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INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

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Abstract: On a locally finite point set, a navigation defines a path through the point set from a point to an other. The set of paths leading to a given point defines a tree, the navigation tree. In this article, we analyze the properties of the navigation tree when the point set is a Poisson point process on \mathbb{R}^d . We examine the distribution of stable functionals, the local weak convergence of the navigation tree, the asymptotic average of a functional along a path, the shape of the navigation tree and its topological ends. We illustrate our work in the small world graphs, and new results are established. This work is motivated by applications in computational geometry and in self-organizing networks.

Key-words: geometric random trees, local weak convergence, small world phenomenon, stochastic geometry

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Navigation sur un processus ponctuel de Poisson

Résumé : Sur un ensemble de points localement fini, une navigation construit un chemin sur cet ensemble qui relie un point à un autre. L'ensemble des chemins aboutissant à un point donné définit un arbre : l'arbre de navigation. Dans ce article, nous analysons les propriétés de l'arbre de navigation lorsque l'ensemble de points est un processus ponctuel de Poisson sur \mathbb{R}^d . Nous examinons la distribution de fonctionnelles stables, la convergence faible locale de l'arbre de navigation, la moyenne asymptotique d'une fonctionnelle le long d'un chemin, la forme de l'arbre de navigation et ses fins topologiques. Nous illustrons notre travail sur les graphes de type "small world" et nous y établissons de nouveaux résultats. Ce travail est motivé par des applications en géométrie computationnelle et dans les réseaux auto-organisés.

Mots-clés : arbres géométriques aléatoires, convergence faible locale, phénomène "Small World", géométrie stochastique

1 Introduction

1.1 Navigation: definition and perspective

In this work, we examine decentralized navigation algorithms on random graphs. Let N be a locally finite point set and O a point in \mathbb{R}^d , taken as the origin. For $x, y \in \mathbb{R}^d$, $|x|$ will denote the Euclidian norm and $\langle x, y \rangle$ the usual scalar product. $B(x, r)$ is the open ball of radius r and center x , and $S^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$ is the d -dimensional hyper-sphere.

Definition 1 Assume that $O \in N$, a navigation (with root O) is a mapping \mathcal{A} from N to N such that for all X in N there exists a finite k satisfying $\mathcal{A}^k(X) = O$. A navigation on a graph $\mathcal{G} = (N, E)$ is a navigation such that $(X, \mathcal{A}(X)) \in E$.

From a navigation with root O , we can define a navigation with root Y by $\mathcal{A}_Y(X; N) = Y + \mathcal{A}(X - Y; S_{-Y} \circ N)$ where S_x is the translation by x : if $B \subset \mathbb{R}^d$, $S_x B = \{y : y - x \in B\}$.

The aim of this work is to analyze the *decentralized navigation* algorithms. For a navigation defined on a graph \mathcal{G} , a decentralized navigation is such that $\mathcal{A}_Y(X)$ depends only on X, Y and the set of vertices adjacent to X in \mathcal{G} . A navigation is always decentralized on the complete graph, so the meaning of this definition is unclear and it is not intrinsic to \mathcal{A} , we will give later a better definition.

Navigation algorithms have emerged recently in papers in four different classes of problems (at least). A first class of problem which has recently drawn much attention is the small world phenomenon. As it is pointed by Kleinberg [18], the small world phenomenon relies on the existence of shortcuts in a decentralized navigation on a small world graph. Extension and refinements of his results have been carried out by Franceschetti and Meester [12], Ganesh et al. [14], [8].

A second field of application is computational geometry. Kranakis, Singh and Urrutia [20] have introduced the compass routing (some numerous variants exist). The Ph.D. Thesis of Morin [24] gives a review of this class of problems. Computer scientists do not analyze the probabilistic properties of navigation algorithms, they rather examine if a given algorithm is a proper navigation, that is if it converges in a finite number of hops to its root.

The ideas of computational geometry may benefit the design of real world networks. A first field of application is sensor and ad-hoc networks, see for example the survey of Akyildiz et al. [1] or the work of Ko and Vaidya [19]. A second application is self-organized overlay and peer-to-peer networks. Each node in the network receives a virtual coordinate in some naming space, and the messages are routed along a geometric navigation algorithm, see Plaxton, Rajaraman and Richa [29], Liebeherr, Nahas and Si [22] or Kermarrec, Massoulié and Ganesh [13].

Lastly, in the probabilistic literature a few authors have examined decentralized navigation algorithms (under other names). Baccelli, Tchoumatchenko and Zuyev [4] have analyzed a navigation on the Delaunay graph. Others examples include the Poisson Forest of Ferrari, Landim and Thorisson [11] and the Directed Spanning Forest introduced by Gangopadhyay Roy and Sarkar [15] (see also Penrose and Wade [25] and Baccelli and Bordenave [9]).

The aim of the present work is to find a unified approach to these problems. The mathematical material used in this work is a natural extension of the ideas developed in [9].

We give three canonical examples of navigation. Among those three, only the last will draw our attention. These examples are nevertheless useful to understand the context better.

A natural navigation is the *shortest path* on a graph $\mathcal{G} = (N, E)$. Let g be a functional on E (g is a cost function) and let $\Pi(X, Y)$ denote the set of paths in \mathcal{G} from X to Y , i.e. the finite sequences of vertices in N (X_0, \dots, X_k) such that $X_0 = X$, $(X_i, X_{i+1}) \in E$ $0 \leq i \leq k-1$ and $X_k = Y$. Provided that it is well defined the shortest path is given by:

$$\pi^*(X, Y) = \arg \inf_{(X_0, \dots, X_k) \in \Pi(X, Y)} \sum_{i=0}^{k-1} g(X_i, X_{i+1}).$$

If $\pi^*(X, Y) = (X_0^*, \dots, X_k^*)$, the shortest path navigation is given by $\mathcal{A}_Y^*(X) = X_1^*$. For $g = 1$, the shortest path is the path which minimizes the number of hops from X to Y in \mathcal{G} : it corresponds to the graph distance between X and Y . Shortest paths on specific graphs have drawn much attention. In particular Vahidi-Asl and Wierman ([31],[32]) have studied the shortest path on the Delaunay graph of a Poisson point process for $g = 1$, see also Pimentel [28]. On the complete graph of a Poisson point process and $g(X, Y) = |X - Y|^\beta, \beta > 2$ an in-depth analysis was performed by Howard and Newman in [17]. The shortest path is the continuum analog of the celebrated *first passage percolation* on the regular \mathbb{Z}^d -lattice. The shortest path navigation has poor decentralization properties, nevertheless it gives the best achievable performance of a decentralized navigation.

A *random walk* on \mathcal{G} is a decentralized navigation provided that the random walk is recurrent: the length of the path is the hitting time of Y starting from X . In the recurrent case, this hitting time is almost surely finite for all pairs (X, Y) . However, on an infinite graph, even in the recurrent case, one might expect that the walk is null recurrent: the expectation of the length is infinite. Therefore random walks are not expected to provide interesting navigation algorithms. More efficient decentralized navigation algorithms exist.

An important decentralized navigation is the *maximal progress navigation*. If \mathcal{A} is a navigation, the progress is defined as:

$$P_Y(X) = |X| - |\mathcal{A}_Y(X)|.$$

An appealing class of decentralized navigation is the subclass of navigation such that the progress is non-negative for all X . On a graph $\mathcal{G} = (N, E)$, the maximal progress navigation is the navigation which maximizes the progress: $\mathcal{A}(X) = Y$ if $(X, Y) \in E$ and $|Y|$ is minimal. Note that the maximal progress navigation will not be a proper navigation on all graphs, some additional properties on the graph (or on the point set) have to be added. Some breaking ties rules should also be defined to guarantee the uniqueness of this navigation.

1.2 Directed Navigation

A navigation links a point X to another Y . When X is far from Y , the progress made is roughly equal to $\langle X - \mathcal{A}_Y(X), X - Y \rangle / |X - Y|$ that is the progress made along an axis

with direction $Y - X$. Hence in most circumstances, we expect that a navigation behaves asymptotically as a *directed navigation*. Let $e_1 \in S^{d-1}$, a directed navigation with direction e_1 is a mapping \mathcal{A}_{e_1} from N to N such that for all X in N , $\lim_k \langle \mathcal{A}_{e_1}^k(X), e_1 \rangle = +\infty$.

On a graph $\mathcal{G} = (N, E)$, a directed navigation is a directed navigation such that for all $X \in N$, $(X, \mathcal{A}_{e_1}(X)) \in E$.

As pointed above, directed navigation will appear as natural limiting objects for navigation. We will actually see later what type of convergence has to be considered.

The *directed progress* is defined as:

$$P_{e_1}(X) = \langle \mathcal{A}_{e_1} X, e_1 \rangle - \langle X, e_1 \rangle.$$

A few examples of decentralized directed navigation may be found in the literature: directed path on the Delaunay tessellation [4], the Poisson forest [11], the directed spanning forest [15], [25], [9].

On a graph, we also define the *maximal directed progress navigation* as the navigation which maximizes the directed progress. The maximal directed progress navigation is the limiting object for the maximal progress navigation.

1.3 Navigation Tree and Navigation Graph

Assume that $O \in N$, a navigation \mathcal{A} to the origin O defines a graph: the *navigation tree* which will be denoted by $\mathcal{T}_0 = (N, E_0)$. It is defined by

$$(X, Y) \in E_0 \text{ if } \mathcal{A}(X) = Y \text{ or } \mathcal{A}(Y) = X.$$

It is easily checked that \mathcal{T}_0 is actually a tree: if there were a loop it would be contradictory with the assumption that $\mathcal{A}(X)^k = O$ for k large enough. \mathcal{T}_0 is the union of all the paths from $X \in N$ to O . Note that \mathcal{T}_0 is a *spanning tree* of N .

For a directed navigation, we define similarly the *directed navigation forest*, $\mathcal{T}_{e_1} = (N, E_{e_1})$ by

$$(X, Y) \in E_{e_1} \text{ if } \mathcal{A}_{e_1}(X) = Y \text{ or } \mathcal{A}_{e_1}(Y) = X.$$

We check similarly that \mathcal{T}_{e_1} is a forest. We will prove that \mathcal{T}_{e_1} is the natural limit of \mathcal{T}_0 for the local weak convergence of Aldous and Steele [2].

Extending the navigation tree to the origin to any point of N , we can also define the *navigation graph* $\cup_{Y \in N} \mathcal{T}_Y$ and the directed navigation graph $\cup_{e_1 \in S^{d-1}} \mathcal{T}_{e_1}$. These two graphs record the set of possible navigation from one point to another (or in a direction).

At this stage we can state an intrinsic definition for a decentralized navigation:

Definition 2 *A navigation \mathcal{A} (to the root O) is decentralized if $\mathcal{A}(X)$ depends only of X , O and the edges adjacent to X in \mathcal{T}_0 .*

With this definition a shortest path navigation is not a decentralized algorithm, whereas a maximal progress navigation is decentralized.

1.4 Poisson Point Process and Poisson Weighted Infinite Tree

We will pay attention to $\mathcal{A}^k(X)$ on a locally finite point set containing X and 0 , and respectively for a directed navigation, to $\mathcal{A}_{e_1}^k(0)$ where $e_1 \in S^{d-1}$ and 0 is a point of the point set. In our analysis, we will prove convergence results for two types of probabilistic models.

The first model is the usual *Poisson point set*. N is a Poisson point process (PPP) of intensity one on \mathbb{R}^d . We will denote: $N^0 = N + \delta_0$ and $N^{0,X} = N + \delta_X + \delta_0$. From Slyvniak-Mecke Theorem, N^0 (resp. $N^{0,X}$) is a PPP on its Palm version at 0 (resp. $(0, X)$). Intuitively, N^0 (resp. $N^{0,X}$) can be understood as a PPP conditioned on having an atom at 0 (resp. atoms at 0 and X). It is not a restriction to assume that the intensity of the PPP is one, with a proper rescaling, our results extend to any positive intensity. Indeed, if $N = \sum_{n \in \mathbb{N}} \delta_{T_n}$ is a realization a PPP of intensity one, then $N_\lambda = \sum_{n \in \mathbb{N}} \delta_{\lambda^{-1/d} T_n}$ is a PPP of intensity $\lambda > 0$.

The second model is the *Poisson Weighted Infinite Tree model*. Following the brilliant approach of Meester and Franceschetti in [12], we will try to understand the intrinsic behavior of a navigation through a virtual model which is the simplest possible probabilistic model. To this end we build a Poisson weighted infinite tree (PWIT) which is a slight variation of Aldous' PWIT [2]. We fix a root $X \in \mathbb{R}^d$ and define the PWIT $T^{0,X}$ as follows. The points of $N^{0,X} \setminus \{X\}$ are the vertices of first generation in T^X and the weight of the edge $(X, Y), Y \in N^{0,X} \setminus \{X\}$ is equal to $|X - Y|$. $T^{0,X}$ is defined iteratively at each generation: at each vertex Y the subtree rooted at Y consisting of all descendants of Y is a PWIT $T^{0,Y}$ and the Poisson point processes are drawn independently of the others. Note that there is a vertex located at 0 at each positive generation. Thus each generation has a different copy of the origin in order to guarantee that $T^{0,X}$ is indeed a tree.

For a decentralized navigation, it is important to note that the distribution of $(X, \mathcal{A}(X))$ is the same in the PWIT $T^{0,X}$ and in the PPP $N^{0,X}$. However the joint distribution of $(\mathcal{A}^k(X))_{k \in \mathbb{N}}$ is not the same in the PWIT and the PPP. It is much simpler on the PWIT.

For a directed navigation \mathcal{A}_{e_1} , let $X_k = \mathcal{A}_{e_1}^k(X)$ and $\mathcal{F}_k = \sigma\{X_0, \dots, X_k\}$. A key feature of the PWIT is the relation

$$\mathbb{P}((X_k, X_{k+1}) \in \cdot | \mathcal{F}_k) = \mathbb{P}((0, \mathcal{A}_{e_1}(0)) \in \cdot). \quad (1)$$

This last property is the (*spatial*) *memoryless* property of the directed navigation on the PWIT. Similarly, for a navigation \mathcal{A} and $X_k = \mathcal{A}^k(X)$, we have:

$$\mathbb{P}((X_k, X_{k+1}) \in \cdot | \mathcal{F}_k) = \mathbb{P}((X_k, X_{k+1}) \in \cdot | X_k), \quad (2)$$

the sequence $(X_k, X_{k+1}), k \in \mathbb{N}$, is a Markov chain with $(0, 0)$ as absorbing state. With an abuse of terminology we will call also this last property the *memoryless* property of a navigation on a PWIT. More generally for a navigation on a PPP, we introduce the following definitions which are the core of this work.

Definition 3 - *A navigation \mathcal{A} is a memoryless navigation if Equation (2) holds (and respectively for a directed navigation with Equation (1)).*

- A navigation is regenerative if there exists a stopping-time (on a enlarged probability space) $\theta > 0$ such that \mathcal{A}^θ is a memoryless navigation and the distribution of $\theta(X)$ is independent of X for $|X| \geq x_0$ (and respectively for a directed navigation).

The stopping time θ will be called a *regenerative time*. If there exists a regenerative time, there exists an increasing sequence $(\theta_n), n \in \mathbb{N}$, which we will call a *regenerative sequence* such that $\theta_0 = 0$, the distribution of $(\theta_{n+1} - \theta_n)_{n \in \mathbb{N}}$ is iid and for $|X| \geq x_0$

$$\mathbb{P}((X_{\theta_n}, X_{\theta_{n+1}}) \in \cdot | \mathcal{F}_{\theta_n}) = \mathbb{P}((X_{\theta_n}, X_{\theta_{n+1}}) \in \cdot | X_{\theta_n}).$$

Respectively for a directed navigation, we will have $\theta_0 = 0$, the distribution of $(\theta_{n+1} - \theta_n)$ is iid and

$$\mathbb{P}((X_{\theta_n}, X_{\theta_{n+1}}) \in \cdot | \mathcal{F}_{\theta_n}) = \mathbb{P}((X_0, X_{\theta_1}) \in \cdot).$$

A memoryless navigation will be much simpler to analyze. We will prove that under some assumptions that a navigation on a PPP will contain a regenerative sequence, that is an embedded memoryless navigation. This idea is the cornerstone of this work.

All the examples of navigation algorithms we have in mind satisfy the following property:

$$\mathcal{A}(X) \text{ is } \mathcal{F}_{B(0, |X|)}^N \text{-measurable,}$$

where for a Borel set B , \mathcal{F}_B^N is the smallest σ -algebra such that the point set $N \cap B$ is measurable. A sufficient condition for this type of navigation to be memoryless is that for all $t \in \mathbb{N}$ and all Borel sets A :

$$\text{If } A \subset B(0, |X_k|) \text{ then } \mathbb{P}(N(A) = t | \mathcal{F}_k) = \mathbb{P}(N(A) = t), \quad (3)$$

in other word, $N \cap B(0, |X_k|)$ is a PPP of intensity 1.

1.5 Examples

1.5.1 Small world graphs

The *small world graph* is a graph $\mathcal{G} = (N^0, E)$ such that vertices $X \in N^0$ and $Y \in N^0$ are connected with probability $f(|X - Y|)$ independently of the other, and f is a non-increasing function with value in $[0, 1]$. We assume, as t tends to infinity, that:

$$f(t) \sim ct^{-\beta},$$

with $c > 0$ and $\beta > 0$. More formally, we add marks to N to obtain a marked point process: $\overline{N} = \sum_n \delta_{X_n, V_n}$, where $V_n = (V_{nm})_{m \in \mathbb{N}} \in [0, 1]^{\mathbb{N}}$ is independent of the collection N , $(V_{nm})_{m < n}$ is an iid sequence of uniform random variables on $[0, 1]$, $V_{nn} = 1$ and $V_{nm} = V_{mn}$. For X, Y in N , we will write $V(X, Y)$ for V_{nm} where n and m are the index of X and Y . The small world graph is defined by:

$$(X, Y) \in E \text{ if } V(X, Y) \leq f(|X - Y|).$$

Note that the degree of a vertex in the small world graph could not be finite for small choices of β (indeed for $\beta \leq d$).

The maximal progress navigation from $X \in N^0$ to 0 is defined as:

$$\mathcal{A}(X) = \arg \min\{|Y| : (X, Y) \in E\}.$$

As such, the small world graph has isolated points and navigation is ill-defined on non-connected graphs. To circumvent this difficulty three possibilities arise:

1. We enlarge slightly E to ensure a positive progress for $X \in N$. This is the approach followed by Ganesh et al. in [14], [8].
2. The marks V are not anymore independent of N , they are conditioned on the event that a positive progress is feasible at any point X of N .
3. Loops are allowed and the model is unchanged but if $\mathcal{A}(X) = X$ then a new set of neighbors for $X = \mathcal{A}(X)$ is drawn independently of everything else.

We will focus on model 2, models 1 and 3 will also be discussed.

The directed navigation with direction e_1 is defined similarly,

$$\mathcal{A}_{e_1}(X) = \arg \max\{\langle Y, e_1 \rangle : (X, Y) \in E^0\}.$$

Let $\mathcal{H}_{e_1}(x) = \{y \in \mathbb{R}^d : \langle y, e_1 \rangle > \langle x, e_1 \rangle\}$, we have to assume that the set of neighbors of X in \mathcal{H}_{e_1} are a.s. finite for the directed navigation to be properly defined (that is $\beta > d$).

1.5.2 Compass Routing on Delaunay Graph

Compass Routing and its numerous variants is a popular navigation in computer science. It was introduced by Kranakis et al. in [20], see also Morin [24]. Let $\mathcal{G} = (N^0, E)$ denote a locally finite connected graph. Compass routing on \mathcal{G} to 0 is a navigation defined by

$$\mathcal{A}(X) = \arg \max\left\{\left\langle \frac{X}{|X|}, \frac{X - Y}{|X - Y|} \right\rangle : (X, Y) \in E\right\},$$

In words: $\mathcal{A}(X)$ is the neighboring point of X in \mathcal{G} which is the closest in direction to the straight line $0X$. Compass routing is not a proper navigation on any graph, a variant of this routing called Face Routing is a proper navigation. As it is pointed by Liebeherr et al. in [22], on a Delaunay Graph Compass Routing is a proper navigation.

The associated directed navigation is naturally:

$$\mathcal{A}_{e_1}(X) = \arg \max\left\{\left\langle e_1, \frac{X - Y}{|X - Y|} \right\rangle : (X, Y) \in E\right\},$$

i.e. the direction of $(X, \mathcal{A}(X, e_1))$ is the closest from e_1 . The algorithm in Baccelli et al. [4] is closely related (but not equivalent).

1.5.3 Radial Navigation

Radial navigation was introduced in [9]. For $X, Y \in N^0$, $X \neq 0$, $|Y| < |X|$ it is defined as:

$$\mathcal{A}(X) = |Y| \quad \text{if} \quad N(B(X, |X - Y|) \cap B(0, |X|)) = \emptyset.$$

$\mathcal{A}(X)$ is the closest point from X which is closer from the origin. Radial navigation has an a.s. positive progress and $\mathcal{A}(X)$ is a.s. uniquely defined. The directed navigation associated to radial navigation is: if $X, Y \in N$ and $\langle Y - X, e_1 \rangle > 0$

$$\mathcal{A}_{e_1}(X) = Y \quad \text{if} \quad N(B(X, |X - Y|) \cap \mathcal{H}_{e_1}(X)) = \emptyset.$$

That is $\mathcal{A}_{e_1}(X)$ is the closest point from X which has a larger e_1 -coordinate.

The corresponding navigation tree is the radial spanning tree and it has been analyzed in [9]. The directed spanning forest is the directed navigation forest associated with \mathcal{A}_{e_1} . This model has been analyzed in [15], [25] and [9].

1.5.4 Road Navigation

Road navigation models a car on \mathbb{R}^d starting at a point X and driving to a destination point 0. A road $R(X, e)$ is the straight line passing through X with direction $e \in S^{d-1}$. The following model has been introduced by Baccelli (private communication).

