ by the square root of the discriminant of the pair of planes containing the conics). The asymptotic height of $\alpha_1\beta_2 + \alpha_2\beta_1$ is $1 + h_{\mathbf{c}}$, the difference of the asymptotic heights of $b(u, v)$ and of the linear factor. The asymptotic height of $\beta_1\beta_2$ is 1, the difference of the asymptotic height of $c(u, v)$ and of twice the asymptotic height of the linear factor. Hence, the asymptotic height of each β_i is at most 1, and the height of each α_i is at most $1 + h_{\mathbf{c}}$. Solving each factor rationally for s and plugging the solution into the parameterization $\mathbf{X}(u, v, s)$ of Q_R, we get parameterizations of the conics with asymptotic height $1 + h_{\mathbf{c}} + h_R + h_{\mathbf{p}}$. Applying Lemmas 2 and 3 with $r = 3$ and $d = 1$, and the bound on $h_{\mathbf{p}}$ found above, the asymptotic height of the parameterizations of the conics is at most $1 + 15 + 5 + \frac{59}{6} = 30 + \frac{5}{6}$.

Q_R has inertia $(2, 2)$. When a quadric Q_R of inertia $(2, 2)$ is used, the biquadratic equation (6) factors in two factors of bidegree $(1, 1)$ corresponding to the conics. Factoring introduces, as above, the square root of the discriminant of the pair of planes containing the conics. Proceeding as in the proof of Proposition 5, we get that the height of each factor is at most 27 plus 36 times the asymptotic height of the point on Q_R used for parameterizing Q_R.

Moreover, we might have an extra square root in the result if the determinant of R is not a square. Consider for instance

$$\begin{cases} Q_S : x^2 - 2w^2 = 0, \\ Q_T : xy + z^2 = 0. \end{cases}$$

Here, the determinantal equation is $2\lambda\mu^3 = 0$. $\sqrt{2}$ (i.e., the discriminant of the pair of planes) cannot be avoided in the result. The point $\mathbf{p} = (-1, 3, 0, 0)$ is contained in the quadric $3Q_S + Q_T$ of inertia $(2, 2)$ and determinant 6. If this quadric is used to parameterize the intersection, we have an extra square root, namely $\sqrt{6}$. $\square$

6 Implementation

We now move on to a description of the main design choices we made to implement our near-optimal parameterization algorithm.

6.1 Implementation outline

Our implementation builds upon the LiDIA [12] and GMP [8] C/C++ libraries. LiDIA was originally developed for computational number theory purposes, but includes many types of simple parameterized and template classes that are useful for our application. Apart from simple linear algebra routines and algebraic operations on univariate polynomials, we use LiDIA's number theory package and its ability to manipulate vectors of polynomials, polynomials having other polynomials as coefficients, etc. On top of it, we have added our own data structures. We have compiled LiDIA so that it uses GMP multiprecision integer arithmetic. From now on, we refer to the multiprecision integers as `bigint`s, following the terminology of LiDIA.

Our implementation consists of more than 17,000 lines of source code, which is essentially divided into the following chapters:

- *data structures* (1,500 lines): structures for intersections of quadrics, for components of the intersection, for homogeneous polynomials with `bigint` coefficients (coordinates of components), for homogeneous polynomials with `bigint` polynomials as coefficients, and basic operations on these structures, etc.
- *elementary operations* (2,000 lines): computing the inertia of a quadric of `bigint`s, the coefficients of the determinantal equation, the gcd of the derivatives of the determinantal equation, the adjoint of a matrix, the singular space of a quadric, the intersection between two linear spaces, applying Descartes's Sign Rule, the Gauss decomposition of a quadratic form into a sum of squares, isolating the roots of a univariate polynomial using Uspensky's method, etc.
- *number theory and simplifications* (1,500 lines): gcd simplifications of the `bigint` coefficients of a polynomial, a vector or a matrix, simplifications of the coefficients of pairs and triples of vectors, reparameterization of lines so that its representative points have small height, . . .
- *quadric parameterizations* (2,000 lines): parameterization of a quadric of inertia $(2, 2)$ with `bigint` coefficients going through a rational point, of a cone (resp. conic), of a cone (resp. conic) with a rational point, of a pair of planes, etc.
- *intersection parameterizations* (9,000 lines): dedicated procedures for parameterizing the components of the intersection in all possible cases, i.e., when the determinantal equation has no multiple root (1,500 lines), one multiple root (3,000 lines), two multiple roots (1,500 lines) or when it vanishes identically (3,000 lines).
- *printing and debugging* (1,000 lines): turning on debugging information with the `DEBUG` preprocessor directive, checking whether the computed parameterizations are correct, pretty printing the parameterizations, etc.

