Analysis of networked control system with packet drops governed by (m,k)-firm constraint

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ANALYSIS OF NETWORKED CONTROL SYSTEM WITH PACKET DROPS GOVERNED BY (*m,k*)-FIRM CONSTRAINT

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Abstract: In this paper we study the effect of the packet drop process governed by (*m,k*)-firm constraint in the feedback loop of a control system. Our approach is to first analyse the stability and the optimality of such a system, then by considering the amplitude variance of the system state as the system performance, we propose a method to optimally distribute the packets to offer a better system performance.

Keywords: Networked control system, real-time, (*m,k*)-firm, packet loss, stability, optimization

1. INTRODUCTION

Much attention has been paid recently towards the networked control systems (NCS). In such systems, the feedback loops are implemented over a network link. The measurements quantized by the sensors are sent to the controller over the network link. After the computation of the controller based on these measurements, the control output is then sent to the actuators via the network. As the network link is often shared among different applications and control loops, the network overload may occur and must be managed. During network overload, some proposed solutions discard data packets in order to reduce the effective utilization of the system. Among these solutions, the technique often mentioned is (*m,k*)-*firm* policy (Ramanathan99) (Quan00). The (*m,k*)-*firm* guarantee model requires that at least *m* out of any *k* consecutive packets must be delivered by the network. An interesting question is: How to choose the rate of *m* over *k* so that the system controlled remains stable and how to distribute the packet drops in the network so that the performance of system can be improved. This paper provides the answers to these questions.

There has, however, been relatively little work studying the impact of packet drop on the control system performance. Most of the previous work has only focused on the impact of packet drops on overall system stability by assuming certain statistical drop models. In (Walsh02), a protocol to schedule the use of shared resources is proposed and the stability of such a networked control system for a continuous plant and a continuous controller is studied. In (Zhang01), the stability of the control system that is modelled as asynchronous switching system is studied under the network induced delay and dropped packets. An approach for data loss compensation in the network link was proposed in (Nilsson98), where the dropouts are governed by a Markov chain and are treated as vacant sampling. The author then presented an analysis of system stability. In (Seiler01), author investigated the

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stability of a linear networked control system in which the packet drop process is i.i.d (independently and identically distributed). Analysis of the performance with data losses replaced by zeros has been made in (Hadjicostis02). In (Ling02) and (Ling03), performance of the networked control system is characterized by the output signal power. By assuming a certain statistical drop model, the performance of system that is defined as a function of packet loss rate is investigated. In (Liu03), to provide a better system performance that is measured by the output signal power, the author introduced a Markov chain constraint (MC constraint) to represent the optimal dropout process and designed scheduling approaches with respect to this MC constraint. Unfortunately, none of the above solutions answers to our question.

In this work, the system setting is much like the setting in (Hadjicostis02), but the packet drop process is quite different. We are specifically interested in control systems communicating over network link which selectively drop packets according to the (m,k)-*firm* constraint. This selective packet dropping mechanism could be implemented at the communication nodes (switch or router) when a packet switching network is used (e.g. switched industrial Ethernet or IP network). The variance of the system state is used as a criteria in the first part of this paper to investigate the stability and optimality of control system, then we will introduce a metric that relates the system performance to the packet drop sequence, and propose an approach that uses the method of Lagrange multiplier to improve the system performance by rationally organizing the drop distribution.

The rest of this paper is organized as follows. In section 2 we describe and formalize the system under study. Section 3 states the necessary and sufficient condition of the stability and gives the interval of the gain of controller and the parameter of network (represented by (m,k)-firm constraint) in order that the variance of the system state remains finite in long run. Section 4 presents a method for minimizing the variance of the system state. In section 5, we will consider the amplitude variance of the system state as the system performance and give a method based on Lagrange multiplier to improve the system performance. In section 6, we will give some numerical examples. We conclude our work and show the perspective in section 7.

2. SYSTEM MODEL

In this paper, we will consider one-dimensional discrete linear time-invariant system with a communication network in the feedback loop (see Figure 1). The system we consider are of the form:

$$x[t+1] = \alpha x[t] + \beta v[t]$$

where α satisfies $|\alpha| > 1$ and β is a nonzero constant.

