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CONTROL VIA STATE ESTIMATIONS OF
SOME NONLINEAR SYSTEMS

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Abstract: We are proposing state estimators for nonlinear systems. Our techniques extend a previous work on *state reconstructors* for linear systems by the same authors (*Reconstructeurs d'états*, C.R. Acad. Sci. Paris, Série I, **338**, 2004, 91-96), which bypasses some of the classic difficulties related to asymptotic observers and Kalman filtering (lack of robustness and knowledge of statistics). Our viewpoint, which avoids the integration of differential equations and therefore any asymptotic estimation, yields fast implementable algebraic formulae. Two concrete case-studies are presented, which are (differentially) flat. Our state estimation permits a state feedback control around the flatness-based reference trajectory. Convincing simulations are provided which demonstrate the robustness of our control strategy with respect to noises with unknown statistical properties.

Keywords: Nonlinear observability, flat systems, state estimation.

1. INTRODUCTION

Although the utilisation of *(differentially) flat* nonlinear systems (Fliess *et al.*, 1995a, 1999) (see, also, Rudolph, 2003a, Sira-Ramírez & Agrawal, 2004) has become quite widespread in industry, a fundamental question like the estimation of state variables is still far from being fully understood. This communication is proposing a clear cut solution which is extending the *state reconstructors* for linear systems obtained in Fliess & Sira-Ramírez, 2004. Our approach, which does not necessitate the integration of any differential equations, is based on the estimation of the time derivatives of the (flat) output, up to some finite order. The formulae for those derivatives, which are quite different from those based on asymptotic observers (see, *e.g.*, Ibrir, 2004) and from the classic ones in applied analysis (see, *e.g.*, Lanczos, 1957), yield very fast calculations of the state. Our results are, like in Fliess & Sira-Ramírez, 2003, 2004, quite robust with respect to perturbations [3]. Contrarily to the usual probabilistic setting, we do not need any precise statistical knowledge of the noise. We

[3] Compare with the nonlinear estimation techniques based on numerical differentiation (see, *e.g.*, Diop *et al.*, 1994, 2000, 2001, Plestan *et al.*, 1999, Röbenack, 2003). Although we will not try in such a conference paper to give any complete overview of the huge literature devoted to nonlinear state estimation, let us cite nevertheless the excellent book edited by Nijmeijer *et al.*, 1999, which covers several recent trends in nonlinear asymptotic observers.

can therefore handle noises of a rather arbitrary nature [4].

Our paper is organised as follows. Section 2 is devoted to a brief overview of nonlinear system theory via differential algebra. Section 3 describes some salient features of our estimation procedure. Two illustrative examples are presented [5] in section 4. Directions for future research are indicated in a short conclusion. We also discuss there some paradigms in nonlinear control.

Remark 1.1. The exact integral estimators of Fliess & Sira-Ramírez, 2003, 2004, already permit to treat some particular nonlinear systems. See, *e.g.*, Suarez-Castañon *et al.*, 2003, for an application to Chua's systems, which are uncontrolled chaotic systems. The authors are able there to bypass celebrated techniques from dynamical system theory and control theory, such as Lyapunov's theory, Takens' approach, asymptotic observers, Kalman filtering, or adaptive control, which are difficult to implement.

2. THE DIFFERENTIAL ALGEBRAIC APPROACH

2.1 Differential fields

A *differential field* K (see, *e.g.*, Kolchin, 1973, Buium, 1994) will be here a commutative field [6] of characteristic zero, which is equipped with a single *derivation* $\frac{d}{dt} : K \rightarrow K$ such that, for any $a, b \in K$, $\frac{d}{dt}(a+b) = \dot{a} + \dot{b}$, $\frac{d}{dt}(ab) = \dot{a}b + a\dot{b}$. A *constant* of K is an element $c \in K$ such that $\dot{c} = 0$. A *(differential) field of constants* is a differential field which only contains constants. The set of all constant elements of K is a subfield, which is called the *subfield of constants*.

Notation 2.1. For any $a \in K$, $\nu = 0, 1, 2, \ldots$, set $a^{(\nu)} = \frac{d^{\nu}a}{dt^{n}u}$.

