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Abstract: In this paper we consider a problem of asymptotic stabilization for a chemostat, by using a feedback control law and considering a delay $\tau > 0$ on its output. The presence of these delays on the outputs makes more difficult to achieve our asymptotic stabilization objectives.

We build a family of feedback control laws (using the dilution rate as control variable) obtaining sufficient conditions for asymptotic stabilization, given by upper bounds for the delay (which are dependent of the feedback control law). Using some reduction techniques we show that the control problem becomes equivalent to obtain global stability conditions for the zero solution of the scalar differential delay equation:

$$\dot{u}(t) = -G(u(t)) + F(u(t-\tau)).$$

Key-words: Chemostat, Feedback Control, Differential Delay Equations

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Commande en boucle fermée pour un chemostat avec retard dans la sortie

Résumé : Dans cet article, nous considérons un problème de stabilisation asymptotique pour un chemostat en utilisant une loi de commande en boucle fermée et considérant un retard $\tau > 0$ dans la sortie. L'existence de ce retard rend plus difficile notre objectif de stabilisation asymptotique.

On construit une famille des lois de commande en boucle fermée (utilisant le taux de dilution comme variable de commande) et on obtient des conditions suffisantes pour la stabilisation asymptotique, représentées sous la forme de bornes supérieures pour le retard (qui sont independantes de la loi de commande). En utilisant certaines techniques de réduction on démontre que le problème est equivalent à l'obtention des conditions suffisantes pour la stabilité globale asymptotique de l'équation différentielle à retard:

$$\dot{u}(t) = -G(u(t)) + F(u(t-\tau)).$$

Mots-clés : Chemostat, Commande en boucle fermée, Equation différentielle à retard

1 Introduction

The chemostat is a continuous bioreactor used to culture microorganisms with concentration x which consume a limiting substrate s to grow. Mathematical modeling of chemostat has been extensively developed, mainly using ordinary differential equations and several results have been validated experimentally.

The chemostat has several industrial and scientific applications: to culture a biomass for its utilization (yeast), to degrade a pollutant (wastewater treatment), to simulate aquatic environments (culture of phytoplankton).

In this paper we follow an idea developed in [1],[4] and consider the chemostat equations as an *input–output* (I/O) system (see *e.g.* [5],[21]), that means a structure with three elements:

(1) A *Plant* defined by the chemostat equations described in the next section.

(2) An *Output* $y(t)$, given by the measurements which we are able to carry out in the chemostat.

(3) An *Input* or control variable $U > 0$ given by some parameters of the chemostat (*e.g.* the input concentration of substrate s_{in} or the dilution rate D) susceptible to being modified externally.

There exist two types of inputs: First, when they are considered as positive functions of time (*i.e.* $U = U(t) > 0$), we deal with an *open–loop control* approach. Second, when the inputs are considered as positive functions defined on the output $y(t)$, we deal with a *feedback control*.

Feedback control of chemostat models presents several advantages for the applications stated above. In general, we can obtain best results from the point of view of performance and robustness with respect to eventually uncertainties of the model and the measurements.

This last approach has been studied in several works (see for example [1],[4]) but in general, it is assumed that the outputs are available online from the plant. Nevertheless, time delays between inputs and outputs are common phenomena in industrial processes and biological systems. Motivated by this fact, we consider delays on the output.

The presence of delays in the outputs and consequently in the feedback control law has two consequences: either the speed of convergence is slowed down or the convergence is not achieved. In consequence, we need to develop a feedback control law with stronger properties and find sufficient conditions for global stability.

We state a control problem in a similar way that [4], but in this case, we must deal with a system of two differential delay equations. By using some reduction techniques (asymptotically autonomous dynamical systems) we prove that the asymptotic behavior can be studied by working only with the differential delay equation describing the biomass concentration.

By following an idea developed in [11], we build a discrete system who inherits the asymptotic properties of the infinite–dimensional system related to the original control problem.

This report is organized as follows: in section 2 we recall some facts of the chemostat model, state our control problem and build a family of feedback control laws, in section 3 we state our main result of asymptotic stabilization, in section 4 we give the proof of the main result. Some numerical examples are given in section 6.

2 System description and problem statement

The chemostat equations are defined by (see for example [19] and the references therein):

$$\begin{cases} \dot{s}(t) = D\big(s_{in} - s(t)\big) - \alpha f\big(s(t)\big)x(t), \\ \dot{x}(t) = f(s(t))x(t) - Dx(t), \\ s(0) = s_0 \in (0, s_{in}] \quad \text{and} \quad x(0) = x_0 \in (0, x_0^+]. \end{cases} \tag{1}$$

Where $s(t)$ and $x(t)$ are the concentration of the nutrient and the density of the biomass at time t, $s_{in} > 0$ denotes the input concentration of nutrient (in this paper we will suppose that is a constant), $D > 0$ is the dilution rate, $\alpha > 0$ is a growth yield constant. The function $f\colon \mathbb{R}_+ \mapsto \mathbb{R}_+$ represents the growth rate of nutrient of the biomass. In this paper we consider only three types of these functions (for more details on each one see [1],[2],[20] respectively):

$$f_1(s) = \frac{\mu_{\max} s}{k_s + s}, \quad \mu_{\max}, k_s > 0. \tag{2}$$

$$f_2(s) = \frac{\mu_{\max} s}{k_s + s + \frac{s^2}{k_i}}, \quad \mu_{\max}, k_s, k_i > 0. \tag{3}$$

$$f_3(s) = \frac{\mu_{\max} s}{k_s + s^2}, \quad \mu_{\max}, k_s > 0. \tag{4}$$

Remark 1 *It is straightforward to verify that f_1 is strictly increasing, concave and that the functions f_2 and f_3 are unimodal,* i.e. *they have one critical point $s_{\max} > 0$ and moreover, $f_i''(s) < 0$ for any $s \in [0, s_c)$ with $s_c > s_{\max}$ (see figures 1,2 and 3).*

2.1 Control problem statement

The control problem we consider is to globally stabilize the substrate concentration at a given level s^* with a nonnegative feedback control law. This control problem will be considered under the following I/O hypothesis:

(H1) The dilution rate D is the feedback control variable.

(H2) The only output available is the substrate and its measure can be obtained with a delay $\tau > 0$:

$$y(t) = s(t - \tau), \quad \tau > 0. \tag{5}$$

The requirement of a nonnegative feedback control comes from the fact that the control variable (dilution rate) represents an input flow. So, it has to be nonnegative for have a physical meaning. Now, we formalize our objective:

Problem 1 *Find a collection of feedback control laws that stabilizes asymptotically the system* (1)–(5) *in a reference value* $s^* \in (0, \min\{s_{\max}, s_{in}\})$. *That means* $\lim_{t\to+\infty} s(t) = s^*$.

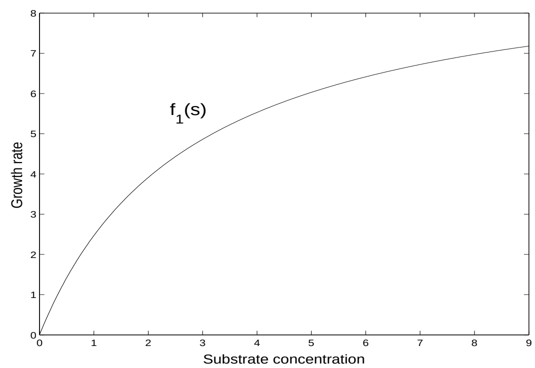

Figure 1: Graph of function f_1 in Eq.(2)

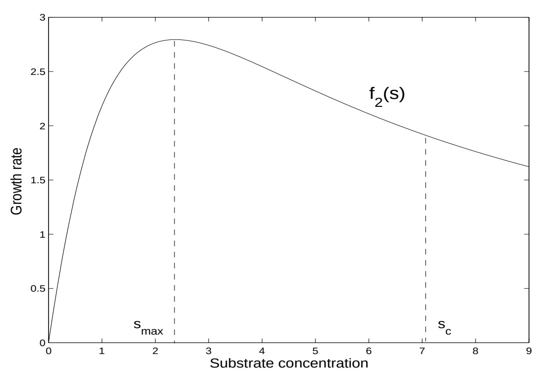

Figure 2: Graph of function f_2 in Eq.(3)

2.2 Motivation

It is clear that by choosing $D = f(s^*)$, the problem is solved immediately when $f = f_1$ and is solved for a wide set of initial conditions (s_0, x_0) dealing with functions $f = f_2, f_3$.

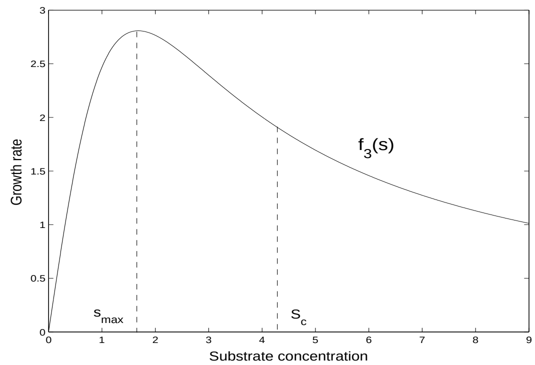

Figure 3: Graph of function f_3 in Eq.(4)

Nevertheless, as we stated before, the introduction of a feedback control law can improve the performances and efficiency of the bioprocess with respect to this "fixed dilution" approach.