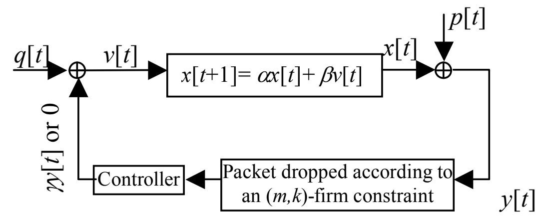

Figure 1: Feedback loop with a network

The measurement of the system state $x[t]$ is quantized as $y[t] = x[t] + p[t]$, where $p[t]$ is the quantization noise. We will assume that this quantized measurement $y[t]$ is sent in a data packet through the network, and the controller has access to the communicated data packet. Specifically, in our work, the data packets are dropped selectively in the communication network during the overload period, and the packet drop process is governed a (m,k)-*firm* constraint, i.e., at least m out of any k consecutive packets sent from the plant must be received by the controller. For analyzing the system, we placed our selves in the worst-case by assuming a long network overload period and the network drops systematically k-m packets every k. So the packet delivery sequence is periodic, i.e., if the n^{th} packet is dropped, the $(n+k)^{\text{th}}$ packet will be also dropped; otherwise, if the n^{th} packet is received by the controller, the $(n+k)^{\text{th}}$ packet will be also received. If the packet containing $y[t]$ is available to the controller, $\gamma y[t]$ for some gain γ will be determined and be sent to the actuator. Otherwise, the controller outputs zero. We assume that there is no communication link from the controller to the actuator. Therefore, the command sent from the controller is received directly by the actuator with a driving noise $q[t]$, i.e., $v[t] = \gamma y[t] + q[t]$.

The system then can be evaluated as following:
If a packet is received by the controller, the system state evolution can be expressed as:

$$\begin{aligned} x[t+1] &= \alpha x[t] + \beta[q[t] + \gamma(x[t] + p[t])] \\ &= (\alpha + \beta\gamma)x[t] + \beta(q[t] + \gamma p[t]) \qquad (1) \end{aligned}$$

When a packet is dropped in the network link, the controller's output is zero, the state evolution becomes:

$$x[t+1] = \alpha x[t] + \beta q[t] \qquad (2)$$

For simplicity, the noises $p[t]$ and $q[t]$ are assumed to be the white noise with zero mean, and have the variance σ_P^2 and σ_Q^2 respectively.

3. CONDITION OF STABILITY

In this section, we will show that the system state $x[t]$ is a zero-mean random variable, and we will discuss the conditions under which the variance of the system state remains finite in long run. We suppose that $p[t]$ and $q[t]$ are uncorrelated (i.e, $E[p[t]q[t+\tau]] = 0$ for all t, τ), so the theorem 1 can be given to describe the property of the system state.

Theorem 1. *Under the assumption that p[t] and q[t] are uncorrelated white noise processes with zero mean, and the system described in the previous section with x[0] = 0, the state variable x[T] is a zero-mean random variable with variance:*

$$\sigma_{x[T]}^2 = \beta^2 \Bigg(\sigma_Q^2 \left(1 + A[T]\right) + \gamma^2 \sigma_P^2 \left(1_{\left[\text{if the } T^{\text{th}} \text{ packet is received}\right]} + A'[T] \right) + \Big(\sigma_Q^2 B[T] + \gamma^2 \sigma_P^2 B'[T] \Big) \frac{1 - \left((a^m b^{k-m})^2\right)^{\left\lfloor \frac{T}{k} \right\rfloor}}{1 - (a^m b^{k-m})^2} - \left(\sigma_Q^2 C[T] + \gamma^2 \sigma_P^2 C'[T] \right) (a^m b^{k-m})^{2\left\lfloor \frac{T}{k} \right\rfloor} \Bigg) \tag{3}$$

where the integer variables m and k represent (m,k)-firm constraint, $a = \alpha+\gamma\beta$, $b = \alpha$ and

$$A[T] = \sum_{i=1}^{T-k\left\lfloor \frac{T}{k} \right\rfloor} \prod_{j=i}^{T-k\left\lfloor \frac{T}{k} \right\rfloor} K_j \qquad A'[T] = \sum_{i=1}^{T-k\left\lfloor \frac{T}{k} \right\rfloor} \left(\left(\prod_{j=i}^{T-k\left\lfloor \frac{T}{k} \right\rfloor} K_j \right) 0_{\left[\substack{\text{if the } (i-1)^{th} \text{ packet} \\ \text{in a period is dropped}}\right]} \right)$$

$$B[T] = \sum_{i=1}^{k} \left(\prod_{j=i}^{k} L_j \prod_{l=1}^{T-k\left\lfloor \frac{T}{k} \right\rfloor} K_l \right) \qquad B'[T] = \sum_{i=1}^{k} \left(\left(\prod_{j=i}^{k} L_j \prod_{l=1}^{T-k\left\lfloor \frac{T}{k} \right\rfloor} K_l \right) 0_{\left[\substack{\text{if the } (i-1)^{th} \text{ packet} \\ \text{in a period is dropped}}\right]} \right)$$

$$C[T] = \prod_{i=1}^{T-k\left\lfloor \frac{T}{k} \right\rfloor} K_i \qquad C'[T] = \prod_{i=1}^{T-k\left\lfloor \frac{T}{k} \right\rfloor} K_i 0_{\left[\substack{\text{if the last packet} \\ \text{in a period is dropped}}\right]}$$

with $K_i = (\alpha+\gamma\beta)^2$, if the $\left(k\left\lfloor \frac{T}{k} \right\rfloor + i\right)^{th}$ packet is received by the controller, if not, $K_i = \alpha^2$; $L_i = (\alpha+\gamma\beta)^2$, if the i^{th} packet is received by the controller, if not, $L_i = \alpha^2$; $B[T]$ and $B'[T]$ are equal to 0 while $\left\lfloor \frac{T}{k} \right\rfloor < 1$.