A *differential field extension* L/K is given by two differential fields K, L, such that the derivation of $K \subseteq L$ is the restriction to K of the derivation of L. An element of L is said to be *differentially algebraic* over K if, and only if, it satisfies an algebraic differential equation with coefficients in K. It is said to be *differentially transcendental* if, and only

if, it is not differentially algebraic. The extension L/K is said to be *differentially algebraic* if, and only if, any element of L is differentially algebraic over K. An extension which is not differentially algebraic is said to be *differentially transcendental*. A set $\{\xi_{\iota} \mid \iota \in I\}$ of elements in L is said to be *differentially algebraically independent over K* if, and only if, the set $\{\xi_{\iota}^{(\nu)} \mid \iota \in I, \nu \geq 0\}$ of derivatives of any order is algebraically independent over K. If a set is not differentially algebraically independent over K, it is *differentially algebraically dependent over K*. Such an independent set which is maximal with respect to inclusion is called a *differential transcendence basis*. The cardinalities of two such bases are equal. This cardinality is the *differential transcendence degree* of the extension L/K. Note that this degree is 0 if, and only if, L/K is differentially algebraic.

Notation 2.2. Write $K\langle S \rangle$, where S is a subset of L, the differential subfield of L generated by K and S.

2.2 Nonlinear systems

Let k be a given differential ground field. A *system* [7] is a finitely generated differential extension K/k. A *dynamics* is a system where a finite subset $\mathbf{u} = (u_1, \ldots, u_m) \subset K$ of *control* variables has been distinguished, such that the extension $K/k\langle \mathbf{u} \rangle$ is differentially algebraic. The control variables are said to be *independent* if, and only if, $\mathbf{u}$ is a differential transcendence basis of K/k. An *input-output system* is a dynamics where a finite subset $\mathbf{y} = (y_1, \ldots, y_p) \subset K$ of output variables has been distinguished.

Remark 2.1. The presentation of all these system theoretic notions may also be given in a differential geometric framework by utilising *diffieties*, *i.e.*, prolongations and jets of infinite order (see, *e.g.*, Fliess *et al.*, 1999, and the references therein).

2.3 Observability

A system variable $\chi \in K$ is said to be *observable* (see Diop & Fliess, 1991a, 1991b) if, and only if, it is algebraic over $k\langle \mathbf{u}, \mathbf{y} \rangle$. An input-output system K/k is said to be *observable* if, and only if, the extension $K/k\langle \mathbf{u}, \mathbf{y} \rangle$ is algebraic.

Example 2.1. Set $k = \mathbf{R}$. Take the following state variable representation

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{F}(\mathbf{x}, \mathbf{u}) \\ \mathbf{y} &= \mathbf{h}(\mathbf{x}) \end{aligned} \qquad (1)$$

[4] See Fliess, Mboup *et al.*, 2003, Fliess, Join, Mboup & Sira-Ramírez, 2004, Fliess, Join & Sira-Ramírez, 2004, for applications of similar techniques to signal processing and linear fault diagnosis.
[5] See Sira-Ramírez & Fliess, 2004, for another example, and Fliess, Join & Mounier, 2004, for an extension to nonlinear fault diagnosis.
[6] See, *e.g.*, Atiyah *et al.*, 1969, for basic field-theoretic notions.
[7] See, *e.g.*, Delaleau, 2002, Fliess *et al.*, 1995a, Rudolph, 2003a, Sira-Ramírez & Agrawal, 2004, for more details.

where $\mathbf{x} = (x_1, \ldots, x_n)$ is the n-uple of state variables, $\mathbf{F} = (F_1, \ldots, F_n)$, $\mathbf{h} = (h_1, \ldots, h_p)$ are respectively n-uples and p-uples of polynomial functions of their arguments. Our observability property means that any state variable is algebraic over $k\langle\mathbf{u}, \mathbf{y}\rangle$, $i.\,e.$, is an algebraic function of the components of $\mathbf{u}$, $\mathbf{y}$ and of a finite number of their derivatives. It is known that this definition is equivalent to the classic observability rank condition (see, $e.\,g.$, Isidori, 1989, Nijmeijer & van der Schaft, 1990, Sastry, 1999) for system (1) (see Diop & Fliess 1991a, 1991b, for full details).