6.2 Implementation variants

Three variants of our implementation are available and using one rather than the other might depend on the context or the application (see Section 7). They are:

- *unsimplified*: nothing is done to simplify the coefficients either during the computations or in the parameterizations computed;
- *mildly simplified*: some gcds are performed at an early stage (optimization of the coefficients and of the roots of the determinantal equation, optimization of the coordinates of singular and rational points, etc.) to avoid hampering later calculations with unnecessarily big numbers;
- *strongly simplified*: mildly simplified, plus extraction of the square factors of some `bigint`s (like in the smooth quartic case, where $\sqrt{\det R}$ can be replaced by $b\sqrt{a}$ if $\det R = a\, b^2$) and gcd simplifications of the coefficients of the final parameterizations.

For the extraction of the square factors of an integer n, the strongly simplified variant finds all the prime factors of n up to $\min\left(\lceil \sqrt[3]{n} \rceil, \texttt{MAXFACTOR}\right)$, where `MAXFACTOR` is a predefined global variable.

Let us finally mention that we tried a fourth variant of our implementation where the extraction of the square factors is done by fully factoring the numbers (using the Elliptic Curve Method and the Quadratic Sieve implemented in LiDIA [12]). But this variant is almost of no interest: for small input coefficients, the strongly simplified variant already finds all the necessary factors, and for medium to large input coefficients, integer factoring becomes extremely time consuming.

6.3 Generating random intersections

Our implementation has been tested both on real and random data (see Section 7). Generating random intersections of a given type, i.e., random pairs of quadrics intersecting along a curve of prescribed topology, is however difficult. We discuss this issue here.

In the smooth quartic case, random examples can be generated by taking input quadrics with random coefficients. Indeed, given two random quadrics, the intersection is a smooth quartic or the empty set with probability one. (Of course, this does not allow to distinguish between the different morphologies of a real smooth quartic, i.e., one or two, affinely finite or infinite, components.)

When the desired intersection is not a smooth quartic, one way to proceed is to start with a canonical pair of quadrics intersecting in a curve of the prescribed type and apply to this pair a random transformation. More precisely, given a canonical pair S, T, four random coefficients r_1, r_2, r_3, r_4, with $r_1 r_4 - r_2 r_3 \neq 0$, and a random projective transformation P, we consider the "random" quadrics with matrices S' and T':

$$S' = P^T (r_1 S + r_2 T) P, \quad T' = P^T (r_3 S + r_4 T) P.$$

If we take the r_i and the coefficients of P randomly in the range $[-\lceil \sqrt[3]{10^s} \rceil, \lceil \sqrt[3]{10^s} \rceil]$, then the quadrics S' and T' have asymptotic expected size s (the size of the canonical pair S, T can be neglected).

The problem here is two-fold. First, since we want the matrices S' and T' to have integer coefficients (because that is what our implementation takes), we have to assume that the r_i and the coefficients of P are integers. But then the above procedure certainly does not reflect a truly random distribution in the space of quadrics with integer coefficients. Indeed, quadrics S' and T' with integer coefficients intersecting in the prescribed curve might exist without P having integer coefficients. Consider for instance the two pairs of quadrics

$$\begin{cases} Q_S : x^2 - w^2 = 0, \\ Q_T : xy + z^2 = 0, \end{cases} \qquad\qquad \begin{cases} Q_{S'} : x^2 - 2w^2 = 0, \\ Q_{T'} : xy + z^2 = 0. \end{cases}$$

The first pair is a canonical form for the case of an intersection made of two real tangent conics. Both pairs generate an intersection of the same type. But the second form cannot be generated from the first using a transformation matrix P with integer coefficients.

As for the second issue, consider the determinantal equation of the pencil generated by S', T':

$$\det R'(\lambda, \mu) = \det(\lambda S' + \mu T') = (\det P)^2 \det\big((\lambda r_1 + \mu r_3) S + (\lambda r_2 + \mu r_4) T\big).$$

In other words, since P is now assumed to have integer entries, the coefficients of the determinantal equation all have a common integer factor, $(\det P)^2$. So, after simplification by this common factor, the coefficients have asymptotic height $\frac{4}{3}$, instead of 4, with respect to S', T'. This explains why the asymptotic heights are not reached.

Note that the same problems appear when working the reverse way, i.e., start with the canonical parameterization $\mathbf{X}$ of a required type of intersection, apply a random transformation P, recover the pencil of quadrics $R'(\lambda, \mu)$ containing the curve parameterized by $P\mathbf{X}$ and filter them according to the height of their coefficients. Indeed, in that case, $R'(\lambda, \mu) = P^T R(\lambda, \mu) P$, where $R(\lambda, \mu)$ is the pencil of quadrics through the curve parameterized by $\mathbf{X}$.

Effectively generating random pairs of quadrics with a prescribed intersection type is an open problem.

7 Experimental results

We now report on some experimental results and findings from our implementation.