To illustrate this idea we will consider two examples:

Example 1: We can consider the chemostat as a depollution machine. This process consists in a chemostat in which toxic contaminants (*e.g.* phenol, toluene) are pumped into with a fixed concentration s_{in} higher than an acceptable level $s^+ > s^*$ relatively near to zero and fixed by environmental authorities. This chemostat also contains a microorganism (for instance *Pseudomonas Putida*) which can resists the adverse effects of organic solvents and is capable of decontamining the tank because it is able to utilize the toxic contaminants as limiting substrate.

The introduction of a feedback control law can drastically modify the outputs of the model. Indeed, as we can see in Figure 4 a depollution process carried on by using a fixed dilution $D = f(s^*)$ could fail (washout of *P. putida*) for some set of initial conditions; nevertheless if we introduce an appropriate feedback control law[1]), the phenol concentration is convergent toward s^* and the depollution goal can be obtained. Nevertheless, if we take into account the delay in the measurements, we can obtain periodic outputs (if the delay exceed certain threshold) impeding our stabilization goal (see Figure 8 in section 6.2).

Example 2: We can use the chemostat to study the phytoplankton in a simulated marine environment (for more details see section 6.2): indeed, several features of marine environments such as light intensity, pH, temperature, can be reproduced externally. Moreover the use of the chemostat makes possible to reproduce several levels s^* of limiting substrate and in consequence we can study the metabolism of phytoplanktonic algae.

[1]Numerical values are shown in section 6.1

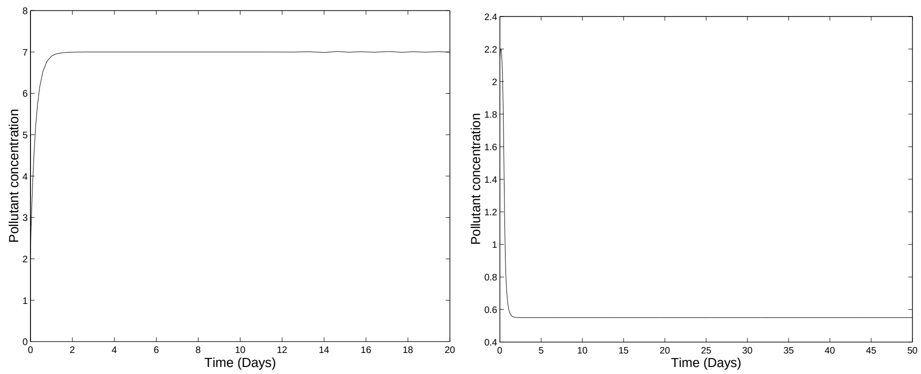

Figure 4: Pollutant concentration in the chemostat: use of fixed dilution $D = f(s^*)$ (Left) and use of a feedback control (Right). Notice the difference between the asymptotic behavior: washout of the biomass (left) and asymptotic stabilization in $s^* = 0.55$ (right).

Figure 5 shows the culture of *Dunaniella tertiolecta* using nitrate as limiting substrate[2]. Notice that the use of a feedback control law can drastically improve the speed of the convergence (with respect a fix dilution strategy) towards the wanted level s^*. Nevertheless, as it has been pointed out in [13], there exist delays in the measure of substrate. So, if we take into account these delays, we can obtain periodic outputs (if the delay exceed certain threshold) impeding our stabilization goal (see Figures 10 and 11 in section 6.2).

2.3 Feedback control law

Let us build the family of feedback control laws:

$$D(y(t)) = h(s^* - s(t - \tau)), \tag{6}$$

where the function $h\colon \mathbb{R} \mapsto \mathbb{R}_+$ is at least $\mathcal{C}^3$ and satisfy the following properties:

(P1) h is increasing, bounded, positive and $h(0) = f(s^*)$.

(P2) The value s^* is the only root of the equation $h(s^* - s) - f(s) = 0$ on the interval $[0, s_{in}]$.

Remark 2 *Notice that when f is described by Eq.*(2)*, the property* **(P2)** *is automatically satisfied. Moreover, if $\tau = 0$ is a trivial exercise to solve the Problem* (1) *replacing D by* (6) *and studying the asymptotic properties of the resulting system.*

[2] Numerical values are shown in section 6.2

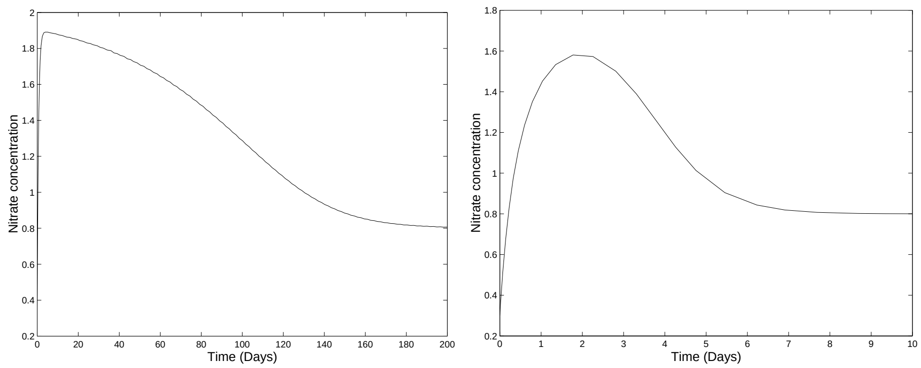

Figure 5: Nitrate concentration in the phytoplankton culture: use of fixed dilution $D = f(s^*)$ (Left) and use of a feedback control (Right). Notice the difference (days) between the speed of convergence

Remark 3 *Moreover, if $\tau = 0$ our feedback control law is a nonlinear version of a **proportional regulator** (see* e.g. [4] *for more details), indeed Eq.*6) *can also be viewed as*:

$$D(y(t)) = D^* + \lambda g(s^* - s(t)), \quad \lambda > 0$$

where $D^* = f(s^*)$. By **(P1)** *we have that $g: \mathbb{R} \mapsto \mathbb{R}$ is an increasing function verifying $g(0) = 0$ and $g(s) > -\lambda^{-1} D^*$ for any $s \in \mathbb{R}$. Finally,* **(P2)** *implies that if $s(t) < s^*$ then $D(y(t)) > D^*$ and s is increasing. On the other hand, if $s(t) > s^*$ then $D(y(t)) < D^*$ and s is decreasing.*

2.4 Some preliminary notations, definitions and results

We point out that if we replace D by the feedback control law (6), we deal with a system of differential delay equations; hence a way to solve the Problem 1 is to find sufficient conditions for global attractivity of the critic point s^*.

In the following, we shall make use of some results of dissipative dynamical systems theory and the Schwarz derivative of a real function that will be useful for the study of asymptotic properties of our control system. For the convenience of the reader we present some basic definitions adapted from [6],[7],[18].

Let (X, d) be a complete metric space, we define a continuous semiflow as a continuous function $\phi: \mathbb{R}_+ \times X \mapsto X$ verifying the properties $\phi(0, \vec{\varphi}) = \vec{\varphi}$ and $\phi(t+s, \vec{\varphi}) = \phi(t, \phi(s, \vec{\varphi}))$. Moreover, we use the notation $\phi(t, \vec{\varphi}) = \phi_t(\vec{\varphi})$.

Definition 1 ([7, chapt.3]) *The semiflow ϕ_t is* point dissipative *on X if there exist a bounded set B that attracts each point of X.*

Definition 2 ([7, chapt.3]) *The semiflow* ϕ_t *is* conditionally completely continuous *for* $t \geq t_1$ *if, for each* $t \geq t_1$ *and each bounded set* $B \subset X$ *for which* $\phi_s(B)$ (*with* $s \in [0,t]$) *is bounded, we have that* $\phi_t(B)$ *is precompact for any* $t > t_1$. *Moreover, a semiflow* ϕ_t *is* completely continuous *for* $t \geq t_1$ *if it is conditionally completely continuous and, for each* $t \geq 0$, *the set* $\phi_s(B)$ (*with* $s \in [0,t]$) *is bounded.*

Definition 3 *The* Schwarz derivative *of a* $\mathcal{C}^3$ *function* $F\colon \mathbb{R} \mapsto \mathbb{R}$ *is defined* (*for any* $r \in \mathbb{R}$ *such that* $F'(r) \neq 0$) *by*:

$$(SF)(r) = \frac{F'''(r)}{F'(r)} - \frac{3}{2}\left(\frac{F''(r)}{F'(r)}\right)^2.$$

Remark 4 *We can easily verify that*:

$$(Sf_1)(s) = 0, \quad (Sf_2)(s) = \frac{-6k_s k_i}{(s^2 - k_s k_i)^2} < 0 \quad \text{and} \quad (Sf_3)(s) = \frac{-6k_s}{(s^2 - k_s)^2} < 0$$