2.4 Flat systems

A system K/k is said to be *(differentially) flat* if, and only if, there exists a differential transcendence basis $\mathbf{z} = (z_1, \ldots, z_m)$ such that the algebraic closures of K and $k\langle\mathbf{z}\rangle$ are the same. The set $\mathbf{z}$ is called a *flat* output. It means that

- any system variable is a function of the components of the flat outputs and of their derivatives up to some finite order,
- any component of the flat output is a function of the system variables and of their derivatives up to some finite order.

The next properties are easy:

Proposition 2.1. Take a flat dynamics with independent control variables, then the cardinalities of $\mathbf{z}$ and $\mathbf{u}$ are equal.

Proposition 2.2. Assume that system (1) is flat and that $\mathbf{y}$ is a flat output. System (1) is then observable.

3. ESTIMATING DERIVATIVES

3.1 Polynomial time signals [8]

Consider the polynomial $x_N(t) = \sum_{\nu=0}^{N} x^{(\nu)}(0)\frac{t^\nu}{\nu!}$ $\in \mathbf{R}[t]$ of degree N. The usual rules of symbolic calculus yield in distributions theory (Schwartz, 1966)

$$x_N^{(N+1)}(t) = x_N(0)\delta^{(N)} + \ldots + x_N^{(N)}(0)\delta$$

where δ is the Dirac measure at 0. From $t\delta = 0$, $t\delta^{(\alpha)} = -\alpha\delta^{(\alpha-1)}$, $\alpha \geq 1$, we obtain the following triangular system of linear equations

for determining the values [9] of the derivatives [10] $x_N^{(\nu)}(0)$:

$$t^\alpha x_N^{(N+1)}(t) = -t^\alpha \left(x_N(0)\delta^{(N)} \right.$$
$$+\dot{x}_N(0)\delta^{(N-1)} + \ldots + x_N^{(N)}(0)\delta \Big) \qquad (2)$$
$$\alpha = 0, \ldots, N$$

The time derivatives of $x(t)$, $t \neq 0$, are removed by integrating with respect to time both sides of equation (2) at least N times:

$$\int^{(\nu)} \tau_1^\alpha x_N^{(N+1)}(\tau_1) = -\int^{(\nu)} \tau_1^\alpha \left(x_N(0)\delta^{(N)}+ \right.$$
$$\dot{x}_N(0)\delta^{(N-1)} + \ldots + x_N^{(N)}(0)\delta \Big) \qquad (3)$$
$$\nu \geq N$$

where $\int^{(\nu)} = \int_0^t \int_0^{\tau_{\nu-1}} \cdots \int_0^{\tau_1}$.

3.2 General time signals

Consider a real-valued time function $x(t)$ which is assumed to be analytic around $t = 0$. Introduce its truncated Taylor expansion

$$x(t) = \sum_{\nu=0}^{N} x^{(\nu)}(0)\frac{t^\nu}{\nu!} + O(t^{\nu+1})$$

Approximate $x(t)$ in the interval $(0, \varepsilon)$, $\varepsilon > 0$, by a polynomial $x_N(t) = \sum_{\nu=0}^{N} x^{(\nu)}(0)\frac{t^\nu}{\nu!}$ of degree N. A "good" estimate [11] of the first derivatives is obtained by replacing $x_N(t)$ in the left hand sides of equations (2) and (3) by $x(t)$.

3.3 Some practical rules

Our estimations of the flat output time derivatives and of the states are based on the following practical rules:

(1) We adopt a sufficiently high order time polynomial approximation of the output signals
(2) "High frequency" perturbations are attenuated by utilising adequate low pass filters like elementary iterated time integrals. Contrarily to the usual probabilistic framework, we do not need any precise statistical knowledge of the noises. See Fliess & Sira-Ramírez, 2003, 2004, Fliess, Mboup *et al.*, 2003, for more details.
(3) We are periodically resetting the calculations as the adopted time approximation becomes non-valid.

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[8] See Fliess, Mboup *et al.*, 2003, Fliess, Join, Mboup & Sira-Ramírez, 2004, for a presentation *via* operational calculus, which we believe to be clearer.