Proof: The proof is inspired from that in (Hadjicostis02) for the calculation of variance of the system state.

Since the initial state $x[0]$ is equal to 0, from (1) and (2), the state $x[T]$ of the system at time step T is given by

$$x[T] = \beta \left(\sum_{i=0}^{T-1} C_{i+1} n[i] + \gamma \sum_{i=0}^{T-1} C_{i+1} I_i m[i] \right)$$

where $C_i = A_{T-1} A_{T-2} \ldots A_{i+2} A_{i+1} A_i$ (specifically $C_T = 1$) with A_i and I_i (for $0 \le i \le T-1$) satisfying:

$I_i = 1$ and $A_i = \alpha+\gamma\beta$, if the $(i+1)^{\text{th}}$ packet is received by the controller;

$I_i = 0$ and $A_i = \alpha$, if the $(i+1)^{\text{th}}$ packet is lost.

We can find that the expected value of $x[T]$ is zero, because the variables $p[t]$ and $q[t]$ are zero-mean random processes. Since $p[t]$ and $q[t]$ are uncorrelated, the variance of $x[T]$ can be calculated as follow.

$$\begin{aligned}
\sigma_{x[T]}^2 &= E\left[\beta \left(\sum_{i=0}^{T-1} C_{i+1} q[i] + \gamma \sum_{i=0}^{T-1} C_{i+1} I_i p[i] \right) \beta \left(\sum_{j=0}^{T-1} C_{j+1} q[j] + \gamma \sum_{j=0}^{T-1} C_{j+1} I_j p[j] \right) \right] \\
&= E\left[\beta^2 \left(\sum_{i=0}^{T-1} C_{i+1}^2 q^2[i] + \gamma^2 \sum_{i=0}^{T-1} C_{i+1}^2 I_i^2 p^2[i] \right) \right] \\
&= \beta^2 \left(\sigma_Q^2 \sum_{i=0}^{T-1} C_{i+1}^2 + \gamma^2 \sigma_P^2 \sum_{i=0}^{T-1} C_{i+1}^2 I_i^2 \right)
\end{aligned} \tag{4}$$

In our system, exactly m out of k consecutive packets are received by the controller, therefore, we get

$$\begin{aligned}
\sum_{i=0}^{T-1} C_{i+1}^2 &= 1 + A[T] + B[T] \sum_{i=0}^{\left\lfloor \frac{T}{k} \right\rfloor - 1} (a^m b^{k-m})^{2i} - C[T](a^m b^{k-m})^{2\left\lfloor \frac{T}{k} \right\rfloor} \\
&= 1 + A[T] + B[T] \frac{1 - \left((a^m b^{k-m})^2\right)^{\left\lfloor \frac{T}{k} \right\rfloor}}{1 - (a^m b^{k-m})^2} \\
&\quad - C[T](a^m b^{k-m})^{2\left\lfloor \frac{T}{k} \right\rfloor}
\end{aligned} \tag{5}$$

Similarly,

$$\begin{aligned}
\sum_{i=0}^{T-1} C_{i+1}^2 I_i^2 &= 1_{\left[\text{if the T}^{\text{th}} \text{ packet is received}\right]} + A'[T] + B'[T] \sum_{i=0}^{\left\lfloor \frac{T}{k} \right\rfloor - 1} (a^m b^{k-m})^{2i} \\
&\quad - C'[T](a^m b^{k-m})^{2\left\lfloor \frac{T}{k} \right\rfloor} \\
&= 1_{\left[\text{if the T}^{\text{th}} \text{ packet is received}\right]} + A'[T] \\
&\quad + B'[T] \frac{1 - \left((a^m b^{k-m})^2\right)^{\left\lfloor \frac{T}{k} \right\rfloor}}{1 - (a^m b^{k-m})^2} - C'[T](a^m b^{k-m})^{2\left\lfloor \frac{T}{k} \right\rfloor}
\end{aligned} \tag{6}$$