We consider a family of probability distributions on S^{d-1} , $\{\Pi_X\}$, $X \in \mathbb{R}^d$. The starting point X is on a road R_0 with random direction $e(X)$ with distribution Π_X . It drives to the closest point on R_0 of 0: the orthogonal projection of 0 on R_0 . From this new point, say X_1 , a new road R_1 starts with direction independently drawn and distribution Π_{X_1} . The driver goes to X_2 , the closest point on R_1 of 0 and so on until it finally reaches its destination (if he ever does).

Note that if $\Pi_X(X^\perp) = 0$, where $X^\perp = \{e \in S^{d-1} : \langle e, X \rangle = 0\}$ then the road navigation has an a.s. positive progress. To be sure that the driver will finally manage to reach its destination we have to assume at least that there exists x_0 such that $\Pi_X(X/|X|) > 0$ for $|X| \leq x_0$.

Our work covers the particular case when the distribution $|\langle e(X), X/|X| \rangle|$ converges weakly as $|X|$ tends to infinity.

Generalizations of this model include higher dimensional roads (as hyperplanes) or even successive projections of the origin on more complex sets than straight lines. Note that adding more roads at each point and choosing the road with the best possible direction is already included in the original model.

Road navigation is not properly a navigation since its maps a point in \mathbb{R}^d to another point in \mathbb{R}^d . All the results presented for regular navigation also apply to road navigation. Road navigation is clearly memoryless.

Remark 1 In §7.1 (in Appendix), two other examples of navigation are given.

1.6 Overview of the Results

Our results are extension of the work [9] on the radial navigation. Throughout this paragraph, we will illustrate some of the results with the small world model 2. We will denote by

$$\pi_d = \frac{\pi^{d/2}}{\Gamma(d/2 + 1)} \quad \text{and} \quad \omega_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)},$$

the d -dimensional volume of $B(0, 1)$ and the d -dimensional surface area measure of S^{d-1} .

1.6.1 Local Weak Convergence of the Navigation Tree

In Section 2, we will state some general conditions under which the navigation tree converges to the directed navigation forest for the local weak convergence on graphs as it defined by Aldous and Steele in [2].

For a graph $\mathcal{G} = (N, E)$, we define $S_x \circ \mathcal{G} = (S_x N, E)$ as the graph obtained by translating all vertices N by x and keeping the same edges.

As an example, on the small world graph, let $\mathcal{T}_{e_1}(N)$ denote the directed navigation forest built on the point set N and $\mathcal{T}_0(N)$ the navigation tree built on the point set N .

Proposition 1 *Assume $\beta > d$ in the small world graph. If $|X_n|$ tends to $+\infty$ and $X_n/|X_n|$ to e_1 then $S_{-X_n} \circ \mathcal{T}_0(N^{0, X_n})$ converges to $\mathcal{T}_{-e_1}(N^0)$ for the local weak convergence.*

Let $F_X(t) = \mathbb{P}(P(X) \leq t)$ be the distribution function of the progress at X , and for $\beta > d$, let $F(t) = \mathbb{P}(P_{e_1}(0) \leq t)$ denote the distribution function of the directed progress. we will also show how to compute the distribution of local Using some basic stochastic geometry tools, we obtain the following proposition.

Proposition 2 *For the Model 3, assume $d \geq 2$, the following properties hold:*

1. *If $\beta > d$, as t goes to infinity:*

$$\bar{F}(t) \sim \frac{2c\omega_{d-2}}{\beta - d} t^{d-\beta} \int_0^{\pi/2} \cos^{\beta-d} \theta d\theta.$$

2. *If $\beta > d$, F_X converges weakly to F .*
3. *If $\beta = d$, let \tilde{F}_X be the distribution of $\tilde{P}(X) = -\ln(1 - P(X)/X) \in [0, +\infty]$, \tilde{F}_X converges weakly to \tilde{F} with $\int \tilde{F}(s) ds = \tilde{\mu} \in (0, +\infty)$.*
4. *If $d - 2 < \beta < d$, the distribution of $|\mathcal{A}(X)|/|X|^{1-\frac{d-\beta}{2}}$ converges weakly to a non degenerated distribution.*

The limit distribution in statements 3 and 4 are computed explicitly. For $d \geq 3$ and $0 < \beta < d - 2$, the same method can be used to prove a convergence of the properly scaled progress. The computation for $d = 1$ is simpler and the same result holds with different constants. Finally this proposition implies a similar result on Model 2, in statement 1, it suffices to divide by $\mathbb{P}(P(0) = 0) = \exp(-\int_{\mathcal{H}_{e_1}(0)} f(y)dy)$ and statements 2, 3, and 4 hold without change.

1.6.2 Path Average

The path from X to 0 in the navigation tree \mathcal{T}_0 is given by a sequence of vertices $\pi(X) = (X_0 = X, \dots, X_{H(X)} = 0)$ where $H(X)$ is the generation of X in \mathcal{T}_0 :

$$H(X) = \inf\{k : \mathcal{A}^k(X) = 0\}.$$

Let g be a measurable function from $\mathbb{R}^d \times \mathbb{R}^d$ to \mathbb{R} , $G(0) = 0$ and

$$G(X) = \sum_{k=0}^{H(X)-1} g(X_k, X_{k+1}) = g(X, \mathcal{A}(X)) + G(\mathcal{A}(X)). \quad (4)$$

In Section 3, we will state the various convergence results that can be expected for Equation (4) for a memoryless navigation. This amounts to analyze a non-homogeneous random walks.

In Section 4, analogous results for regenerative navigation will be obtained as corollaries.

In the PWIT model on the small world graph, Proposition 2 will imply a result on the convergence of $H(X)$ for all $\beta > d - 2$.

Proposition 3 - If $\beta > d + 1$ and $\mu = \int rF(dr)$, a.s.

$$\lim_{|X| \rightarrow \infty} \frac{H(X)}{|X|} = \frac{1}{\mu}.$$

- If $\beta = d + 1$ a.s.

$$\lim_{|X| \rightarrow \infty} \frac{H(X) \ln |X|}{|X|} = \frac{1}{c}.$$

- If $\beta \in (d, d + 1)$

$$\liminf_{|X| \rightarrow \infty} \mathbb{E} \frac{H(X)}{|X|^{\beta-d}} > 0 \quad \text{and} \quad \limsup_{|X| \rightarrow \infty} \mathbb{E} \frac{H(X)}{|X|^{\beta-d}} < \infty.$$

- If $\beta = d$ and $\tilde{\mu}$ as in Proposition 2 a.s.

$$\lim_{|X| \rightarrow \infty} \frac{H(X)}{\ln |X|} = \frac{1}{\tilde{\mu}}.$$

- If $d - 2 < \beta < d$, a.s.

$$\lim_{|X| \rightarrow \infty} \frac{H(X)}{\ln \ln |X|} = -\frac{1}{\ln(1 - \frac{d-\beta}{2})}.$$

1.6.3 How to prove that a navigation is regenerative ?

In Section 4, we explain a general method to prove that a navigation algorithm is regenerative. This original method relies on geometric properties of the navigation and tail bounds in the GI/GI/ ∞ queue.

As an example, we will prove that the small world navigation on a PPP has good regenerative properties for $\beta \leq d$ and $\beta > d + 2$. This will enable us to prove that $H(X)/|X|$ converges a.s. for $\beta > d + 2$, $H(X)/\ln |X|$ converges a.s. for $\beta = d$ and $H(X)/\ln \ln |X|$ converges a.s. for $d - 2 < \beta < d$. For $d - 2 < \beta \leq d$ we are able to compute the constant explicitly. A similar result would hold for $d \geq 3$ and $\beta \leq d - 2$. Our method fails in the case $d < \beta \leq d + 2$.

Proposition 4 - *If $\beta > d + 1$, \mathcal{A} is regenerative and the regenerative time has finite expectation.*

- *If $\beta > d + 2$ there exists $\mu > 0$ such that a.s.*

$$\lim_{|X| \rightarrow \infty} \frac{H(X)}{|X|} = \frac{1}{\mu}.$$

- *If $\beta = d$ and $\tilde{\mu}$ as in Proposition 2 a.s.*

$$\lim_{|X| \rightarrow \infty} \frac{H(X)}{\ln |X|} = \frac{1}{\tilde{\mu}}.$$

- *If $d - 2 < \beta < d$, a.s.*

$$\lim_{|X| \rightarrow \infty} \frac{H(X)}{\ln \ln |X|} = -\frac{1}{\ln(1 - \frac{d-\beta}{2})}.$$

This proposition implies that the *PWIT model gives the exact order of magnitude for $H(X)$* . It is also worth to mention that our method has enabled us to determine the exact asymptotic limit for $\beta \in (d - 2, d]$.

1.6.4 Path Deviation and Tree Topology

In Section 5, we examine the path from X to O in the navigation tree. For regenerative navigation algorithms, we establish an upper bound on the maximal deviation of this path with respect to the straight line \overline{OX} :

$$\Delta(X) = \max_{0 \leq k \leq H(X)} |X_k - U_k|,$$

with $U_k = \langle X_k, X/|X| \rangle X/|X|$ is the projection of X_k on the straight line \overline{OX} .

Using the terminology of Howard and Newman in [17], this bound will enable us to find conditions under which \mathcal{T}_0 is an f -straight tree. In particular, it will characterize the semi-infinite paths of the navigation tree.

On the small world navigation, we obtain the following proposition.

Proposition 5 *There exists $C \geq 1$, such that if $\gamma\beta > (\gamma + C)d + C$, then for some $\eta > 0$, there exists $C_1 > 0$ such that*

$$\mathbb{P}(\Delta(X) \geq |X|^\gamma) \leq C_1 |X|^{-d-\eta}.$$

and \mathcal{T}_0 is f -straight with $f(x) = |x|^{\gamma-1}$.

A bound for the constant C could be explicitly computed. We only point out that for a small world navigation on a PWIT, $C = 1$.

1.6.5 Shape of the Navigation Tree

Finally, in Section 6 we will state a shape theorem for regenerative navigation algorithms. We define

$$\mathcal{T}_0(k) = \{X \in N : \mathcal{A}^k(X) = 0\}.$$

We restrict our attention to the following case: a.s.,

$$\lim_{|X| \rightarrow \infty} \frac{H(X)}{|X|} = \frac{1}{\mu},$$

where $\mu > 0$ will be the asymptotic directed mean progress.

Under some additional assumptions, we will state that for all $\epsilon > 0$ there exists a.s. K such that if $k \geq K$:

$$N \cap B(0, (1 - \epsilon)k\mu) \subset \mathcal{T}_0(k) \subset B(0, (1 + \epsilon)k\mu),$$

and moreover a.s. and in L^1 :

$$\frac{|\mathcal{T}_0(k)|}{\pi_d k^d} \rightarrow \mu^d,$$

where π_d is the volume of d -dimensional unit ball.

On the small world graph, we will obtain the following proposition.

Proposition 6 *Let μ (resp. $\tilde{\mu}$) as in Proposition 6 (resp. Proposition 2).*

- *There exists $C \geq 1$ such that if $\beta > (C + 1)d + 2C$, for all $\epsilon > 0$ there exists a.s. K such that if $k \geq K$:*

$$N \cap B(0, (1 - \epsilon)k\mu) \subset \mathcal{T}_0(k) \subset B(0, (1 + \epsilon)k\mu).$$

Moreover a.s. and in L^1 :

$$\frac{|\mathcal{T}_0(k)|}{\pi_d k^d} \rightarrow \mu^d,$$

- If $\beta = d$, for all $\epsilon > 0$ there exists a.s. K such that if $k \geq K$:

$$N \cap B(0, e^{(1-\epsilon)k\tilde{\mu}}) \subset \mathcal{T}_0(k) \subset B(0, e^{(1+\epsilon)k\tilde{\mu}}).$$

Moreover a.s. and in L^1 :

$$\frac{\ln |\mathcal{T}_0(k)|}{k} \rightarrow d\tilde{\mu}.$$

- For $d - 2 < \beta < d$, let $\alpha = 1 - (d - \beta)/2$, for all $\epsilon > 0$ there exists a.s. K such that if $k \geq K$:

$$N \cap B(0, \exp(\alpha^{(1-\epsilon)k})) \subset \mathcal{T}_0(k) \subset B(0, \exp(\alpha^{(1+\epsilon)k})).$$

Moreover a.s. and in L^1 :

$$\frac{\ln \ln |\mathcal{T}_0(k)|}{k} \rightarrow \ln \alpha.$$

Again, a bound for the constant C could be computed. In the PWIT model $C = 1$.

1.7 Notations and Preliminaries

If not otherwise mentioned, for $|x| \in \mathbb{R}^d$, $|x|$ will denote the Euclidian norm. $B(X, r)$ will denote the open ball of radius r and center x , $S^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$ is the d -dimensional hyper-sphere. If A is a set, $|A|$ will denote the cardinal of A and \overline{A} the closure of this set for the underlying topology. Throughout this work, we will denote by

$$\pi_d = \frac{\pi^{d/2}}{\Gamma(d/2 + 1)} \quad \text{and} \quad \omega_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)},$$

the d -dimensional volume of $B(0, 1)$ and the d -dimensional surface area measure of S^{d-1} . ℓ^0 will denote the set of measurable $\mathbb{R}_+ \rightarrow \mathbb{R}_+$ functions tending to 0 at $+\infty$.

If N is a countable set of points in \mathbb{R}^d with no accumulation points, we write for all bounded sets A : $N(A) = \sum_{x \in N} \mathbb{1}(x \in A)$. A set of points N of \mathbb{R}^d is said to be *non-equidistant* if there do not exist points x, y, z, t of N such that $\{x, y\} \neq \{z, t\}$ and $|x - y| = |z - t|$. We define a *nice* point set as a non-equidistant locally finite point set, a Poisson point process (PPP) of positive intensity is a.s. nice. It is good to note that on nice sets, the maximal progress navigation and the radial navigation are well defined.

Let M be a complete metric space and \mathcal{N} the space of all counting measures on $\mathbb{R}^d \times M$, defined on the Borel σ -field $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(M)$. We assume that the restriction to \mathbb{R}^d of each measure in \mathcal{N} is nice that is if $N \in \mathcal{N}$, the support of $N(\cdot \times M)$ is a nice point set. We endow the space \mathcal{N} with its usual topology (see, for instance, the book by Daley and Vere-Jones [7] for the details). A (locally finite) marked point process on \mathbb{R}^d with marks in M is a measurable mapping $N : \Omega \rightarrow \mathcal{N}$ defined on some probability space (Ω, \mathcal{F}, P) . Any point process on \mathbb{R}^d with marks in M can be represented as

$$N = \sum_{n \in \mathbb{Z}} \delta_{(T_n, V_n)},$$

where $\delta_{(t,v)}$, $(t, v) \in \mathbb{R}^d \times M$, is the Dirac's measure on $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(M)$.

Let $\widehat{N} = \sum_n \delta_{T_n}$ be a point process on \mathbb{R}^d and let $U(x)$, $x \in \mathbb{R}^d$, be a stochastic process with value in a complete metric space M and independent of \widehat{N} such that $U(x)$ and $U(x')$ are independent and identically distributed (iid) for $x \neq x'$. We define the marked point process: $N = \sum_n \delta_{(T_n, U(T_n))}$. A point process $\widehat{N} = \sum_n \delta_{T_n}$ on \mathbb{R}^d is *stationary* if $S_x \circ \widehat{N} = \sum_n \delta_{T_n+x}$ and \widehat{N} have the same distribution. Note that if \widehat{N} is a stationary point process then $N = \sum_n \delta_{(T_n, U(T_n))}$ is also stationary for the shift $S_{x,x} \circ N = \sum_n \delta_{(T_n+x, U(T_n+x))}$. Since \widehat{N} is independent of the process U , for the simplicity of notations, we will skip the marks $U(T_n)$ in N and identify \widehat{N} and N : we will write simply $N = \sum_n \delta_{T_n}$.

Several qualitative results of the present paper involve constants. For the sake of clarity, we will use C_0 to denote a positive constant to be thought of as small and C_1 to denote a positive constant to be thought of as large. The exact value of C_0 and C_1 may change from one line to the other and we could for example write : $C_0/C_1 = C_0$. The important point is that C_0 and C_1 are constants that may depend on the dimension d but they will never depend on other parameters of the problem.

2 Convergence of navigation to directed navigation

2.1 Stable Functionals and Local Weak Convergence

In this paragraph we prove that under some conditions that the navigation tree tends weakly to the directed navigation tree, for the local weak convergence. We consider a navigation \mathcal{A} with non-negative progress on a PPP N of intensity 1. Proving the convergence of the navigation tree is not a difficult task, provided that we use the right concepts.

We introduce an important class of functional, the *stable functionals*. This class was first introduced by Lee [21] and it was further developed by Penrose and Yukich (see for example [26], [27]); it is slightly modified here to suit to our framework.

Definition 4 Let $F(X, N)$ be a measurable function valued in a complete separable metric space. F is stable on N if for all $X \in \mathbb{R}^d$ there exists a random variable $R(X) > 0$ such that $F(X, N)$ is $\mathcal{F}_{B(X, R(X))}^N$ -measurable and $R(X)$ is stochastically upper bounded uniformly in X .

A graph $\mathcal{G} = (N, E)$ is a stable graph if for all $X \in N$, $V(X, N) = \{Y \in N : (X, Y) \in E\}$ (i.e. the set of vertices adjacent to X in \mathcal{G}) is a stable functional. Note that $R(X)$ need not to be a stopping-time for the filtration $\mathcal{F}_{B(X, t)}^N$. In particular in the Small World graph \mathcal{G} , $V(X, N)$ is not $\mathcal{F}_{B(X, R)}^N$ -measurable for some stopping-time R .

$$\mathcal{A} \text{ is the maximal progress navigation on a stable graph } \mathcal{G} = (N, E). \quad (5)$$

This condition is still quite general since a navigation with a positive progress is always a maximal progress navigation on its associated navigation tree. We defined the maximal

directed progress navigation with direction $e_1 \in S^{d-1}$ as

$$\mathcal{A}_{e_1}(X) = \arg \max\{\langle Y - X, e_1 \rangle : Y \in N, (X, Y) \in E\}.$$

Let \mathcal{G}^0 the graph built on N^0 and \mathcal{G} the graph built on \mathcal{G} . Note that we define the navigation \mathcal{A} on \mathcal{G}^0 and the directed navigation \mathcal{A}_{e_1} on \mathcal{G} .

Lemma 1 *Let $X \in \mathbb{R}^d \setminus \{0\}$ and $e_1 \in S^{d-1}$ with $\cos \theta = \langle X/|X|, e_1 \rangle$. Under the foregoing Assumption (5) there exists a function $\epsilon \in \ell^0$ with*

$$\mathbb{P}(\mathcal{A}(X) \neq \mathcal{A}_{-e_1}(X)) \leq \epsilon(|X|) + \epsilon(1/\theta). \quad (6)$$

Proof. By Equation (5) there exists $h \in \ell^0$ such that for all X , $\mathbb{P}(\rho(X) \geq t) \leq h(t)$, and

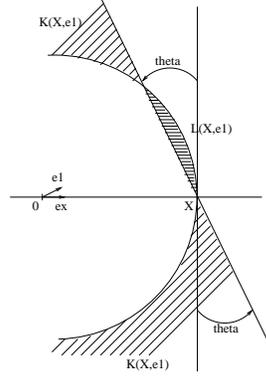


Figure 1: The sets $L(X, e_1)$ and $K(X, e_1)$.

$V(X, N)$ is $\mathcal{F}_{B(X, \rho(X))}^N$ -measurable. Without loss of generality we suppose $X = xe_x$, $x > 0$ and $\theta > 0$. Let $K(X, e_1) = \{Y \in \mathbb{R}^d : |Y| \geq |X|, \langle Y - X, e_1 \rangle \leq 0\}$ and $L(X, e_1) = \{Y \in \mathbb{R}^d : |Y| \leq |X|, \langle Y - X, e_1 \rangle \geq 0\}$, the sets $L(X, e_1)$ and $K(X, e_1)$ are depicted in Figure 1. Let $S(X) = N \cap B(X, \rho(X))$, $\mathcal{A}(X) = \arg \min\{|Y| : Y \in S(X)\}$ and $\tilde{\mathcal{A}}_{e_1}(X) = \arg \min\{\langle Y, e_1 \rangle : Y \in S(X)\}$. If $\rho(X) + \rho(0) < |X|$, $S(X) \cap \{K(X, e_1) \cup L(X, e_1)\} = \emptyset$ and $\tilde{\mathcal{A}}_{e_1}(X) = \tilde{\mathcal{A}}(X)$ then $\mathcal{A}_{e_1}(X) = \mathcal{A}(X)$. It follows

$$\begin{aligned} \mathbb{P}(\mathcal{A}_{-e_1}(X) \neq \mathcal{A}(X)) &\leq \mathbb{P}(\rho(X) + \rho(0) \geq |X|) + \mathbb{P}(S(X) \cap L(X, e_1) \neq \emptyset) \\ &\quad + \mathbb{P}(S(X) \cap K(X, e_1) \neq \emptyset) + \mathbb{P}(\tilde{\mathcal{A}}_{e_1}(X) \neq \tilde{\mathcal{A}}(X)). \end{aligned} \quad (7)$$

The first term of Equation (7) is easily computed : $\mathbb{P}(\rho(X) + \rho(0) \geq |X|) \leq h(|X|/2)$.

We now upper bound the second term of Equation (7). We notice that $L(X, e_1)$ is contained in a cone of apex θ (see Figure 1). Let \mathcal{C}_θ be a cone issued from 0 with apex θ ,

we have:

$$\begin{aligned} \mathbb{P}(S(X) \cap L(X, e_1) \neq \emptyset) &\leq \mathbb{P}(N \cap \mathcal{C}_\theta \cap B(0, \theta^{-1/2}) \neq \emptyset) + \mathbb{P}(\rho(X) \geq \theta^{-1/2}) \\ &\leq 1 - \exp(-C_0 \theta^{1/2}) + h(\theta^{-1/2}). \end{aligned}$$

The third term of Equation (7) is upper bounded similarly. Let $K_+(X, e_1)$ be the largest half of $K(X, e_1)$; we have

$$\begin{aligned} \mathbb{P}(S(X) \cap K(X, e_1) \neq \emptyset) &\leq 2\mathbb{P}(S(X) \cap K_+(X, e_1) \neq \emptyset) \\ &\leq 2\mathbb{P}(N \cap \mathcal{C}_{\arcsin(\frac{t}{2|X|})+\theta} \cap B(0, t) \neq \emptyset) + 2\mathbb{P}(\rho(X) \geq t) \\ &\leq 2(1 - \exp(-C_0 t(\arcsin(\frac{t}{2|X|}) + \theta)) + h(t)). \end{aligned}$$

If $1/\sqrt{|X|} \leq \theta$, we chose $t = \sqrt{|X|}$ else we pick $t = 1/\sqrt{\theta}$.