The experiments were made on a Dell Precision 360 with a 2.60 GHz Intel Pentium CPU. LiDIA, GMP and our code were compiled with g++ 3.2.2.

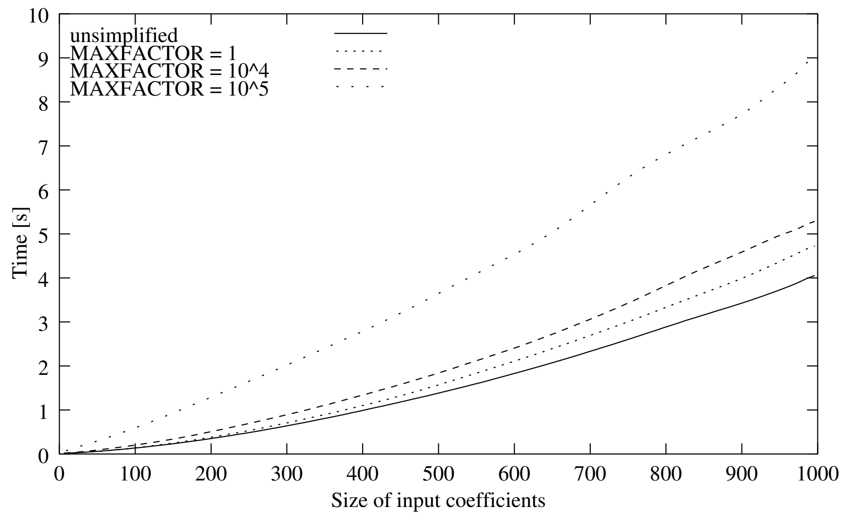

Figure 5: Evolution of execution time in the smooth quartic case as a function of the size of the input for very large input sizes.

7.1 Random data

Let us first discuss the impact of the `MAXFACTOR` variable (see Section 6.2) on the output. Figure 5 shows that values of 10^5 and higher have a dramatic impact on computation time while all values less than 10^4 are acceptable. We have determined that the best compromise between efficiency and complexity of the output is obtained by setting `MAXFACTOR` to 10^3, which we assume now.

Figure 6 shows the evolution of the aggregate computation time in the smooth quartic case, which is the most computationally demanding case, with the three variants outlined above. We infer from these plots that the computation times for the unsimplified and mildly simplified variants are very similar, while we observe (see Figure 2) a dramatic improvement in the height of the output coefficients with the mildly simplified variant for reasonably small inputs. This explains our choice of putting the mild simplifications in the form of a preprocessor directive, not a binary argument: they might as well have been called *mandatory simplifications*.

A second lesson to be learned from Figures 2 and 6 is that for an input with coefficients ranging from roughly 5 to 60 digits, the computation time is roughly 30% larger for the strongly simplified variant than for the mildly simplified. At the same time, the height of the output is on average between 20% (input size of 5) and 5% (input size of 60) smaller. For large values of the input size, the difference in computation time between the mildly simplified and the strongly simplified variants drops to less than 10% (see the two curves in Figure 5 with `MAXFACTOR` equal to 1 and 10^4), but not much is gained in terms of height of the output (see Figure 2).

Another interesting piece of information inferred from Figure 2 is that the standard deviation of the height of the output coefficients is large for small input size in the strongly simplified variant. This means that in the good cases the height of the output is dramatically smaller than the height in the mildly simplified case, and in the bad cases is similar to it.

Deciding to spend time on simplification essentially depends on the application. For most real-world applications, where the size of the input quadrics is small by construction, we believe simplifying is important: it should be kept in mind that the computed parameterizations are often the input to a later processing step (like in boundary evaluation) and limiting the growth of the coefficients at an early stage makes good sense.

A last comment that can be made looking at Figure 5 concerns the efficiency of our implementation. Indeed, those plots show that we can compute the parameterization of the intersection of two quadrics with coefficients having 400 digits in 1 second and 1,000 digits in 5 seconds (on average).

Efficiency can be measured in a different way. In Figure 7, we have plotted the total computation time, with the strongly simplified variant, for a file containing 120 pairs of quadrics covering all intersection situations over the reals. The “random” quadrics were generated as in Section 5.2.2. For an input size $s = 500$, the total computation