Finally, if we plug (5) and (6) into (4), we get

$$\begin{aligned}
\sigma_{x[T]}^2 &= \beta^2 \Bigg(\sigma_Q^2 \left(1 + A[T] + B[T] \frac{1 - \left((a^m b^{k-m})^2\right)^{\left\lfloor \frac{T}{k} \right\rfloor}}{1 - (a^m b^{k-m})^2} - C[T](a^m b^{k-m})^{2\left\lfloor \frac{T}{k} \right\rfloor} \right) \\
&\quad + \gamma^2 \sigma_P^2 \left(1_{\left[\text{if the T}^{\text{th}} \text{ packet is received}\right]} + A'[T] + B'[T] \frac{1 - \left((a^m b^{k-m})^2\right)^{\left\lfloor \frac{T}{k} \right\rfloor}}{1 - (a^m b^{k-m})^2} \right. \\
&\quad \left. - C'[T](a^m b^{k-m})^{2\left\lfloor \frac{T}{k} \right\rfloor} \right) \Bigg) \\
&= \beta^2 \Bigg(\sigma_Q^2 \left(1 + A[T]\right) + \gamma^2 \sigma_P^2 \left(1_{\left[\text{if the T}^{\text{th}} \text{ packet is received}\right]} + A'[T] \right) \\
&\quad + \left(\sigma_Q^2 B[T] + \gamma^2 \sigma_P^2 B'[T] \right) \frac{1 - \left((a^m b^{k-m})^2\right)^{\left\lfloor \frac{T}{k} \right\rfloor}}{1 - (a^m b^{k-m})^2} \\
&\quad - \left(\sigma_Q^2 C[T] + \gamma^2 \sigma_P^2 C'[T] \right) \left(a^m b^{k-m}\right)^{2\left\lfloor \frac{T}{k} \right\rfloor} \Bigg)
\end{aligned}$$

□

The variables $A[t]$, $B[t]$, $A'[t]$ and $B'[t]$ take their values periodically in a finite set of length k in long run, thus γ can be chosen so that the variance remains finite as T goes to infinity, i.e., $\left((\alpha+\gamma\beta)^m \alpha^{k-m}\right)^2 < 1$.

We have therefore the following lemma:

Lemma 1. *Given a system described in the previous section with x[0] = 0, under the same assumptions in the theorem 1, the gain γ can be chosen so that the variance $\sigma^2_{x[T]}$ remains finite as T goes to infinity if and only if*

$$\left((\alpha+\gamma\beta)^m \alpha^{k-m}\right)^2 < 1 \qquad (7)$$

Note that the above condition for system stability (in the mean square sense) is necessary and sufficient for our system. By equation (1), we can find that the system stability depends on not only the choice of the gain γ but also the (m,k)-firm constraint. I.e., the variance of the system state can be bounded in long run by adjusting the gain γ for a fixed (m,k)-firm constraint, or by adjusting the rate of m over k for a fixed gain γ. Furthermore, note that the distribution of packet drops in a packet delivery sequence has no influence upon the system stability.

Given an (m,k)-firm constraint, the gain γ should be chosen by considering the parity of m, the sign of β and α^{k-m} for (7) being justified. If we suppose that β is positive, by a bit of algebraic manipulation, the following intervals of γ can be obtained for each corresponding condition.

$$\frac{\left(-\sqrt[m]{\frac{1}{\alpha^{k-m}}}-\alpha\right)}{\beta} < \gamma < \frac{\left(\sqrt[m]{\frac{1}{\alpha^{k-m}}}-\alpha\right)}{\beta}$$

if $\alpha^{k-m} > 0$ and m is a even number

(8.1)

$$\frac{\left(\sqrt[m]{-\frac{1}{\alpha^{k-m}}}-\alpha\right)}{\beta} < \gamma < \frac{\left(\sqrt[m]{\frac{1}{\alpha^{k-m}}}-\alpha\right)}{\beta}$$

if $\alpha^{k-m} > 0$ and m is a odd number

(8.2)

$$\frac{\left(-\sqrt[m]{-\frac{1}{\alpha^{k-m}}}-\alpha\right)}{\beta} < \gamma < \frac{\left(\sqrt[m]{-\frac{1}{\alpha^{k-m}}}-\alpha\right)}{\beta}$$

if $\alpha^{k-m} < 0$ and m is a even number

(8.3)

$$\frac{\left(\sqrt[m]{\frac{1}{\alpha^{k-m}}}-\alpha\right)}{\beta} < \gamma < \frac{\left(\sqrt[m]{-\frac{1}{\alpha^{k-m}}}-\alpha\right)}{\beta}$$

if $\alpha^{k-m} < 0$ and m is a odd number

(8.4)

If β is negative number, it is enough to replace the sign 'less than' by the sign 'greater than' for each above obtained interval.

If the gain γ and the parameter k of the (m,k)-firm constraint are given, we can derive the following intervals from (7) for the choice of m that justify the inequality (7): if $(\alpha+\gamma\beta)\alpha^{-1} > 0$ and k is a even number; or if $(\alpha+\gamma\beta)\alpha^{-1} < 0$, m is a even number and k is a even number; or if $(\alpha+\gamma\beta)\alpha^{-1} > 0$, $\alpha > 0$ and k is a odd number; or if $(\alpha+\gamma\beta)\alpha^{-1} < 0$, m is a even number, $\alpha > 0$ and k is a odd number; or if $(\alpha+\gamma\beta)\alpha^{-1} < 0$, m is a odd number, $\alpha < 0$ and k is a odd number, then

$$m > \log_{(\alpha+\gamma\beta)\alpha^{-1}} 1/\alpha^k$$

else

$$m > \log_{(\alpha+\gamma\beta)\alpha^{-1}} \left(-1/\alpha^k\right)$$

Obviously, the adjustable range of the rate of m over k for guaranteeing the system stability is not as great as that of the gain γ because the variable m must be integer and be located between 1 and k.