It remains to bound the last term of Equation (7). For $Y \in B(X, \rho(X))$, let $K'(Y, X, e_1) = \{Z \in B(X, \rho(X)) : |Z| \geq |Y|, \langle Z - Y, e_1 \rangle \leq 0\}$, that is the set of points with a larger norm but a smaller projection on e_1 ; we have $K'(X, X, e_1) = K(X, e_1)$. We can then bound the last term as we have bounded the third term:

$$\begin{aligned} \mathbb{P}(\tilde{\mathcal{A}}_{e_1}(X) \neq \tilde{\mathcal{A}}(X)) &\leq \mathbb{P}(\exists Y \in S(X) : K'(Y, X, e_1) \cap N \neq \emptyset) \\ &\leq \mathbb{P}(\rho(X) \geq t) + \mathbb{P}(N(B(X, t)) \geq n) + 2n\mathbb{P}(N \cap \mathcal{C}_{\arcsin(\frac{t}{2(|X|-t)})+\theta} \cap B(0, t) \neq \emptyset) \end{aligned}$$

We pick $t = \min(|X|, 1/\theta)^{1/3d}$, $n = |X|^{1/2}$, then using the inequality $\mathbb{P}(N(B(X, t)) > n) \leq \exp(-n \ln \frac{n}{C_1 t^d})$, we get the required bound. \square

Let \mathcal{T}_{e_1} denote the directed navigation forest associated to \mathcal{A}_{e_1} and \mathcal{T}_0 the navigation tree associated to \mathcal{A} . A functional is stable on a graph \mathcal{G} if it is stable on its vertex set.

Theorem 1 *Let F be a stable functional on \mathcal{T}_{-e_1} . If Equation (6) holds then as x tends to $+\infty$, the distribution of $F(xe_1, \mathcal{T}_0)$ converges in total variation toward the distribution of $F(0, \mathcal{T}_{-e_1})$.*

Proof. We set $X = xe_1$, $x > 0$ and we build \mathcal{T}_0 and \mathcal{T}_{-e_1} on the same PPP. For all $r > 0$, we define the event $J_t(X) = \{\mathcal{T} \cap B(X, t) = \mathcal{T}_{-e_1} \cap B(X, t)\}$. F is a stable functional on \mathcal{T}_{-e_1} for a radius $R(X)$, we have:

$$\begin{aligned} \mathbb{P}(F(X, \mathcal{T}_0) \neq F(X, \mathcal{T}_{-e_1})) &\leq \mathbb{P}(J_{R(X)}(X)^c) \\ &\leq \mathbb{P}(R(X) > t) + \mathbb{P}(J_t(X)^c) \\ &\leq \mathbb{P}(R > t) + \mathbb{P}(\cup_{Y \in N \cap B(X, t)} \mathcal{A}(Y) \neq \mathcal{A}_{-e_1}(Y)) \\ &\leq \mathbb{P}(R > t) + \mathbb{P}(N(B(X, t)) \geq n) + n\epsilon(x - t) + n\epsilon((x - t)/t), \end{aligned}$$

where we have used Equation (6).

For $\eta > 0$, we fix t such that $\mathbb{P}(R > t) \leq \eta$. Note also that $\mathbb{P}(N(B(X, t)) > n) \leq \exp(-n \ln \frac{n}{C_1 t^d})$. Hence taking, $n = \lfloor 1/\sqrt{f(x)} \rfloor$, we deduce that

$$\limsup_x \mathbb{P}(F(xe_1, \mathcal{T}_0) \neq F(xe_1, \mathcal{T}_{-e_1})) \leq \eta,$$

and it follows $\lim_x \mathbb{P}(F(xe_1, \mathcal{T}_0) \neq F(xe_1, \mathcal{T}_{-e_1})) = 0$.

To complete the proof, notice that \mathcal{T}_{-e_1} is stationary: $F(xe_1, \mathcal{T}_{-e_1})$ and $F(0, \mathcal{T}_{-e_1})$ have the same distribution. \square

Remark 2 It is easy to check that the vector $X - \mathcal{A}(X)$ (and hence the progress $P(X)$) or the degree at X are stabilizing functional for \mathcal{T}_{-e_1} . So are the first k segments of the path from X to the origin in \mathcal{T}_{-e_1} , for all finite k , or the subtree of the directed navigation forest rooted in X and of depth k .

Theorem 1 has to be related to the convergence of graphs as it is defined for the Objective Method (refer to [2]). Let S_x denote the natural translation on geometric graphs induced by the translation by x of point sets. As an immediate corollary we have:

Corollary 1 *If Equation (6) holds and if $|X_n|$ tends to $+\infty$ and $X_n/|X_n|$ to e_1 then $S_{-X_n} \circ \mathcal{T}_0(N^0, X_n)$ converges to $\mathcal{T}_{-e_1}(N^0)$ for the local weak convergence. Moreover if $(e_1^n), n \in \mathbb{N}$ converges to e_1 then $\mathcal{T}_{e_1^n}$ converges to \mathcal{T}_{e_1} for the local weak convergence.*

2.2 Sketch of Application: Spatial Average

In this paragraph, we explain how it is possible to state a spatial average result on the navigation tree. It is not in the scope of this paragraph to state a precise result but rather to point out the good references, for a detailed application on the radial spanning tree see [9]. Let F be a stable functional on the navigation tree with value on \mathbb{R}_+ . We consider the sum

$$T(r) = \sum_{X \in \mathcal{N}} \mathbf{1}(X \in B(0, r)) F(X, \mathcal{T}_0).$$

We assume that the conclusion of Theorem 1 holds, that $m_X = \mathbb{E}F(X, \mathcal{T}_0)$ depends only on $|X|$ and that m_{xe_1} is uniformly bounded and converges to $m = \mathbb{E}F(0, \mathcal{T}_{e_1})$. From Slyvniack-Mecke Theorem, we deduce:

$$\mathbb{E}T(r) = \omega_{d-1} \int_0^r m_{xe_1} x^{d-1} dx,$$

With the change of variable : $t = \frac{x}{r}$, this leads to :

$$\mathbb{E} \frac{T(r)}{r^d} = \omega_{d-1} \int_0^1 t^{d-1} m_{tre_1} dt,$$

The dominated convergence theorem together with $\lim_{x \rightarrow +\infty} m_{x e_1} = m$ gives:

$$\lim_{x \rightarrow \infty} \mathbb{E} \frac{T(r)}{r^d} = \omega_{d-1} m \int_0^1 t^{d-1} dt = m \pi_d.$$

Indeed we can hope to prove a stronger result: the almost sure and weak convergence of $\frac{T(r)}{r^d}$ to $m \pi_d$, that is the convergence of $T(r) = 1/\mathbb{E}N(B(0, r)) \sum_{X \in N} \mathbf{1}(X \in B(0, r)) F(X, \mathcal{T}_0)$ toward $1/\mathbb{E}N(B(0, r)) \sum_{X \in N} \mathbf{1}(X \in B(0, r)) F(X, \mathcal{T}_{e_1})$.

The weak convergence could be derived from Theorem 2.1 of Penrose and Yukich [27]. To prove an almost sure convergence, we consider a slightly different problem, we draw independently N points uniformly and get a finite point set $F_N = \{0, X_1, \dots, X_N\}$ and we build the navigation tree $\mathcal{T}_0^{F_N}$ on the point set F_N . We define $T(F_N) = \sum_{n=1}^N F(X_n, \mathcal{T}_0^{F_N})$, assume for example that F is homogeneous of order α , $T(\lambda F) = \lambda^\alpha T(F)$ for $\lambda > 0$. $N(B(0, r))/r^d$ tends a.s. toward π_d hence by homogeneity, to prove that $\frac{T(r)}{r^d}$ converges almost surely toward $\pi_d m$ it is necessary and sufficient to prove that $\frac{T(F_N)}{N^{(d-\alpha)/d}}$ tends to $m \pi^{\alpha/d}$. Since we have already computed the mean of $\frac{T(r)}{r^d}$, it is sufficient to prove that $\frac{T(F_N)}{N^{d-\alpha/d}}$ converges a.s. toward a constant. If T satisfies a smoothness property, this will be a consequence of Rhee and Talagrand concentration inequalities, refer to the monographs of Yukich [33] and Steele [30].

It is important to notice that spatial average and path average have no reason to be equal when the navigation is not memoryless: $\lim_{r \rightarrow \infty} 1/N(B(0, r)) \sum_{X \in B(0, r)} F(X, \mathcal{T}_{e_1})$ could differ from $\lim_{|X| \rightarrow \infty} 1/H(X) \sum_{k=0}^{H(X)-1} F(\mathcal{A}^k(X), \mathcal{T}_{e_1})$.

2.3 Example: Progress Distribution in the Small World

In this paragraph, we give an example of a computation of the distribution of a local functional of \mathcal{T}_{e_1} and \mathcal{T}_0 .

We consider the navigation on the Small World \mathcal{A} and the directed navigation depicted \mathcal{A}_{e_1} in §1.5.1, Model 3. We will soon check that the directed navigation is defined if and only if $\beta > d$. Let F denote the distribution function of the directed progress in the Small World $P_{e_1}(0) = \langle \mathcal{A}_{e_1}(X), e_1 \rangle$ and F_X the distribution function of the progress at X in the Small World $P(X) = |X| - |\mathcal{A}(X)|$.

Lemma 2 *Assume $d \geq 2$, the following properties hold:*

1. *If $\beta > d$, as t goes to infinity:*

$$\overline{F}(t) \sim \frac{2c\omega_{d-2}}{\beta-d} t^{d-\beta} \int_0^{\pi/2} \cos^{\beta-d} \theta d\theta.$$

2. *If $\beta > d$ then for all function $\epsilon \in \ell^0$*

$$\lim_{|X| \rightarrow +\infty} \sup_{t \leq |X| \epsilon(|X|)} t^{\beta-d} |F_X(t) - F(t)| = 0.$$

3. If $d - 2 < \beta < d$, the distribution of $|\mathcal{A}(X)|/|X|^{1-\frac{d-\beta}{2}}$ converges weakly and

$$\sup_X \mathbb{E}(|\ln \frac{|\mathcal{A}(X)|}{|X|^{1-\frac{d-\beta}{2}}}| | \mathcal{A}(X) \neq 0) < \infty.$$

4. If $\beta = d$, let \tilde{F}_X be the distribution $\tilde{P}(X) = -\ln(1-P(X)/X) \in [0, +\infty]$, \tilde{F}_X converges weakly to \tilde{F} with $\int \tilde{F}(s)ds = \tilde{\mu} \in (0, +\infty)$. Moreover, for all ϵ in ℓ^0

$$\lim_{|X| \rightarrow +\infty} \sup_{t \leq \ln |X|^{\epsilon(|X|)}} e^{2t} |\tilde{F}_X(t) - \tilde{F}(t)| = 0.$$

As we will see in the next section, the weak convergence results given in statements 2, 4 and 3 of this lemma will be used to derive various limits.

For $d \geq 3$ and $0 < \beta < d-2$ the reader should be convinced that similar weak convergence results hold. To avoid longer computations, we will not try to compute the explicit scaling at which the distribution of $|\mathcal{A}(X)|$ exhibits a non-degenerated limit.

The computation for $d = 1$ are simpler and the same result holds with different constants. Note that this Lemma implies a similar result on Model 2, in statement 1, it suffices to rescale by $\mathbb{P}(P(0) = 0) = \exp(-\int_{\mathcal{H}(0)} f(y)dy)$ and statements 2, 4, and 3 hold without change.

All the distributions can be computed explicitly. The distribution \tilde{F} in statement 4 is given by Equation (9) and the weak limit of $|\mathcal{A}(X)|/|X|^{1-\frac{d-\beta}{2}}$ has a distribution obtained in Equation (8).

Proof. The proof relies on explicit computations and does not involve any subtle argument, we skip most details.

Statement 1.

Let $\mathcal{G} = (N, E)$ denote the Small World graph and $V(X) = \{Y : (X, Y) \in E\}$ the set of neighbors of X in the graph G , $V(X)$ is a non-homogenous Poisson point process of intensity $f(|X - x|)dx$. We have

$$\begin{aligned} \mathbb{P}(P(0) > t) &= \mathbb{P}(V(0) \cap \mathcal{H}(t) \neq \emptyset) \\ &= 1 - \exp(-\int_{\mathcal{H}(t)} f(y)dy) \\ &\sim \int_{\mathcal{H}(t)} f(y)dy, \end{aligned}$$

as t tends to infinity. Let $\Lambda_t = \int_{\mathcal{H}(t)} f(y)dy$, writing $y = r \cos \theta e_1 + r \sin \theta e_2$ with $\langle e_1, e_2 \rangle = 0$ and $e_2 \in S^{d-1}$, we obtain

$$\begin{aligned} \Lambda_t &= 2\omega_{d-2} \int_0^{\pi/2} \int_{t/\cos \theta}^{\infty} f(r)r^{d-1}drd\theta \\ &\sim 2\omega_{d-2} \int_0^{\pi/2} \int_{t/\cos \theta}^{\infty} cr^{d-\beta-1}drd\theta \\ &\sim 2\frac{2c\omega_{d-2}}{\beta-d} \int_0^{\pi/2} \left(\frac{t}{\cos \theta}\right)^{d-\beta}d\theta \\ &\sim \frac{2c\omega_{d-2}}{\beta-d} t^{d-\beta} \int_0^{\pi/2} \cos^{\beta-d}\theta d\theta. \end{aligned}$$

Statement 2.

We can suppose without loss of generality that $X = -xe_1$, with $x > 0$. By definition, for $t < x$:

$$\begin{aligned} \mathbb{P}(P(X) > t) &= \mathbb{P}(V(X) \cap B(0, x-t)(t) \neq \emptyset) \\ &= 1 - \mathbb{P}(0 \in V(X)) \exp\left(-\int_{B(0, x-t)} f(|X-y|)dy\right) \\ &= 1 - (1 - f(x)) \exp\left(-\int_{B(0, x-t)} f(|X-y|)dy\right) \end{aligned}$$

In \mathbb{R}^2 for $u \in (0, 1)$ and $0 \leq \theta < \arcsin(1-u)$, the straight line with equation $y = \tan \theta$ intersects the sphere of radius u and center $(1, 0)$ at two points of respective norms $A(\theta, u)$ and $B(\theta, u)$. A direct computation leads to

$$\begin{aligned} A(\theta, u) &= \cos \theta \left(1 - \sqrt{1 - \frac{u(2-u)}{\cos^2 \theta}}\right) = \frac{u}{\cos \theta} + o\left(\frac{u}{\cos \theta}\right) \\ B(\theta, u) &= \cos \theta \left(1 + \sqrt{1 - \frac{u(2-u)}{\cos^2 \theta}}\right) = 2 \cos \theta - \frac{u}{\cos \theta} + o\left(\frac{u}{\cos \theta}\right). \end{aligned}$$

Let $\Lambda_t(x) = \int_{B(0, x-t)} f(|X-y|)dy$, we get as t, x tend to infinity and t/x tends to 0:

$$\begin{aligned} \Lambda_t(x) &= 2\omega_{d-2} \int_0^{\arcsin(1-t/x)} \int_{xA(\theta, t/x)}^{xB(\theta, t/x)} f(r)r^{d-1}drd\theta \\ &\sim 2\omega_{d-2} \int_0^{\arcsin(1-t/x)} \int_{xA(\theta, t/x)}^{xB(\theta, t/x)} cr^{d-\beta-1}drd\theta \\ &\sim \frac{2c\omega_{d-2}}{\beta-d} \int_0^{\arcsin(1-t/x)} (xA(\theta, t/x))^{d-\beta} - (xB(\theta, t/x))^{d-\beta}d\theta \\ &\sim \overline{F}(t). \end{aligned}$$

It follows also

$$\begin{aligned} |\Lambda_t(x) - \Lambda(t)| &\leq \int_{\arcsin(1-t/x)}^{\pi/2} f(r)r^{d-1} dr d\theta + \int_0^{\arcsin(1-t/x)} \int_{t/\cos(\theta)}^{xA(\theta,t/x)} f(r)r^{d-1} dr d\theta \\ &\quad + \int_0^{\arcsin(1-t/x)} \int_{xB(\theta,t/x)}^{\infty} f(r)r^{d-1} dr d\theta \end{aligned}$$

and if $t = x^{d-\beta}\epsilon(x)$, with $\epsilon \in \ell^0$, we easily get that $t^{\beta-d}|\Lambda_t(x) - \Lambda(t)|$ tends to 0.

Statement 3.

Let $U(X) = |\mathcal{A}(X)|/x^\alpha = (x - P(X))/x^\alpha$ with $|X| = x$ and $\alpha = 1 - (d - \beta)/2 \in (0, 1)$. Let $0 < s < x^{1-\alpha}$, we have

$$\begin{aligned} \mathbb{P}(U(X) < s) &= 1 - (1 - f(x)) \exp\left(-\int_{B(0, sx^\alpha)} f(|X - y|) dy\right), \\ &= 1 - (1 - f(x)) \exp(-\Lambda_{x-sx^\alpha}(x)), \end{aligned}$$

with as x tends to $+\infty$, uniformly in $s < x^{1-\alpha'}$, $\alpha' > \alpha$:

$$\begin{aligned} \Lambda_{x-sx^\alpha}(x) &\sim 2\omega_{d-2} \int_0^{\arcsin(sx^{\alpha-1})} \int_{xA(\theta, 1-sx^{\alpha-1})}^{xB(\theta, 1-sx^{\alpha-1})} cr^{d-\beta-1} dr d\theta \\ &\sim \frac{2c\omega_{d-2}}{d-\beta} \int_0^{\arcsin(sx^{\alpha-1})} (xB(\theta, 1-sx^{\alpha-1}))^{d-\beta} - (xA(\theta, 1-sx^{\alpha-1}))^{d-\beta} d\theta. \end{aligned}$$

We have $B(\theta, 1-sx^{\alpha-1}) = \cos\theta(1 + \sqrt{s^2x^{2(\alpha-1)}/\cos^2\theta - \tan^2\theta}) = \cos\theta(1 + \sqrt{s^2x^{\beta-d}/\cos^2\theta - \tan^2\theta})$ and $A(\theta, 1-sx^{\alpha-1}) = \cos\theta(1 - \sqrt{s^2x^{\beta-d}/\cos^2\theta - \tan^2\theta})$. Hence as x tends to ∞ :

$$(xB(\theta, 1-sx^{\alpha-1}))^{d-\beta} - (xA(\theta, 1-sx^{\alpha-1}))^{d-\beta} \sim 2(d-\beta)x^{d-\beta} \cos^{d-\beta}\theta \sqrt{s^2x^{\beta-d}/\cos^2\theta - \tan^2\theta}.$$

and we obtain:

$$\begin{aligned} \Lambda_{x-sx^\alpha}(x) &\sim 4c\omega_{d-2} \int_0^{\arcsin(sx^{(\beta-d)/2})} x^{d-\beta} \cos^{d-\beta}\theta \sqrt{s^2x^{\beta-d}/\cos^2\theta - \tan^2\theta} d\theta \\ &\sim 4c\omega_{d-2}s^2. \end{aligned}$$

Finally we have proved that uniformly in $s < x^{(d-\beta)/2-\eta}$ (for some $\eta > 0$):

$$\lim_{|X| \rightarrow \infty} \mathbb{P}(U(X) > s) = \exp(-4c\omega_{d-2}s^2). \quad (8)$$

and this concludes the proof of statement 3.

Statement 4. Similarly, we still suppose that $X = -xe_1$, with $x > 0$, let $s > 0$ and $u = 1 - \exp(-s) \in (0, 1)$:

$$\begin{aligned} \mathbb{P}(\tilde{P}(X) > s) &= \mathbb{P}(P(X) > xu) \\ &= 1 - (1 - f(x)) \exp\left(-\int_{B(0, (1-u)x)} f(|X - y|) dy\right), \end{aligned}$$

as above with $\Lambda_t(x) = \int_{B(0, x-t)} f(|X - y|) dy$:

$$\begin{aligned} \Lambda_{ux}(x) &= 2\omega_{d-2} \int_0^{\arcsin(1-u)} \int_{xA(\theta, u)}^{xB(\theta, u)} f(r) r^{d-1} dr d\theta \\ &\sim 2\omega_{d-2} \int_0^{\arcsin(1-u)} \int_{xA(\theta, u)}^{xB(\theta, u)} c/r dr d\theta \\ &\sim 2c\omega_{d-2} \int_0^{\arcsin(1-u)} \ln \frac{B(\theta, u)}{A(\theta, u)} d\theta \end{aligned}$$

We define

$$\begin{aligned} \tilde{F}(s) &= 1 - \exp\left(-2c\omega_{d-2} \int_0^{\arcsin(\exp(-s))} \ln \frac{B(\theta, 1 - \exp(-s))}{A(\theta, 1 - \exp(-s))} d\theta\right) \\ &= 1 - \exp\left(-c \int_{B(0, \exp(-s))} |e_1 - y|^{-d} dy\right). \end{aligned} \quad (9)$$

A direct analysis shows that, as s tends to $+\infty$:

$$\tilde{F}(s) \sim 4c\omega_{d-2} e^{-2s}. \quad (10)$$

The statement 4 follows. \square

3 Path average for memoryless navigation

In this section we assume that \mathcal{A} is a memoryless navigation and we derive various results on the asymptotic of $H(X)$, the generation of X in the navigation tree \mathcal{T}_0 .