We can define the Schwarz derivative in terms of the *Pre–Schwarz derivative* which is defined as:

$$(PF)(r) = \frac{F''(r)}{F'(r)}$$

and in consequence we can deduce:

$$(SF)(r) = \frac{d}{dr}(PF)(r) - \frac{1}{2}\big[(PF)(r)\big]^2.$$

Lemma 1 *If* F *and* G *are functions which are at least* $\mathcal{C}^3$, *then*
(i) $S(F \circ G)(r)$ *and* $P(F \circ G)(r)$ *are defined by*:

$$P(F \circ G)(r) = (PF)(G(r))\{G'(r)\} + (PG)(r).$$

$$S(F \circ G)(r) = (SF)(G(r))\{G'(r)\}^2 + (SG)(r).$$

(ii) *Let* $\chi = F_1 - F_2$, *hence* $(S\chi)(r)$ *is defined by*:

$$(S\chi) = \frac{1}{(\chi')^2}\left\{(SF_1)F_1'\chi' - (SF_2)F_2'\chi' - \frac{3}{2}F_1'F_2'\left[\Big(\frac{F_1''}{F_1'}\Big) - \Big(\frac{F_2''}{F_2'}\Big)\right]^2\right\}.$$

Proof: Property (i) can be checked by direct computation. The proof for the property (ii) is given in the Appendix. □

The following propositions will play an important role:

Proposition 1 ([18]) *Let* $\chi\colon [\alpha,\beta] \mapsto [\alpha,\beta]$ *be a* $\mathcal{C}^2$ *with third derivative continuous, unless a finite set of points* $\{a_1,\ldots,a_n\} \subset [\alpha,\beta]$ *and decreasing map with a unique fixed point* γ. *If* γ *is locally asymptotically stable,* $\chi''(a_j) \neq 0$ *for any integer* $j \in \{1,\ldots,n\}$ *and the Schwarz derivative* $(S\chi)(r) < 0$ *verifies for all* r, *then* γ *is a global attractor of* χ.

Proof: The proof is similar to proof stated in [10, Proposition 3.3]. □

Proposition 2 ([11, lemma 2.1]) *Let* $g\colon \mathbb{R} \mapsto \mathbb{R}$ *a* $\mathcal{C}^3$ *function satisfying the properties*:
(a) $rg(r) > 0$ *for any* $r \neq 0$, $g'(0) > 0$ *and* $g''(0) < 0$,
(b) g *is upperly bounded and can have at most one critical point* r^* *which is a local maximum,*
(c) $(Sg)(r) < 0$ *for any* $r \in \mathbb{R}$ *such that* $g'(r) \neq 0$.
Hence there exists a function $\sigma\colon [2g'(0)g''(0)^{-1}, +\infty] \mapsto \mathbb{R}$ *defined by*:

$$\sigma(r) = \frac{2g'(0)^2 r}{2g'(0) - g''(0)r}$$

such that we have $\sigma(r) > g(r)$ *for any* $r > 0$ *and* $\sigma(r) < g(r)$ *for any* $r \in (2g'(0)g''(0)^{-1}, 0)$. *Moreover, it follows that* $\sigma'(0) = g'(0)$ *and* $\sigma''(0) = g''(0)$.

3 Main Results

As we pointed out in Remark 3, if $\tau = 0$ then the feedback control law (6) satisfying assumptions **(P1)**–**(P2)** stabilizes asymptotically the chemostat in s^*. Hence, it is reasonable to suppose that taking account the influence of delays in the measurements, this nonlinear proportional regulator could loss a big part of its effectiveness or became useless. On the other hand, there exists a widespread consensus summarized as *small delays are harmless*. For these reasons we are interested in finding upper bounds for the delays in the measurements for which the feedback control law (6) is still effective. We introduce more assumptions on h with the hope to solve Problem 1:

(P3) For any couple of continuous functions ψ_i $(i = 1, 2)$, there exists a constant $L_0 > 0$ such that for any $t \geq 0$:

$$\sup_{\theta \in [-\tau,0]} |h(s^* - \psi_1(t+\theta)) - h(s^* - \psi_2(t+\theta))| \leq L_0 \sup_{\theta \in [-\tau,0]} |\psi_1(\theta) - \psi_2(\theta)|.$$

(P4) $(Sh)(r) < 0$ for any $r \in \mathbb{R}$ and $h(0)$ and its derivatives in $r = 0$ satisfy the inequality:

$$\frac{[f'(s^*) + h'(0)]^2 - 3\big[(Ph)(0) + (Pf)(s^*)\big]^2 f'(s^*)h'(0)}{[f'(s^*) + h'(0)]\big\{(Sh)(0)h'(0) + (Sf)(s^*)f'(s^*)\big\}} < 2(s_{in} - s^*)$$

Remark 5 Property **(P3)** is a technical assumption which ensures the existence and uniqueness of the solutions of the I/O system defined by Eqs.(1) and (5). On the other hand **(P4)** is satisfied for a wide family of control functions h.

Notice that if we replace D in the system (1) for any feedback control law satisfying properties **(P)**, we have that $s(t) > 0$ for any $t \in \mathbb{R}_+$. Nevertheless, as in several articles which study chemostat models by using a dynamical system approach, it is necessary to build a prolongation for the functions f_i for any $r<0$; constant (*i.e.* $f_i(r)=0$) or odd prolongation $f_i(r)=-f_i(-r)$ have been used in some articles.

Notice that the functions f_i are still defined in an interval $[-a,0]$ (with $a>0$), using this fact we will define a prolongation $\mu_i\colon\mathbb{R}\mapsto\mathbb{R}$ for the functions f_i ($i=1,\ldots,3$) as follows:

$$\mu_i(r)=\begin{cases}\rho_i(r) & \text{if } r\in(-\infty,-a),\\ f_i(r) & \text{if } r\in[-a,+\infty)\end{cases}$$

where $\rho_i\colon(-\infty,-a)\mapsto\mathbb{R}$ is a monotone function verifying $(S\rho_i)(r)<0$ and $\rho_i^{(k)}(-a)=f_i^{(k)}(-a)$ $(k=1,2)$.

For example, we can build a function:

$$\rho_i(r)=f_i(-a)+f_i'(-a)(s+a)+f_i''(-a)\frac{(s+a)^2}{2}\quad i=1,\ldots,3$$

which is increasing in $(-\infty,-a)$ and $(S\rho_i)(r)=-\frac{3}{2}\left(\frac{f''(-a)}{f'(-a)+f''(-a)(s+a)}\right)^2$.

Notice that the functions $\mu_i\colon\mathbb{R}\mapsto\mathbb{R}$ are $\mathcal{C}^2$ with continuous third derivative except the point $r=-a$. Moreover, $(S\mu_i)(r)<0$ for any $r\in\mathbb{R}$.

Let us now introduce some notation and make precise the mathematical setting: we will build a discrete dynamical system that inherits some asymptotic properties of the chemostat model. To build this discrete system we must introduce some auxiliary functions related with μ_i and h (we will make the distinction between the functions μ_1,μ_2 and μ_3) described by:

$$g_{1_j}(r)=\begin{cases}(\mu_j\circ\lambda_1)(r) & \text{if } r<0,\\ \dfrac{2[\kappa(0)f_j'(s^*)]^2r}{2\kappa(0)f_j'(s^*)+\kappa(0)[f_j'(s^*)-f_j''(s^*)]r} & \text{if } r>0\end{cases}\qquad (j=2,3),$$

$$g_{1_1}(r)=(\mu_1\circ\lambda_1)(r)\quad\text{and}\quad g_2(r)=(h\circ\lambda_2)(r)$$

where $\kappa,\lambda_j\colon\mathbb{R}\to\mathbb{R}$ are defined by

$$\kappa(r)=[s_{in}-s^*]e^{-r},\quad \lambda_1(r)=s_{in}-\kappa(r),\quad\text{and}\quad \lambda_2(r)=\kappa(r)-\kappa(0).$$

To simplify the notation, we will write g_1 instead of g_{1_j} $(j=1,2,3)$. We will give more details with respect to this notation, if it is necessary.

