A method of Lagrange–Galerkin of second order in time

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Une méthode de Lagrange–Galerkin d'ordre deux en temps

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Abstract

The Lagrange–Galerkin method is the coupling of a finite element method for space discretisation with the method of characteristics for the discretisation of the material derivative in some parabolic problems. We propose a new scheme of second-order accuracy in time, which in contrast with previous methods does not require a correction term. Numerical examples, including Burgers equation, illustrate the convergence rate and low computational cost of the method. *To cite this article: J. Étienne, C. R. Acad. Sci. Paris, Ser. I 336 (2003).*

Résumé

La méthode de Lagrange–Galerkin consiste à coupler une discrétisation aux éléments finis en espace avec la méthode des caractéristiques pour la discrétisation de la dérivée matérielle dans certains problèmes paraboliques. Nous proposons un nouveau schéma, d'ordre deux en temps, ne faisant pas intervenir de terme de correction comme les méthodes précédentes. Des exemples numériques, dont l'équation de Burgers, illustrent le taux de convergence et l'efficacité en temps de calcul de la méthode. *Pour citer cet article : J. Étienne, C. R. Acad. Sci. Paris, Ser. I 336 (2003).*

1. Introduction

The method of characteristics consists in using the characteritic curves of an advection field $\boldsymbol{a}$ in order to rewrite the transport operator $\left(\frac{\partial}{\partial t} + \boldsymbol{a} \cdot \boldsymbol{\nabla}\right)$ in terms of a partial derivative in time. Indeed, if one introduces the characteristics $\boldsymbol{X}$ defined for any $(\boldsymbol{x}, s) \in \Omega \times [0, T]$ by:

$$\begin{aligned} \frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t}\{\boldsymbol{x}, s; t\} &= \boldsymbol{a}(\boldsymbol{X}\{\boldsymbol{x}, s; t\}, t) \qquad \text{for } t \in [0, T], \\ \boldsymbol{X}\{\boldsymbol{x}, s; s\} &= \boldsymbol{x} \end{aligned} \tag{1}$$

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one notes that by the chain rule,

$$\frac{\partial \boldsymbol{u}(\boldsymbol{X}\{\boldsymbol{x},s;t\},t)}{\partial t} = \left[\left(\frac{\partial}{\partial t} + \boldsymbol{a}\cdot\boldsymbol{\nabla}\right)\boldsymbol{u}\right](\boldsymbol{X}\{\boldsymbol{x},s;t\},t) \tag{2}$$

If the advection field is interpreted as the velocity of a continuous medium, this transform is the passage from a Eulerian frame to the Lagrangian frame.

This method was employed in the context of finite elements by Bercovier and Pironneau [1] as a particular upwinding method, where the left hand side derivative of (2) is discretised by a backward Euler finite difference and the instant s is chosen as the current time step t_n, so that $\boldsymbol{X}\{\cdot,s;t_n\}$ is the identity. For a parabolic problem of the type

$$\begin{cases} \left(\dfrac{\partial}{\partial t} + \boldsymbol{a}\cdot\boldsymbol{\nabla}\right)\boldsymbol{u} + A\boldsymbol{u} = f & \text{in } \Omega, \\ \boldsymbol{u}(\cdot,t_0) = \boldsymbol{u}_0, & \boldsymbol{u}\big|_{\partial\Omega\times[0,T]} = 0, \end{cases}$$

a time-step by the method of characteristics is:

$$\frac{\boldsymbol{u}^n - \boldsymbol{u}^{n-1}\circ\boldsymbol{X}\{\cdot,t_n;t_{n-1}\}}{\Delta t} + A\boldsymbol{u}^n = f(\cdot,t_n), \tag{3}$$

and allows to calculate an approximation $\boldsymbol{u}^n$ of $\boldsymbol{u}(\cdot,t_n)$ if the characteristic mapping $\boldsymbol{X}\{\cdot,t_n;t_{n-1}\}$ and an approximation $\boldsymbol{u}^{n-1}$ of $\boldsymbol{u}(\cdot,t_{n-1})$ are known, with a first order accuracy (see e.g. [2]).