3.1 Finite Mean Progress

3.1.1 Non-Homogeneous Renewal Equation

In this paragraph, we assume that the distribution of $g(X, \mathcal{A}(X))$ and $P_0(X)$ is invariant by rotations with center 0. Let $x > 0$, $z(x) = \mathbb{E}g(xe_1, \mathcal{A}(xe_1))$, $Z(x) = \mathbb{E}G(xe_1)$ and F_x the distribution of $P_0(xe_1)$, $x > 0$. from the independence property of the memoryless navigation, we deduce immediately:

$$Z(x) = z(x) + \int_0^x Z(x-r)F_x(dr), \quad (11)$$

where $F_x(dr)$, $0 \leq r \leq x$, is the measure of progress. and $z(x)$ is the mean weight of the link to the ancestor. We define also the mean generation, $U(x) = \mathbb{E}H(x)$:

$$U(x) = 1 + \int_0^x U(x-r)F_x(dr), \quad (12)$$

We assume:

- (i) $Z(x)$ is finite for all x .
- (ii) F_x converges in f -norm to F the *directed distribution of progress* with $f(r) = 1+r$.
- (iii) $\mu = \int_0^\infty rF(dr) \in (0, +\infty)$.
- (iv) z is a bounded Riemann function and $l = \lim_{x \rightarrow +\infty} z(x)$ exists and is positive.

Theorem 2 *Let $a < b$, under the foregoing assumptions, as x tends to infinity,*

$$Z(x) \sim \frac{l}{\mu}x. \quad \text{and} \quad U(x+a) - U(x+b) \rightarrow \frac{b-a}{\mu}.$$

The proof relies on the Renewal Theorem and on a few technical lemmas.

Lemma 3 *There exists a positive constant C_1 such that*

$$Z(x) \leq C_1(x+1).$$

Proof. Let $M(x) = \sup_{0 \leq t \leq x} Z(t)$ and L an upper bound for z . $M(x)$ is finite (Assumption (i)), M is non-decreasing and

$$M(x) \leq L + \int_0^x M(x-r)F_x(dr).$$

From Assumption (ii), the sequence of measures F_x converges in total variation and $F(\{0\}) < 1$ (indeed $\mu > 0$ by Assumption (iii)). Thus we may find $x_0 \geq \rho > 0$ such that for all $x \geq x_0$, $F([0, \rho], x) \leq 1/2$. We deduce for $x \geq x_0$:

$$M(x) \leq L + \frac{1}{2}M(x) + \frac{1}{2}M(x-\rho),$$

which in turn implies:

$$M(x) \leq \frac{2L}{\rho}x + M(x_0).$$

□

Lemma 4 Let $\delta(x) = C_1 \int_0^x (x-r+1)|F_x(dr) - F(dr)|$. δ is a bounded Riemann function and

$$\lim_{x \rightarrow \infty} \delta(x) = 0.$$

Proof. Since $\delta(x) \leq C_1 \int_0^\infty (r+1)|F_x(dr) - F(dr)|$ this lemma follows immediately from Assumption (ii). \square

We state a straightforward corollary of the Renewal Theorem as it is stated in [10].

Theorem 3 If y is a bounded Riemann function, the solution Y of the renewal equation:

$$Y(x) = y(x) + \int_0^x Y(x-r)F(dr)$$

satisfies as x tends to infinity:

$$Y(x) \sim \frac{1}{\mu} \int_0^x y(r)dr.$$

Now we can turn to the proof of Theorem 2.

Proof. Let \tilde{Z} be the solution of the renewal equation given by:

$$\tilde{Z}(x) = z(x) + \int_0^x \tilde{Z}(x-r)F(dr).$$

Assumption (iv) and Theorem 3 (applied to $y(x) = z(x)$) imply that $\tilde{Z}(x) \sim lx/\mu$. Note also:

$$\begin{aligned} |Z(x) - \tilde{Z}(x)| &\leq \int_0^x Z(x-r)|F_x(dr) - F(dr)| + \int_0^x |Z(x-r) - \tilde{Z}(x-r)|F(dr) \\ &\leq \delta(x) + \int_0^x |Z(x-r) - \tilde{Z}(x-r)|F(dr), \end{aligned}$$

where $\delta(x)$ is a bounded Riemann function in view of Lemmas 3 and 4. From Lemma 4 and Theorem 3 (applied to $y(x) = \delta(x)$):

$$\lim_{x \rightarrow \infty} \frac{|Z(x) - \tilde{Z}(x)|}{x} = 0.$$

Hence:

$$Z(x) \sim \tilde{Z}(x) \sim \frac{l}{\mu}x.$$

It remains to prove the more precise statement for $U(x)$. Let \tilde{U} the solution of Equation (12) where F_x is replaced by F . We δ as previously for the function $z = 1$. Let $0 \leq a < b$ and a non-decreasing function g with $g(x) \leq x$ and $g(x)/x$ tends to 0 at infinity. Now assume that $z(x) = \mathbf{1}(a, b)(x - g(x))$ will be chosen later and define Z and \tilde{Z} as above. Note that $Z(x) = U(x - g(x) - a) - U(x - g(x) - b)$ (resp. for \tilde{Z} and \tilde{U}).

Note that F cannot be arithmetic since F is the limit of a distance between two points in a PPP. Then the Renewal Theorem (first form) (§XI.1 in [10]) implies:

$$\lim_{x \rightarrow \infty} \tilde{Z}(x) = \lim_{x \rightarrow \infty} \tilde{U}(x-a) - \tilde{U}(x-b) = \frac{b-a}{\mu}. \quad (13)$$

Moreover:

$$\begin{aligned} |Z(x) - \tilde{Z}(x)| &\leq \int_0^x |Z(x-r)|F_x(dr) - F(dr)| + \int_0^x |Z(x-r) - \tilde{Z}(x-r)|F(dr) \\ &\leq \delta(x) + \int_0^x |Z(x-r) - \tilde{Z}(x-r)|F(dr) \\ &\leq \delta(x) + \int_0^{x-g(x)} |Z(x-r) - \tilde{Z}(x-r)|F(dr), \end{aligned}$$

indeed $Z(x-r)$ and $\tilde{Z}(x-r)$ are equal to 0 for $x-r-g(x-r) \leq a \leq 0$ and thus for $r \geq x-g(x)$.

We deduce (Equation §XI.1 (1.5) in [10]):

$$|Z(x) - \tilde{Z}(x)| \leq \int_{x-g(x)}^x \delta(r)\tilde{U}(dr). \quad (14)$$

Let $I_n = [n, n+1)$, $n \in \mathbb{N}$, from Assumption (ii) $\sup_{x \in I_n} \delta(x) = m_n < \infty$, let $\tilde{\delta}(x) = \sum_n m_n \mathbf{1}(x \in I_n)$. $\tilde{\delta}$ is a bounded Riemann function and $\lim_{x \rightarrow \infty} \tilde{\delta}(x) = \lim_{n \rightarrow \infty} m_n = 0$. By Lemma 24 (in Appendix), we may suppose that g has been chosen so that $\int_{x-g(x)}^x \tilde{\delta}(r)dr$ tends to 0 as x tends to infinity.

Equation §XI.1 (1.17) in [10] stated for Equation (14) asserts:

$$\lim_{x \rightarrow \infty} |Z(x) - \tilde{Z}(x)| \leq \frac{1}{\mu} \lim_{x \rightarrow \infty} \int_{x-g(x)}^x \tilde{\delta}(r)dr = 0.$$

The theorem follows then from Equation (13). \square

Remark 3 \tilde{Z} is the expectation of the sum of length of the links that connect a point added to the PPP at $(x, 0)$ to the axis $x = 0$ in the directed spanning forest model. We have therefore proved that the $Z(x)$ is equivalent to the asymptotic model as x tends to ∞ .

3.1.2 Law of Large Numbers

We now prove an almost sure convergence result for $H(X)$ on the path $\pi(X)$ from X to 0 in the memoryless navigation with non-negative progress.

Proposition 7 *Assume that F_X converges weakly as $|X|$ tends to infinity to F and that (F_X) is uniformly integrable then a.s.*

$$\lim_{|X| \rightarrow +\infty} \frac{H(X)}{|X|} = \frac{1}{\mu},$$

where $\mu = \int_0^\infty rF(dr) < \infty$.

Before proving this proposition, we will state two lemmas. The first lemma will be often used.

Lemma 5 *Let \mathcal{A} is a navigation with a.s. positive progress on a PPP or a PWIT. Let $x_0 \geq 0$, $\tau(X) = \inf\{k \geq 0 : |\mathcal{A}^k(X)| \leq x_0\}$, and let $\ell(X)$ be a positive $\mathbb{R}^d \rightarrow \mathbb{R}_+$ function tending to ∞ as $|X|$ tends to ∞ . If a.s. (resp. in L^p) $\tau(X)/\ell(X)$ converges to Z then a.s. (resp. in L^p) $H(X)/\ell(X)$ converges to Z .*

Proof. We have:

$$\tau(X) \leq H(X) \leq \tau(X) + \sup_{Y \in B(0, x_0) \cap N^{0, X_{\tau(X)}}} H(Y),$$

hence it is sufficient to prove that, for s small enough:

$$\mathbb{E} \sup_{Y \in N \cap B(0, x_0)} \exp(sH(Y)) < \infty.$$

The progress is non-negative: $\mathcal{A}(X) \in B(0, |X|)$, it follows that for a navigation on a PPP: $\mathbb{E} \sup_{Y \in N \cap B(0, x_0)} \exp(sH(Y)) \leq \mathbb{E} \exp(sN(B(0, x_0))) < \infty$. For a navigation on a PWIT, the vertices in $T(Y)$ of first generation is a PPP, and $|\mathcal{A}(Y)| \leq t(Y)$ where $t(Y) = \arg \sup_{Z \in N \cap B(0, |Y|)} |Z|$. Let $C(X, t) = B(0, |X|) \setminus B(0, |X| - t)$, $t(X)$ is a stopping time for the filtration $\{\mathcal{F}_{C(X, t)}^N\}_{0 \leq t \leq |X|}$. Since $N \cap C(X, t)$ is independent of $N \cap B(0, t)$, we obtain: $\mathbb{E} \exp(sH(Y)) \leq \mathbb{E} \exp(sN(B(0, |Y|))) < \infty$ for s small enough. Let $(U_k), k \in \mathbb{N}$, be an iid sequence of r.v. with the distribution of $\exp(sN(B(0, x_0)))$. From the independency of the subtrees in a PWIT and using Wald's formula, we get

$$\mathbb{E} \sup_{Y \in N \cap B(0, x_0)} \exp(sH(Y)) \leq \mathbb{E} \sum_{k=0}^{N(B(0, x_0))} U_k = \pi_d x_0^d \mathbb{E} U_1 < \infty.$$

□

We consider the following property:

$$\text{If } \epsilon \in \ell^0 \text{ then } \lim_{|X| \rightarrow \infty} \int_0^{|X|^{\epsilon(|X|)}} |F_X(t) - F(t)| dt = 0. \quad (15)$$

From Markov Inequality, Property (15) is implied by the stronger property, for some $\alpha > 1$:

$$\text{If } \epsilon \in \ell^0 \text{ then } \lim_{|X| \rightarrow \infty} \sup_{t \leq |X|^{\epsilon(|X|)}} t^\alpha |F_X(t) - F(t)| dt = 0,$$

which has already appeared in Lemma 2. The next lemma asserts that Property (15) is implied by the assumptions done in Proposition 7.

Lemma 6 *If F_X converges in f -norm to F with $f(r) = r$ then Property (15) holds. Conversely, if F_X converges weakly to F and $\{F_X\}_{X \in \mathbb{R}^d}$ is uniformly integrable then F_X converges in f -norm to F with $f(r) = r$.*

Proof. Let $\eta > 0$, there exists x_0 such that for all X , $|X| \geq x_0$,

$$\sup_{\psi: |\psi(r)| \leq r} \left| \int \psi(r) F_X(dr) - \int \psi(r) F(dr) \right| \leq \eta.$$

Let $\Delta_X(t) = 1 - |F_X(t) - F(t)|$ and $I = \{t : F_X(t) > F(t)\}$, we have

$$\begin{aligned} \int \Delta_X(t) dt &= \int |\overline{F}_X(t) - \overline{F}(t)| dt \\ &= \int_I \int_t^\infty F_X(dr) dt - \int_I \int_t^\infty F(dr) dt - \int_{I^c} \int_t^\infty F_X(dr) dt + \int_{I^c} \int_t^\infty F(dr) dt \\ &= \int \phi(r) F_X(dr) - \int \phi(r) F(dr) \\ &\leq \eta, \end{aligned}$$

where we have used Fubini's Theorem and set $\phi(r) = r \mathbf{1}(r \in I) - r \mathbf{1}(r \in I^c)$, $|\phi(r)| \leq r$.

We now prove the converse statement, the hypothesis imply that for all $\eta > 0$ there exists T such that for all X $\int_T^{+\infty} F_X(t) dt \leq \eta$ and $\int_T^{+\infty} F(t) dt \leq \eta$. Let $\psi(r) \leq r$, $I = \{t : F_X(t) > F(t)\}$ and $\phi(r) = r \mathbf{1}(r \in I) - r \mathbf{1}(r \in I^c)$. As above:

$$\begin{aligned} \left| \int \psi(r) F_X(dr) - \int \psi(r) F(dr) \right| &\leq \int \phi(r) F_X(dr) - \int \phi(r) F(dr) \\ &\leq \int |F_X(t) - F(t)| dt \\ &\leq 2\eta + \int_0^T |F_X(t) - F(t)| dt. \end{aligned}$$

and the second term tends to 0 by assumption. \square

We turn to the proof of Proposition 7.

Proof. We first assume that $\mu > 0$. Let $0 < \eta < \mu/2$ and $\epsilon(X) = 1/\sqrt{|X|}$, by Lemma 6 we may find x_0 and a function f such that if $|X| \geq x_0$:

$$\mathbf{1}(t \leq \sqrt{|X|})(\overline{F}(t) - f(t)) \leq \overline{F}_X(t) \leq \overline{F}(t) + f(t).$$

where $\int f(t) dt \leq \eta$, $f(t) \leq \overline{F}(t)$ and $\int_0^{\sqrt{|X|}} \overline{F}(t) - f(t) dt \geq \mu - 2\eta$.

Let $\tau(X) = \inf\{n : |X_n| \leq x_0\}$ and $(U_n), n \in \mathbb{N}$, (resp. $(V_n), n \in \mathbb{N}$) be an iid sequence of variables with tail distribution $1 \wedge (\overline{F} + f)$ (resp $\overline{F} - f$). We now define: $Y_n = |X| - \sum_{k=0}^{n-1} U_k$, $Z_n = |X| - \sum_{k=0}^{n-1} V_k \mathbf{1}(V_k \leq \sqrt{x_0})$,

$$\tau_+(X) = \inf\{n : |Y_n| \leq x_0\} \quad \text{and} \quad \tau_-(X) = \inf\{n : |Z_n| \geq x_0\}.$$

From Lemma 25 (in Appendix):

$$\mathbf{1}(\tau(X) > n)Z_n \stackrel{st}{\leq} \mathbf{1}(\tau(X) > n)|X_n| \stackrel{st}{\leq} \mathbf{1}(\tau(X) > n)Y_n$$

We deduce that:

$$\tau_-(X) \stackrel{st}{\leq} \tau(X) \stackrel{st}{\leq} \tau_+(X). \quad (16)$$

We have $\mathbb{E}U_n \leq \mu + \eta$ and $\mathbb{E}V_n \mathbf{1}(V_n \leq \sqrt{x_0}) \geq \mu - 2\eta$. By the elementary renewal Theorem, a.s.:

$$\liminf_X \frac{\tau_-(X)}{|X|} \geq \frac{1}{\mu + \eta}, \text{ and } \limsup_X \frac{\tau_+(X)}{|X|} \leq \frac{1}{\mu - 2\eta}. \quad (17)$$

From Equations (16) and (17) we get a.s.: $\liminf_X \tau(X)/|X| \geq 1/(\mu + \eta)$ and $\limsup_X \tau(X)/|X| \leq 1/(\mu - 2\eta)$. Then by Lemma 5, $H(X)/|X|$ tends a.s. to $1/\mu$. For $\mu = 0$ the same proof works but we consider only $\tau_-(X)$. \square

3.2 α -Stable Model

We now turn to the case, let $0 < \alpha < 1$, $c > 0$:

$$\bar{F}(t) \sim_{t \rightarrow +\infty} \frac{c}{t^\alpha}. \quad (18)$$

In this model, the directed progress is a.s. finite but it has an infinite mean. This case is slightly more complex than the previous. The tail of F is very large and due to some large jumps, the directed navigation differs significantly from the navigation. In view of Lemma 2, the extra assumption is

$$\text{If } \epsilon \in \ell^0 \text{ then } \lim_{|X| \rightarrow \infty} \sup_{t \leq |X| \epsilon(|X|)} t^\alpha |F_X(t) - F(t)| = 0. \quad (19)$$

This assumption is a uniformity assumption on the convergence of F_X to F to guarantee that the tail of F_X converges uniformly to the tail of F .

Proposition 8 *Let χ_α a random variable with α -stable distribution: $\mathbb{E} \exp(-s\chi_\alpha) = \exp(-s^\alpha)$. If Equations (18) and (19) hold then*

$$\lim_{|X| \rightarrow \infty} \mathbb{P}\left(\frac{H(X)}{|X|^\alpha} \geq t\right) \geq \mathbb{P}(\chi_\alpha^{-1/\alpha} \geq \Gamma(1 - \alpha)ct)$$

and

$$\limsup_{|X| \rightarrow \infty} \frac{\mathbb{E}H(X)}{|X|^\alpha} \leq 2 \frac{1 - \alpha}{c}.$$

This proposition is somewhat disappointing: we have not managed to prove that $H(X)/|X|^\alpha$ converges in law. Equation (19) is the best convergence that we can hope to prove however it is not sufficient: directed navigation and navigation do not have the same exact asymptotic behavior.

Proof. The proof of this theorem is similar to the proof of Theorem 7. Let $\epsilon(X) \in \ell^0$ such that $|X|\epsilon(|X|)$ tends to infinity and fix $\eta \in (0, c)$, by Equation (19), we may find x_0 such that for all X with $|X| \geq x_0$:

$$\mathbf{1}(t \leq |X|\epsilon(|X|))(\overline{F}(t) - \eta f(t)) \leq \overline{F}_X(t) \leq \overline{F}(t) + \eta f(t), \quad (20)$$

where $0 \leq \eta f(t) \leq \overline{F}(t)$ and $f(t) \sim t^{-\alpha}$.

Let $\tau(X) = \inf\{n : |X_n| \leq x_0\} \leq H(X)$. Following the proof of Theorem 7 and using Lemma 27 (in Appendix), the right hand side of Equation (20) gives:

$$\liminf_X \mathbb{P}\left(\frac{\tau(X)}{|X|^\alpha} \geq t\right) \geq \mathbb{P}(\chi_\alpha^{-1/\alpha} \geq \Gamma(1-\alpha)(c+\eta)t).$$

The proof of the left hand side of Equation (20) uses Lemma 30 (in Appendix):

$$\limsup_X \frac{\mathbb{E}\tau(X)}{|X|^\alpha \epsilon(|X|)^{\alpha-1}} \leq \frac{1-\alpha}{c-\eta}.$$

Since this last equation holds for all function $\epsilon(X)$ tending to 0 at infinity we deduce:

$$\limsup_X \frac{\mathbb{E}\tau(X)}{|X|^\alpha} < \infty.$$

Then using Lemma 5 we deduce that $\limsup \mathbb{E}H(X)/|X|^\alpha < \infty$. We can go one step further, using Remark 6, we have: $\limsup_X \mathbb{E}H(X)/|X|^\alpha \leq 2(1-\alpha)/c$. Indeed, fix $\eta > 0$ and let $\epsilon_k(X) = |X|^{1/k}$, for $|X|$ large enough, for all k : $\mathbb{E}H(X)/|X|^{1/k+\alpha(k-1)/k} \leq 2(1-\alpha)/c + \eta$. (from Equation (67)). \square

Remark 4 If we had supposed instead that the directed progress tail was equivalent to $l(t)/t^\alpha$ for a slowly varying function l then the same type of convergence result holds with $|X|^\alpha$ replaced by $|X|^\alpha/l(|X|)$.

3.3 Relatively Stable Model

We now turn to a limit case for some $c > 0$:

$$\overline{F}(t) \sim_{t \rightarrow +\infty} \frac{c}{t} \quad (21)$$

Proposition 9 *If Equation (21) holds and*

$$\text{If } \epsilon \in \ell^0 \text{ then } \lim_{|X| \rightarrow \infty} \sup_{t \leq |X|\epsilon(|X|)} t|F_X(t) - F(t)| = 0, \quad (22)$$

then a.s.

$$\lim_{|X| \rightarrow \infty} \frac{H(X) \ln |X|}{|X|} = \frac{1}{c}$$

Proof. The proof is similar to the proof of Proposition 8. We fix a function $\epsilon(X)$ tending to 0 with $|X|\epsilon(X)$ tending to infinity. Equation (20) still holds with $f(t) \sim 1/t$. Using Lemmas 28 and 30, we deduce:

$$\text{a.s. } \liminf_{|X| \rightarrow \infty} \frac{H(X) \ln |X|}{|X|} \geq \frac{1}{c} \quad \text{and} \quad \limsup_{|X| \rightarrow \infty} \frac{\mathbb{E}H(X) \ln |X| \epsilon(X)}{|X|} \leq \frac{1}{c}$$

If $\epsilon(X) = |X|^{1/n}$, we have: $\limsup_{|X| \rightarrow \infty} \frac{\mathbb{E}H(X) \ln |X|}{|X|} \leq n/(c(n-1))$. This last equation holds for all n , hence: $\limsup_{|X| \rightarrow \infty} \frac{\mathbb{E}H(X) \ln |X|}{|X|} \leq 1/c$. \square

3.4 Scaled Progress

We discuss in this paragraph cases when $P(X)$ does not converge toward an asymptotic progress but rather $|X|^{-\alpha}(|X| - P(X))$ for some $0 < \alpha \leq 1$.