Let us define the interval $I_\tau=[\alpha\tau,\{g_2(\alpha)-g_1(\alpha)\}\tau]$ where:

$$\alpha=\begin{cases}h(s^*-s_{in})-f(s_{in}) & \text{if } f=f_1,\\ h(s^*-s_{in})-h(0)-\dfrac{2(s_{in}-s^*)f'(s^*)^2}{f'(s^*)-(s_{in}-s^*)f''(s^*)} & \text{if } f=f_2,f_3\end{cases}.$$

Notice that λ_1 is increasing and λ_2 is decreasing. By consequence the function g_1 is increasing and g_2 is decreasing. Moreover, it is straightforward to verify that $\alpha<0$ and

$g_2(\alpha) - g_1(\alpha) > 0$. By using Remark 4 and properties **(P)**, we can see that the set of intervals I_τ where the inequality

$$\frac{\Big[g_2'(r) - g_1'(r)\Big]\Big[\sum_{i=1}^{2}(Sg_i)(r)g_i'(r)\Big]}{\Big[\sum_{i=1}^{2}(-1)^i(Pg_i)(r)\Big]^2} < \frac{3}{2}\prod_{i=1}^{2} g_i'(r), \tag{7}$$

is verified is not empty and we can define the number:

$$\tau_a^* = \sup\Big\{\tau > 0\colon \text{Inequality (7) is verified in } I_\tau\Big\}.$$

We are now in position to state our main results:

Theorem 1 *Let f be a function defined by f_i $(i = 1, \ldots, 3)$. If properties* **(P1)–(P5)** *are fulfilled and the delay τ satisfies*:

$$\tau < \min\left\{\frac{1}{\kappa(0)[h'(0) + f'(s^*)]}, \tau_a^*\right\} \tag{8}$$

then the feedback control law (6) *stabilizes asymptotically the output in s^*.*

Theorem 2 *Let f be a function defined by f_i $(i = 1, 2, 3)$. If properties* **(P1)–(P3)** *are fulfilled and the delay τ satisfies*:

$$\tau < \tau_b^* = \sup\Big\{\tau > 0\colon \tau\big|g_2'(r) - g_1'(r)\big| < 1 \quad \text{is verified in } I_\tau\Big\}. \tag{9}$$

then the feedback control law (6) *stabilizes asymptotically the output in s^*.*

4 Proof of Theorem 1

Firstly we will study the closed–loop system, replacing D in the system (1) by the feedback control law (6), the closed–loop system becomes:

$$\begin{cases} \dot{s} = h(s^* - s(t-\tau))(s_{in} - s) - \alpha f(s)x, \\ \dot{x} = x\big(f(s) - h(s^* - s(t-\tau))\big), \\ x(0) \geq 0 \quad 0 \leq s(\theta) = \varphi_1(\theta) \leq s_{in} \quad \text{for any } \theta \in [-\tau, 0], \end{cases} \tag{10}$$

where φ_1 is a nonnegative continuous and upperly bounded function on the interval $[-\tau, 0]$.

Let us define by

$$C = C\big([-\tau, 0], \mathbb{R}^2\big) \quad \text{and} \quad C_+ = C\big([-\tau, 0], \mathbb{R}_+^2\big)$$

the Banach space of scalar continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^2$ and the cone of nonnegative continuous functions, respectively. C is equipped with the supremum norm and C_+ becomes a complete metric space (C_+, d) under the induced metric.

The initial conditions of the system (10) are in the space $C_+ \times \mathbb{R}$ and can be embedded in the space $X = C_+ \times C_+$. Using **(P1)** and **(P3)**, it can be easily proved (see *e.g.* theorems 2.3 and 3.2 from [6]) that all solutions of system (10) define a semiflow $\phi: \mathbb{R}_+ \times X \mapsto X$, where $\phi_t(\varphi_1, \varphi_2) = (s_t, x_t)$ with $s_t(\theta) = s(t+\theta)$ and $x_t(\theta) = x(t+\theta)$ for any $\theta \in [-\tau, 0]$ and $t \geq 0$.

By using **(P2)**, it is straightforward to prove that the equilibria of system (10) are given by $E_0 = (s_{in}, 0)$ and $E_1 = (s^*, \alpha^{-1}[s_{in} - s^*])$. We will prove that E_1 is a globally attractive equilibria for any nonnegative initial condition.

The proof will be divided into three steps. Firstly (in subsection 4.1), we will prove that the critical point E_0 cannot be attractive. Secondly (in subsection 4.2) we will prove that the asymptotic behavior of this system is –under some suitable assumptions– equivalent to the asymptotic behavior of a scalar differential delay equation. Finally (subsection 4.3), we will build a discrete dynamical system that inherits some asymptotic properties of the infinite dimensional dynamical system defined by the scalar delay equation constructed before.

4.1 Uniform persistence of system (10)

The goal of this section is to prove that the critical point $E_0 = (s_{in}, 0)$ is a repeller, that is equivalent to prove that the biomass $x(t)$ is uniformly persistent, *i.e.* there exists a number $\delta_0 > 0$ (independent of initial conditions) such that

$$\liminf_{t \to \infty} x(t) > \delta_0.$$

In order to prove this property, we will present some compactness and invariance properties of the semiflow ϕ_t.

Lemma 2 *There exists a global attractor set $A \subset X$ for the semiflow ϕ_t. That means, a maximal compact invariant set A which attracts each bounded set in X.*

Proof: We will prove that the semiflow ϕ_t is point dissipative and completely continuous for $t > \tau$. Hence the Lemma is a consequence of Theorem 3.4.8 from [7].

Firstly we will prove that the semiflow is point dissipative: we take some initial condition (φ_1, φ_2) verifying:

$$|\varphi_1(\theta) + \alpha\varphi_2(\theta) - s_{in}| \leq K \quad \text{for any } \theta \in [-\tau, 0].$$

Moreover, let us build the functional:

$$v(t) = s(t) + \alpha x(t) - s_{in}$$

where (s_t, x_t) is a solution of the system (10). It is straightforward to prove that $v(t)$ satisfies the following differential equation:

$$\dot{v}(t) = -h\big(s^* - s(t-\tau)\big)v(t), \quad \text{for } t > 0,$$

$$v(\theta) = \eta(\theta) = \varphi_1(\theta) + \alpha\varphi_2(\theta) - s_{in}, \quad \theta \in [-\tau, 0].$$

It is a simple exercise to prove that for any $t \geq 0$ it follows that:

$$|v(t)| = |\varphi_1(0) + \alpha\varphi_2(0) - s_{in}| \exp\Big(-\int_0^t h(s^* - s(r-\tau))\,dr\Big).$$

By using **(P1)**, we can prove that there exists $\rho > 0$ such that:

$$||s_t + \alpha x_t - s_{in}||_\infty \leq K e^{-\rho t} \quad \text{for any } t > 0 \text{ and } \theta \in [-\tau, 0]. \tag{11}$$

Now, letting $t \to \infty$ we have that that for any initial condition (φ_1, φ_2) it follows that $\lim_{t\to+\infty} d\big(\phi_t(\varphi_1, \varphi_2), K_0\big) = 0$, where the bounded set K_0 is defined by:

$$K_0 = \Big\{(\varphi_1, \varphi_2) \in X\colon \varphi_1 + \alpha\varphi_2 = s_{in}\Big\}$$

which implies point dissipativity.

Secondly, we will prove that the semiflow ϕ_t is completely continuous for any $t > \tau$. Indeed, we take any initial condition (φ_1, φ_2) in a bounded set $B \subset X$, we will see that the orbits of system (10) are a precompact set for any $t \geq \tau$.

By using point dissipativity properties, we define the constants K_1 and K_2:

$$K_1 = \sup_{t\geq 0}\Big\{||s_t||_\infty\colon s_0 = \varphi_1 \in B\Big\} \quad \text{and} \quad K_2 = \sup_{t\geq 0}\Big\{||x_t||_\infty\colon x_0 = \varphi_2 \in B\Big\}.$$

Notice that, the set $\phi_t(B)$ is equicontinuous for any $t \geq \tau$. Indeed, there exists a number $\delta(\varepsilon) = \varepsilon/L$ where L is defined by:

$$L = \min\Big\{\max_{|u|\leq K_1}[h(u)s_{in} + \alpha f(u)K_2], K_2 \max_{|u|\leq K_1} f(u) - h(s^* - u)\Big\}$$

such that for any couple $\theta', \theta'' \in [-\tau, 0]$ satisfying $|\theta' - \theta''| < \delta$ we have that $|s_t(\theta') - s_t(\theta'')| < \varepsilon$ and $|x_t(\theta') - x_t(\theta'')| < \varepsilon$.

By the Arzelà–Ascoli Theorem, it follows that the set $\phi_t(B)$ is precompact for any $t \geq \tau$, which implies that ϕ_t is completely continuous. □

Lemma 3 *The biomass $x(t)$ is uniformly persistent.*

Proof: Without loss of generality, we can suppose in this proof that the initial conditions of the system (10) are in the compact set A.

Let us define the subset $A_0 = \{(\varphi_1, \varphi_2) \in A \colon \varphi_2 = 0\}$ and notice that the set A_0 is positively invariant under the semiflow ϕ_t. We will prove that A_0 is a repeller that implies the uniform persistence.

Firstly, notice that for any initial condition in A_0, the semiflow (s_t, x_t) can be studied as a solution of the following integral equation:

$$\left\{ \begin{array}{l} s(t) = \varphi_1(0) \exp\Big(-\int_0^t h(s^* - s(r-\tau))\, dr \Big) + s_{in}, \quad \text{for any } t \geq 0, \\ s(\theta) = \varphi_1(\theta) \quad \text{for any } \theta \in [-\tau, 0]. \end{array} \right. \tag{12}$$

Secondly, let us build the functional $P \colon A \mapsto \mathbb{R}$ defined by $P(\phi_t(\vec{\varphi})) = x_t(0)$. This functional satisfy the following properties:
(a) $P(\phi_t(\vec{\varphi})) \equiv 0$ if $\vec{\varphi} \in A_0$ and $P(\phi_t(\vec{\varphi})) > 0$ if $\vec{\varphi} \in A \setminus A_0$.
(b) $\dot{P} = \Psi(\phi_t(\vec{\varphi}))P$ where $\Psi \colon A \mapsto \mathbb{R}$ is a continuous function defined by:

$$\Psi(\phi_t(\vec{\varphi})) = f(s_t(0)) - h(s^* - s_t(-\tau)).$$

(c) It follows from **(P1)**–**(P2)** and Eq.(12) that $\Psi(\phi_t(E_0)) = f(s_{in}) - h(s^* - s_{in}) > 0$ and for any initial condition in A_0 we have that:

$$\lim_{t \to +\infty} (s_t, x_t) = E_0.$$

Notice that properties (a)-(b) imply that P is an average Lyapunov function (see *e.g.* [9]). Using the fact that ϕ_t is a semiflow defined on a compact metric space combined with property (c) and corollary 2 from [9], we can deduce that the set A_0 is a repeller set and the lemma follows. □

Remark 6 *Applying Eq.*(11) *and Lemma* (3) *we can prove that any solution satisfy* $s(t) \leq s_{in}$ *after a finite time and in consequence, we will consider only initial conditions verifying* $||\varphi_1||_\infty \leq s_{in}$.