The transport problem in Eq. (3) is hidden in the term $\boldsymbol{u}^{n-1}\circ\boldsymbol{X}\{\cdot,t_n;t_{n-1}\}$. The calculation of this term cannot be considered separately from the discretisation in space employed, and thus we now introduce our finite elements discretisation. First we write a variational problem to solve,

Given $\boldsymbol{u}^{n-1}\in V$, find $\boldsymbol{u}^n\in V$ such that

$$\frac{1}{\Delta t}\left(\boldsymbol{u}^n,\,\boldsymbol{v}\right) + a(\boldsymbol{u}^n,\boldsymbol{v}) = f(\cdot,t_n) + \frac{1}{\Delta t}\left(\boldsymbol{u}^{n-1}\circ\boldsymbol{X}\{\cdot,t_n;t_{n-1}\},\,\boldsymbol{v}\right) \qquad \forall\boldsymbol{v}\in V \tag{4}$$

where V is an appropriate subspace of $H_0^1(\Omega)$, and $a(\boldsymbol{v},\boldsymbol{v}') = (A\boldsymbol{v},\,\boldsymbol{v}')$ for all $\boldsymbol{v},\,\boldsymbol{v}'\in V$. Let $V_h\subset V$ be a finite element space of degree k based on a triangulation $\mathcal{T}_h$, equiped with a projection operator Π_h such that for $f\in V$, $s=0$ or 1, $\|f-\Pi_h f\|_{s,\Omega}\leq Ch^{k-s+1}\|f\|_{k+1,\Omega}$ Then the discrete problem is,

Given $\boldsymbol{u}_h^{n-1}\in V_h$, find $\boldsymbol{u}_h^n\in V_h$ such that

$$\frac{1}{\Delta t}\left(\boldsymbol{u}_h^n,\,\boldsymbol{v}_h\right) + a(\boldsymbol{u}_h^n,\boldsymbol{v}_h) = \Pi_h f(\cdot,t_n) + \frac{1}{\Delta t}\left(\boldsymbol{u}_h^{n-1}\circ\boldsymbol{X}\{\cdot,t_n;t_{n-1}\},\,\boldsymbol{v}_h\right) \qquad \forall\boldsymbol{v}_h\in V_h \tag{5}$$

where Π_h is a projection operator from V onto V_h. The *advected field transfer*, that is, the calculation of the term $\left(\boldsymbol{u}_h^{n-1}\circ\boldsymbol{X}\{\cdot,t_n;t_{n-1}\},\,\boldsymbol{v}_h\right)$ is not standard, because in general $\boldsymbol{u}_h^{n-1}\circ\boldsymbol{X}\{\cdot,t_n;t_{n-1}\}\notin V_h$, and can be done in two main different ways [3], that we will briefly describe. In one approach, $\boldsymbol{u}_h^{n-1}\circ\boldsymbol{X}\{\cdot,t_n;t_{n-1}\}$ is treated as a function analytically known at every point of the domain, and quadrature formulae are used either directly, or after interpolating $\boldsymbol{u}_h^{n-1}\circ\boldsymbol{X}\{\cdot,t_n;t_{n-1}\}$ by a function in V_h. Alternatively, advantage can be taken of the fact that $\boldsymbol{u}_h^{n-1}$ belongs to V_h by calculating a triangulation $\mathcal{T}_\cap = (\mathcal{T}_h\circ\boldsymbol{X}_h\{\cdot,t_{n-1};t_n\})\cap\mathcal{T}_h$, where $\boldsymbol{X}_h\{\cdot,t_{n-1};t_n\}$ is an approximation of $\boldsymbol{X}\{\cdot,t_{n-1};t_n\}$, and the projection $\tilde{\boldsymbol{u}}_h\in V_h$ can be calculated such that:

$$(\tilde{\boldsymbol{u}}_h,\,\boldsymbol{v}_h) = \sum_{K\in\mathcal{T}_\cap}\int_K \boldsymbol{v}_h\boldsymbol{u}_h^{n-1}\circ(\boldsymbol{X}_h\{\cdot,t_{n-1};t_n\})^{-1}\mathrm{d}\boldsymbol{x} \qquad \forall\boldsymbol{v}_h\in V_h. \tag{6}$$

because both $\boldsymbol{v}_h$ and $\boldsymbol{u}_h^{n-1}\circ(\boldsymbol{X}_h\{\cdot,t_{n-1};t_n\})^{-1}$ are polynomial on $K\in\mathcal{T}_\cap$.