3.4.1 Scale Free Progress

A case which will have an important impact in applications is $\alpha = 1$: the *scaled distribution of progress*, $P(X)/|X|$, converges weakly to a non-degenerate limit. Following Meester and Franceschetti in [12], we say that the navigation is *scale free* if the distribution of $P(X)/|X|$ does not depend on X . Similarly the distribution is *asymptotically scale free* if the distribution of $P(X)/|X|$ converges weakly to a non-degenerate limit. Let $\tilde{P}(X) = -\ln(1 - P(X)/|X|) \in \mathbb{R}_+ \cup \{+\infty\}$ and $\tilde{F}_X(t) = \mathbb{P}(\tilde{P}(X) \geq t)$. Note that $\mathbb{P}(\tilde{P}(X) = \infty)$ may be positive. We assume that \tilde{F}_X converges weakly to some limit distribution \tilde{F} . We can deduce from the finite mean case an almost sure convergence result of $H(X)$. Indeed, define for $i < H(X) - 1$, let $\tilde{P}_i = -\ln(1 - P(X_i)/|X_i|)$, we have $|X_k| = |X| \prod_{i=0}^{k-1} (1 - P(X_i)/|X_i|)$ and $\ln |X_k| = \ln |X| - \sum_{i=0}^{k-1} \tilde{P}_i$. The corresponding path in $\mathbb{R} \cup \{-\infty\}$ is $\tilde{\pi}(X) = \{\ln |X|, \ln |X| - \tilde{P}_0, \dots, -\infty\}$. Let $\tau(X) = \sup\{n : \ln |X_n| < 0\}$, from Lemma 5 a.s. $\tau(X)$ and $H(X)$ are equivalent as $|X|$ tends to infinity (provided that they tend to infinity). We may apply Proposition 7 to the path $\tilde{\pi}(X)$ up to $\ln |X_{\tau(X)}|$. We immediately deduce the following result:

Proposition 10 *If \tilde{F}_X converges weakly to \tilde{F} as $|X|$ tends to infinity and (\tilde{F}_X) is uniformly integrable then a.s.*

$$\lim_{|X| \rightarrow +\infty} \frac{H(X)}{\ln |X|} = \frac{1}{\tilde{\mu}},$$

where $\tilde{\mu} = \int s \tilde{F}(ds) < \infty$.

3.4.2 Subcritical Case

We study the case when $(|X| - P(X))|X|^{-\alpha}$ is non-degenerate for some $0 < \alpha < 1$. Let $U(X) = |\mathcal{A}(X)||X|^{-\alpha} = (|X| - P(X))|X|^{-\alpha}$.

Proposition 11 *Assume that $\sup_{X \in \mathbb{R}^d} \mathbb{E}(|\ln U(X)| | U(X) \neq 0) < +\infty$ then a.s.*

$$\lim_{|X| \rightarrow +\infty} \frac{H(X)}{\ln \ln |X|} = -\frac{1}{\ln \alpha}.$$

Note that the proposition does not require any weak convergence of $U(X)$ toward a non-degenerate limit.

Proof. For $1 \leq k \leq H(X)$, let $U_k = |X_k||X_{k-1}|^{-\alpha}$. If $k < H(X)$, $\ln |X_k| = \alpha^k \ln |X| + \sum_{i=1}^k \alpha^{k-i} \ln U_i$ hence:

$$\ln |X_k| = \alpha^k \ln |X| + R_k, \quad (23)$$

with $|R_k| \leq Z_k = \sum_{i=1}^k \alpha^{k-i} |\ln U_i|$. With the convention that $Z_k = 0$ for $k \geq H(X)$, $(Z_k, X_k), k \in \mathbb{N}$, is a Markov chain and

$$Z_{k+1} = \alpha Z_k + |\ln U_{k+1}|.$$

Let $0 < \beta < 1 - \alpha$, by assumption there exists C_1 such that $\sup_{X \in \mathbb{R}^d} \mathbb{E} \mathbf{1}(U(X) \neq 0) |\ln U(X)| \leq C_1$ (with the convention " $0 \times \infty = 0$ "). It follows

$$\begin{aligned} \mathbb{E}(\mathbf{1}(H(X) > k+1)(Z_{k+1} - Z_k) | Z_k = z) &\leq -(1 - \alpha)z + C_1 \\ &\leq -\beta z + C_1 \mathbf{1}(z \in C), \end{aligned} \quad (24)$$

with $C = \{z \in \mathbb{R}_+ : z \leq C_1/(1 - \alpha - \beta)\}$. Equation (24) is a geometric drift condition on a Markov chain (see (V4), p371 in Meyn and Tweedie [23]). Let $K = \inf\{k \geq 1 : Z_k \in C\}$, by Theorem 15.2.5 in [23], for some $s > 0$,

$$\sup_{z \in C} \mathbb{E}(e^{s(K \wedge H(X))} | Z_0 = z) < \infty. \quad (25)$$

Let $x_0 = \exp(1 + C_1/(1 - \alpha - \beta))$, By Lemma 5, it is sufficient to show that a.s.

$$\lim_{|X| \rightarrow +\infty} \frac{\tau(X)}{\ln \ln |X|} = -\frac{1}{\ln \alpha},$$

where $\tau(X) = \inf\{k \geq 0 : |X_k| \leq x_0\}$. We fix $\epsilon > 0$ and let $(X^n), n \in \mathbb{N}$, be a sequence in \mathbb{R}^d such that $|X^n|$ tends to infinity. We define $K(n, \epsilon) = \lfloor -(1 + \epsilon)(\ln \ln |X^n|)/(\ln \alpha) \rfloor$ and $K'(n, \epsilon) = H(X) \wedge \inf\{k \geq K(n, \epsilon) : Z_k \in C\}$. From Borel-Cantelli Lemma and Equation (25), a.s. for n large enough $K'(n, \epsilon/3) \leq 2K(n, \epsilon/3) \leq K(n, \epsilon)$. Therefore for n large enough, from Equation (23):

$$\ln |X_{K(n, \epsilon)}| \leq \ln |X_{K'(n, \epsilon/3)}| \leq (\ln |X^n|)^{-\epsilon/3} + C_1/(1 - \alpha - \beta) \leq \ln x_0,$$

and it follows that a.s.

$$\limsup_X \frac{\tau(X)}{\ln \ln |X|} \leq -\frac{1}{\ln \alpha}.$$

The same computation can be done with $K(n, -\epsilon)$ and we deduce easily the statement of the proposition. \square

3.5 Average along a Path

We have so far taken interest only in $H(X)$, more generally we may try to find some almost sure convergence results for

$$G(X) = \sum_{i=0}^{H(X)-1} g(X_i, X_{i+1}) = g(X, \mathcal{A}(X)) + G(\mathcal{A}(X)).$$

$H(X)$ is the case $g = 1$. This is straightforward to generalize our results to $G(X)$. The same analysis can be done and we obtain for example:

Lemma 7 *Assume that $H(X)$ tends almost surely to infinity, that $(g(X, \mathcal{A}(X)))_{X \in \mathbb{R}^d}$ converges weakly as $|X|$ tends to infinity and $(g(X, \mathcal{A}(X)))_{X \in \mathbb{R}^d}$ is uniformly integrable then a.s.:*

$$\lim_{|X| \rightarrow \infty} \frac{G(X)}{H(X)} = \nu(g),$$

where $\nu(g) = \lim_{|X| \rightarrow +\infty} \mathbb{E}g(X, \mathcal{A}(X))$.

The proof of this lemma is omitted since it is identical to the proof of Proposition 7.

It is possible to get a convergence result even when no weak convergence holds.

Lemma 8 *Assume that $H(X)$ tends almost surely to infinity, that $\nu(g) = \lim_{|X| \rightarrow +\infty} \mathbb{E}g(X, \mathcal{A}(X))$ exists and that $\sup_{X \in \mathbb{R}^d} \mathbb{E}|g(X, \mathcal{A}(X))|^2$ is finite, then a.s.*

$$\lim_{|X| \rightarrow \infty} \frac{G(X)}{H(X)} = \nu(g).$$

Proof. As above \mathcal{F}_n denotes the σ -algebra generated by X_0, \dots, X_n . Since $\mathbb{E}(|g(X_k, X_{k+1})|^2) < M < \infty$, from Theorem VII.9.3 of [10] a.s.

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{k=0}^{H(X)-1} g(X_k, X_{k+1}) - \mathbb{E}(g(X_k, X_{k+1}) | \mathcal{F}_k) = 0. \quad (26)$$

For a memoryless navigation $\mathbb{E}(g(X_k, X_{k+1}) | \mathcal{F}_k) = \mathbb{E}(g(X_k, X_{k+1}) | X_k)$, thus our assumption implies a.s. $\lim_{|X| \rightarrow +\infty} \mathbb{E}(g(X_k, X_{k+1}) | \mathcal{F}_k) = \nu(g)$ and consequently a.s.:

$$\lim_{|X| \rightarrow +\infty} \frac{1}{H(X)} \sum_{k=0}^{H(X)-1} \mathbb{E}(g(X_k, X_{k+1}) | \mathcal{F}_k) = \nu(g). \quad (27)$$

Equations (26) and (27) lead to:

$$\lim_{|X| \rightarrow +\infty} \frac{G(X)}{H(X)} = \lim_{|X| \rightarrow +\infty} \frac{1}{H(x)} \sum_{k=1}^{H(x)} g(X_k, X_{k+1}) = \nu(g).$$

□

Remark 5 If the assumptions of the lemma hold with $g(X, Y) = |X| - |Y|$, $g(X, \mathcal{A}(X)) = P(X)$ then we obtain, a.s.:

$$\frac{|X|}{H(X)} \rightarrow \mu.$$

We thus deduce an alternative proof for the convergence of $H(X)/|X|$ (under different conditions).

4 Path Average for Navigation

We now turn to a more challenging question: the analysis of decentralized navigation on a PPP. The analysis in the PWIT or in a memoryless navigation was greatly simplified by the fact that the progress $P(X_k)$ was depending on the past history $\mathcal{F}_k = \sigma(X_0, \dots, X_k)$ on the sole position of the X_k . This property enabled us to rewrite the path as a non-homogeneous random walk.

As it pointed in [8] there is a technical issue to cope with in a navigation \mathcal{A} on the PPP: the dependency structure is much more complicated. In the Small World navigation for example, if $\mathcal{A}(X) = Y$ then it implies that there are (stochastically) fewer points of N^0 in $B(0, |Y|)$. So the navigation along these edges will not have the nice property of the navigation on the PWIT. We will circumvent this difficulty by a coupling argument, more precisely we will prove that the navigation in the Small World is regenerative.

4.1 Path Average for Regenerative Navigation

In this paragraph, we exhibit some sufficient conditions for a regenerative navigation to have converging path averages. Let \mathcal{A} be a regenerative navigation and θ its regenerative time: \mathcal{A}^θ is a memoryless navigation. We define $P(X) = |X| - |\mathcal{A}(X)|$ and $P^\theta(X) = |X| - |\mathcal{A}^\theta(X)|$. Let $\theta_0 = 0$, $\theta_1 = \theta$ and $\theta_k = \theta(X_{k-1})$, for $k \geq 1$. We define $H^\theta(X) = \inf\{k \geq 0 : \mathcal{A}^{\theta_k}(X) = 0\}$. The next lemma is elementary but nevertheless useful.

Lemma 9 *Let $\ell(X)$ be a positive $\mathbb{R}^d \rightarrow \mathbb{R}_+$ function tending to ∞ as $|X|$ tends to ∞ . We assume:*

1. $H^\theta(X)/\ell(X)$ tends a.s. to $1/\mu$, $\mu > 0$.
2. $\mathbb{E}\theta < \infty$.

Let $\lim_{|X| \rightarrow +\infty} \mathbb{E}\theta(X) = \bar{\theta}$, the following limit holds a.s.

$$\lim_{|X| \rightarrow +\infty} \frac{H(X)}{\ell(X)} = \frac{\bar{\theta}}{\mu}.$$

Proof. Note that $\theta_{H^\theta(X)-1} < H(X) \leq \theta_{H^\theta(X)}$ hence:

$$\frac{\theta_{H^\theta(X)-1}}{\ell(X)} < \frac{H(X)}{\ell(X)} \leq \frac{\theta_{H^\theta(X)}}{\ell(X)}.$$

Let $\tilde{\mathcal{A}} = \mathcal{A}^\theta$, we can apply Lemma 7 to $g(X, \tilde{\mathcal{A}}(X)) = \theta(X)$: we get that $\theta_{H^\theta(X)}/H^\theta(X)$ converges almost surely to $\bar{\theta}$. \square

This simple lemma states that the behavior on regenerative navigation relies on the behavior of its embedded memoryless navigation.

4.2 Directed Navigation on the Discrete Lattice Small World

The aim of this paragraph is pedagogical: on the simplest example we build a regenerative sequence. The point set N is a thinned one dimensional lattice. More precisely, let $(U_i), i \in \mathbb{Z}$, be a sequence of iid Bernoulli random variable, $\mathbb{P}(U_i = 1) = p > 0$. A point $i \in \mathbb{Z}$ is in N if $U_i = 1$:

$$N = \sum_{i \in \mathbb{Z}} U_i \delta_i.$$

The connection graph $\mathcal{G} = (N, E)$ is a Small World graph: vertices i and j are connected with probability: $|i - j|^{-\beta}$, $\beta > 1$, independently of the other conditioned on the event that i has at least one neighbor on its right. Let $(V_{ij}), i < j, i, j \in \mathbb{Z}$, be a sequence of random variable on $[0, 1]$ and $V_{ij} = V_{ji}$, $V_{ii} = 0$. There is an edge between i and j if $V_{ij} \leq |i - j|^{-\beta}$. We consider a maximal directed progress navigation to the right:

$$\mathcal{A}(i) = \sup\{j \in N : (i, j) \in E\} = \sup\{j \in N : V_{ij} \leq |i - j|^{-\beta}\}.$$

We assume that $(V_{ij}), i < j, i, j \in \mathbb{Z}$ is independent and uniformly distributed conditioned on the event: $\Omega = \cup_i \Omega_i$ and $\Omega_i = \{\sum_{j \geq 1} U_j \mathbf{1}(V_{ij} \leq |j - i|^{-\beta}) > 0\}$. In the computations $(\tilde{V}_{ij}), i < j, i, j \in \mathbb{Z}$ will denote a sequence of iid variables uniform on $[0, 1]$. We have $\mathbb{P}(\sum_{j \geq 1} U_j \mathbf{1}(\tilde{V}_{ij} \leq |j - i|^{-\beta}) > 0) = 1 - \prod_{j \geq 1} (1 - \frac{p}{j^\beta}) = \lambda^{-1}$.

As above the directed progress is defined by $P(i) = \mathcal{A}(i) - i > 0$. For $t \geq 1$:

$$\begin{aligned} \mathbb{P}(P(i) = t) &= \lambda \mathbb{P}(\{U_t \mathbf{1}(\tilde{V}_{i, i+t} \leq t^{-\beta}) = 1\} \bigcap_{k \geq t+i+1} \{U_k \mathbf{1}(\tilde{V}_{ik} > |k - i|^{-\beta}) = 0\}) \\ &= \lambda \frac{p}{t^\beta} \prod_{j > t} (1 - \frac{p}{j^\beta}) \end{aligned}$$

From similar computations, we also obtain for $\beta > 1$, as t tends to infinity:

$$\mathbb{P}(P(i) \geq t) = 1 - \lambda \left(1 - \prod_{j < t} \left(1 - \frac{p}{j^\beta}\right)\right) \prod_{j \geq t} \left(1 - \frac{p}{j^\beta}\right) \sim_t \frac{p}{\beta - 1} t^{1-\beta}. \quad (28)$$

It follows that $P(i)$ is almost surely finite for $\beta > 1$ and that $\mathbb{E}P(i)$ is finite for $\beta > 2$.

We define $X_0 = 0$, $X_k = \mathcal{A}^k(0)$, $P_k = P(X_k) = X_{k+1} - X_k$. Let $N_j = N \cap [j+1, +\infty] - j$, $N_{X_k} = \sum_{i \geq 1} U_{X_k+i} \delta_i$, N_{X_k} is the future of the navigation sequence. In particular $N_0 = \sum_{i \geq 1} U_i \delta_i$, where (U_i) are iid Bernoulli variable. The distribution of N_{X_k} , $k \geq 1$, is not as simple as the distribution of N_0 : X_k and $(U_i)_{i > X_k}$ are correlated: *the navigation is anticipating with respect to the natural spatial filtration*.

Consider the same navigation on \mathbb{Z} with the *same* V_{ij} : $\bar{\mathcal{A}}(i) = \sup\{j \in \mathbb{Z} : V_{ij} \leq |i - j|^{-\beta}\}$. Similarly we can compute the distribution of $\bar{P}(i) = \bar{\mathcal{A}}(i) - i$. It is important to notice that $\bar{P}(i) \geq P(i)$ and for $t \in \mathbb{N}$:

$$\mathbb{P}(\bar{P}(i) = P(i)) = \lambda p \quad \text{and} \quad \mathbb{P}(\bar{P}(i) - P(i) \geq t) \leq C_1 t^{2(1-\beta)}, \quad (29)$$

for some positive constant C_1 . The first equality comes from the fact that $\mathbb{P}(\bar{P}(i) = P(i)) = \mathbb{P}(U_{\bar{\mathcal{A}}(i)} = 1) = \lambda \mathbb{P}(U_{\bar{\mathcal{A}}(i)} = 1, \sum_{j > i} U_j \tilde{V}_{ij} > 0) = \lambda p$. The second inequality in Equation (29) stems for the fact that $\bar{\mathcal{A}}(i) = \max(\mathcal{A}(i), \mathcal{A}^*(i))$, where $\mathcal{A}^*(i)$ is the ancestor of i in the dual point set: $U_i^* = 1 - U_i$. As in Equation (28), the tail of $\mathcal{A}^*(i) - i$ is equivalent to $(1-p)t^{1-\beta}/(\beta-1)$. For $t \geq 1$, a direct computation leads to $\mathbb{P}(\bar{P}(i) - P(i) \geq t) = \mathbb{P}(\mathcal{A}^*(i) - \mathcal{A}(i) \geq t) = \mathbb{P}(\sum_{j \geq t} (1 - U_{\mathcal{A}(i)+j}) \mathbf{1}(\tilde{V}_{0, \mathcal{A}(i)+j} \leq |\mathcal{A}(i) + j|^{-\beta}) > 0) \leq C_1 t^{2(1-\beta)}$, indeed on the event $\{U_j = 0\}$, V_j and \tilde{V}_j have the same distribution.

Let $X_0 = Y_0 = 0$, $Y_k = \bar{\mathcal{A}}(X_{k-1})$. \mathcal{F}_k is σ -algebra generated by the variables $(X_1, Y_1), \dots, (Y_k, X_k)$

Theorem 4 *If $\beta > 2$, \mathcal{A} is regenerative: there exists a.s. a $\{\mathcal{F}_k\}_{k \in \mathbb{N}}$ -stopping time $\theta \geq 1$ such that:*

$$N_{X_\theta} | \mathcal{F}_\theta \stackrel{\mathcal{L}}{=} N_0.$$

If $\beta > 3$, $\mathbb{E}\theta < \infty$.

This theorem may appear weird: the process X_k is anticipating but there exists a stopping time θ which guarantee a regenerative property. This paradox vanishes if we remind that θ is a stopping time with respect to the filtration *in time* \mathcal{F}_k and includes the variable Y_k which looks forward X_k *in space*. \mathcal{F}_k is thus the horizon of spatial anticipation at time k .

In order to prove Theorem 4, we will use a coupling between X and Y .

Lemma 10 *Assume that there exists an a.s. finite time $\theta \in \mathbb{N}^*$ such that*

$$X_\theta = \max_{k \leq \theta} Y_k,$$

then θ is a regenerative time.

Proof. Let $Z_n = \max_{k \leq n} Y_k$. By definition $Y_n = \sup\{j \in \mathbb{Z} : V_{X_n, j} \leq |i - j|^{-\beta}\}$ hence $Z_n = \sup\{j \geq Z_{n-1} : V_{X_n, j} \leq |i - j|^{-\beta}\}$.

Assume now that $X_\theta = Z_\theta$, $\theta \geq 1$: $N_{X_\theta} = \sum_{i \geq 1} U_{X_\theta + i} \delta_i = \sum_{i \geq 1} U_{Z_\theta + i} \delta_i$. Let $I = \{i_1, \dots, i_n\}$ be a finite subset in $\mathbb{N} \setminus \{0\}$ and $N(I) = \sum_{i \in I} U_i$. The smallest σ -algebra such that N_j is measurable is the σ -algebra generated by the collection $(N(j + I) = t)$, $I \subset \mathbb{N} \setminus \{0\}$, $t \in \mathbb{N}$. It thus suffices to prove $\mathbb{P}(N(X_\theta + I) = t | \mathcal{F}_\theta) = \mathbb{P}(N(I) = t)$ for all finite subsets I of $\mathbb{N} \setminus \{0\}$ and $t \in \mathbb{N}$. To this end, we write:

$$\begin{aligned} \mathbb{P}(N(X_\theta + I) = t | \mathcal{F}_\theta) &= \mathbb{P}(N(Z_\theta + I) = t, \bigcap_{0 \leq l \leq \theta} \bigcap_{j \in I} \{V_{X_l, j+Z_\theta} > (j + Z_\theta - X_l)^{-\beta}\} | \mathcal{F}_\theta) \\ &= \mathbb{P}(N(Z_\theta + I) = t | \bigcap_{0 \leq l \leq \theta} \bigcap_{j \in I} \{V_{X_l, j+Z_\theta} > (j + Z_\theta - X_l)^{-\beta}\}, \mathcal{F}_\theta) \\ &= \mathbb{P}(N(I) = t) \end{aligned} \quad (31)$$

Equation (30) holds since for all $j > 0$, $\mathbb{P}(\bigcap_{0 \leq l \leq n} \{V_{X_l, j+Z_n} > (j + Z_n - X_l)^{-\beta}\}) = 1$ and Equation (31) comes from the fact that, given $\bigcap_{0 \leq l \leq n} \{V_{X_l, j+Z_n} > (j + Z_n - X_l)^{-\beta}\}$, U_{j+Z_n} is a Bernoulli variable with parameter p . \square

We need a natural definition to compare two point sets. Let $N^1 = \sum_{i \geq 1} U_i^1 \delta_i$ and $N^2 = \sum_{i \geq 1} U_i^2 \delta_i$ we say that $N^1 \stackrel{st}{\leq} N^2$ if for all finite subset of positive integers I and $t \in \mathbb{N}$: $\mathbb{P}(\sum_{i \in I} U_i^1 \geq t) \leq \mathbb{P}(\sum_{i \in I} U_i^2 \geq t)$.