It is straightforward to prove that the system (10) is equivalent to the following system

$$\left\{ \begin{array}{l} \dot{s}(t) = [h(s^* - s(t-\tau)) - f(s(t))](s_{in} - s(t)) - f(s(t))v(t), \\ \dot{v}(t) = -h(s^* - s(t-\tau))v(t), \\ v(\theta) = \eta(\theta), \quad s(\theta) = \varphi_1(\theta) \leq s_{in} \quad \text{for any } \theta \in [-\tau, 0]. \end{array} \right. \tag{13}$$

4.2 Reduction of system

As we stated above, the asymptotic behavior of the systems (10) and (13) can be described by studying only the substrate equation. In this subsection we will formalize this idea.

Let us insert the solution $v(t)$ of system (13) into the equation $\dot{s}$. Then, for each initial condition η, we obtain the nonautonomous differential delay equation:

$$\left\{ \begin{array}{l} \dot{s}(t) = [h(s^* - s(t-\tau)) - f(s(t))](s_{in} - s(t)) - f(s(t))v(t) = k(t, s_t), \\ s(\theta) = \varphi_1(\theta) \leq s_{in} \quad \text{for any } \theta \in [-\tau, 0]. \end{array} \right. \tag{14}$$

where $v(t)$ is a solution of the system (13).

Using the results of asymptotically autonomous theory (see *e.g.* [15],[22]) it can be proved that the solutions of Eq.(14) define a nonautonomous continuous semiflow $\Phi\colon \Delta \times C_+ \mapsto A$ where $\Delta = \{(t,s)\colon 0 \leq s \leq t < +\infty\}$ asymptotically autonomous to the semiflow defined by the scalar autonomous differential delay equation:

$$\left\{ \begin{array}{l} \dot{s}(t) \; = [h(s^* - s(t-\tau)) - f(s(t))](s_{in} - s(t)) = g(s_t), \\ s(\theta) = \varphi_1(\theta) \leq s_{in}. \end{array} \right. \tag{15}$$

It is straightforward to verify that all the solutions of system (15) are upperly bounded by s_{in}.

Lemma 4 *If the critical point s^* is a globally attractive solution of Eq.*(15) *then is also a globally attractive solution of Eq.*(14)*.*

Proof: Notice that using Eq.(11) the functionals $\mu(t, s_t)$ and $g(s_t)$ defined in the systems (14) and (15) respectively, we can verify that the solution of Eq.(14) define a nonautonomous semiflow $\Phi(t, t_0, \varphi_1)$ asymptotically autonomous with limit semiflow $\Theta_t(\varphi_1)$ defined by the solution of Eq.(15).

Notice that s_{in} and s^* are isolated and invariant subsets of A. Moreover, Lemma 3 implies that s_{in} is a repeller and as we have that s^* is a global attractor we can conclude that the existence of a Θ–cyclical chain of Θ–equilibria is not possible. Finally, using theorem 4.2 from [22] the Lemma follows. □

The way of the proof is now clear: if we find sufficient conditions for the global attractivity of the critical point s^* in Eq.(15), then Lemma 4 implies that s^* is a critical point globally attractive of system (10). This reduction enables us to employ the extensive literature on differential delay equations of type

$$\dot{u}(t) = \mathcal{F}(u(t), u(t-1)),$$

where the continuous function $\mathcal{F}$ verifies $\mathcal{F}(0,0) = 0$ and is decreasing with respect to $u(t-1)$, see for example [12] where it was proved that the Poincaré–Bendixson theorem holds and by consequence asymptotic periodicity is the "most complicated" type of behavior.

4.3 End of proof

Making the transformations:

$$u(t) = \ln\left(\frac{s_{in} - s^*}{s_{in} - s(t\tau)}\right),$$

$$F(r) = \tau[h([s_{in} - s^*][e^{-r} - 1]) - h(0)] = \tau[g_2(r) - h(0)],$$

$$G(r) = \tau[f(s_{in} - [s_{in} - s^*]e^{-r}) - f(s^*)] = \tau[g_1(r) - f(s^*)].$$

the system (15) becomes:

$$\begin{cases} \dot{u}(t) = -G(u(t)) + F(u(t-1)) & \text{for any } t \geq 0, \\ u(\theta) = \varphi(\theta) & \text{for any } \theta \in [-1, 0]. \end{cases} \tag{16}$$

Notice that $G\colon \mathbb{R} \mapsto \mathbb{R}$ is a $\mathcal{C}^2$ function with third derivative continuous, except in the point $r_1 = \ln\left(\frac{s_{in}-s^*}{s_{in}+a}\right)$, $F\colon \mathbb{R} \mapsto \mathbb{R}$ is a $\mathcal{C}^3$ function, moreover the following properties are straightforward:

(a) $rF(r) < 0$ and $rG(r) > 0$ for any $r \in \mathbb{R} \setminus \{0\}$,

(b) F is decreasing and G can be increasing (when $f = f_1$) or unimodal (when $f = f_2, f_3$) with maximum in $r = \ln\left(\frac{s_{in}-s^*}{s_{in}-s_{\max}}\right) > 0$. Moreover Remark 1 implies that $G''(0) < 0$.

(c) $F(r) \to \tau[h(-\Lambda) - h(0)]$ and $G(r) \to \tau[f(s_{in}) - f(s^*)]$ as $r \to +\infty$.

(d) By Lemma 1, Remark 4 and **(P3)**, it follows that $(SF)(r) < 0$ and $(SG)(r) < 0$.

By property (a) it follows that $u(t) \equiv 0$ is an equilibrium of Eq.(16). We will prove that this solution is globally attractive.

First of all, we notice that if the solution of (16) is non-oscillatory, then the following result stands

Lemma 5 *If the solution $u(t)$ is non–oscillatory (that means, when exists a finite number $\tilde{t} > 0$ such that $u(t)$ has a constant sign), it follows that* $\lim_{t\to+\infty} u(t) = 0$.

Proof: Without loss of generality, we suppose that $u(t) > 0$ for any $t > \tilde{t} + 1$. Hence, by the properties of F and G stated above, we have that $u'(t) < 0$ for any $t > \tilde{t} + 1$ and consequently:

$$\lim_{t\to+\infty} u(t) = l \geq 0.$$

We will prove that $l = 0$. To obtain a contradiction, let us suppose that $l > 0$, integrating the equation (16) between $T > \tilde{t} + 1$ and t we have that:

$$u(t) = u(T) + \int_T^t F(u(r-1))\, dr - \int_T^t G(u(r))\, dr$$

$$u(t) \leq u(T) + (t-T) \underbrace{\left(\max_{r\in[u(T-1),l]} F(r) - \min_{r\in[u(T),l]} G(r) \right)}_{<0}.$$

Letting $t \to +\infty$, it follows that $l < -\infty$ and we obtain a contradiction. Hence, $l = 0$ and the lemma follows. □

By virtue of Lemma 5, we have only to consider the case when solutions of Eq.(16) are oscillatory, that means, there exists a sequence $\{v_n\} \to +\infty$ when $n \to +\infty$ verifying $u(v_n) = 0$ for any integer $n > 1$.

If the solution $u(t)$ is oscillatory, we can suppose that

$$\liminf_{t\to+\infty} u(t) = m \leq 0 \leq M = \limsup_{t\to+\infty} u(t).$$

We will prove that $m = M = 0$. To obtain a contradiction, let us suppose that $m < 0$ and $M > 0$.

By the fluctuations lemma (see *e.g.* [8, Lemma 4.2]) there exist two sequences of real numbers $\{t_n\}$,$\{s_n\} \to +\infty$ when $n \to +\infty$ such that $u'(t_n) = u'(s_n) = 0$ for any integer $n \geq 1$ and moreover:

$$\lim_{n\to+\infty} u(t_n) = M \quad \text{and} \quad \lim_{n\to+\infty} u(s_n) = m.$$

Integrating Eq.(16) between $t_n - 1$ and t_n, it follows that:

$$M_n = u(t_n) = u(t_n - 1) + \int_{t_n-1}^{t_n} F(u(r-1))\,dr - \int_{t_n-1}^{t_n} G(u(r))\,dr. \tag{17}$$

Without loss of generality, we can suppose that $M_n > 0$ and $m_n < 0$ for any integer $n \geq 0$. Furthermore, by using Eq.(17) combined with properties of the sequence $u(t_n)$ and functions F and G we obtain that $u(t_n - 1) < 0$ which implies:

$$M_n = u(t_n) \leq \int_{t_n-1}^{t_n} F(u(r-1))\,dr - \int_{t_n-1}^{t_n} G(u(r))\,dr.$$

Let us build the auxiliary function $R\colon \mathbb{R} \mapsto \mathbb{R}$ defined by

$$R(r) = \begin{cases} G(r) & \text{if } r \in [-\infty, 0], \\ H(r) & \text{if } r \geq 0. \end{cases}$$

Where H is defined as

$$H(r) = \begin{cases} G(r) & \text{if } f = f_1, \\ \dfrac{2G'(0)^2 r}{2G'(0) - G''(0)r} & \text{if } f = f_2, f_3. \end{cases}$$

Notice that R verify the following properties:

(i) Proposition 2 implies that $G(r) \leq R(r)$ for any $r \geq 0$,

(ii) R is increasing. Moreover $R(0) = 0$,$G'(0) = R'(0)$ and $G''(0) = R''(0)$,

(iii) $(SR)(r) \leq 0$ for any $r \in \mathbb{R}$.