The characteristic feet $\boldsymbol{X}\{\cdot,t_n;t_{n-1}\}$ (for the first approach of the advected field transfer) or $\boldsymbol{X}\{\cdot,t_{n-1};t_n\}$ (for the second) are not known exactly in general, and need to be numerically integrated from Problem (1). For a first order scheme, a Euler mehod is enough to obtain the desired accuracy, with:

$$\boldsymbol{X}\{\boldsymbol{x},t_{n-1};t_n\} = \boldsymbol{x} + \Delta t\,\boldsymbol{a}(\boldsymbol{x},t_{n-1}) + O(\Delta t^2), \qquad \boldsymbol{X}\{\boldsymbol{x},t_n;t_{n-1}\} = \boldsymbol{x} - \Delta t\,\boldsymbol{a}(\boldsymbol{x},t_n) + O(\Delta t^2). \tag{7}$$

The space discretisation has to be done in a finite element space of degree larger than k. Thus, if $k \geq 2$, isoparametric elements elements of degree k are needed if one wants to calculate $\mathcal{T}_\cap$. In the case of nonlinear transport ($\boldsymbol{a} = \boldsymbol{u}$), the approximation $\boldsymbol{u}_h^n$ of $\boldsymbol{a}(\cdot,t_n)$ is not known by the time when we need to approximate $\boldsymbol{X}\{\cdot,t_n;t_{n-1}\}$, but one notes that using $\boldsymbol{u}_h^{n-1}$ instead does not affect the order of accuracy in Eq. (7). This will not be the case anymore when considering a second order scheme.

It is then easy to check formally that $\frac{1}{\Delta t}(\boldsymbol{u}_h^n - \tilde{\boldsymbol{u}}_h)$ is a discretisation of $\left(\frac{\partial}{\partial t} + \boldsymbol{a}\cdot\boldsymbol{\nabla}\right)\boldsymbol{u}$ to the first order in time, and it is possible to prove an error estimate in $h^{k+s-1} + \Delta t$ as is done in [2] for the incompressible Navier–Stokes equations. We will now consider the extension of this scheme to second-order time accuracy.

2. Second order schemes

The difficulty of extending the Lagrange–Galerkin scheme to second-order time accuracy was pointed out in [4]. Indeed, a time-scheme such as:

$$\frac{\boldsymbol{u}^n - \boldsymbol{u}^{n-1}\circ\boldsymbol{X}\{\cdot,t_n;t_{n-1}\}}{\Delta t} + \frac{1}{2}(A\boldsymbol{u}^{n-1} + A\boldsymbol{u}^n) = \frac{1}{2}(f(\cdot,t_{n-1}) + f(\cdot,t_n)) \tag{8}$$

is not second order accurate, because we have:

$$\left(\frac{\partial}{\partial t} + \boldsymbol{a}\cdot\boldsymbol{\nabla}\right)\boldsymbol{u}(\boldsymbol{X}\{\cdot,t_n;t_{n-1/2}\},t_{n-1/2}) = \frac{1}{\Delta t}(\boldsymbol{u}^n - \boldsymbol{u}^{n-1}\circ\boldsymbol{X}\{\cdot,t_n;t_{n-1}\}) + O(\Delta t^2),$$

but the point $\boldsymbol{X}\{\cdot,t_n;t_{n-1/2}\}$ is at a distance of order Δt of x, and we have only:

$$\left(\frac{\partial}{\partial t} + \boldsymbol{a}\cdot\boldsymbol{\nabla}\right)\boldsymbol{u}(x,t_{n-1/2}) = \frac{1}{\Delta t}(\boldsymbol{u}^n - \boldsymbol{u}^{n-1}\circ\boldsymbol{X}\{\cdot,t_n;t_{n-1}\}) + O(\Delta t).$$

Thus a correct second order scheme writes,

$$\frac{\boldsymbol{u}^n - \boldsymbol{u}^{n-1}\circ\boldsymbol{X}\{\cdot,t_n;t_{n-1}\}}{\Delta t} + \frac{1}{2}(A\boldsymbol{u}^{n-1}\circ\boldsymbol{X}\{\cdot,t_n;t_{n-1}\} + A\boldsymbol{u}^n) = \frac{1}{2}(f(\boldsymbol{X}\{\cdot,t_n;t_{n-1}\},t_{n-1}) + f(\cdot,t_n)) \tag{9}$$

where $A\boldsymbol{u}^{n-1}\circ\boldsymbol{X}\{\cdot,t_n;t_{n-1}\}$ is understood as $(A\boldsymbol{u}^{n-1})\circ\boldsymbol{X}\{\cdot,t_n;t_{n-1}\}$. The problem is that, if one wants to apply the Green formula to the variational form of Eq. (9), the change of variable $\boldsymbol{X}\{\cdot,t_n;t_{n-1}\}$ produces an additional correction term. If $-A$ is the Laplace operator, this term is $\Delta t\left(J^n\boldsymbol{\nabla}\boldsymbol{u}^{n-1}\circ\boldsymbol{X}\{\cdot,t_n;t_{n-1}\},\,\boldsymbol{v}_h\right)$, where J^n is the Jacobian matrix of $\boldsymbol{a}(\cdot,t_{n-1})$. Although calculations are possible with this discretisation [4], we show in this Note that a two-step time discretisation allows to write a simpler scheme, which has the advantage to be independent of the form of operator A, and can be naturally extended to the nonlinear transport case $\boldsymbol{a} = \boldsymbol{u}$.