4. OPTIMAL CHOICE OF THE GAIN CONSTANT

By equation (3), we can find that the more the quantity of m is, the more the quantities of $A[t]$, $B[t]$, $A'[t]$, $B'[t]$ and $\left((\alpha+\gamma\beta)^m \alpha^{k-m}\right)^2$ decrease. Therefore, given a gain γ, the variance of the system state is minimized when m is equal to k. Obviously, the quantity of m must be chosen by considering the parameters of network link. In this section, we will discuss how to choose γ so that the variance of the system state is minimized as T goes to infinity.

If $\left((\alpha+\gamma\beta)^m \alpha^{k-m}\right)^2 < 1$, when T goes to infinity, the expression of the variance of system state is transformed to:

$$\sigma^2_{x[T]} = \beta^2 \Bigg(\sigma_Q^2 \left(1 + A[T]\right) + \gamma^2 \sigma_P^2 \Big(1_{\left[\text{if the T}^{\text{th}} \text{ packet is received}\right]} + A'[T] \Big) + \left(\sigma_Q^2 B[T] + \gamma^2 \sigma_P^2 B'[T] \right) \frac{1}{1-(a^m b^{k-m})^2} \Bigg) \qquad (9)$$

Clearly, when $\gamma = \pm\infty$, the quantity of the term $\left(\sigma_Q^2 B[T] + \gamma^2 \sigma_P^2 B'[T]\right)\frac{1}{1-(a^m b^{k-m})^2}$ in equation (9) is minimized and is equal to zero (since $1-(a^m b^{k-m})^2$ becomes infinite). Within one interval S of γ defined in (8), the quantity of $\left(\sigma_Q^2 B[T] + \gamma^2 \sigma_P^2 B'[T]\right)\frac{1}{1-(a^m b^{k-m})^2}$ is finite and tends to $+\infty$ as we reach the limits of S from within S (since the quantity $\sigma_Q^2 B[T] + \gamma^2 \sigma_P^2 B'[T]$ is always positive and that $1-(a^m b^{k-m})^2$ tends to zero). On the other hand, the term $\gamma^2 \sigma_P^2 \left(1_{\left[\text{if the T}^{\text{th}} \text{ packet is received}\right]} + A'[T]\right)$ in (9) forms a symmetrical concave curve by considering γ as a variable. Therefore, the value of γ that minimizes the quantity of the sum of these two terms $\left(\sigma_Q^2 B[T] + \gamma^2 \sigma_P^2 B'[T]\right)\frac{1}{1-(a^m b^{k-m})^2}$ and $\gamma^2 \sigma_M^2 \left(1_{\left[\text{if the T}^{\text{th}} \text{ packet is received}\right]} + A'[T]\right)$ is not certainly located in a interval defined in (8). I.e., one can not simply take the derivative of the expression in equation (9) with respect to γ to derive the optimal gain. So to find a reasonable value of the gain γ, a feasible method is to examine all the values in a interval defined in (8) by fixing a acceptable search precision and select the best one that minimizes the quantity of equation (9). Obviously, the value of the gain γ is not unique for the different values of $A[t]$, $B[t]$, $A'[t]$ and $B'[t]$. The variables $A[t]$, $B[t]$, $A'[t]$ and $B'[t]$ take their values periodically in a finite set of length k, therefore there are k gains that need to be switched by the controller to generate the control data.

Notice that when σ_P^2 is equal to zero, the value of gain γ that minimizes the quantity of equation (9) is the one that makes $(a^m b^{k-m})^2$ zero, i.e., $\gamma = -\frac{\alpha}{\beta}$. On the other hand, when σ_Q^2 is equal to zero, equation (9) is equal to zero if the value of gain γ is zero.

5. OPTIMAL PACKET DELIVERY SEQUENCE

In section 3, we have discussed how to guarantee that the variance of the system state $x[t]$ remains finite as t goes to infinity, but in many actual situation, the bounded variance of control system is not enough to guarantee a stable operation of system. The high amplitude variation of the system state may be very troubling for it indicates that even though the variance of the system state is bounded, the large signal amplitudes may be unacceptable. Therefore the variation level of control system state is often considered as an important metric of system performance. A way to optimize this system performance is to minimize the variation of the system state as we have discussed in the previous section. Furthermore, in our work, the fact that the packet dropout sequence is governed by the (m,k)-firm constraint makes a further optimization possible for the positions of the packet drops in a packet delivery sequence has a important influence to the amplitude variation of system state. In this section, we will present a metric that directly relates control system performance to the packet drop sequence, then an approach based on this metric to deduce an optimal packet delivery sequence.

In the rest of the paper, we will use the number 0 and 1 to facilitate the presentation of the packet delivery sequence. I.e. 1 represents that the packet is received by the controller, and zero represents a packet drop. E.g., the sequence 10 represents that the first packet is received by the controller and the second is dropped.

To focus on the relationships between the packets, the effects of the noises will be eliminated from the system. Therefore the system is transformed as following.