[9] The multiplication by $-t$ corresponds in operational calculus to the *algebraic derivative* $\frac{d}{ds}$ (see Fliess & Sira-Ramírez, 2003, and the references therein).
[10] Those quantities are *linearly identifiable* (see Fliess & Sira-Ramírez, 2003, Fliess, Mboup *et al.*, 2003).
[11] See Fliess, Join, Mboup & Sira-Ramírez, 2004, for more details.

4. TWO CONCRETE CASE STUDIES

Remark 4.1. Some of the system equations below are not differentially algebraic since they involve sine or cosine functions of a system variable χ. It is nevertheless possible to recover the above formalism by introducing $\tan\frac{\chi}{2}$, as in Fliess *et al.*, 1995a. Our mathematical formalism is therefore applicable.

4.1 Control of a unicycle [12]

The following kinematic model is flat

$$\dot{x} = u_1 \cos\theta, \quad \dot{y} = u_1 \sin\theta, \quad \dot{\theta} = u_2$$

It represents a unicycle contained on a vertical plane and rolling on a horizontal coordinate plane (x, y) with orientation angle θ with respect to the x-axis. We are measuring the coordinates (x, y) of the flat output, which is the wheel point of contact with the plane. A first order extension of the control input u_1 is required by the trajectory tracking dynamic feedback linearising controller

$$\begin{pmatrix} \dot{u}_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \dfrac{[\dot{x}]_e}{\sqrt{([\dot{x}]_e)^2 + ([\dot{y}]_e)^2}} & \dfrac{[\dot{y}]_e}{\sqrt{([\dot{x}]_e)^2 + ([\dot{y}]_e)^2}} \\ -\dfrac{[\dot{y}]_e}{([\dot{x}]_e)^2 + ([\dot{y}]_e)^2} & \dfrac{[\dot{x}]_e}{([\dot{x}]_e)^2 + ([\dot{y}]_e)^2} \end{pmatrix}$$
$$\times \begin{pmatrix} \ddot{x}^*(t) - 2\xi\omega_n([\dot{x}]_e - \dot{x}^*(t)) - \omega_n^2(x - x^*(t)) \\ \ddot{y}^*(t) - 2\xi\omega_n([\dot{y}]_e - \dot{y}^*(t)) - \omega_n^2(y - y^*(t)) \end{pmatrix}$$

where $x^*(t)$, $y^*(t)$ represent a desired flatness-based trajectory. Following section 3 the velocity estimates $[\dot{x}]_e$, $[\dot{y}]_e$ are given by

$$[\dot{x}(t)]_e = \frac{1}{(t - t_i)^3}\left[6(t - t_i)^2 x(t)\right.$$
$$\left. -18\int_{t_i}^{t}(\sigma - t_i)x(\sigma)d\sigma + 6\int_{t_i}^{t}\int_{t_i}^{\sigma}x(\lambda)d\lambda d\sigma\right]$$

$$[\dot{y}(t)]_e = \frac{1}{(t - t_i)^3}\left[6(t - t_i)^2 y(t)\right.$$
$$\left. -18\int_{t_i}^{t}(\sigma - t_i)y(\sigma)d\sigma + 6\int_{t_i}^{t}\int_{t_i}^{\sigma}y(\lambda)d\lambda d\sigma\right]$$

During the re-initialisation intervals, $[t_i, t_i + \delta]$, we set the estimated derivatives to take *a constant value*, represented by the corresponding derivative value at time t_i^-, *i.e.*, at the end of the preceding estimation interval $[t_{i-1}, t_i]$. The simulations below depict the response of the kinematic model of the car. We have chosen here to track, in a counterclockwise sense, a circular trajectory of radius equal to one, with angular velocity $\omega = 0.15$ [rad/s]. The controller parameters were set to be

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[12] This simplified model of unicycle has often been studied in the literature (see, *e.g.*, Sira-Ramírez & Agrawal, 2004).

$\xi = 0.81, \omega_n = 0.7$ [rad/s]. The small interval of time δ, after each re-initialisation instant, prescribed for allowing the numerical processor to obtain a precise quotient calculation, was taken to be $\delta = 0.05$ [s]. The spikes in the control are largely due to those resettings.