Lemma 11

$$N_{X_k} | \mathcal{F}_k \stackrel{st}{\leq} N_0. \quad (32)$$

Proof. Let I be a subset of $\mathbb{N} \setminus \{0\}$, as in the proof of Lemma 10, we write:

$$\begin{aligned} \mathbb{P}(N(X_k + I) = t | \mathcal{F}_k) &= \mathbb{P}(N(X_\theta + I) = t, \bigcap_{0 \leq l \leq k} \bigcap_{j \in I} \{V_{X_l, j+X_k} > (j + X_k - X_l)^{-\beta}\} | \mathcal{F}_k) \\ &\leq \mathbb{P}(N(X_k + I) = t | \bigcap_{0 \leq l \leq k} \bigcap_{j \in I} \{V_{X_l, j+X_k} > (j + X_k - X_l)^{-\beta}\}, \mathcal{F}_k) \\ &\leq \mathbb{P}(N(I) = t), \end{aligned}$$

indeed, as already pointed, for $j > 0$ given $\bigcap_{0 \leq l \leq k} \{V_{X_l, j+X_k} > (j + X_k - X_l)^{-\beta}\}$, U_{j+X_k} is a Bernoulli variable with parameter p . \square

We can now prove Theorem 4.

Proof. Let $Z_n = \max_{k \leq n} Y_k$ and $W_n = Z_n - X_n \geq 0$, $W_0 = 0$. By Lemma 10 it suffices to prove that there exists a time $\theta \geq 1$ such that $W_\theta = 0$. We rewrite W_n as:

$$\begin{aligned}
W_n &= \max_{1 \leq i \leq n} (Y_i - X_n) \\
&= \max_{1 \leq i \leq n} ((Y_i - X_i) + (X_i - X_n)) \\
&= \max_{1 \leq i \leq n} ((Y_i - X_i) - \sum_{k=i}^{n-1} P_k),
\end{aligned}$$

(where by convention $\sum_{k=n}^{n-1} \cdot = 0$). We note that $P_k \geq 1$. Let $(\sigma_k)_{k \in \mathbb{N}}$ be a sequence of iid copies of variables with distribution Y_1 , we deduce:

$$W_n \leq \left(\max_{2 \leq i \leq n-1} (\sigma_{i-1} - \sum_{k=i-1}^{n-1} 1) \right)^+$$

That is W_n is upper bounded by the largest residual service time in a GI/GI/ ∞ queue (see Appendix 7.4). By Lemma 32: for $\beta > 2$ θ is a.s. finite and for $\beta > 3$ and $\mathbb{E}\theta < \infty$. \square

As a consequence of Theorem 4 we have the following:

Corollary 2 *If $\beta > 3$, there exists a constant $\mu > 0$ such that:*

$$\frac{\mathcal{A}^k(0)}{k} \rightarrow \frac{1}{\mu}.$$

Proof. Assume $\beta > 3$, by Theorem 4 there exists an increasing sequence $(\theta_n), n \in \mathbb{N}, \theta_0 = 0, \theta_1 = \theta, \theta_{n+1} - \theta_n$ is iid and $\mathbb{E}(\theta_{n+1} - \theta_n) < \infty$. This sequence satisfies: $N_{X_{\theta_n}} \stackrel{\mathcal{L}}{=} N_0$. We define:

$$P_k^\theta = X_{\theta_{k+1}} - X_{\theta_k} = \sum_{l=\theta_k}^{\theta_{k+1}-1} P_l.$$

The sequence $(P_k^\theta), k \in \mathbb{N}$, is iid. Let $F^\theta(t)$ denote the distribution function of P_0^θ .

$$\mathbb{P}(P_n \geq t | \mathcal{F}_n) \leq \mathbf{1}(t \geq W_n) \bar{F}(t) + \mathbf{1}(t < W_n). \quad (33)$$

As t tends to infinity, from Equation (28), we have $\bar{F}(t) = \mathbb{P}(P_0 \geq t) \sim \frac{p}{\beta-1} t^{1-\beta}$. It follows

$$\mathbb{E}P_0^\theta \leq \mathbb{E}\theta \mathbb{E}P_0 + \mathbb{E} \sum_{l=0}^{\theta-1} P_l < \infty.$$

Therefore, Corollary 2 follows from the strong law of large numbers. \square

Our method fails for $\beta \in (1, 3)$. For example for $1 < \beta < 2$, we expect that $\mathcal{A}^k(0)$ is of order of magnitude $k^{1/(\beta-1)}$. Nevertheless, this approach paves the way to proofs of asymptotic results to directed navigation on PPP.

4.3 Directed Navigation on a Small World

In this paragraph, we extend the results established on the discrete lattice to Poisson Point Processes on \mathbb{R}^d . The method is exactly the same but the coupling is different.

We recall the model introduced in §1.5.1. The navigation graph if $\mathcal{G} = (N^0, E)$, N is PPP on \mathbb{R}^d . We mark N to obtain a marked point process: $\bar{N} = \sum_n \delta_{X_n, V_n}$, where $V_n = (V_{ni})_{i \in \mathbb{N}} \in [0, 1]^{\mathbb{N}}$. For X, Y in N , we will write $V(X, Y)$ for V_{ni} where n and i are the index of X and Y . Let f be a measurable function from \mathbb{R}_+ to $[0, 1]$ such that for a constant $c > 0$ and $\beta > d$:

$$f(t) \sim_{t \rightarrow +\infty} \frac{c}{t^\beta}.$$

The Small World Graph is defined by:

$$(X, Y) \in E \text{ if } V(X, Y) \leq f(|X - Y|).$$

We fix a direction e_1 in S^{d-1} . The maximal progress navigation from $X \in N$ to 0 is defined as:

$$\mathcal{A}_{e_1}(X) = \arg \max\{ \langle Y, e_1 \rangle : (X, Y) \in E \}.$$

We define $\mathcal{H}(X) = \{x \in \mathbb{R}^d : \langle x - X, e_1 \rangle \geq 0\}$, the directed progress is positive if $\mathcal{A}(X, e_1) \in \mathcal{H}(X)$. We assume that the marks $(V_{ij}), i < j, i, j \in \mathbb{N}$ are independent and uniformly distributed conditioned on the event: $\Omega = \{\forall X \in N : \int_{\mathcal{H}(X)} \mathbf{1}(V(X, x) \leq |x - X|^{-\beta}) N(dx) > 0\}$.

Let F be the distribution of the directed progress: $P_{e_1}(X) = \langle X, e_1 \rangle - \langle \mathcal{A}(X, e_1), e_1 \rangle$ (which does not depend on e_1). In this section $P_{e_1}(X)$ will be denoted for short by $P(X)$ and $P_k = P(X_k, e_1)$

The aim of this section is to prove the following theorem:

Theorem 5 *The following assertions holds:*

- If $\beta > d + 1$, \mathcal{A} is regenerative.
- If $\beta > d + 2$, there exists a constant $\mu > 0$ such that:

$$\frac{\langle \mathcal{A}^k(0), e_1 \rangle}{k} \rightarrow \frac{1}{\mu}.$$

The remaining part of this subsection is devoted to the proof of Theorem 5. The proof is parallel to the proof in the lattice case and relies on the existence of a regenerative sequence. *Proof.* The set of neighbors of in G to X is denoted by $V(X)$. It is a thinning a N and $V(X)$ is a non-homogeneous Poisson point process with intensity: $f(|X - x|)dx$ conditioned on $\{V(X) \cap \mathcal{H}(X) \neq \emptyset\}$. In the next computation $\tilde{V}(X)$ will denote a PPP of intensity $f(|X - x|)dx$ obtained by an independent thinning of N . If A is a Borel set in $\mathcal{H}(X)$, $0 \notin A$

then $\mathbb{P}(\tilde{V}(X)(A) = 0) = \exp(-\int_A f(|X-x|)dx)$ and $\mathbb{P}(V(X)(A) = 0) = \mathbb{P}(\tilde{V}(X)(A) = 0)\mathbb{P}(\tilde{V}(X)(\mathcal{H}(X) \cap A^c) > 0)/\mathbb{P}(\tilde{V}(X)(\mathcal{H}(X)) > 0)$. We then write:

$$\begin{aligned} \mathbb{P}(N(A) = k | A \subset \mathcal{H}(X_1)) &= \mathbb{P}(\tilde{V}(X)(A) = 0 | N(A) = k) \mathbb{P}(N(A) = k) / \mathbb{P}(\tilde{V}(X)(A) = 0) \\ &= \left(1 - \frac{\int_A f(|X-x|)dx}{|A|}\right)^k \frac{|A|^k}{k!} \exp(-|A| + \int_A f(|X-x|)dx) \\ &= (|A| - \int_A f(|X-x|)dx)^k \frac{1}{k!} \exp(-|A| + \int_A f(|X-x|)dx). \end{aligned}$$

In other words: $N \cap \mathcal{H}(X_1)$ is a Poisson point process of intensity: $(1 - f(|X-x|))dx$. Since f tends to 0, far from X_1 , the distribution $N \cap \mathcal{H}(X_1)$ and $N \cap \mathcal{H}(0)$ are close. We formalized this idea with the next lemma.

Lemma 12 *For all X there exists a random variable $Y(X) \geq \langle X, e_1 \rangle + P(X)$ such that for all Borel sets A with $A \subset \mathcal{H}(Y(X)e_1)$, $t \in \mathbb{N}$:*

$$\mathbb{P}(N(A) = t | Y(X)) = \mathbb{P}(N(A) = t). \quad (34)$$

$$\mathbb{P}(Y(X) - \langle X, e_1 \rangle \geq t) \leq C_1 t^{d-\beta}. \quad (35)$$

This lemma states that there exists an a.s. finite length $Y(X)$ such that beyond $Y(X)$, N given X_1 is distributed as an homogeneous PPP.

Proof. $N \cap \mathcal{H}(X + P(X)e_1)$ is a PPP of intensity $(1 - f(|X-x|))dx$. We build a coupling to retrieve a PPP of intensity 1. Let $\tilde{V}(X)$ be a PPP with intensity $f(|X-x|)dx$ and independent of N . Since $\tilde{V}(X) \cap \mathcal{H}(X + P(X)e_1)$ and $N \cap \mathcal{H}(X + P(X)e_1)$ are independent: $(\tilde{V}(X) + N) \cap \mathcal{H}(X + P(X)e_1)$ is a PPP of intensity 1 on $\mathcal{H}(X + P(X)e_1)$. $\tilde{V}(X)$ is a.s. a finite point set. Let $\rho(X)$ be the radius of the smallest ball containing $\tilde{V}(X)$, we have:

$$\mathbb{P}(\rho(X) \geq t) = 1 - \exp(-\int_{B(0,t)^c} f(x)dx) \leq C_1 t^{d-\beta}.$$

We define:

$$Y(X) = \langle X, e_1 \rangle + \max(P(X), \rho(X)),$$

and clearly Equation (35) holds. If A is a Borel set in $\mathcal{H}(Y(X)e_1)$ then $(\tilde{V}(X) + N)(A) = N(A)$. Since $(\tilde{V}(X) + N) \cap \mathcal{H}(X + P(X)e_1)$ is a PPP of intensity 1, we deduce Equation (34). \square

We build a non-increasing sequence (Z_k) such that for $A \subset \mathcal{H}(Z_k)$,

$$\mathbb{P}(N(A) = t | X_0, \dots, X_k, Z_k) = \mathbb{P}(N(A) = t). \quad (36)$$

We set $Z_1 = Y_1 = Y(X)$, from Lemma 12, given Z_1 , $N \cap \mathcal{H}(Z_1e_1)$ is a PPP of intensity 1. $N \cap \mathcal{H}(X_2)$ is a thinning of N . Hence, given Z_1 and X_2 , $N \cap \mathcal{H}(\max(X_2, Z_1e_1))$ is a PPP $(1 - f(|X_1-x|))dx$. As in the proof of Lemma 12, let $\tilde{V}(X_1)$ be a PPP with intensity

$f(|X_1 - x|)dx$ and independent of N . Let ρ_1 denotes the smallest ball which contains $\tilde{V}(X_1)$, and $Y_2 = \langle X_1, e_1 \rangle + \max(\rho_1, P_1)$. We define:

$$Z_2 = \max(Y_1, Y_2).$$

Equation (36) for $k = 2$. Similarly we define iteratively, $Y_k = \max(\langle X_{k-1}, e_1 \rangle + \rho_{k-1}, \langle X_k, e_1 \rangle)$, and $Z_k = \max_{l \leq k} Y_l$. Equation (36) holds for this sequence. Let \mathcal{F}_k be σ -algebra generated by $((X_1, Y_1), \dots, (X_k, Y_k))$. Since ρ_k is independent of N we have

$$\mathbb{P}(Y_k - \langle X_k, e_1 \rangle \geq t | \mathcal{F}_k) \leq \mathbb{P}(\rho_k \geq t) \leq C_1 t^{d-\beta}. \quad (37)$$

The coupling we have build for a PPP on \mathbb{R}^d is different for the coupling that we have used on the \mathbb{Z} lattice. Let $N_X = N \cap \mathcal{H}(X) - X$. We endow the set of point processes of the natural partial order relation: $N_1 \stackrel{st}{\leq} N_2$ if for all Borel sets A and $t \in \mathbb{N}$, $\mathbb{P}(N_1(A) \geq t) \leq \mathbb{P}(N_2(A) \geq t)$. The next Lemma is similar to Lemma 11.

Lemma 13

$$N_{X_k} | \mathcal{F}_k \stackrel{st}{\leq} N_0 \quad (38)$$

and consequently for some $C_0 > 0$,

$$(i) \quad \mathbb{P}(P_k \geq 1 | \mathcal{F}_k) \geq C_0. \quad .$$

$$(ii) \quad \mathbb{P}(Y_k = \langle X_k, e_1 \rangle | \mathcal{F}_k) \geq C_0.$$

Proof. Equation (38) is a direct consequence of the fact that N_{X_k} is a non-homogeneous PPP of intensity $\prod_{l=0}^{k-1} (1 - f(|x + X_k - X_l|)) \leq 1$.

Assertion (i) stems from the fact that the progress is a.s. positive. Indeed, let $\mathcal{A}'(X) = \arg \min\{Y, e_1\} : Y \in N \cap \mathcal{H}(X), (X, Y) \in E\}$. The set of vertices in $V(X)$ is a.s. finite hence a.s. $\langle \mathcal{A}'(X) - X, e_1 \rangle > 0$ and a direct computation shows that $\mathbb{P}(\langle \mathcal{A}'(0), e_1 \rangle > t)$ is positive for all t . From Equation (38), there are stochastically fewer points in N_{X_k} given \mathcal{F}_k than in a PPP of intensity 1. We thus have the lower bound $\mathbb{P}(P_k \geq t | \mathcal{F}_k) \geq \mathbb{P}(\langle \mathcal{A}'(0), e_1 \rangle > t)$.

Statement (ii) follows from:

$$\mathbb{P}(Y_k = \langle X_k, e_1 \rangle | \mathcal{F}_k) \geq \mathbb{P}(\rho_k \leq 1) \mathbb{P}(P_k \geq 1 | \mathcal{F}_k) \geq C_0.$$

□

Note that Equation (37) and Statement (ii) in Lemma 13 imply that there exists a variable σ such that:

$$Y_k - X_k | \mathcal{F}_k \stackrel{st}{\leq} \sigma, \quad \mathbb{P}(\sigma = 0) > 0 \quad \text{and} \quad \mathbb{P}(\sigma \geq t) \leq C_1 t^{d-\beta}. \quad (39)$$

As in the lattice case, we define $W_n = \langle X_n, e_1 \rangle - Z_n \geq 0$, $W_0 = 0$. With the convention that inf over an empty set is $+\infty$, let $\theta_0 = 0$, $\theta_{n+1} = \inf\{k > \theta_n : W_k = 0\}$. We have:

$$W_n \stackrel{st}{\leq} \left(\max_{2 \leq i \leq n-1} (\sigma_{i-1} - \sum_{k=i-1}^{n-1} \tau_k) \right)^+,$$

where $(\sigma_k)_{k \in \mathbb{N}}$ is a sequence of iid copies of σ given in Lemma 12 and $(\tau_k)_{k \in \mathbb{N}}$ is a sequence of iid copies of τ with $\mathbb{P}(\tau \geq t) = \exp(-C_0 t^d)$, as in Lemma 13, Assertion (i). W_n is upper bounded by the largest residual service time in a GI/GI/ ∞ queue (see Appendix 7.4). The remainder of the proof is then as in Corollary 2. \square

4.4 Navigation in a Small World

4.4.1 Main Result

We continue our analysis to the model 2 introduced in §1.5.1. As in the §4.3, let F be the distribution of the directed progress: $P_{e_1}(X) = \langle X, e_1 \rangle - \langle \mathcal{A}(X, e_1), e_1 \rangle$ and F_X the distribution of the progress $P(X) = |X| - |\mathcal{A}(X)|$. Let $H(X)$ be the generation of X that is: $H(X) = \inf\{k \geq 0 : \mathcal{A}^k(X) = 0\}$. Let $X_k = \mathcal{A}^k(X)$ and $P_k = P(X_k) = |X_k| - |X_{k+1}|$.

Theorem 6 - If $\beta > d + 1$, \mathcal{A} is regenerative.

- If $\beta > d + 2$ and μ as in Theorem 5 a.s.

$$\lim_{|X| \rightarrow \infty} \frac{H(X)}{|X|} = \frac{1}{\mu}.$$

- If $\beta = d$, a.s. and $\tilde{\mu}$ as in Lemma 2 a.s.

$$\lim_{|X| \rightarrow \infty} \frac{H(X)}{\ln |X|} = \frac{1}{\tilde{\mu}}.$$

- If $d - 2 < \beta < d$, a.s.

$$\lim_{|X| \rightarrow \infty} \frac{H(X)}{\ln \ln |X|} = -\frac{1}{\ln(1 - \frac{d-\beta}{2})}.$$

In view of Lemma 2 this theorem is of the same type than the convergence results we have proved in the PWIT. The *PWIT approximation* gives the exact order of magnitude for $H(X)$. It is also worth to mention that our method has enabled us to determine the exact asymptotic limit for $\beta \in (d - 2, d]$.

4.4.2 Proof of Theorem 6: $\beta > d$

Step One: Regenerative Sequence

For $\beta > d + 2$, we build a sequence (θ_n) of stopping time on an enlarged filtration of (X_0, \dots, X_n) . The proof is close to the proof of Theorem 5. We will only focus on the differences.

As $N \cap B(0, X_1)$ is a Poisson point process of intensity: $(1 - f(|X - x|))dx$ (under its Palm version at 0).

Lemma 14 For all X there exists a random variable $0 \leq Y(X) \leq |X| - P(X)$ such that for all Borel sets A with $A \subset B(0, Y(X))$, $t \in \mathbb{N}$:

$$\mathbb{P}(N(A) = t | R(X)) = \mathbb{P}(N(A) = t). \quad (40)$$

Moreover for all X :

$$\mathbb{P}(|X| - Y(X) \geq t) \leq C_1 t^{d-\beta}. \quad (41)$$

The proof uses the same coupling than Lemma 12

Proof. Let $\tilde{V}(X)$ be a PPP with intensity $f(|X-x|)dx$ and independent of N . Since $\tilde{V}(X) \cap B(0, |X| - P(X))$ and $N \cap B(0, |X| - P(X))$ are independent: $(\tilde{V}(X) + N) \cap B(0, |X| - P(X))$ is a PPP of intensity 1 on $B(0, |X| - P(X))$ in its Palm version at 0. $\tilde{V}(X)$ is a.s. a finite point set. Let $\rho(X)$ be the radius of the smallest ball containing $\tilde{V}(X)$, we have for a some $C_1 > 0$ (not depending on X):

$$\mathbb{P}(\rho(X) \geq t) = 1 - \exp\left(-\int_{B(0,t)^c} f(x)dx\right) \leq C_1 t^{d-\beta}.$$

We then define: $Y(X) = (|X| - \max(P(X), \rho(X)))^+$. □

Let ρ_k denote the smallest ball which contains $\tilde{V}(X_k)$, where $\tilde{V}(X_k)$ is a PPP with intensity $f(|X_k - x|)dx$ and independent of N . We define $Y_0 = |X|$ and $Y_k = (\min(|X_{k-1}| - \rho_k, |X_k|))^+$ and \mathcal{F}_k be σ -algebra generated by $(X_1, Y_1), \dots, (X_k, Y_k)$, Let $Z_0 = |X|$ and

$$Z_k = \min(Z_{k-1}, |X_{k-1}| - \rho_k, |X_k|) = \min_{0 \leq l \leq k} Y_l.$$

For $A \subset B(0, Z_k)$, we have:

$$\mathbb{P}(N(A) = t | \mathcal{F}_k) = \mathbb{P}(N(A) = t). \quad (42)$$

The next lemma is the analog of Lemma 13.

Lemma 15 For all Borel set $A \subset B(0, |X_k|)$, $t \in \mathbb{N}$:

$$\mathbb{P}(N(A) \geq t | \mathcal{F}_k) \leq \mathbb{P}(N(A) \geq t). \quad (43)$$

and consequently:

- (i) if $|X_k| \geq 2$, $\mathbb{P}(P_k \geq 1 | \mathcal{F}_k) \geq C_0$,
- (ii) if $|X_k| \geq 2$, $\mathbb{P}(X_{k+1} = Y_{k+1} | \mathcal{F}_k) \geq C_0$.