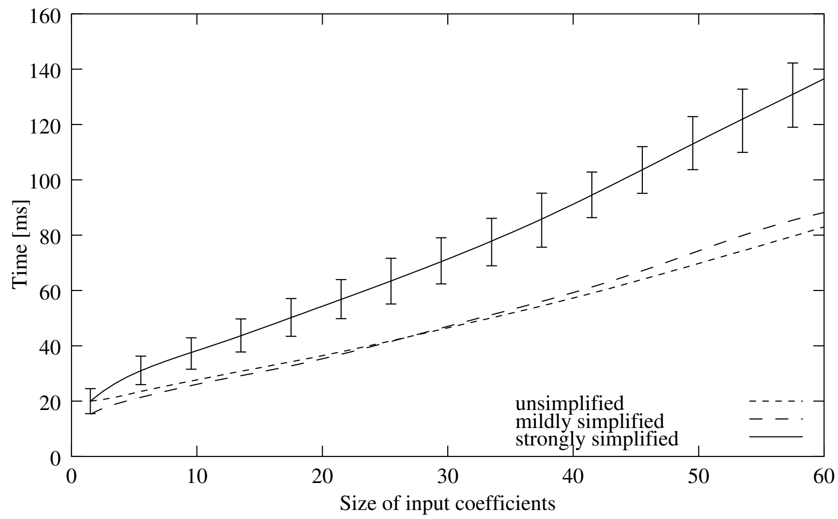

Figure 6: Evolution of execution time in the smooth quartic case as a function of the input size, with the standard deviation shown on the simplified plot.

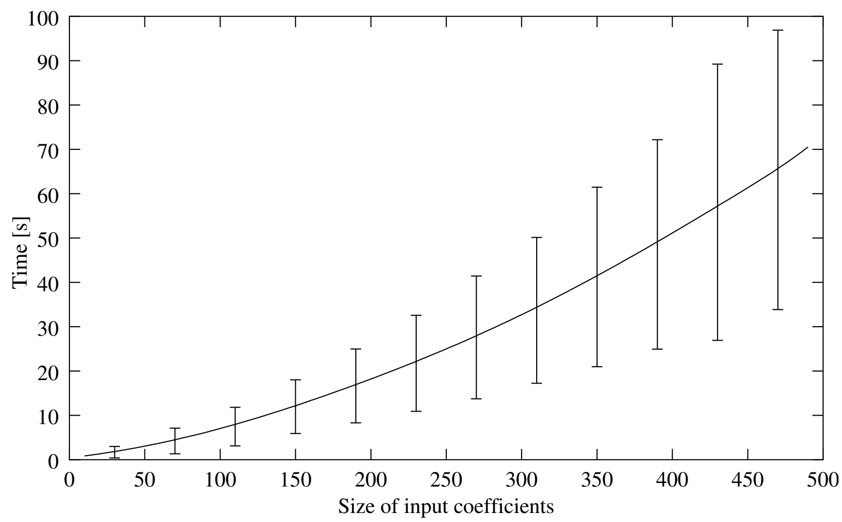

Figure 7: Computation time for 120 pairs of quadrics covering all intersection cases, with standard deviation.

time is roughly 72 seconds, on average, for the 120 pairs of quadrics, i.e., 0.6 second per intersection. This should be compared to the 1.7 seconds on average needed to compute the intersection in the smooth quartic case for the same size of input (Figure 5). This difference is simply explained by the fact that very degenerate intersections (like when the determinantal equation vanishes identically, which represents 36 of the 120 quadrics in the file) are usually much faster to compute.

Our last word will be on memory consumption. Our implementation consumes very little memory. In the smooth quartic case, the total memory chunks allocated sum up to less than 64 kilobytes for input sizes up to 20. It takes input coefficients of more than 700 digits to get to the 1 MB range of used memory.

7.2 Real data

Our intersection code has also been tested on real solid modelling data. Our three test scenes are the teapot, the pencil box, and the chess set (Figure 8). They were modelled with the SGDL modelling kernel [18]. The chess set was rendered with a radiosity algorithm using the virtual mesh paradigm [1]. All computations were made with the strongly simplified variant of our implementation.