(iv) R is $\mathcal{C}^2$ with third derivative continuous, except in the points $r_1 = -a$ and 0. We can choose r_1 such that $F''(r_1) \neq R''(r_1)$.

Let us build the auxiliary function $\chi\colon \mathbb{R} \mapsto \mathbb{R}$ defined as

$$\chi(r) = F(r) - R(r).$$

By the definition of M and the properties of F,G and R stated above, it follows that for any $\varepsilon > 0$ there exist a number $T(\varepsilon) > 0$ such that for any $t > T + 2$ we have the following inequalities:

$$R(m-\varepsilon) \leq \min_{u\in[m-\varepsilon,0]} G(u) \leq G(u(t)) \leq \max_{u\in[0,M+\varepsilon]} G(u) \leq R(M+\varepsilon),$$

$$F(M+\varepsilon) \leq F(u(t)) \leq F(m-\varepsilon).$$

We thus deduce that for any $t_n > T(\varepsilon) + 2$ we have the inequality

$$M_n = u(t_n) \leq F(m-\varepsilon) - R(m+\varepsilon).$$

Letting $\varepsilon \to 0$ and $n \to +\infty$ we obtain:

$$M \leq F(m) - R(m) = \chi(m).$$

Analogously, using the sequence $\{s_n\}$ we have the inequality:

$$m \geq F(M) - R(M) = \chi(M).$$

Notice that $\chi(+\infty)$ is defined by:

$$\chi(+\infty) = \begin{cases} \tau[h(s^* - s_{in}) - f(s_{in})] & \text{if } f = f_1, \\ \tau\left[h(s^* - s_{in}) - h(0) - \dfrac{2(s_{in} - s^*)f'(s^*)^2}{f'(s^*) - (s_{in} - s^*)f''(s^*)}\right] & \text{if } f = f_2, f_3. \end{cases}$$

Let $I = [\chi(+\infty), \chi^2(+\infty)]$, it follows that $\chi^2(+\infty) > 0$, hence the map $\chi\colon I \mapsto I$ is well defined.

Using the fact $\chi(M) \leq m$ and $M \leq \chi(m)$ we obtain that $m, M \in I$ and it can be proved by mathematical induction that $[m, M] \subset \chi^k([m, M])$ for any integer $k \geq 1$.

Moreover, by inequality (8) we have that $|\chi'(0)| < 1$ which implies that 0 is a locally stable (and unique) fixed point of χ.

To conclude, we will prove that $(S\chi)(r) < 0$. Firstly, we consider $f_1 = f$. Moreover, notice that statement (ii) of Lemma 1 combined with $F_1 = F$ (*i.e.* $g_1 = g_{1_1}$) and $F_2 = R = G$ implies that:

$$(S\chi)\chi'^2 = \underbrace{(SF)F'\chi' - (SG)G'\chi'}_{K_1(r)} - \frac{3}{2}\underbrace{F'G'\Big[\frac{F''}{F'} - \frac{G''}{G'}\Big]^2}_{K_2(r)}$$

Moreover, by using the fact that $(S\lambda_1)(r)=(S\lambda_2)(r)=1/2$ (see statement (i) of Lemma 1) we have that:

$$
\begin{aligned}
(SF)(r)F'(r) &= -\tau\kappa(r)h'(\lambda_2(r))\left[(Sh)(\lambda_2(r))\kappa(r)^2-\frac{1}{2}\right],\\
&= \tau(Sg_2)(r)g_2'(r).\\
(SG)(r)G'(r) &= \tau\kappa(r)f'(\lambda_1(r))\left[(Sf)(\lambda_1(r))\kappa(r)^2-\frac{1}{2}\right],\\
&= \tau(Sg_1)(r)g_1'(r).
\end{aligned}
$$

It is easy to see that $\chi'(r)=\tau[g_2'(r)-g_1'(r)]$ and we obtain:

$$
K_1(r)=\tau^2\Big[g_2'(r)-g_1'(r)\Big]\left(\sum_{i=1}^{2}(Sg_i)(r)g_i'(r)\right).
$$

By using the definition of Pre–Schwarz derivative we can deduce that $(P\lambda_1)(r)=(P\lambda_2)(r)=1$. Moreover, by using statement (i) of Lemma 1 we can see that:

$$
\begin{aligned}
K_2(r) &= -\tau^2\kappa^2(r)\mu'(\lambda_1(r))h'(\lambda_2(r))\left[\frac{h''(\lambda_2(r))}{h'(\lambda_2(r))}+\frac{f''(\lambda_1(r))}{f'(\lambda_1(r))}\right]^2,\\
&= \tau^2\left[\prod_{i=1}^{2}g_i'(r)\right]\left[\sum_{i=1}^{2}(-1)^i(Pg_i)(r)\right]^2.
\end{aligned}
$$

In consequence $(S\chi)(r)<0$ for any $r\in I$ if and only if $K_1(r)<\frac{3}{2}K_2(r)$ for any $r\in I_\tau$ which is equivalent to inequality (7).

Now, we consider $f=f_2,f_3$ (which imply $g_1=g_{1_2},g_{1_3}$). Moreover, notice that statement (ii) of Lemma 1 combined with $F_1=F$ and $F_2=R$ implies that:

$$
(S\chi)\chi'^2=\underbrace{(SF)F'\chi'-(SR)R'\chi'}_{K_1(r)}-\frac{3}{2}\underbrace{F'R'\Big[\frac{F''}{F'}-\frac{R''}{R'}\Big]^2}_{K_2(r)}
$$

As $R=G$ for any $r\leq 0$, we will consider only the case $r>0$. By using the fact that $(S\lambda_1)(r)=(S\lambda_2)(r)=1/2$ (see statement (i) of Lemma 1) we have that:

$$
\begin{aligned}
(SF)(r)F'(r) &= -\tau\kappa(r)h'(\lambda_2(r))\left[(Sh)(\lambda_2(r))\kappa(r)^2-\frac{1}{2}\right],\\
&= \tau(Sg_2)(r)g_2'(r).
\end{aligned}
$$

$$(SR)(r)R'(r) = \tau(Sg_1)(r)g_1'(r) = 0.$$

By using the fact that $\chi(r) = \tau[g_2(r) - g_1(r)]$, it is easy to see that:

$$K_1(r) = \tau^2\Big[g_2'(r) - g_1'(r)\Big]\left(\sum_{i=1}^{2}(Sg_i)(r)g_i'(r)\right).$$

By using the definition of Pre-Schwarz derivative we can deduce that $(P\lambda_1)(r) = (P\lambda_2)(r) = 1$. Moreover by using statement (i) of Lemma 1 we can see that:

$$\frac{F''(r)}{F'(r)} = -1 - \kappa(r)\frac{h''(\lambda_2(r))}{h'(\lambda_2(r))} = (Pg_2)(r),$$

$$\frac{R''(r)}{R'(r)} = -\frac{2G''(0)}{2G'(0) - G''(0)r} = (Pg_1)(r).$$

Moreover:

$$\begin{aligned} F'(r)R'(r) &= -\tau^2\kappa(r)h'(\lambda_2(r))\frac{4G'(0)^3}{[2G'(0) - G''(0)r]^2}, \\ &= -\tau^2\frac{4\kappa(r)h'(\lambda_2(r))[\kappa(0)f'(s^*)]^3}{\kappa(0)^2[2f'(s^*) - \{f'(s^*) - f''(s^*)r\}]^2}, \\ &= \tau^2\prod_{i=1}^{2}g_i'(r) \end{aligned}$$

It is easy to see that

$$\begin{aligned} K_2(r) &= \tau^2\left[\prod_{i=1}^{2}g_i'(r)\right]\left[1 + \kappa(r)\frac{h''(\lambda_2(r))}{h'(\lambda_2(r))} - \frac{G''(0)}{2G'(0) - G''(0)r}\right]^2, \\ &= \tau^2\left[\prod_{i=1}^{2}g_i'(r)\right]\left[\sum_{i=1}^{2}(-1)^i(Pg_i)(r)\right]^2. \end{aligned}$$

In consequence $(S\chi)(r) < 0$ for any $r \in I$ if and only if $K_1(r) < \frac{3}{2}K_2(r)$ for any $r \in I_\tau$ which is equivalent to inequality (7).