Let $\text{rect}(t) = \text{comp}(t) - 0.5$ be a zero mean random process, where $\text{comp}(t)$ is a computer generated random processes [13] synthesised on the basis of a piecewise constant random variable uniformly distributed in the interval $[0, 1]$. Set $\eta_1 = 0.01\,\text{rect}(t)$, $\eta_2 = 0.01\,\text{rect}(t)$, $\eta_3 = -0.01\,\text{rect}(t)$, and $\nu_1(t) = 0.0001\,\text{rect}(t)$, $\nu_2(t) = 0.0002\,\text{rect}(t)$. The robustness was tested with

$$\dot{z} = u_1\cos\theta + \eta_1(t)$$
$$\dot{q} = u_1\sin\theta + \eta_2(t)$$
$$\dot{\theta} = u_2 + \eta_3(t)$$
$$x = z + \nu_1(t)$$
$$y = q + \nu_2(t)$$

4.2 Control of a dc motor pendulum

Consider, following Fan & Arcak, 2003, a flat system [14] consisting of a DC-motor joined to an inverted pendulum through a torsional spring

$$J_m\ddot{\theta}_m = \kappa(\theta_l - \theta_m) - B\dot{\theta}_m + K_\tau u$$
$$J_l\ddot{\theta}_l = -\kappa(\theta_l - \theta_m) - mgh\sin(\theta_l)$$
$$y = \theta_l$$

θ_m represents the angular deviation of the motor shaft. The flat output $y = \theta_l$, where θ_l is the inverted pendulum angular position, is measured. Asymptotic tracking of a given smooth reference trajectory $y^*(t) = \theta_l^*(t)$ is achieved by the feedback controller

$$u = \frac{1}{K_\tau}\left\{\frac{J_m}{\kappa}\left[J_l v + \kappa\ddot{y}_e\right.\right.$$
$$\left. + mgh\left(\ddot{y}_e\cos y - (\dot{y}_e)^2\sin(y)\right)\right]$$
$$+ J_l\ddot{y}_e + mgh\sin(y)$$
$$\left. + \frac{B}{\kappa}\left[J_l y_e^{(3)} + \kappa\dot{y}_e + mgh\,\dot{y}_e\cos y\right]\right\}$$
$$v = [y^*(t)]^{(4)} - \gamma_4(y_e^{(3)} - [y^*(t)]^{(3)})$$
$$- \gamma_3(\ddot{y}_e - \ddot{y}^*(t)) - \gamma_2(\dot{y}_e - \dot{y}^*(t))$$
$$- \gamma_1(y - y^*(t))$$

where the subscript "e" denotes the estimated value of the derivative. The design parameters $\gamma_1, \cdots, \gamma_4$ are chosen so that the resulting characteristic polynomial is Hurwitz. The availability of $y = \theta_l$ and of its first three time derivatives $\dot{y}$, $\ddot{y}$, $y^{(3)}$ readily allows one to implement the proposed