Let us consider the two-step scheme,

$$\frac{\boldsymbol{u}^* - \boldsymbol{u}^{n-1}}{\frac{1}{2}\Delta t} = -A\boldsymbol{u}^{n-1} + f(\cdot,t_{n-1}), \tag{10a}$$

$$\frac{\boldsymbol{u}^n - \boldsymbol{u}^*\circ\boldsymbol{X}\{\cdot,t_n;t_{n-1}\}}{\frac{1}{2}\Delta t} + A\boldsymbol{u}^n = f(\cdot,t_n). \tag{10b}$$

No $A\boldsymbol{u}^{n-1}\circ\boldsymbol{X}\{\cdot,t_n;t_{n-1}\}$ appears in Eqs. (10a) and (10b), so the Green formula can be applied as usual to their variational form, without the Jacobian of the transform $\boldsymbol{X}\{\cdot,t_n;t_{n-1}\}$ appearing. On the other hand, if we compose the first equation by $\boldsymbol{X}\{\cdot,t_n;t_{n-1}\}$ and sum the two, we obtain exactly Eq. (9), which implies that $\boldsymbol{u}^n$ constructed in this way is a second-order accurate approximation of $\boldsymbol{u}(\cdot,t_n)$.

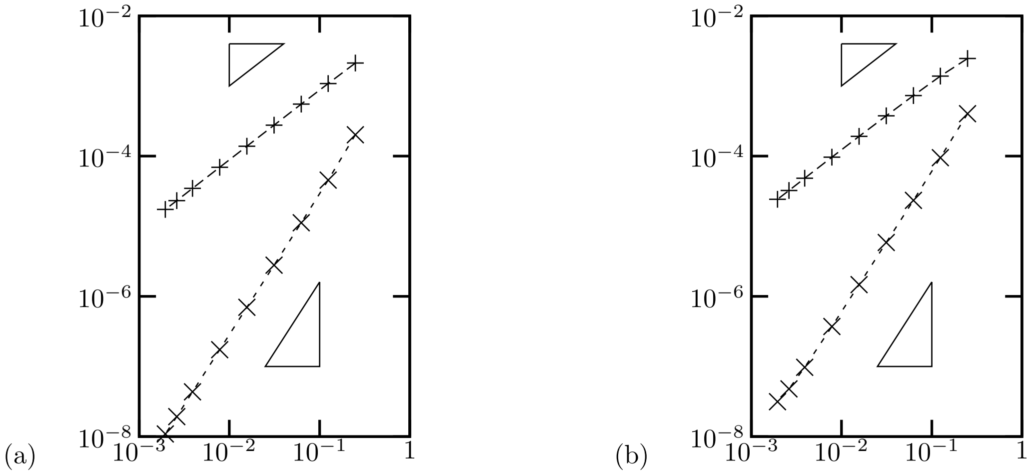

Figure 1. Error $\max_n \left\| \boldsymbol{u}_h^n - \boldsymbol{u}(\cdot, t_n) \right\|_0$ versus Δt for (a), problem (11) and (b), problem (12). $- + -$, algorithm (3), and -×-, algorithm (10). The triangles have a ratio of 1 and 2, respectively.

Note that the first step of this scheme is only the explicit calculation of some auxiliary variable $\boldsymbol{u}^*$, and is computationally very cheap. This auxiliary variable can be interpreted as $\boldsymbol{u}^* = \boldsymbol{u}(\cdot, t_{n-1}) + \frac{\Delta t}{2} \left(\frac{\partial}{\partial t} + \boldsymbol{a} \cdot \boldsymbol{\nabla} \right) \boldsymbol{u}(\cdot, t_{n-1}) + O(\Delta t)$. This is interesting, because an approximation of appropriate order of the mapping $\boldsymbol{X}\{\cdot, t_{n-1}; t_n\}$ writes:

$$\boldsymbol{X}\{\boldsymbol{x}, t_{n-1}; t_n\} = \boldsymbol{x} + \Delta t\, \boldsymbol{a}(\boldsymbol{x}, t_{n-1}) + \frac{\Delta t^2}{2} \left(\frac{\partial}{\partial t} + \boldsymbol{a} \cdot \boldsymbol{\nabla} \right) \boldsymbol{u}(\cdot, t_{n-1}) + O(\Delta t^3),$$

and thus in the nonlinear transport case $\boldsymbol{a} = \boldsymbol{u}$, we can calculate the charateristic mapping very easily [1] with:

$$\boldsymbol{X}\{\boldsymbol{x}, t_{n-1}; t_n\} = \boldsymbol{x} + \Delta t\, \boldsymbol{u}^* + O(\Delta t^3).$$