Applying Proposition 1 (see Appendix D) to map $\chi\colon I \mapsto I$ we conclude that 0 is a global attractor of χ, hence as $[m, M] \subset \chi^k([m, M]) \to \{0\}$ when $k \to +\infty$ implies that $m = M = 0$ and the Theorem follows.

5 Proof of Theorem 2

The proof is similar to the proof of Theorem 1 until the definition of the auxiliary decreasing function $\chi\colon \mathbb{R} \mapsto \mathbb{R}$. As before, it is straightforward to verify that $\chi(+\infty) < 0$ and $\chi^2(+\infty) > 0$, hence the map $\chi\colon I \mapsto I$ (where $I = [\chi(+\infty), \chi^2(+\infty)]$) is well defined.

As before, it can be proved by mathematical induction that:

$$[m, M] \subset \chi([m, M]) \subset \ldots \subset \chi^k([m, M])$$

for any integer $k \geq 1$.

By using the inequality $\tau < \tau_b^*$ for any $r \in I_\tau$, it follows that $|\chi'(r)| < 1$ and consequently we have that:

$$[m, M] \subset \lim_{k \to +\infty} \chi^k([m, M]) = 0$$

which implies $m = M = 0$ and the theorem follows.

Remark 7 *A careful reading of our proof of theorems* 1 *and* 2 *shows that we can generalize our result for any* $\mathcal{C}^3$ *function* f *satisfying the following properties*:

- $f(0) = 0$, $f'(0) > 0$,$f''(0) < 0$,
- f *can have at most one maximum* $s_{\max} > 0$ *and only one inflection point* $s_c > s_{\max}$,
- $(Sf)(r) < 0$ *for any* $r \neq s_{\max}$.

6 Numerical Examples

Let us come back to the asymptotic stabilization problems stated in the section 2.2.

6.1 Depollution of phenol in the water

We will consider biological degradation of phenol in the water by using *Pseudomonas putida*, which growth is described by the function:

$$f(s) = \mu_{max} \frac{s}{k_s + s^2}$$

where the parameters are defined in the Figure 6.1 (see also [20]):

Our goal is to stabilize the phenol concentration in a neighborhood of $s^* = 0.55$mg/L, for this task we build the feedback control law:

$$h(y(t)) = 4.1357 + 4.13 \tanh\big(s^* - s(t - \tau)\big).$$

It is straightforward to verify that **(P1)**–**(P5)** are satisfied, indeed notice that $(Sh)(r) = -2$ (for any $r \in \mathbb{R}$) and a simple computation shows:

$$([s_{in} - s^*][h'(0) + f'(s^*)])^{-1} = 0.889.$$

Parameter	Value	Units
$\mu_{\max}$	15.96	Day^{-1}
k_s	1.82	mg/L
s_{in}	7	mg/L
α	1	non–dimensional

Figure 6: Parameters for depollution problem

We can see with the help of computer that $0.889 < \tau_a^*$, that leads the sufficient condition $\tau < 0.889$ to solve Problem 1.

Numerical simulations were carried out using DDE23 [17] (we only show the results for nitrate concentration) and considering initial conditions $(\varphi_1, \varphi_2) = (2.14, 0.14)$. We give some results considering several delays τ. Notice that our sufficient condition can be improved because for delays $\tau \in [0.889, \tau_0)$ (with $\tau_0 \approx 1.03729$) the solution E_1 is still globally stable. Nevertheless, notice that when the size of delay increases, the speed of convergence towards s^* becomes slow (see Figure 7).

Moreover, when τ passes through the critical value τ_0, the point E_1 loses its stability and a periodic solution appears (see Figure 8).

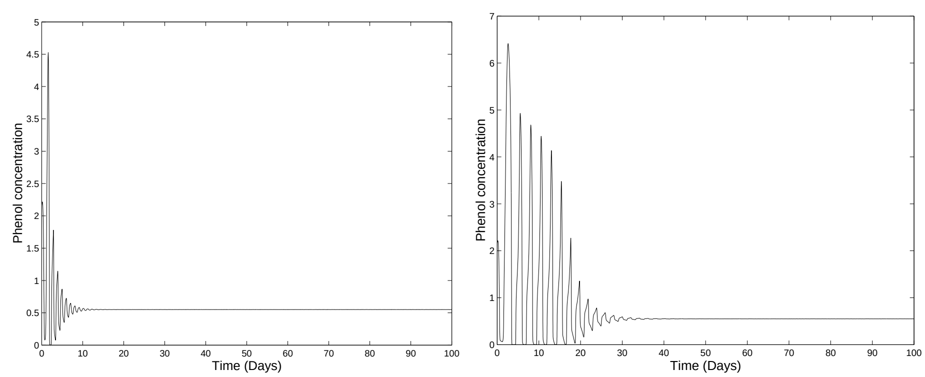

Figure 7: Output of system: $\tau = 0.5$ (left) and $\tau = 1.0$ (right)

6.2 Culture of Phytoplankton

We will consider *Dunaniella tertiolecta* growth in a chemostat by using nitrate as limiting substrate. We will work with a growth function given by the Michaelis–Menten function

$$f(s) = \mu_m \frac{s}{k_s + s}$$

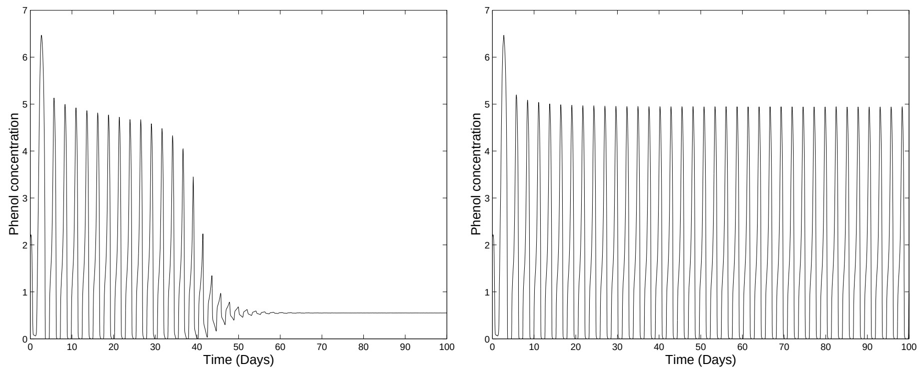

Figure 8: Outputs of system: $\tau = 1.03$ (left) and $\tau = 1.04$ (right)

where the parameters are shown in Figure 6.2 (see also [23]):

Parameter	Value	Units
$\mu_{\max}$	1.6	Day^{-1}
k_s	0.02	μatg/L
s_{in}	2	μatg/L
α^{-1}	1	non–dimensional

Figure 9: Parameters for the culture of phytoplankton problem

Our goal is to stabilize the nitrate concentration in a neighborhood of $s^* = 0.8$ (μatg/Liter), for this task we build the feedback control law:

$$h(y(t)) = 1.561 + \tanh\big(s^* - s(t-\tau)\big).$$

It is straightforward to verify that **(P1)**–**(P4)** are satisfied, indeed notice that $(Sh)(r) = -2$ (for any $r \in \mathbb{R}$) and a simple computation shows:

$$([s_{in} - s^*][h'(0) + f'(s^*)])^{-1} = 0.795.$$

We can see with the help of computer that $0.795 < \tau_a^*$, that leads the sufficient condition $\tau < 0.795$ to solve Problem 1.

Numerical simulations were carried out using DDE23 [17] (we only show the results for nitrate concentration) and considering initial conditions $(\varphi_1, \varphi_2) = (0.3, 0.1)$. We give some results considering several delays τ. Notice that our sufficient condition can be improved because for delays $\tau \in [0.592, \tau_0)$ (with $\tau_0 \approx 1.349$) the solution E_1 is still globally stable.

Nevertheless, notice that when the size of delay increases, the speed of convergence towards s^* is lower (see Figure 10).

When τ passes through the critical value τ_0, the point E_1 loses its stability and a periodic solution appears (see Figure 11). From this example, it is obvious that to improve our sufficient condition (upper bound for the delay τ) is of utmost importance to solve our asymptotic stabilization problem.