If a packet is received by the controller, the system state evolution can be expressed as:

$$x[t+1] = \alpha x[t] + \gamma x[t]$$
$$= (\alpha + \gamma) x[t] \qquad (10)$$

When a packet is dropped in the communication link, the state evolution becomes:

$$x[t+1] = \alpha x[t]. \qquad (11)$$

The square value of the difference between two peak values of system state determine a level of amplitude variation of these two peaks: the greater this square value is, the greater the distance between these two peaks is. Therefore the sum of the square values of the difference between any two peak values can be used as a metric to measure the control system performance. Since the packet delivery sequence is periodic (i.e. the same situation will be repeated in each period of the packet delivery sequence), the measurement of performance will be calculated in a single period of the packet delivery sequence. Obviously, by increasing the dimension of this calculation (i.e. the squares of the difference between two peak values in two different period is also taken into account), a more precise measurement of system performance can be obtained. However, such an increase of dimension can also augment the calculation complexity. So in this work, the measurement of system performance is taken in a period of the packet delivery sequence. The control system performance is therefore characterized by the following equation.

$$\sum_{i=1}^{2m-1} \sum_{j=i+1}^{2m} \left(x[t_i] - x[t_j] \right)^2 \qquad (12)$$

where t_i is a peak moment (called peak moment in the following) that the last packet drop in the consecutive packet drops occurs or the moment that a packet is received by the controller.

Example 1. Given a (3,9)-firm constraint, if the system states in a period of packet delivery sequence are $x[1], x[2], .., x[9]$, and a period of packet delivery sequence can be described as 100100010, then the control system performance is represented by $\sum_{i=1}^{5} \sum_{j=i+1}^{6} \left(x[t_i] - x[t_j] \right)^2$ with t_1=1, t_2=3, t_3=4 t_4=7 t_5=8 t_6=9.

In our work, we suppose that the first packet in a period of the delivery sequence is received (the reason will be detailed in the following). So (12) can be transformed to:

$$x[t_1 - 1] \sum_{i=1}^{2m-1} \sum_{j=i}^{2m} \left(a^{\left\lceil \frac{i}{2} \right\rceil} b^{\sum_{l=1}^{\left\lceil \frac{i-1}{2} \right\rceil} P_l} - a^{\left\lceil \frac{j}{2} \right\rceil} b^{\sum_{l=1}^{\left\lceil \frac{j-1}{2} \right\rceil} P_l} \right)^2 \qquad (13)$$

where $a = \alpha+\gamma$, $b = \alpha$, P_j represents the number of the packets dropped between two consecutive packets received by the controller:

$$\sum_{i=1}^{m} P_i = k - m \tag{14}$$

The problem now becomes to determining the value of P_i so that the quantity of expression (13) is minimized. I.e. we will try to find the extreme value of the following expression (15) subject to the constraint (14).

$$\sum_{i=1}^{2m-1}\sum_{j=i}^{2m}\left(a^{\left\lceil \frac{i}{2}\right\rceil} b^{\sum_{l=1}^{\left\lceil \frac{i-1}{2}\right\rceil} P_l} \text{-} a^{\left\lceil \frac{j}{2}\right\rceil} b^{\sum_{l=1}^{\left\lceil \frac{j-1}{2}\right\rceil} P_l} \right)^2 \tag{15}$$

One way to accomplish this objective is to use the method of Lagrange multiplier. We will show an example to illustrate this.

Example 2. Given a system described by (10) and (11), the packet drop process is governed by (2, k)-firm constraint ($k > 2$ and $k \in$Z). To find an optimal distribution of packet drops, we set up firstly the metric describing the system performance. According to the formulas (14) and (15), we get:

$$\begin{aligned} f(p_1,p_2) = {} & 3a^2 - 2a^2b^{p_1} + 3a^2b^{2p_1} - 2a^3b^{p_1} + 3a^4b^{2p_1} \\ & - 2a^3b^{p_1+p_2} + 3a^4b^{2(p_1+p_2)} - 2a^3b^{2p_1} \\ & - 2a^3b^{(p_1+p_2)} - 2a^4b^{(2p_1+p_2)} \end{aligned} \tag{16}$$

$$g(p_1,p_2) = p_1 + p_2 = k-2 \tag{17}$$

Therefore, equation (16) is the formula that we try to minimize subject to (17).