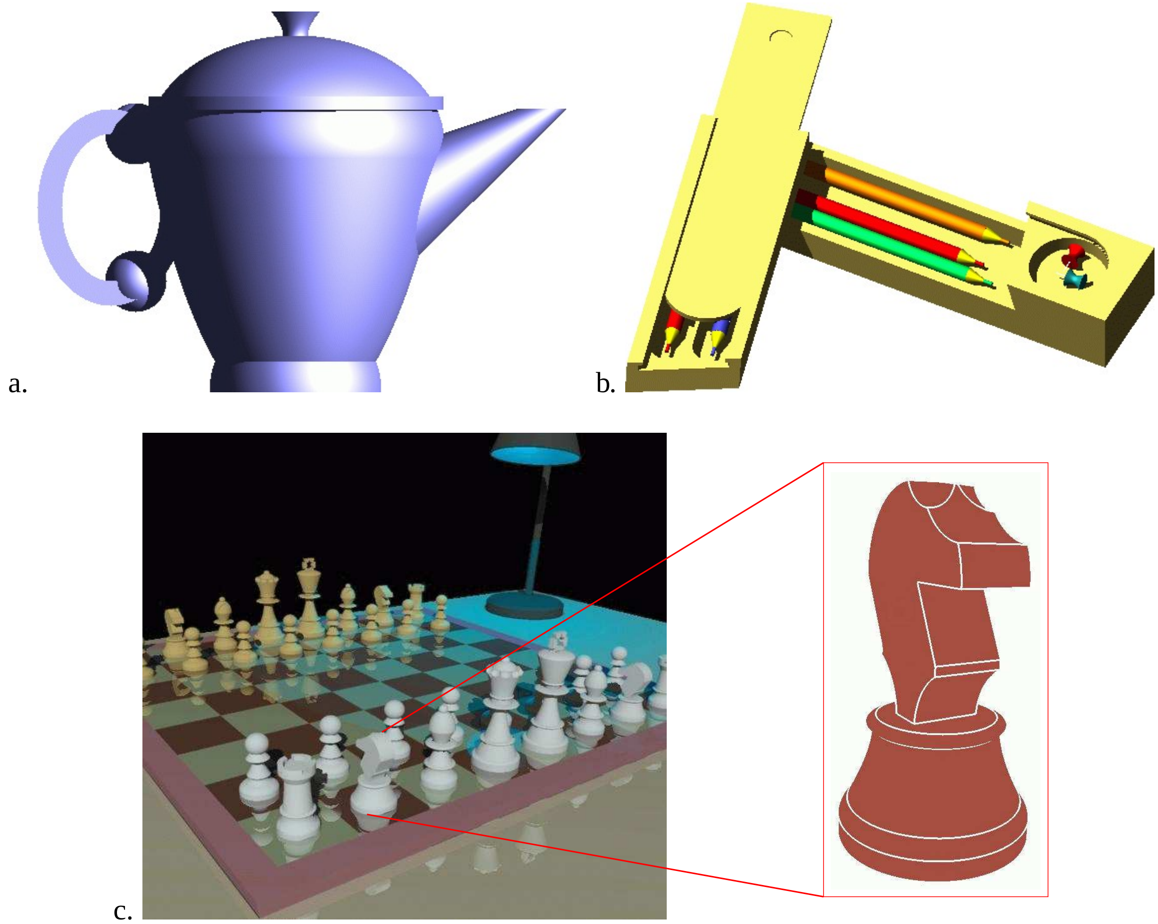

Figure 8: Three CSG models made entirely of quadrics (models courtesy of SGDL Systems, Inc.). a. A teapot. b. A pencil box. c. A chess set, with a close-up on the knight.

The teapot (Figure 8.a) is made of 18 distinct quadrics (one hyperboloid of one sheet, one cone, one circular cylinder, two elliptic cylinders, two ellipsoids, four spheres, and seven pairs of planes). The coefficients of each input quadric have between 2 and 5 digits. The 153 intersections (i.e., pairs of quadrics) are computed in 450 milliseconds, or 2.9 ms on average per intersection. They consist in 51 real smooth quartics, 31 nodal quartics, 35 cuspidal quartics, 65 conics, 101 lines, and 9 points. The height of the output never exceeds 6 in terms of the input.

The pencil box (Figure 8.b) is made of 61 quadrics, most of which are pairs of planes. The input size for each quadric is between 2 and 5 for most quadrics, with four quadrics having a size of 18. The 1,830 intersections are computed in 6.25 s, or 3.4 ms per intersection on average. They consist in 65 smooth quartics, 356 nodal quartics, 119 cubics, 612 conics, 2,797 lines, and 139 points. The height of the output reaches 11 for some smooth quartics.

In the chess set (Figure 8.c), the pawn, the bishop, the knight, the rook, the king, and the queen are respectively made of 12, 14, 20, 18, 19, and 25 quadrics. Most of the quadrics have coefficients with between 2 and 7 digits, except for a small number having 15 digits (the crown of the queen has for instance been generated by rotations of $\pi/10$ applied to a sphere). The intersections were computed for each piece separately. They consist in 86 smooth quartics, 123 nodal quadrics, 360 cuspidal quartics, 284 conics, 484 lines, and 13 points. In total, the 971 intersections were computed in 3.33 s, or 3.4 ms per intersection on average. The height of the output never exceeds 8.