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[13] The noises here and in the next example are not usual in the traditional mathematical theory of nonlinear filtering.
[14] This single input system is of course static state feedback linearisable.
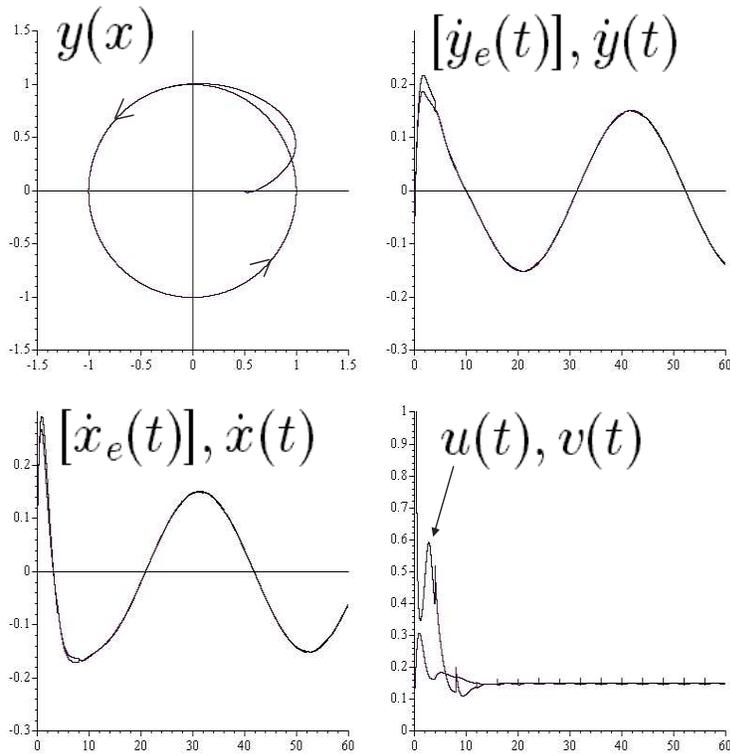
Fig. 1. Closed loop response of the unicycle model to dynamic feedback control based on the algebraic velocities estimations.
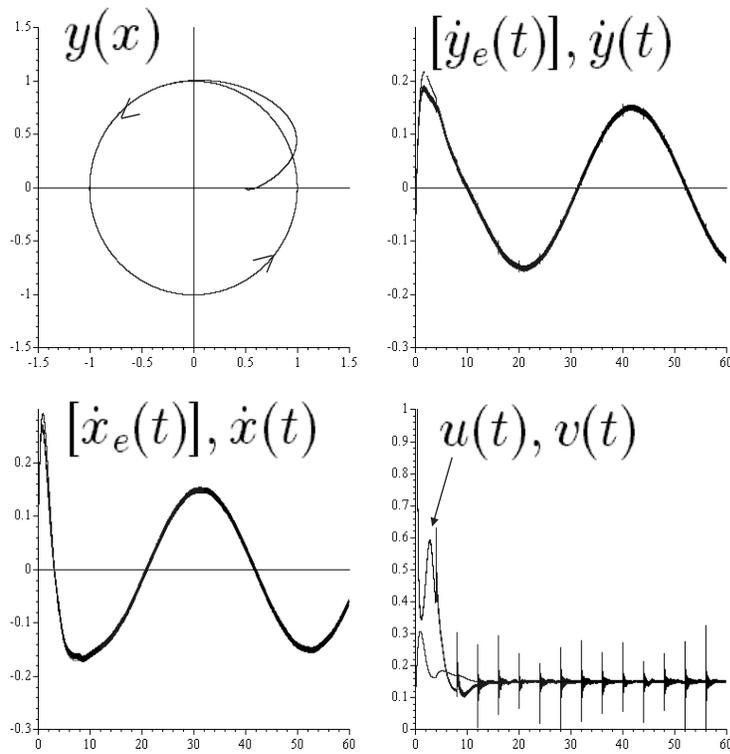

Fig. 2. Closed loop response of the perturbed unicycle model to dynamic feedback control based on the algebraic velocities estimations with noisy measurements.

flatness based trajectory tracking feedback control law. Note that the unmeasured motor shaft angular position would be estimated as

$$\widehat{\theta}_m = \frac{1}{\kappa}\left[J_l\ddot{y}_e + \kappa y + mgh\sin y\right]$$

Similar computations to those with the first example yield

$$\dot{y}_e(t) = $$
$$\frac{1}{(t-t_i)^7}\left[42(t-t_i)^6 y - 882(\int_{t_i} (t-t_i)^5 y)\right.$$
$$\overset{(2)}{+7350(\int_{t_i}(t-t_i)^4 y)}$$
$$\overset{(3)}{-29400(\int_{t_i}(t-t_i)^3 y)}$$
$$\overset{(4)}{+52920(\int_{t_i}(t-t_i)^2 y)} \overset{(5)}{-35280(\int_{t_i}(t-t_i)y)}$$
$$\left.\overset{(6)}{+5040(\int_{t_i}y)}\right]$$

$$\ddot{y}_e(t) = $$
$$\frac{1}{(t-t_i)^7}\left[35(t-t_i)^6\dot{y}_e - 630(t-t_i)^5 y\right.$$
$$\overset{(2)}{+7350(\int_{t_i}(t-t_i)^{(4)}y)} - 29400(\int_{t_i}(t-t_i)^3 y)$$
$$\overset{(3)}{+52920(\int_{t_i}(t-t_i)^2 y)} \overset{(4)}{-35280(\int_{t_i}(t-t_i)y)}$$
$$\left.\overset{(5)}{+5040(\int_{t_i}y)}\right]$$

$$y_e^{(3)}(t) = $$
$$\frac{1}{(t-t_i)^7}\left[28(t-t_i)^6\ddot{y}_e - 420(t-t_i)^5\dot{y}_e\right.$$
$$+4200(t-t_i)^4 y - 29400(\int_{t_i}(t-t_i)^3 y)$$
$$\overset{(2)}{+52920(\int_{t_i}(t-t_i)^2 y)} \overset{(3)}{-35280(\int_{t_i}(t-t_i)y)}$$
$$\left.\overset{(4)}{+5040(\int_{t_i}y)}\right]$$