3. Numerical examples

We present numerical examples for one-dimensional test problems and compare the result to their analytical solution. The finite element space V_h chosen is piecewise linear ($k = 1$) on a 1 000-elements regular mesh and the advected field transfer is done by projection. The first problem is linear, with the right-hand-side f chosen such that the exact solution is $u(x, t) = x(1 - x)(1 + \cos t) \notin V_h$:

$$\frac{\partial u}{\partial t}(x, t) + x(1 - x) \sin(t) \frac{\partial u}{\partial x}(x, t) + \frac{\partial^2 u}{\partial x^2}(x, t) = f(x, t) \qquad (x, t) \in (0, 1) \times [0, 1] \tag{11}$$

$$u(x, 0) = x(1 - x), \qquad\qquad u(0, t) = u(1, t) = 0$$

The convergence of algorithms (3) and (10) is shown in Fig. 1(a).

[1] This is the case only for the advected field transfer by projection, for the quadrature method, it is necessary to obtain a second-order approximation $\boldsymbol{X}_h^1\{\boldsymbol{x}, t_n; t_{n-1}\}$ of $\boldsymbol{X}\{\boldsymbol{x}, t_n; t_{n-1}\}$, for instance with Eq. (7), and then to calculate $\boldsymbol{X}\{\boldsymbol{x}, t_n; t_{n-1}\} = \boldsymbol{x} + \Delta t\, \boldsymbol{u}^* \circ \boldsymbol{X}_h^1\{\boldsymbol{x}, t_n; t_{n-1}\} + O(\Delta t^3)$.

The second problem is known as Burger's viscous equation, which involves a nonlinear transport term, and this time the right-hand-side f is such that the exact solution is $u(x,t) = x(1-x)t$:

$$\frac{\partial u}{\partial t}(x,t) + u(x,t)\frac{\partial u}{\partial x}(x,t) + \frac{\partial^2 u}{\partial x^2}(x,t) = f(x,t) \qquad (x,t) \in (0,1) \times [0,1] \tag{12}$$

with the same initial and boundary conditions. The convergence of algorithms (3) and (10) is shown in Fig. 1(b). For the same timestep, the CPU time is found to be only around 30% more with the second order accurate method. This overcost should be even lower in dimensions higher than one, as the resolution of the linear system in the implicit step increases faster than the one of other operations.

4. Perspectives

This novel method of second-order in time for Lagrange–Galerkin discretisation of parabolic problems does not require to introduce a correction term but only the explicit calculation of an auxilliary variable. The case of nonlinear transport is treated, and a method making use of the same auxilliary variable is introduced to approximate the characteristic curves in that case. Numerical results confirm the order of convergence of the method. This method can be directly applied to much more complicated problems, such as the Navier–Stokes equations. It can also be used for free-boundary flow problems, since the application of this technique in the context of the Arbitrary Lagrangian–Eulerian method (ALE) is staightforward. A detailed analysis of the second-order ALE method will be presented in a forthcoming paper [5].

References

[1] M. Bercovier and O. Pironneau. Characteristics and the finite elements method. In Tadahiko Kawai, editor, *Finite element flow analysis, proceedings of the 4th Int. Symp. on Finite Elements Methods in Flow Problems, Tokyo*, pages 67–73. North-Holland, 26–29 July 1982.

[2] E. Süli. Convergence and nonlinear stability of the Lagrange-Galerkin method for the Navier-Stokes equation. *Numer. Math.*, 53:459–483, 1988.

[3] A. Priestley. Exact projections and the Lagrange-Galerkin method: A realistic alternative to quadrature. *J. Comput. Phys.*, 112:316–333, 1993.

[4] H. Rui and M. Tabata. A second order characteristic finite element scheme for convection-diffusion problems. *Numer. Math.*, 92:161–177, 2001.

[5] J. Étienne. The second order Lagrange–Galerkin method for free surface flows. in preparation.