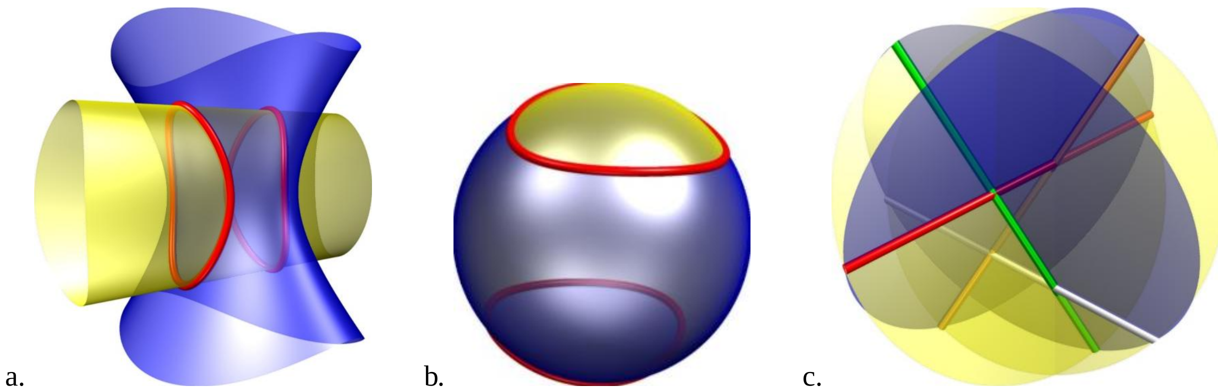

Figure 9: Further examples of intersection. a. b. Smooth quartics. c. Four skew lines.

8 Examples

We now give four examples of parameterizations computed by our algorithm. Other examples can be tested by querying our parameterization server.

Comparing our results with the parameterizations computed with other methods does not make much sense since our implementation is the first to output exact parameterizations in all cases. However, for the sake of illustration, our first two examples are taken from the paper describing the plane cubic curve method of Wang, Joe, and Goldman [23].

8.1 Example 1: smooth quartic

Our first example is Example 4 from [23]. The two quadrics are a quadric of inertia $(2,1)$ (an elliptic cylinder) and a quadric of inertia $(2,2)$ (a hyperboloid of one sheet). The curve of intersection C has implicit equation

$$\begin{cases} 4\,x^2 + z^2 - w^2 = 0, \\ x^2 + 4\,y^2 - z^2 - w^2 = 0. \end{cases}$$

A rendering of the intersection is given in Figure 9.a.

In [23], the authors find the following parameterization for C:

$$\mathbf{X}(u,v) = \mathbf{X}_1(u,v) \pm \mathbf{X}_2(u,v)\sqrt{\Delta(u,v)}, \quad (u,v) \in \mathbb{P}^1(\mathbb{R}), \tag{14}$$

with

$$\mathbf{X}_1(u,v) = \begin{pmatrix} 0.0 \\ 1131.3708\,u^3 - 5760.0\,u^2v + 10861.1602\,uv^2 - 8192.0\,v^3 \\ -1600.0\,u^3 + 10861.1602\,u^2v - 21504.0\,uv^2 + 11585.2375\,v^3 \\ 1600.0\,u^3 + 3620.2867\,u^2v + 5120.0\,uv^2 + 11585.2375\,v^3 \end{pmatrix}, \quad \mathbf{X}_2(u,v) = \begin{pmatrix} -80.0\,u + 1181.0193\,v \\ 0.0 \\ 0.0 \\ 0.0 \end{pmatrix},$$

and $\Delta(u,v) = 905.0967\,u^3v - 3328.0\,u^2v^2 + 2896.3094\,uv^3$. The authors report a computation error on this example (measured as the maximum distance from a sequence of sample points on the curve to the input quadrics) of order $O(10^{-7})$.

Our implementation outputs the following exact and simple result in less than 10 ms:

$$\mathbf{X}(u,v) = \begin{pmatrix} 2\,u^3 - 6\,uv^2 \\ 7\,u^2v + 3\,v^3 \\ 10\,u^2v - 6\,v^3 \\ 2\,u^3 + 18\,uv^2 \end{pmatrix} \pm \begin{pmatrix} -2\,v \\ u \\ 2\,u \\ 2\,v \end{pmatrix} \sqrt{-3\,u^4 + 26\,u^2v^2 - 3\,v^4}, \quad (u,v) \in \mathbb{P}^1(\mathbb{R}).$$

The polynomials involved in the parameterization are defined in $\mathbb{Z}[u,v]$, which means we are in the lucky case where the intermediate quadric of inertia $(2,2)$ found to parameterize the intersection has a square as determinant. So the parameterization obtained is optimal (in the extension of $\mathbb{Z}$ on which its coefficients are defined).