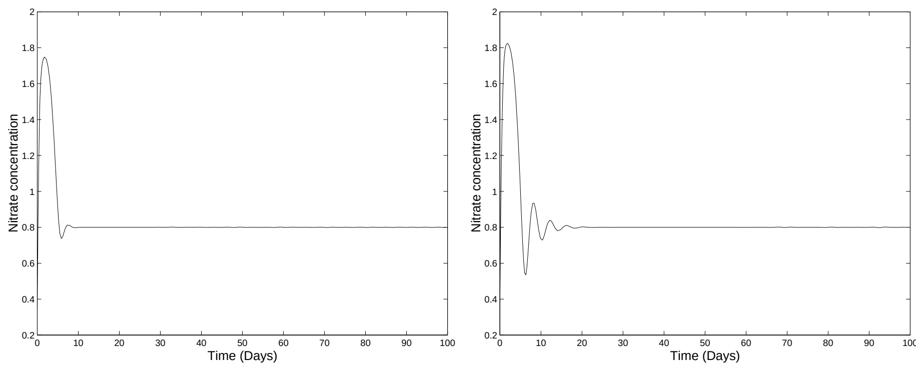

Figure 10: Output of system: $\tau = 0.592$ (left) and $\tau = 0.9$ (right)

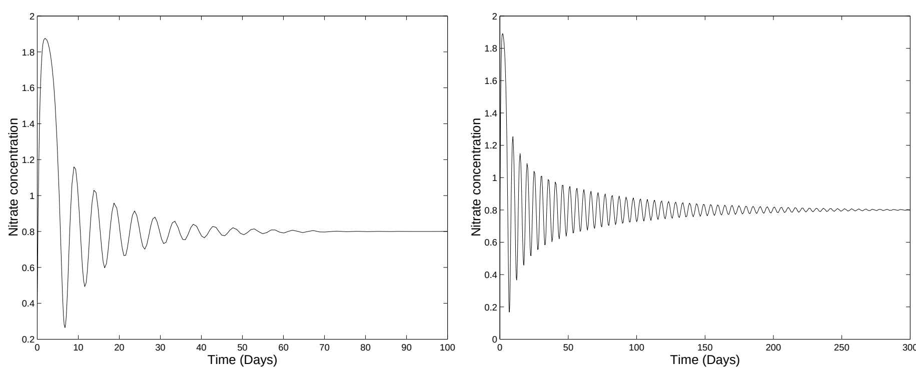

Figure 11: Output of system: $\tau = 1.2$ (left) and $\tau = 1.32$ (right)

7 Discussion and future work

A problem of asymptotic stabilization for a chemostat model with delays in its outputs has been considered. The model is described by Eq.(1) and a sufficient condition (for the global asymptotic stability of a substrate concentration s^*) is obtained. Moreover, there exist some interesting mathematical and control issues related to this work.

From a mathematical point of view, we pointed out before that the solutions of Eq.(16) satisfies the Poincaré–Bendixson theorem, we must rule out the existence of periodic solutions. In this direction, the study of the characteristic equation of Eq.(16):

$$\lambda + G'(0) - F'(0)e^{-\lambda} = 0 \tag{18}$$

suggests to us some interesting problems:

(a) There exist a relationship between the size of delay $\tau > 0$ and the convergence toward s^*. As we pointed out in section 3.6, the sufficient condition is not optimal. A possible optimal condition could be given by the study of roots of (18): if all roots have negative real part, are the solutions of Eq.(16) convergent to 0?.

(b) Moreover, considering the practical applications of this control problem, we are interested in to find sufficient conditions to ensure a fast convergence toward s^*, in this sense the study of the roots of Eq.(18) could help us: if all roots have only real part (property related with the size of τ), are the solutions non oscillatory? are the solutions super exponential? (see *e.g.* [3]).

(c) As we pointed out in the section 1.5 of the Introduction, the estimation of parameters for growth functions f_i is a difficult task, in general they are not well known which trigger the study of delay equations of type

$$\dot{u}(t) = -G(u(t)) + F(u(t-1)) + w(t, u(t)),$$

where the function w reflect the uncertainties in the estimation of the parameters of functions f_i. This type of equations has been studied in several works (see *e.g.* [14] and the references therein) and it will be extremely desirable to extend our result working on robust stabilization problems.

From a point of view of control theory, we can see that some classic control strategies as proportional regulators are still effective –up to a threshold– against delays in the outputs. Nevertheless, we must take into account the following problems.

(a) It is necessary to find sufficient conditions relating speed of convergence with some delays.

(b) It is necessary to generalize our approach for outputs of type:

$$y(t) = s(t-\tau)[1 + \Delta_1] + \Delta_2$$

where Δ_i $(i = 1, 2)$ are perturbations (deterministic or stochastic).

(c) It is suggested in [4, Chapt. 6] that the implementation of proportional integral (PI) and proportional integral differential (PID) regulators could be employed in this problem. Moreover, there exists other approaches to solving this problem, mainly the use of Smith predictors (see *e.g.*[16] and the references given there). It will be interesting to compare the efficiency of these approaches.

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Appendix: Proof of statement (ii) of Lemma 1

Let $\chi(r) = F_1(r) - F_2(r)$, we will prove that the Schwarz derivative for χ is defined by:

$$(S\chi) = \frac{1}{(\chi')^2}\left\{(SF_1)F_1'\chi' - (SF_2)F_2'\chi' - \frac{3}{2}F_1'F_2'\left[\left(\frac{F_1''}{F_1'}\right) - \left(\frac{F_2''}{F_2'}\right)\right]^2\right\}. \tag{19}$$

Indeed, by definition we have that

$$(S\chi) = \frac{F_1''' - F_2'''}{F_1' - F_2'} - \frac{3}{2}\left[\frac{F_1'' - F_2''}{F_1' - F_2'}\right]^2.$$

Hence

$$\begin{aligned}
(S\chi) &= \frac{1}{(\chi')^2}\left\{\big(F_1''' - F_2'''\big)\big(F_1' - F_2'\big) - \frac{3}{2}\big(F_1'' - F_2''\big)^2\right\} \\
&= \frac{1}{(\chi')^2}\left\{F_1'''F_1' - F_1'''F_2' - F_2'''F_1' + F_2'''F_2' - \frac{3}{2}\left[(F_1'')^2 - 2F_1''F_2'' + (F_2'')^2\right]\right\} \\
&= \frac{1}{(\chi')^2}\left\{\underbrace{F_1'''F_1' - \frac{3}{2}(F_1'')^2}_{\text{(I)}} + \underbrace{F_2'''F_2' - \frac{3}{2}(F_2'')^2}_{\text{(II)}} - \underbrace{\left[F_1'''F_2' + F_2'''F_1'\right]}_{\text{(III)}} + 3F_1''F_2''\right\}.
\end{aligned}$$

We will study the terms (I),(II) and (III). Notice that (I) and (II) are respectively equivalent to

$$F_1'''(r)F_1'(r) - \frac{3}{2}\big(F_1''(r)\big)^2 = \left[F_1'''(r)F_1'(r) - \frac{3}{2}\big(F_1''(r)\big)^2\right]\left(\frac{F_1'(r)}{F_1'(r)}\right)^2 = (SF_1)(r)[F_1'(r)]^2,$$

$$F_2'''(r)F_2'(r) - \frac{3}{2}\big(F_2''(r)\big)^2 = \left[F_2'''(r)F_2'(r) - \frac{3}{2}\big(F_2''(r)\big)^2\right]\left(\frac{F_2'(r)}{F_2'(r)}\right)^2 = (SF_2)(r)[F_2'(r)]^2.$$

Finally, notice that (III) is equivalent to

$$F_1'''F_2' + F_2'''F_1' = F_1'''F_2'\Big[\frac{F_1'}{F_1'}\Big] + F_2'''F_1'\Big[\frac{F_2'}{F_2'}\Big] = F_1'F_2'\Big\{\frac{F_1'''}{F_1'} + \frac{F_2'''}{F_2'}\Big\}$$

$$= F_1'F_2'\Big\{SF_1 + SF_2 + \frac{3}{2}\Big[\Big(\frac{F_1''}{F_1'}\Big)^2 + \Big(\frac{F_2''}{F_2'}\Big)^2\Big]\Big\}$$

Hence, replacing (I),(II) and (III) we obtain that

$$(S\chi) = \frac{1}{(\chi')^2}\left\{(SF_1)[F_1']^2 + (SF_2)[F_2']^2 - F_1'F_2'(SF_1 + SF_2) - \frac{3}{2}F_1'F_2'\Big[\Big(\frac{F_1''}{F_1'}\Big)^2 + \Big(\frac{F_2''}{F_2'}\Big)^2\Big] + 3F_1''F_2''\right\}$$

$$= \frac{1}{(\chi')^2}\left\{(SF_1)[F_1'](\chi') - (SF_2)[F_2'](\chi') - \underbrace{\left(\frac{3}{2}F_1'F_2'\Big[\Big(\frac{F_1''}{F_1'}\Big)^2 + \Big(\frac{F_2''}{F_2'}\Big)^2\Big] - 3F_1''F_2''\right)}_{\text{(IV)}}\right\}.$$

Notice that (IV) is equivalent to

$$\frac{3}{2}F_1'F_2'\Big[\Big(\frac{F_1''}{F_1'}\Big)^2 + \Big(\frac{F_2''}{F_2'}\Big)^2\Big] - 3F_1''F_2'' = \frac{3}{2}F_1'F_2'\Big[\Big(\frac{F_1''}{F_1'}\Big)^2 + \Big(\frac{F_2''}{F_1'}\Big)^2\Big] - 2\frac{3}{2}\frac{F_1''F_2''}{F_1'F_2'}F_1'F_2'$$

$$= \frac{3}{2}F_1'F_2'\left[\Big(\frac{F_1''}{F_1'}\Big)^2 - 2\frac{F_1''}{F_1'}\frac{F_2''}{F_2'} + \Big(\frac{F_2''}{F_2'}\Big)^2\right]$$

$$= \frac{3}{2}F_1'F_2'\left[\Big(\frac{F_1''}{F_1'}\Big) - \Big(\frac{F_2''}{F_2'}\Big)\right]^2$$

and the Eq.(19) follows.

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