Proof. We omit most the proof which is similar to the proof of Lemma 13. We only explain statement (i). Let $\mathcal{A}'(X) = \arg \max\{|Y| : Y \in N \cap B(0, |X|), (X, Y) \in E\}$. The progress is a.s. positive and the set of vertices in $V(X)$ is a.s. finite. Hence $|X| - |\mathcal{A}'(X)| > 0$ and a direct computation shows that $\mathbb{P}(|X| - |\mathcal{A}'(X)| > x_0) \geq \mathbb{P}(V(X) \cap (B(0, |X|) \setminus B(0, |X| - x_0)) = \emptyset) > C_0$ for $|X| \geq 2x_0$. From Equation (43), there are stochastically fewer points

in $N \cap B(0, |X_k|)$ given \mathcal{F}_k than in a PPP of intensity 1. We thus have the lower bound $\mathbb{P}(P_k \geq x_0 | \mathcal{F}_k) \geq \mathbb{P}(|X| - |\mathcal{A}'(X)| > x_0 | X = X_k)$. \square

Since ρ_k and X_k are independent, $\mathbb{P}(|X_k| - Y_k \geq t | \mathcal{F}_k) \leq \mathbb{P}(\rho_k \geq t) \leq C_1 t^{d-\beta}$. This last equation and statement (ii) in Lemma 15 imply that there exists a variable σ such that, if $|X| \geq 2$:

$$(Y_k - X_k) | \mathcal{F}_k \stackrel{st}{\leq} \sigma, \quad \mathbb{P}(\sigma = 0) > 0 \quad \text{and} \quad \mathbb{P}(\sigma \geq t) \leq C_1 t^{d-\beta}. \quad (44)$$

Exactly as in the lattice case, we define $W_n = |X_n| - Z_n \geq 0$, $W_0 = 0$ and for $n \geq H(X)$, $W_n = 0$. We have:

$$W_n \stackrel{st}{\leq} \left(\max_{2 \leq i \leq n-1} (\sigma_{i-1} - \sum_{k=i-1}^{n-1} \tau_k) \right)^+, \quad (45)$$

where $(\sigma_k)_{k \in \mathbb{N}}$ is a sequence of iid copies of σ given in Equation (44) and $(\tau_k)_{k \in \mathbb{N}}$ is a sequence of iid copies of τ with $\tau = 1$ with probability C_0 and 0 otherwise, as in Lemma 15.

By Equation (45), W_n is upper bounded by the largest residual service time in a GI/GI/ ∞ queue (see Appendix 7.4). Let \tilde{W}_n be the right hand side of Equation (45) and $\theta = \inf\{k \geq 1 : \tilde{W}_k = 0\}$. By Lemma 32 (in Appendix): if $\beta > d + 1$, θ is a.s. finite and if $\beta > d + 2$: $\mathbb{E}\theta < \infty$. By Equation (42), θ is a regenerative time for the the small world navigation.

Step Two: Embedded memoryless navigation

\mathcal{A}^θ is a memoryless navigation (for $|X| \geq 2$). We define:

$$P^\theta(X) = |X| - |X_\theta| = \sum_{k=0}^{\theta-1} P_k.$$

$$\mathbb{P}(P_k \geq t | \mathcal{F}_k) \leq \bar{F}_X(t) \mathbf{1}(t \geq W_k) + \mathbf{1}(t < W_k), \quad (46)$$

where $W_k \stackrel{st}{\leq} M$, $\mathbb{P}(M \geq t) \leq C_1 t^{-2(\beta-d)+1}$ and $\bar{F}_X(t) \leq C_1 t^{d-\beta}$. If $(U_k)_{k \in \mathbb{N}}$ denotes an iid sequence of variables such that $\mathbb{P}(U_k \geq t) = 1 \wedge C_1 t^{\beta-d}$ with U_k independent of \mathcal{F}_k , we have from Equation (46):

$$P^\theta(X) \stackrel{st}{\leq} Q = \sum_{k=0}^{\theta-1} (U_k + W_k).$$

We have $\mathbb{E}Q = \mathbb{E}\theta \mathbb{E}U + \mathbb{E} \sum_0^\theta W_k < \infty$ (from cycle formula, see Baccelli and Brémaud §3.1 [3]). It follows also that $(P^\theta(X))_{X \in \mathbb{R}^d}$ is uniformly integrable.

The next step is to identify $\lim_{|X| \rightarrow \infty} \mathbb{E}P^\theta(X)$. For the directed navigation with direction e_1 , the same sequence regenerative time θ was defined and Theorem 5 gives:

$$\mathbb{E}P_{e_1}^\theta(0) = \mu \mathbb{E}\theta.$$

$P_{e_1}^\theta(X)$ is a stabilizing functional of the Small World Graph and the distribution of $P_{e_1}^\theta(0)$ does not depend on e_1 . Hence from Theorem 1, $P^\theta(X)$ converges weakly to $P_{e_1}^\theta(0)$. Since $(P^\theta(X))_{X \in \mathbb{R}^d}$ is uniformly integrable, we obtain:

$$\lim_{|X| \rightarrow \infty} \mathbb{E}P^\theta(X) = \mathbb{E}P_{e_1}^\theta(0) = \mu \mathbb{E}\theta.$$

Thus we can apply Proposition 7 and Lemma 9 and we deduce that $H(X)/|X|$ tends a.s. to $1/\mu$.

4.4.3 Proof of Theorem 6: $\beta = d$

We define the scaled free progress as $\tilde{P}_k = -\ln(1 - P_k/|X_k|)$, we have $\ln|X_k| = \ln|X| - \sum_{i=0}^{k-1} \tilde{P}_i$. The proof follows step by step the proof of the case $\beta > d$ with a major difference: we need to consider scaled variables. We need also to be careful with the event $\{\tilde{P}(X) = \infty\} = \{P(X) = |X|\}$: in this paragraph, we will use the convention " $\ln \frac{0}{0} = 0$ ". We define $Y(X) = \min(|\mathcal{A}(X)|, \sup\{t : B(0, t) \cap \tilde{V}(X) = \emptyset\})$, where $\tilde{V}(X)$ is a PPP intensity $f(|x - X|)$ and independent of everything else, as in Lemma 14 we obtain:

Lemma 16 *There exists a random variable $0 \leq Y(X) \leq |\mathcal{A}(X)|$ such that for all Borel sets A with $A \subset B(0, Y(X))$, $t \in \mathbb{N}$:*

$$\mathbb{P}(N(A) = t|Y(X)) = \mathbb{P}(N(A) = t).$$

Moreover for all X :

$$\mathbb{P}(\ln \frac{|\mathcal{A}(X)|}{Y(X)} \geq s | Y(X) > 0) \leq C_1 \exp(-2s).$$

We define the sequence (Y_k) and (Z_k) as previously. Equation (42) still holds, and the analog of Lemma 15 reads:

Lemma 17 *For all Borel sets $A \subset B(0, |X_k|)$, $t \in \mathbb{N}$:*

$$\mathbb{P}(N(A) \geq t | \mathcal{F}_k) \leq \mathbb{P}(N(A) \geq t).$$

At this point of the proof an obstacle shows up, $\mathbb{P}(P_k \geq 1 | \mathcal{F}_k) \geq C_0 > 0$ does not implies the same statement on the scaled progress \tilde{P}_k . We thus have to circumvent this problem. To this end we define the event:

$$\Omega_\epsilon = \{\forall X \in \mathbb{R}^d : N^0 \cap B(0, |X|) \geq \epsilon |X|^d\}.$$

It is easy to check that $\mathbb{P}(\Omega_\epsilon) > 0$ for $\epsilon < \epsilon_0$ and $\lim_{\epsilon \rightarrow 0^+} \mathbb{P}(\Omega_\epsilon) = 1$.

Let $\mathbb{P}_\epsilon(\cdot)$ denote the conditional probability given Ω_ϵ . Under \mathbb{P}_ϵ , Lemmas 16 and 17 still hold. Moreover since a point in $B(0, t|X|)$ is at most at distance $(1+t)|X|$ of $|X|$, we have:

$$\begin{aligned} \mathbb{P}_\epsilon(\tilde{P}_k \geq 1 | \mathcal{F}_k) &= \mathbb{P}(P_k \geq |X_k| e^{-1} | \mathcal{F}_k) \\ &\geq 1 - (1 - (1 - f((1 + e^{-1})|X|)))^{\epsilon |X|^d} \\ &\geq 1 - \exp(-\epsilon |X|^d f((1 + e^{-1})|X|)), \end{aligned}$$

where we have used the assumption that f is non-increasing. Then, since $f(t) \sim ct^{-d}$ we deduce that there exists x_0 such that for all $X \in \mathbb{R}^d$, $|X| \geq x_0$:

$$\mathbb{P}_\epsilon(\tilde{P}_k \geq 1 | \mathcal{F}_k) \geq C_\epsilon.$$

Similarly, by Lemma 16 and 17, there exists a r.v. σ such that:

$$P_\epsilon(\ln \frac{|X_{k+1}|}{Y_{k+1}} \geq s | \mathcal{F}_k, Y_{k+1} > 0) \leq \mathbb{P}(\sigma \geq s) \quad \text{with } \mathbb{P}_\epsilon(\sigma = 0) > 0 \quad \text{and } \mathbb{P}_\epsilon(\sigma \geq s) \leq C_1 \exp(-2s). \quad (47)$$

We define $H'(X) = \inf\{k \geq 0, Y_k = 0\}$. $H(X)/\ln|X|$ converges a.s. is equivalent to $H'(X)/\ln|X|$ converges a.s.. Indeed ρ_k is independent of X_k and $|X_{H'(X)}| \leq \rho_{H'(X)}$, hence $|H'(X) - H(X)| \leq N(B(0, \rho_{H'(X)}))$. For $n \leq H'(X)$, $W_n = \ln(|X_n|/Z_n) \geq 0$, we have

$$W_{n+1} = \max(W_n - \ln \frac{|X_n|}{|X_{n+1}|}, \ln \frac{|X_{n+1}|}{Y_{n+1}}),$$

and it follows

$$W_n \leq \left(\max_{2 \leq i \leq n-1} (\sigma_{i-1} - \sum_{k=i-1}^{n-1} \tau_k) \right)^+, \quad (48)$$

where $(\sigma_k)_{k \in \mathbb{N}}$ is a sequence of iid copies of σ given in Equation (47) and $(\tau_k)_{k \in \mathbb{N}}$ is a sequence of iid copies of τ with $\mathbb{P}_\epsilon(\tau \geq 1) \geq C_\epsilon$. W_n is upper bounded by the largest residual service time in a GI/GI/ ∞ queue (see Appendix 7.4). Let θ be the first positive time at which the queue appearing on the left hand side of Equation (48) is empty. By Lemma 31 (in Appendix): θ is a.s. finite and for some $C_\epsilon > 0$:

$$\mathbb{E}_\epsilon \exp(C_\epsilon \theta) < \infty. \quad (49)$$

We define $\tilde{P}^\theta(X) = \sum_{k=0}^{\theta-1} \tilde{P}_k$. Using Equations (49), (46) and Lemma 2, we deduce that for all $0 < \epsilon < \epsilon_0$ there exists C_0 and C_1 such that $\mathbb{E}_\epsilon[\mathbf{1}(\tilde{P}^\theta(X) < \infty) \exp(-C_0 \tilde{P}^\theta(X))] \leq C_1$. We assume for the while that \tilde{P}^θ converges weakly and we define $\tilde{\mu}_\epsilon = \lim_{|X| \rightarrow \infty} \mathbb{E}_\epsilon \tilde{P}^\theta(X) / \mathbb{E}_\epsilon(\theta)$. From Proposition 10 we obtain

$$\mathbb{P}_\epsilon - a.s. \quad \lim_{|X| \rightarrow \infty} \frac{H(X)}{\ln|X|} = \frac{1}{\tilde{\mu}_\epsilon},$$

Since $\Omega_\epsilon \subset \Omega'_\epsilon$ for $\epsilon > \epsilon'$, $\tilde{\mu}_\epsilon$ does not depend on ϵ and we drop the ϵ in its expression, $\tilde{\mu}_\epsilon = \tilde{\mu}'$. Notice also that $\{\Omega_{1/n}\}_{n \in \mathbb{N}}$ is an increasing sequence of events and $\cup_n \Omega_{1/n} = \Omega$, so finally

$$\mathbb{P} - a.s. \quad \lim_{|X| \rightarrow \infty} \frac{H(X)}{\ln|X|} = \frac{1}{\tilde{\mu}'}$$

It remains to prove that \tilde{P}^θ converges weakly. For simplicity, we will only consider Model 3, with obvious change, the proof applies also to Model 2. We cannot apply Theorem 1 and instead we prove this fact directly. Let $\tilde{F}_{k,X}$ denote the distribution of $\tilde{P}_k(X)$. Lemma 2 asserts that $\tilde{F}_{0,X}$ converges weakly to \tilde{F} with distribution given by Equation (9). As already pointed, $N \cap B(0, |X_1|)$ is a Poisson Point process of intensity $\lambda_{X_0}(y) dy = (1 - f(|X_0 - y|)) dy$. We thus have

$$\mathbb{P}(\tilde{P}_1(X) \geq s | \mathcal{F}_1) = 1 - (1 - f(|X_1|)) \exp\left(- \int_{B(0, |X_1| e^{-s})} f(|X_1 - y|) \lambda_{X_0}(y) dy\right),$$

with the change of variable $z = y/|X_1|$ and $e_i = X_i/|X_i|$, we end up with:

$$\mathbb{P}(\tilde{P}_1(X) \geq s | \mathcal{F}_1) = 1 - (1 - f(|X_1|)) \exp\left(- \int_{B(0, e^{-s})} |X_1|^d f(|X_1| |e_1 - z|) (1 - f(|X_1| |e_0 e^{\tilde{P}_0} - z|)) dz\right),$$

Using the fact that $|X_1| = |X|e^{-\tilde{P}_0}$, we obtain that $\mathbb{P}(\tilde{P}_0(X) + \tilde{P}_1(X) \geq s)$ is equal to:

$$1 - \mathbb{E}(1 - f(|X|e^{-\tilde{P}_0})) \exp\left(- \int_{B(0, e^{-s-\tilde{P}_0})} e^{-d\tilde{P}_0} |X|^d f(e^{-\tilde{P}_0} |X| |e_1 - z|) (1 - f(|X| |e_0 - ze^{-\tilde{P}_0}|)) dz\right),$$

Letting $|X|$ tends to infinity and finally we deduce that:

$$\tilde{P}_0(X) + \tilde{P}_1(X) \Rightarrow \tilde{Q}_0 + \tilde{Q}_1,$$

where $(\tilde{Q}_k)_{k \in \mathbb{N}}$ is a iid sequence of variables with common distribution function \tilde{F} . Similarly for all $n \in \mathbb{N}$ we have:

$$\sum_{k=0}^{n-1} \tilde{P}_k(X) \Rightarrow \sum_{k=0}^{n-1} \tilde{Q}_k.$$

From Wald's Formula, $\mathbb{E}_\epsilon \sum_{k=0}^{\theta-1} Q_k = \mathbb{E}_\epsilon \theta \mathbb{E} Q_1$. Since the sequence $(\sum_{k=0}^{\theta-1} \tilde{P}_k(X))$ is uniformly integrable we deduce that $\lim_{|X| \rightarrow \infty} \sum_{k=0}^{\theta-1} \tilde{P}_k(X) = \mathbb{E}_\epsilon \theta \mathbb{E} Q_1$ and it follows

$$\tilde{\mu} = \tilde{\mu}'.$$

4.4.4 Proof of Theorem 6: $d - 2 < \beta < d$

The proof follows from Proposition 11 and the argument used in the case $\beta = d$. Let $\alpha = 1 - (d - \beta)/2$, we define for $1 \leq k \leq H(X)$, $U_k = |X_k|/|X_{k-1}|^\alpha$ and $U_k = 0$ for $k > H(X)$.

Let $|X_k| - \rho_k = \sup\{t : B(0, t) \cap \tilde{V}(X_k) = \emptyset\}$, where $\tilde{V}(X_k)$ is a PPP intensity $f(|x - X_k|)$ and independent of everything else. We define the sequences (Y_k) and (Z_k) as usual : $Y_k = \min((|X_{k-1}| - \rho_{k-1})^+, |X_k|)$ and $Z_k = \min_{k \leq l} Y_l$. Let $s > 0$, we have:

$$\begin{aligned} \mathbb{P}(\ln \frac{|X_1|}{|Y_1|} \geq s | Y_1 > 0) &\leq \mathbb{P}(\ln \frac{|X_1|}{(|X_1| - \rho_0)^+} \geq s | Y_1 > 0) \\ &\leq \mathbb{P}(\ln U_1 - \ln \frac{(|X_1| - \rho_0)^+}{|X_1|^\alpha} \geq s | Y_1 > 0) \\ &\leq \mathbb{P}(|\ln U_1| \geq s | X_1 \neq 0) + \mathbb{P}(|\ln \frac{|X_1| - \rho_0}{|X_1|^\alpha}| \geq s | \rho_0 < |X_1|) \\ &\leq C_1 \exp(-C_0 s^2). \end{aligned}$$

Let $H'(X) = \inf\{k \geq 1 : Y_k = 0\}$ and $W_n = \ln(|X_n|/Z_n) \geq 0$. The remainder of the proof is as in §4.4.3 with obvious changes.

4.5 Decentralized Navigation

4.5.1 How to prove that a navigation is regenerative ?

We now turn to more general decentralized navigation. In this paragraph we generalize the coupling method used in Small World graphs. We will only write down the method we have applied to the Small World and see under which conditions it applies to other navigation schemes. We consider a navigation \mathcal{A} on a PPP N which satisfies the hypothesis of Theorem 1. The associated directed navigation is denoted by \mathcal{A}_{e_1} , and we assume that the distribution of $P_{e_1}(0) = \langle \mathcal{A}_{e_1}(0) - 0, e_1 \rangle$ does not depend on e_1 (the directed progress distribution is isotropic). We define $H(X) = \inf\{k : \mathcal{A}^k(X) = 0\}$ and as usual our aim is to prove that:

$$\frac{H(X)}{|X|} = \frac{1}{\mu},$$

where $\mu = \lim_{k \rightarrow \infty} \frac{\mathbb{E}\langle \mathcal{A}_{e_1}^k(0), e_1 \rangle}{k} \in (0, +\infty)$. A convenient way to state this result is to find an answer to the question:

Under which conditions a navigation is regenerative ?

Step One : Regenerative time on the directed navigation.

We start by the directed navigation $\mathcal{A}_{e_1}(0)$. We define $X_0 = 0$ and $X_k = \mathcal{A}_{e_1}^k(0)$, $P_{e_1, k} = P_{e_1}(X_k)$. Let $E(X) = (X, \mathcal{A}_{e_1}(X))$ be the edge vector and $E_k = E(X_k)$. $\mathcal{H}(X) = \{Y : \langle Y - X, e_1 \rangle > 0\}$ and $N_X = N \cap \mathcal{H}(X) - X$.

The first assumption is:

- (i) *If N_0 is a PPP of intensity $\lambda(x)dx$, $N_{\mathcal{A}_{e_1}(0)}$ is a PPP of intensity $(1 - \phi_{E(0)}(x))\lambda(x - \mathcal{A}_{e_1}(0))$, with $x \rightarrow \phi_{E(0)}(x)$ measurable and taking value in $[0, 1]$.*

Let $Y_0 = 0$ and $\mathcal{F}_0 = \sigma\{X_0, Y_0\}$, by induction we define a non-decreasing sequence Y_k and a filtration \mathcal{F}_k . Let \tilde{N}^0 be a PPP with intensity $\phi_{E_0}(x)dx$ and independent of N given E_0 . Then $(\tilde{N}^0 + N_{X_1})$ is a PPP of intensity 1 on $\mathcal{H}(0)$. Our coupling method will only work if \tilde{N}^0 is an a.s. finite set. We will assume for each k :

- (ii) *For some $\alpha > 2$, $\mathbb{E}((\int_{\mathcal{H}(0)} \phi_{E_k}(x)dx)^\alpha | \mathcal{F}_k) \leq C_1$.*

From assumption (ii), \tilde{N}^0 is a.s. a finite point set. Let ρ_0 be the radius of the smallest ball containing \tilde{N}^0 , we have:

$$\mathbb{P}(\rho_0 \geq t) = 1 - \mathbb{E} \exp\left(- \int_{B(0,t)^c \cap \mathcal{H}(0)} \phi_{E_0}(x)dx\right) \leq C_1 t^{-\alpha}.$$

We define:

$$Y_1 = \max(\langle X_0, e_1 \rangle + \rho_0, \langle X_1, e_1 \rangle) \quad \text{and} \quad \mathcal{F}_1 = \mathcal{F}_0 \vee \sigma\{X_1, Y_1\}.$$

Using Assumption (ii), we have, for $t > 0$:

$$\mathbb{P}(Y_1 - \langle X_1, e_1 \rangle > t) \leq \mathbb{P}(\rho_0 > t) \leq C_1 t^{-\alpha}$$

Similarly

$$\mathbb{P}(Y_1 = \langle X_1, e_1 \rangle) \geq \mathbb{E} \exp\left(-\int_{\mathcal{H}(0)} \phi_{E_0}(x) dx\right) \geq \exp(-C_1) = C_0.$$

Now if A is a Borel set in $\mathcal{H}(Y_1)$ then $(S_{X_1} \tilde{N}^0 + N)(A) = N(A)$ where $S_x N = \sum_n \delta_{T_n+x}$ if $N = \sum_n \delta_{T_n}$. Since $(\tilde{N}^0 + N_{X_1}) \cap \mathcal{H}(Y_1 - X_1)$ is a PPP of intensity 1, we deduce:

$$N_{Y_1} | \mathcal{F}_1 \stackrel{\mathcal{L}}{=} N_0.$$

Assume now that we have built a sequence $(Y_k)_{0 \leq k \leq n-1}$ and a filtration $(\mathcal{F}_k)_{0 \leq k \leq n-1} = \vee_{k=0}^{n-1} \sigma\{X_k, Y_k\}$ such that for all $k \leq n-1$:

$$N_{Z_k} | \mathcal{F}_k \stackrel{\mathcal{L}}{=} N_0 \quad \text{with} \quad Z_k = \max_{0 \leq l \leq k} Y_l. \quad (50)$$

$$0 \leq (Y_k - \langle X_k, e_1 \rangle) | \mathcal{F}_k \leq \sigma \quad \text{with} \quad \mathbb{P}(\sigma > t) \leq C_1 t^{-\alpha} \quad \text{and} \quad \mathbb{P}(\sigma = 0) > C_0. \quad (51)$$

From Assumption (i), N_{X_n} is a PPP of intensity $\lambda_n(x) dx$ where:

$$\lambda_n(x) = (1 - \phi_{E_n}(x)) \lambda_{n-1}(x - X_n + X_{n-1}) = \prod_{k=0}^n (1 - \phi_{E_k}(x - X_n + X_k)).$$

Let \tilde{N}^{n-1} be a PPP with intensity $\phi_{E_n}(x) \lambda_{n-1}(x - X_n + X_{n-1}) dx$ and independent of N given (E_n, \mathcal{F}_{n-1}) . $(\tilde{N}^{n-1} + N) \cap \mathcal{H}(X_n)$ is a PPP of intensity $\lambda_{n-1}(x - X_n + X_{n-1}) dx$. We define ρ_{n-1} as the radius of the smallest ball containing \tilde{N}^{n-1} and:

$$Y_n = \max(\langle X_{n-1}, e_1 \rangle + \rho_{n-1}, \langle X_n, e_1 \rangle) \quad \text{and} \quad \mathcal{F}_n = \mathcal{F}_{n-1} \vee \sigma\{X_n, Y_n\}.$$

Since $\lambda_{n-1} \leq 1$, we check as we did for $k=1$ that the tail inequality in Equation (51) holds for $k=n$. Moreover we have:

$$\mathbb{P}(Y_n = \langle X_n, e_1 \rangle | \mathcal{F}_{n-1}) \geq \mathbb{E} \exp\left(-\int_{\mathcal{H}(0)} \phi_{E_{n-1}}(x) dx\right) | \mathcal{F}_{n-1} \geq C_0.$$

Equation (50) follows also from the same reasoning. Indeed assume that A is a Borel set in $\mathcal{H}(Z_n)$ then $(S_{X_n} \tilde{N}^{n-1} + N)(A) = N(A)$ and we conclude as we did for $n=1$.