QI output 1 Execution trace for Example 2.

```
>> quadric 1: 19*x^2 + 22*y^2 + 21*z^2 - 20*w^2
>> quadric 2: x^2 + y^2 + z^2 - w^2

>> launching intersection
>> determinantal equation: - 175560*l^4 - 34358*l^3*m - 2519*l^2*m^2 - 82*l*m^3 - m^4
>> gcd of derivatives of determinantal equation: 1
>> number of real roots: 4
>> intervals: ]-14/2^8, -13/2^8[, ]-26/2^9, -25/2^9[, ]-25/2^9, -24/2^9[, ]-3/2^6, -2/2^6[
>> picked test point 1 at [ -13 256 ], sign > 0 -- inertia [ 2 2 ] found
>> picked test point 2 at [ -3 64 ], sign > 0 -- inertia [ 2 2 ] found
>> quadric (2,2) found: - 16*x^2 + 5*y^2 - 2*z^2 + 9*w^2
>> decomposition of its determinant [a,b] (det = a^2*b): [ 12 10 ]
>> a point on the quadric: [ 3 0 0 4 ]
>> param of quadric (2,2): [0, - 24*s*u - 24*t*v, 0, 0] + sqrt(10)*[3*t*u + 6*s*v, 0, 12*s*u
   - 12*t*v, - 4*t*u + 8*s*v]
>> status of smooth quartic param: near-optimal
>> end of intersection

>> complex intersection: smooth quartic
>> real intersection: smooth quartic, two real bounded components
>> parameterization of smooth quartic, branch 1:
 [(72*u^3 + 4*u*v^2)*sqrt(10) + 3*v*sqrt(10)*sqrt(Delta), - 340*u^2*v + 10*v^3 - 24*u*sqrt(Delta),
  (- 118*u^2*v + 5*v^3)*sqrt(10) + 12*u*sqrt(10)*sqrt(Delta), (96*u^3 - 12*u*v^2)*sqrt(10)
  - 4*v*sqrt(10)*sqrt(Delta)]
>> parameterization of smooth quartic, branch 2:
 [(72*u^3 + 4*u*v^2)*sqrt(10) - 3*v*sqrt(10)*sqrt(Delta), - 340*u^2*v + 10*v^3 + 24*u*sqrt(Delta),
  (- 118*u^2*v + 5*v^3)*sqrt(10) - 12*u*sqrt(10)*sqrt(Delta), (96*u^3 - 12*u*v^2)*sqrt(10)
  + 4*v*sqrt(10)*sqrt(Delta)]
>> Delta = 20*u^4 - 140*u^2*v^2 + 5*v^4
>> size of input: 2.3424, height of Delta: 1.3431

>> time spent: < 10 ms
```

8.2 Example 2: smooth quartic

Our second example is Example 5 from [23]. It is the intersection of a sphere and an ellipsoid that are very similar (see Figure 9.b):

$$\begin{cases} 19\,x^2 + 22\,y^2 + 21\,z^2 - 20\,w^2 = 0, \\ x^2 + y^2 + z^2 - w^2 = 0. \end{cases}$$

In [23], the authors compute the parameterization (14) with

$$\mathbf{X}_1(u,v) = \begin{pmatrix} -0.72\,u^3 - 0.72\,u^2v + 0.08\,uv^2 + 0.08\,v^3 \\ 0.0 \\ 0.72\,u^3 - 1.2\,u^2v - 0.72\,uv^2 - 0.08\,v^3 \\ 1.0182\,u^3 + 0.3394\,u^2v + 0.3394\,uv^2 + 0.1131\,v^3 \end{pmatrix}, \quad \mathbf{X}_2(u,v) = \begin{pmatrix} 0.0 \\ 1.697\,u + 0.5656\,v \\ 0.0 \\ 0.0 \end{pmatrix},$$

and $\Delta(u,v) = 0.48\,u^3v - 0.32\,u^2v^2 - 0.16\,uv^3$.

Our implementation gives the result displayed in Output 1. Since the polynomials of $\mathbf{X}(u,v)$ involve a square root $\sqrt{10}$, the quadric Q_R of inertia $(2,2)$ used to parameterize the intersection is such that its determinant is not a square. As explained in Section 3.1, the parameterization is thus only near-optimal in the sense that it is possible, though not necessary, that the square root can be avoided in the coefficients.

It turns out that in this particular example it can be avoided. Consider the cone Q_R corresponding to the rational root $(-1, 21)$ of the determinantal equation:

$$Q_R : -Q_S + 21\,Q_T = 2x^2 - y^2 - w^2.$$

Q_R contains the obvious rational point $(1,1,0,1)$, which is not its singular point. This implies that it can be rationally parameterized. Plugging this parameterization in the equation of Q_S or Q_T gives a simple parameterization