Figure 3 depicts the closed loop performance, in a rest-to-rest trajectory tracking task, when the flatness based controller is synthesised on the basis of the estimated derivatives of the flat output. The estimated motor shaft angular position $\widehat{\theta}_m$ is also shown. The controller parameters were set so that the characteristic polynomial of the closed loop linearised system coincided with $(s^2 + 2\zeta\omega_n s + \omega_n^2)^2$, $\zeta = 0.81$, $\omega_n = 7.5$ [rad/s]. The parameter

ϵ representing the interval in which we block the singularity off the calculation was chosen to be $\epsilon = 0.04$ [s]. The estimation period T for the derivatives calculations was chosen as $T = 0.3$ [s]. The desired rest to rest maneuver for the pendulum angular position takes θ_l from $\theta_l(t_{\text{init}}) = 0$ [rad], with $t_{\text{init}} = 0.5$ [s] to $\theta_l(t_{\text{final}}) = 1.5$ [rad], with $t_{\text{final}} = 1.5$ [s]. The jumps in the figure are due to resettings and to the adoption of a constant value while the initial singularity is overcome. To test the robustness we have included an additive perturbation term in the output measurement equation $y = \theta_l + a\eta(t)$, where $\eta(t)$ is a computer generated piecewise constant noise comprised of a sequence of pseudo random variables exhibiting an uniform probability distribution in the interval $[-0.5, 0.5]$ [rad], at each small integration step of value 10^{-3} [s]. The amplitude of the noise was chosen to be $a = 0.5 \times 10^{-5}$ [rad]. Figure 4 depicts the computer simulations.

5. CONCLUSION

The extension to state estimation when the measured output is non-flat and/or for non-flat systems is straightforward. Identification of unknown parameters will be done in future publications along the same lines [15]. We will also achieve, as in Fliess & Sira-Ramírez, 2003, for linear systems, nonlinear closed loop identification.

This major progress in nonlinear system theory is made possible by a complete change of viewpoint [16] and certainly not by solving the long-standing mathematical problems related to nonlinear asymptotic observers or nonlinear Kalman filtering:

(1) Estimated values are no more given in an asymptotic way with respect to a large time [17]. On the contrary the exact values of the unknown quantities are obtained when $t \to 0$, i.e., for very short time windows.
(2) Those estimated values result from algebraic formulae which are providing good approximations in real time even in a noisy environment.

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[15] See Diop & Fliess, 1991a, 1991b, for an algebraic definition of nonlinear identifiability: An unknown parameter is identifiable if, and only if, it is a function of the control and output variables and their time derivatives, up to some finite order.

[16] Flatness may also be viewed as a new understanding of the controllability concept (see Fliess, 2000). Note that the celebrated state variable representation becomes also much less crucial.