According to the method of Lagrange multipliers, we introduce a variable λ (the Lagrange multiplier) and set up the equations $\frac{\partial f}{\partial p_1} = \lambda \frac{\partial g}{\partial p_1}$ and $\frac{\partial f}{\partial p_2} = \lambda \frac{\partial g}{\partial p_2}$, we get respectively

$$\begin{aligned} & -2a^2b^{p_1}\ln(b) + 6a^2b^{2p_1}\ln(b) - 2a^3b^{p_1}\ln(b) + 6a^4b^{2p_1}\ln(b) \\ & -2a^3b^{(p_1+p_2)}\ln(b) + 6a^4b^{2(p_1+p_2)}\ln(b) - 4a^3b^{2p_1}\ln(b) \\ & -4a^3b^{(2p_1+p_2)}\ln(b) - 4a^4b^{(2p_1+p_2)} = \lambda \end{aligned}$$

and

$$\begin{aligned} & -2a^3b^{(p_1+p_2)}\ln(b) + 6a^4b^{2(p_1+p_2)}\ln(b) - 2a^3b^{(2p_1+p_2)_1}\ln(b) \\ & -2a^4b^{(2p_1+p_2)}\ln(b) = \lambda \end{aligned}$$

By solving these equations with the constraint (17), we get:

$$p_1 = \frac{\ln\left(\dfrac{1+a+ae^{(\ln(b)(k-2))}+a^2e^{(\ln(b)(k-2))}}{3+3a^2-2a}\right)}{\ln(b)} \tag{18}$$

$$p_2 = -\frac{\ln\left(\dfrac{1+a+ae^{(\ln(b)(k-2))}+a^2e^{(\ln(b)(k-2))}}{3+3a^2-2a}\right) - k\ln(b) + 2\ln(b)}{\ln(b)} \tag{19}$$

Obviously, the number of the dropped packets must be a positive integer. From the method of Lagrange multiplier, it's difficult to obtain the integer solution, and sometimes there are some negative numbers in the results. When this happens, we will carry out an approximate calculation as following:

1. replace all the negative numbers by 0, and divide the sum of these negative numbers by the number of the positive numbers in the results;
2. add the result of the division to all the positive numbers;
3. delete all the decimal parts of the positive numbers and add 1 to the number whose decimal part was the largest after step 2.

For example, given $a = 0.03$, $b = 3$ and $k = 9$, from equations (18) and (19), we get $p_1 = 2.866$ and $p_2 = 4.134$. By the method of approximation, we get $p_1 = 3$ and $p_2 = 4$. Therefore, a period of the packet delivery sequence is described as 100010000. If $k = 3$ and the other parameters remain same, we get $p_1 = -0.8773$ and $p_2 = 1.877$. By the method of approximation, we get $p_1 = 0$ and $p_2 = 1$. A period of the packet delivery sequence is thus described as 110.

Before proceeding further, we make several comments about this section. If the number of the packets that must be received by the controller in a period is different from 2, then example 2 gives a procedure for finding the optimal distribution of packet drops. We force the first packet in a period of packet delivery sequence to be delivered, as by doing this, the general level of amplitude variation of system state in a period can be depressed (the reason being that the last peak value in a period is unchangeable). Also note that the effect of the noises is not considered in this section. But if the initial disturbance in a control system described by (1) and (2) is considerable, the packet delivery sequence obtained by our approach can rapidly eliminate the high amplitude bursts caused by this initial disturbance. We will illustrate this in the next section.

6. NUMERICAL EXAMPLES

In this section, we present two examples which exploit the research results obtained in the previous sections. The first example is based on an instance of the system described by (1) and (2). We will firstly calculate the value interval of the gain so that the variance of system state is bounded in long run, then we will give the optimal gain that minimizes the variance of system state. In the next example, we will apply our approach to find an optimal packet drop distribution, then we will compare the performance of this packet delivery sequence with that of the packet delivery sequence obtained by the algorithm in (Ramanathan99) under two different system settings.

Given a system described by (1) and (2) with α =3, β=1, $\sigma_M^2 = 1$, $\sigma_N^2 = 4$, and the packet drop process is governed by (4,7)-firm constraint, we assume that a period of packet delivery sequence is given by 1101010. By inequality (8.1), we find that the value of the gain γ must satisfy $-3.333 < \gamma < -2.667$ so that the variance of the system state remains finite in long run. To calculate a optimal gain γ that minimizes the variance of the system state, we examine all the values in the interval of the gain γ that we have obtained by fixing the search precision as 0.001. Therefore, we get $\gamma_{op}[t_1]$ = -2.974, $\gamma_{op}[t_2]$ = -2.832, $\gamma_{op}[t_3]$ = -2.832, $\gamma_{op}[t_4]$ = -2.962, $\gamma_{op}[t_5]$ = -2.962, $\gamma_{op}[t_6]$ = -2.963, $\gamma_{op}[t_7]$ = -2.963 where t_i represents a moment that the i^{th} packet delivery occurs in a period of the packet delivery sequence. Figure 2 shows the system state trace with the obtained optimal gain constants.