QI output 2 Execution trace for Example 3.

```
>> quadric 1: - 4*x^2 - 56*x*y - 24*x*z - 79*y^2 - 116*y*z + 70*y*w - 85*z^2 - 20*z*w + 9*w^2
>> quadric 2: 6*x^2 + 84*x*y + 36*x*z + 45*y^2 + 160*y*z - 210*y*w + 131*z^2 + 30*z*w - 45*w^2

>> launching intersection
>> determinantal equation: 8*l^4 - 76*l^3*m + 234*l^2*m^2 - 297*l*m^3 + 135*m^4
>> gcd of derivatives of determinantal equation: 4*l^2 - 12*l*m + 9*m^2
>> triple real root: [ -3 -2 ]
>> inertia: [ 1 1 ]
>> rational point on cone: [ 0 0 0 1 ]
>> parameterization of cone with rational point
>> parameterization of pair of planes
>> the two conics are tangent at [ -39 3 6 -5 ]
>> status of intersection param: optimal
>> end of intersection

>> complex intersection: two tangent conics
>> real intersection: two tangent conics
>> parameterization of conic:
 [- 39*u^2 + 443*u*v - 7254*v^2, 3*u^2 - 66*u*v + 1388*v^2, 6*u^2 - 132*u*v + 701*v^2, - 5*u^2
  + 110*u*v - 3005*v^2]
>> cut parameter: (u, v) = [1, 0]
>> size of input: 3.3222, height of output: 1.4631
>> parameterization of conic:
 [- 39*u^2 + 443*u*v - 4004*v^2, 3*u^2 - 66*u*v + 1138*v^2, 6*u^2 - 132*u*v + 201*v^2, - 5*u^2
  + 110*u*v - 1205*v^2]
>> cut parameter: (u, v) = [1, 0]
>> size of input: 3.3222, height of output: 1.3854

>> time spent: 10 ms
```

of the intersection:

$$\mathbf{X}(u,v) = \begin{pmatrix} u^2+2\,v^2 \\ 2\,uv \\ u^2-2\,v^2 \\ 0 \end{pmatrix} \pm \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \sqrt{2\,u^4+4\,u^2v^2+8\,v^4}, \quad (u,v) \in \mathbb{P}^1(\mathbb{R}).$$

8.3 Example 3: two tangent conics

Our next two examples illustrate the fact that our implementation is complete in the sense that it computes parameterizations in all possible cases.

Output 2 shows the execution trace for two quadrics intersecting in two conics that are tangent in one point. As can be seen, our implementation gives information about the incidence between the different components of the intersection: for each component, we give the parameter values (“cut parameters”) at which it intersects the other components of the intersection.

8.4 Example 4: four skew lines

Our final example concerns an intersection made of four skew lines, as depicted in Figure 9.c. Output 3 shows the execution trace for this example, again illustrating the efficiency and completeness of our implementation.

9 Conclusion

We have presented a C++ implementation of an algorithm for parameterizing intersections of quadrics. The implementation is exact, efficient and covers all the possible cases of intersection. This implementation is based on the LiDIA library and uses the multiprecision integer arithmetic of GMP.

QI output 3 Execution trace for Example 4.

```
>> quadric 1: 199*x^2 - 4*x*y + 830*x*z + 1068*x*w - 55*y^2 - 278*y*z - 528*y*w + 587*z^2
   + 1146*z*w + 360*w^2
>> quadric 2: 41*x^2 - 64*x*y + 92*x*z + 108*x*w + 23*y^2 - 32*y*z - 24*y*w + 80*z^2
   + 174*z*w + 72*w^2

>> launching intersection
>> determinantal equation: 49*l^4 - 84*l^3*m + 22*l^2*m^2 + 12*l*m^3 + m^4
>> gcd of derivatives of determinantal equation: 7*l^2 - 6*l*m - m^2
>> ranks of singular quadrics: 2 and 2
>> two real rational double roots: [ -1 -1 ] and [ -1 7 ]
>> status of intersection param: optimal
>> end of intersection

>> complex intersection: four skew lines
>> real intersection: four skew lines
>> parameterization of line:
 [- 42*v, 32*u - 78*v, 28*u, - 25*u + v]
>> cut parameter: (u, v) = [- 19, 8]
>> cut parameter: (u, v) = [- 51, - 22]
>> size of input: 4.0592, height of output: 0.71248
>> parameterization of line:
 [48*v, 64*u + 176*v, 68*u + 76*v, - 47*u - 69*v]
>> cut parameter: (u, v) = [0, 1]
>> cut parameter: (u, v) = [59, - 25]
>> size of input: 4.0592, height of output: 0.79955
>> parameterization of line:
 [6*u, 6*u - 40*v, - 68*v, - 7*u + 111*v]
>> cut parameter: (u, v) = [49, 4]
>> cut parameter: (u, v) = [22, 3]
>> size of input: 4.0592, height of output: 0.75023
>> parameterization of line:
 [- 12*v, 4*u, - 52*u - 60*v, 33*u + 41*v]
>> cut parameter: (u, v) = [67, - 49]
>> cut parameter: (u, v) = [39, - 25]
>> size of input: 4.0592, height of output: 0.68441

>> time spent: 10 ms
```

Future work will be devoted to understanding the gaps between predicted and observed values for the height of the coefficients of the parameterizations, to working out predicates and filters for making the code robust with floating point data (many classes and data structures have already been templated for a future use with floating point coefficients) and to porting our code to the CGAL geometry algorithms library [4].

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