[17] Remember that the feedback stabilisation of a nonholonomic vehicle around a rest point was also given in a non asymptotic manner by Fliess et al., 1995b. This feedback synthesis has often been most successfully implemented in practice.
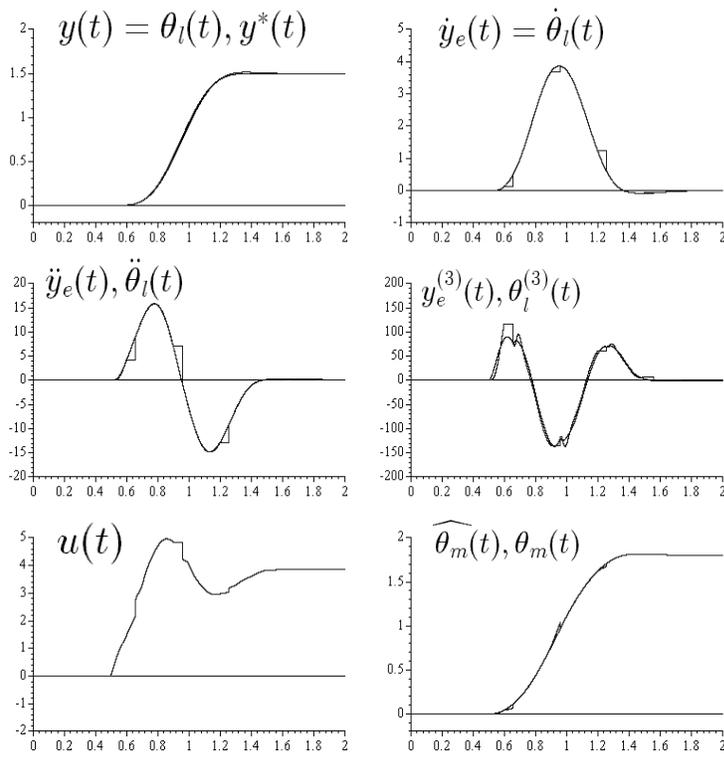
Fig. 3. Closed loop system response to feedback control
based on the estimations of piecewise continuous flat
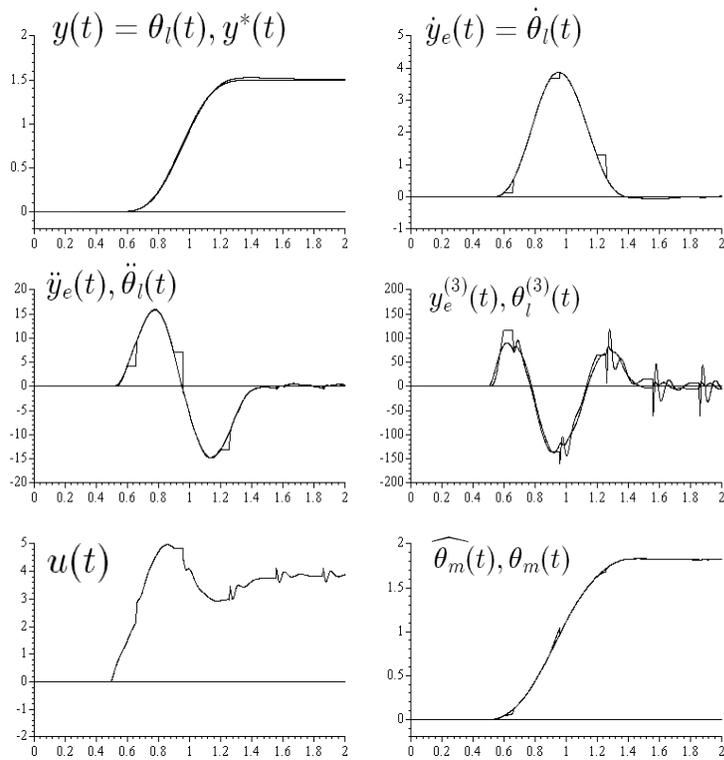output derivatives.

Fig. 4. Closed loop system response to feedback control
based on the estimations of piecewise continuous
flat output derivatives obtained with noisy output
measurements.

(3) We do not need any probabilistic machinery[18]. The noises are viewed as high frequency perturbations which may be attenuated by low pass filters without any precise knowledge of their statistical properties (see, also, Fliess & Sira-Ramírez, 2003, 2004, Fliess, Mboup *et al.*, 2003, Fliess, Join & Sira-Ramírez, 2004).

(4) We do not need any discretisation scheme of the original system[19] nor of a complex asymptotic observer. Our numerical techniques, which will be further developed, are of a complete different nature.

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[18] See, *e.g.*, Jaulin *et al.*, 2001, for another approach to estimation without probability.

[19] Let us recall that flatness-based control does not necessitate for its practical implementation any discretisation of the system (see, *e.g.*, Rudolph, 2003a, Sira-Ramírez & Agrawal, 2004, and the references therein). This applies also to the flatness-based control of infinite-dimensional systems, like delay systems or systems with distributed parameters (see, *e.g.*, Rudolph, 2003a, 2003b, Rudolph *et al.*, 2003, Sira-Ramírez & Agrawal, 2004, and the references therein).