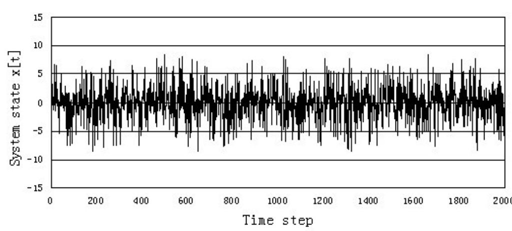

Figure 2. The system state trace under the optimal gain constants

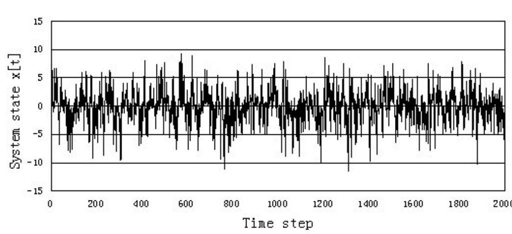

Figure 3. The system state trace under a fixed gain constants

In figure 3, we give the state trace of the system whose gain γ is fixed to –2.8. We can see that the variance is slightly higher than the first one.

In the next experiment, we will take a instance of the system described by (10) and (11) with α =3, γ=-2.97, and the system is under (3,10)-firm constraint (the system is stable in mean square sense, because $(\alpha+\gamma)^3 \alpha^7 < 1$). By the procedure introduced in the section 5, we get the optimal packet delivery sequence that is 1100010000. The initial system state is set to be 5000, the system state traces under the packet delivery sequence obtained by our approach and that obtained by the algorithm in (Ramanathan99) are illustrated in figure 4 and 5.

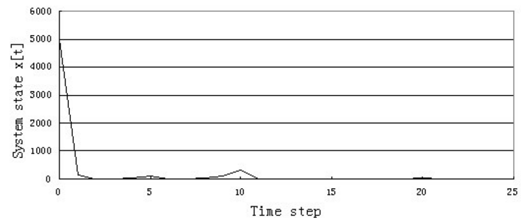

Figure 4. The system state trace with the packet delivery sequence 1100010000

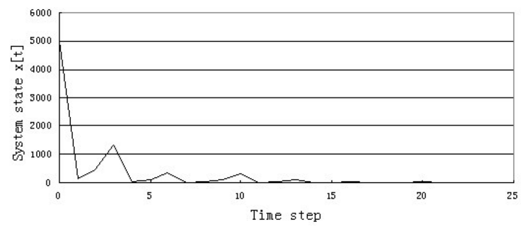

Figure 5. The system state trace with the packet delivery sequence 1001001000

Comparing the figure (4) and (5), we see that the amplitude variance of the system state in figure (5) is much more higher than that in the figure (4), and the system state converges more rapidly to the steady state in figure (4).

Now we introduce a quantization noise with $\sigma_M^2 = 3$ to this control system, the system state traces with these two packet delivery sequence are given in figure 6 and 7.

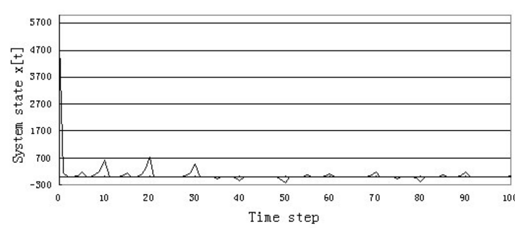

Figure 6. The state trace of the system under the quantization noise with the packet delivery sequence 1100010000

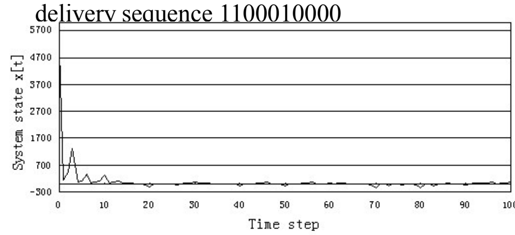

Figure 7. The state trace of the system under the quantization noise with the packet delivery sequence 1001001000

In figure 6, the influence of the initial disturbance can be rapidly eliminated by using the packet delivery sequence obtained by our algorithm, but there still exist occasional high amplitude bursts in long run. In figure 7, the system state slightly changes around zero after a high amplitude burst at the first period.
By the same procedures for the first and second experimentation, we have tested many others independent trials by changing the setup of the system, and all results have confirmed our research results.

7. CONCLUSION AND PERSPECTIVE

This paper studied the networked control system in which packet drop process is governed by the (*m,k*)-firm constraint. We first analyzed the problem of stability of such a system and identified the interval of the gain constant and the network parameter (that is identified by (*m,k*)-firm constraint) over which the variance of the system state is bounded in long run, and a method for minimizing the variance of the system state was proposed. These research results allow to optimally choose the gain of the controller so that the system stability (in mean square sense) can be guaranteed and to specify bounds on the network's QoS (packet drop rate) that improve control system performance. We then considered the amplitude variance of the system state as an important system performance, and associated the system performance with the packet drop strategy. For improving the system performance (i.e. minimizing the amplitude variance of the system state), we proposed a method based on the Lagrange multiplier to optimally lay out the packet drops in the packet delivery sequence.

In our work, the packet drop process is assumed to be governed by (*m,k*)-firm constraint, and the packet delivery sequence is periodic. A future work intends to explore the networked control system possessing a more flexible packet drop process model. This flexible drop process is realized by a judicious choice for the real-time scheduler. One such candidate is a scheduler implementing the Markov chain constraint (Liu03).

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