Step Two : Embedded memoryless directed navigation.

At this point, we introduce a new assumption:

(iii) For some positive constants C_0, C_1 : $\mathbb{P}(P_{e_1, k} \geq C_1 | \mathcal{F}_k) \geq C_0$

We then have built a sequence $(Y_k)_{0 \leq k \leq n-1}$ satisfying Equations (50) and (51). As usual, we define $W_n = \langle X_n, e_1 \rangle - Z_n \geq 0$, and let $\theta = \inf\{k \geq 1 : W_k = 0\}$ (with the convention that \inf over an empty set is $+\infty$). We have:

$$W_n \stackrel{st}{\leq} \left(\max_{2 \leq i \leq n-1} (\sigma_{i-1} - \sum_{k=i-1}^{n-1} \tau_k) \right)^+,$$

where $(\sigma_k)_{k \in \mathbb{N}}$ is a sequence of iid copies of σ and $(\tau_k)_{k \in \mathbb{N}}$ is a sequence of iid copies of τ with $\mathbb{P}(\tau = C_1) = C_0$ and $\mathbb{P}(\tau = 0) = 1 - C_0$, as it is given by Assumption (iii). By Lemma 32 (in Appendix): θ is a.s. finite and $\mathbb{E}\theta < \infty$. The directed navigation is thus regenerative, let $P_{e_1}^\theta(0) = \sum_{k=0}^{\theta-1} P_{e_1, k}$, from the strong law of large numbers, a.s.

$$\frac{\langle \mathcal{A}_{e_1}^k(0), e_1 \rangle}{k} \rightarrow \mu \quad \text{where} \quad \mu = \mathbb{E}P_{e_1}^\theta(0)/\mathbb{E}\theta.$$

Note at this point that μ is positive but may be infinite.

Step Three: Navigation

Now we turn back to the navigation from X to 0, $X_0 = X$, $X_k = \mathcal{A}(X)^k$ and $P_k = X_{k+1} - X_k$. N is a PPP in its Palm version at $(0, X)$. We assume that the set of assumptions (i), (ii) and (iii) extend to the navigation as well:

- (i') If $N \cap B(0, X)$ is a PPP of intensity $\lambda(x)dx$, $N \cap B(0, |\mathcal{A}(X)|)$ is a PPP of intensity $(1 - \phi_{E(X)}(x))\lambda(x - \mathcal{A}(X) + X)$.
- (ii') $\mathbb{E}((\int_{B(0, |X|)} \phi_{E_k}(x)dx)^\alpha | \mathcal{F}_k) \leq C_1$.
- (iii') For $|X| \geq x_0$, $\mathbb{P}(P_k \geq C_1 | \mathcal{F}_k) \geq C_0$.

As we previously did, we define by iteration $Y_0 = |X|$ and $Y_k = \min(|X_{k-1}| - \rho_k, |X_k|)$, where ρ_k is the radius of the smallest ball containing N^{k-1} a PPP with intensity $\phi_{E_k}(x)\lambda_{k-1}(x - X_k + X_{k-1})dx$ and independent of N given (E_k, \mathcal{F}_{k-1}) . Let $Z_k = \min(Z_{k-1}, Y_k) = \min_{0 \leq l \leq k} Y_l$ and $W_n = |X_n| - Z_n$. As long as $|X| \geq x_0$, $W_n \stackrel{st}{\leq} (\max_{2 \leq i \leq n-1} (\sigma_{i-1} - \sum_{k=i-1}^{n-1} \tau_k))^+$, we define θ as the first positive time at which the GI/GI/ ∞ queue is empty. By Lemma 32 (in Appendix), $\mathbb{E}\theta < \infty$. We have proved that the navigation is regenerative. Then we introduce:

$$P^\theta(X) = |X| - |X_\theta| = \sum_{k=0}^{\theta-1} P_k.$$

From Theorem 1, $P^\theta(X)$ converges weakly to $P_{e_1}^\theta(0)$. The following assumption guarantee that the convergence holds also in L^1

- (iv) $P^\theta(X)$ is uniformly integrable.

It remains to apply Proposition 7 and Lemma 9, we finally obtain:

$$a.s. \quad \frac{H(X)}{|X|} \rightarrow \frac{1}{\mu}.$$

4.5.2 Example of Application: Radial Navigation

Radial navigation is an example of application of our method. In dimension 2, the radial navigation is regenerative and θ , its regenerative time satisfies: $\mathbb{E} \exp(s\theta) < \infty$ for all $s > 0$.

Assumptions (i) and (i') hold with

$$\phi_{E(0)}(x) = \mathbf{1}(x \in B(X - \mathcal{A}(X), |X - \mathcal{A}(X)|)).$$

Indeed, on a nice point set N , $Y = \mathcal{A}(X)$ if and only if $|Y| < |X|$ and $N \cap B(0, |X|) \cap B(X, |X - Y|) = \emptyset$.

Let \mathcal{A}_{e_1} denote the directed radial navigation and $X_k = \mathcal{A}_{e_1}^k(0)$. In order to prove assumptions (ii) and (ii'), notice that:

$$\int_{\mathcal{H}(0)} \phi_{E_k}(x) dx = \int \mathbf{1}(x \in \mathcal{H}(0) \cap B(X_k - X_{k+1}, |X_k - X_{k+1}|)) dx \leq \pi_d |X_k - X_{k+1}|^d. \quad (52)$$

Let $D_n = \mathcal{H}(0) \cup_{l=0}^{k-1} B(X_k - X_n, |X_k - X_{k+1}|)$, N_{X_n} is a PPP of intensity $\lambda_n(x) dx = (1 - \mathbf{1}(x \in D_n)) dx$. We assume that the dimension d is 2. We define the cones $c_\alpha = \{x = (r, \theta) \in \mathbb{R}^2 : \theta \in [0, \alpha]\}$ for $\alpha > 0$, and $c_\alpha = \{x = (r, \theta) \in \mathbb{R}^2 : \theta \in (\alpha, 0]\}$ for $\alpha < 0$. The following lemma is proved in [9]

Lemma 18 For all n ,

$$c_{\frac{\pi}{6}} \subset D_n \quad \text{or} \quad c_{-\frac{\pi}{6}} \subset D_n.$$

This lemma implies in particular that:

$$P(|X_k - X_{k+1}| \geq t | \mathcal{F}_k) \leq e^{-\frac{\pi t^2}{12}}.$$

From Equation (52), we deduce, for all $s \in \mathbb{R}$:

$$\mathbb{E}(\exp(s \int_{\mathcal{H}(0)} \phi_{E_k}(x) dx) | \mathcal{F}_k) < \infty.$$

This last equation implies assumptions (ii) and (ii'). It remains to check that assumptions (iii) and (iii') hold. Let $x = (r, \theta)$, expressed in polar coordinates with basis $(0, e_1)$, $r > 0$, $\theta \in [0, \pi/2]$. The set D_k has the following property: if $x \in D_k$ then for all $0 \leq \alpha \leq \theta$, $(r, \alpha) \in D_k$ (and resp. if $\theta \in [-\pi/2, 0]$ for all $\theta \leq \alpha \leq 0$). This last property implies that $P_{e_1, k} | \mathcal{F}_k \stackrel{st}{\geq} P_{e_1, 0}$ (see [9] for details). Assumptions (iii) and (iii') follow.

5 Navigation Tree Topology

5.1 Maximal Deviation, Tree topology and f -straightness

We turn back to a navigation \mathcal{A} with a non-negative progress and we assume that an associated directed navigation \mathcal{A}_{e_1} exists for all $e_1 \in \mathbb{R}^d$. Let $X_k = \mathcal{A}^k(X)$, $P_k = |X_k| -$

$|X_{k+1}|$, $H(X) = \inf\{k : \mathcal{A}^k(X) = 0\}$, $\mathcal{F}_k = \sigma\{X_0, \dots, X_k\}$ and let $U_k = \langle X_k, X/|X| \rangle X/|X|$ be the projection of X_k on the straight line $\overline{0X}$.

The path from X to 0 in the navigation tree \mathcal{T}_0 is denoted by $\Pi(X) = \{X_0, X_1, \dots, 0\}$. $\Pi(X)$ may be seen as a piecewise linear curve in \mathbb{R}^d . The maximal deviation of this curve between is defined as

$$\Delta(X) = \max_{0 \leq k \leq H(X)} |X_k - U_k|. \quad (53)$$

To understand the intrinsic structure of \mathcal{T}_0 we need to characterize its *ends*. An end is a semi-infinite self-avoiding path in \mathcal{T}_0 , starting from the origin: $(0 = X_0, X_1, \dots)$. The set of ends of a tree is the set of distinct semi-infinite, self-avoiding paths (two semi-infinite paths are not distinct if they share an infinite sub-path). A semi-infinite path $(0 = X_0, X_1, \dots)$ has an asymptotic direction if $X_n/|X_n|$ has a limit in the unit sphere S^{d-1} .

Following Howard and Newman in [17], we will derive results on the semi-infinite self-avoiding paths in \mathcal{T}_0 by proving tail bounds on $\Delta(X)$.

For $X \in N$, let $\Pi_{out}(X)$ be the set of offspring of X in the \mathcal{T}_0 , namely the set $X' \in N$ such that $X \in \Pi(X')$. We now state a definition introduced in [17].

Definition 5 Let $f \in \ell^0$, A tree is said to be *f-straight at the origin*, if for all but finitely many vertices :

$$\Pi_{out}(X) \subset C(X, f(|X|)),$$

where for all $X \in \mathbb{R}^d$ and $\epsilon \in \mathbb{R}^+$, $C(X, \epsilon) = \{Y \in \mathbb{R}^d : \theta(X, Y) \leq \epsilon\}$ and $\theta(X, Y)$ is the absolute value of the angle (in $[0, \pi]$) between X and Y .

The following result shows how to relate *f-straightness* to $\Delta(X)$.

Proposition 12 Let \mathcal{T} be a random spanning tree on a PPP with an atom at 0. Let $\gamma \in (0, 1)$ and $\eta > 0$, if $\mathbb{P}(\Delta(X) \geq |X|^\gamma) \leq C_1 |X|^{-d-\eta}$ and $\sup_{X \in \mathbb{R}^d} \mathbb{E}|X - \mathcal{A}(X)|^r < \infty$ for some $r > (d+1)/\gamma$ then \mathcal{T} is *f-straight at the origin* for $f(x) = |x|^{\gamma-1}$.

Proof. We first prove that the number K of points T_n of N such that $\Delta(T_n) \geq |T_n|^\gamma$ is a.s. finite. From Slivnyak-Campbell's Formula :

$$\begin{aligned} \mathbb{E}K &= \mathbb{E} \sum_{T_n \in N} \mathbf{1}(\Delta(T_n) \geq |T_n|^\gamma) \\ &= \omega_{d-1} \int_0^\infty P(\Delta(x) \geq |x|^\gamma) x^{d-1} dx \\ &\leq \omega_{d-1} \int_0^\infty x^{d-1} \min(1, C_1 x^{-d-\eta}) dx < \infty. \end{aligned}$$

We define

$$B_{\gamma, x} = \{\exists X \in N : |X| \leq 2x \text{ and } |X - \mathcal{A}(X)| > x^\gamma\}.$$

Using the inequalities $\mathbb{P}(N(B(0, x)) \geq t) \leq \exp(-t \ln(t/(e\pi_d x^d)))$ and $\sup_{X \in \mathbb{R}^d} \mathbb{E}|X - \mathcal{A}(X)|^r \leq C_1$, we have

$$\begin{aligned} \mathbb{P}(B_{\gamma, x}) &\leq \mathbb{P}(N(B(0, 2x)) \geq e^2 \pi_d 2^d x^d) + e^2 \pi_d 2^d x^d \frac{\mathbb{E}|X - \mathcal{A}(X)|^r}{x^{r\gamma}} \\ &\leq \exp(-e^2 \pi_d 2^d x^d) + e^2 C_1 \pi_d 2^d x^{d-r\gamma} \\ &\leq C_1 x^{d-r\gamma} \end{aligned}$$

From Borel-Cantelli Lemma, it follows that there is some finite random x_0 so that for $X \in N \setminus B(0, x_0)$, $|X - \mathcal{A}(X)| \leq |X|^\gamma$. The rest of the proof uses the same argument as Lemma 2.7 of [17] (with $1 - \delta$ replaced by γ). \square

f -straight trees have a simple topology described by Proposition 2.8 of [17] and restated in Proposition 13.

Proposition 13 *Let \mathcal{T} be an f -straight spanning tree on a PPP. The following set of properties holds almost surely:*

- every semi-infinite path has an asymptotic direction,
- for every $u \in S^{d-1}$, there exists at least one semi-infinite path with asymptotic direction u ,
- the set of u 's of S^{d-1} such that there is more than one semi-infinite path with asymptotic direction u is dense in S^{d-1} .

In the following subsection we prove under some assumptions that \mathcal{T}_0 is f -straight with $f(x) = x^{1-\gamma}$.

5.2 Memoryless Isotropic navigation

We start with the simplest case in order to illustrate the method used to derive bounds on

$$\mathbb{P}(\Delta(X) \geq |X|^\gamma).$$

We assume in this paragraph that the navigation is memoryless with non-negative progress.

Let $e_1, e_2 \in S^{d-1}$, we define $\mathbb{U}(e_1, e_2) = \{R \in \mathbb{U} : R(e_2) = e_1\}$, where \mathbb{U} is the orthogonal group of \mathbb{R}^d .

Definition 6 *A navigation \mathcal{A} is isotropic if for all e_1, e_2 in S^{d-1} , $x \geq 0$ and $R \in \mathbb{U}(e_1, e_2)$:*

$$R\mathcal{A}(xe_2) \stackrel{\mathcal{L}}{=} \mathcal{A}(xe_1),$$

Note that if we consider $X \neq 0$ and e_1, e_2 in S^{d-1} with $\langle e_i, X/|X| \rangle = 0$ for $i = 1, 2$, the definition implies that $\langle \mathcal{A}(X), e_1 \rangle \stackrel{\mathcal{L}}{=} \langle \mathcal{A}(X), e_2 \rangle$. We can apply this fact to $e_1 = -e_2$ and we deduce

$$\mathbb{E}\langle \mathcal{A}(X), e_1 \rangle = 0.$$

All the navigation algorithms we have introduced are isotropic.

Theorem 7 *Let $\gamma \in (1/2, 1)$, if the navigation is isotropic, memoryless, with non-negative progress and:*

- $\mathcal{A}(xe_1) - xe_1$ converges weakly to $\mathcal{A}_{-e_1}(0)$ as x tends to $+\infty$.
- $\sup_{X \in \mathbb{R}^d} \mathbb{E}|X - \mathcal{A}(X)|^r < \infty$ with $r > (d+1)/\gamma$
- For $|X| \geq x_0$, $\mathbb{P}(P(X) \geq c) \geq \epsilon$ with $x_0, c, \epsilon > 0$.

Then for some $\eta > 0$, there exists C_1 such that

$$\mathbb{P}(\Delta(X) \geq |X|^\gamma) \leq C_1 |X|^{-d-\eta}.$$

and \mathcal{T}_0 is f -straight with $f(x) = |x|^{\gamma-1}$.

The second statement follows immediately from the first and Proposition 12. We will see in the proof of Theorem 7 that we may pick any η in $(0, r - (d+1)/\gamma)$, thus as an immediate corollary, we have:

Corollary 3 *If the navigation is isotropic, memoryless and with non-negative progress and:*

- $\mathcal{A}(xe_1) - xe_1$ converges weakly to $\mathcal{A}_{-e_1}(0)$ as x tends to $+\infty$.
- $\sup_{X \in \mathbb{R}^d} \mathbb{E}|X - \mathcal{A}(X)|^r < \infty$ for all $r > 1$.
- For $|X| \geq x_0$, $\mathbb{P}(P(X) \geq c) \geq \epsilon$ with $x_0, c, \epsilon > 0$.

Then for all $\epsilon > 0$ and $n \in \mathbb{N}$, there exists C_1 such that

$$\mathbb{P}(\Delta(X) \geq |X|^{1/2+\epsilon}) \leq C_1 |X|^{-n}.$$

and \mathcal{T}_0 is f -straight with $f(x) = |x|^{-1/2+\epsilon}$.

5.3 Proof of Theorem 7

5.3.1 Navigation in a cone

We fix $\gamma > 1/2$ and we assume first that there exists $0 < \alpha < \pi/2$ such that

$$\mathcal{A}(X) - X \in \mathcal{C}(\alpha, -X/|X|), \tag{54}$$

where $\mathcal{C}(\alpha, e_1) = \{Y \in \mathbb{R}^d \setminus \{0\} : \cos \alpha \leq \langle e_1, Y/|Y| \rangle\} \cup \{0\}$ (that is the cone with apex α and direction e_1).

Let $e_1, e_2 \in S^{d-1}$ with $\langle e_1, e_2 \rangle = 0$ and we assume that $X = |X|e_1$. We take interest to

$$U_k = \langle X_k, e_1 \rangle \quad \text{and} \quad V_k = \langle X_k, e_2 \rangle.$$

Let $F = \text{vect}(e_1, e_2)$, X_k^F the orthogonal projection of X_k on F and $(\cos \theta_k, \sin \theta_k)$ the coordinates of the projection of $X_k^F/|X_k^F|$ on the basis (e_1, e_2) . Let $R_k \in \mathbb{U}$ such that

$R_k X_k = |X_k| e_1$ and $e_2^k = R_k^{-1} e_2$. We define $p_k = \langle X_k^F - X_{k+1}^F, X_k / |X_k| \rangle$ and $q_k = \langle X_{k+1}^F - X_k^F, e_2^k \rangle$ (see figure 2). We have $p_k \geq 0$ (since the navigation has non-negative progress, $X_{k+1} \in \overline{B(0, |X_k|)}$) and

$$\begin{aligned}
 V_{k+1} &= V_k + q_k \cos \theta_k - p_k \sin \theta_k \\
 U_{k+1} &= U_k - p_k \cos \theta_k - q_k \sin \theta_k \\
 \tan \theta_k &= V_k / U_k.
 \end{aligned}$$

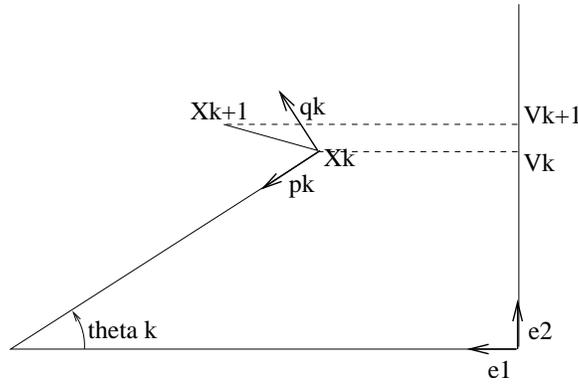


Figure 2: q_k , p_k , θ_k and V_k

If the navigation is isotropic and memoryless then the distribution of p_k and q_k depends only on $|X_k|$. From the isotropy we also have that $\mathbb{E}(q_k \cos \theta_k | \mathcal{F}_k) = \cos \theta_k \mathbb{E}(q_k | \mathcal{F}_k) = 0$. Let $V'_k = V_{k \wedge K}$ with

$$K = \inf\{k : V_k < 0 \text{ or } |U_k| \leq x^\gamma \tan \alpha\},$$

K is a $\{\mathcal{F}_k\}_{k \in \mathbb{N}}$ -stopping time. Since $\theta_k \in (0, \pi)$ for $k < K$ we have:

$$\mathbb{E}(V'_{k+1} | \mathcal{F}_k) = V'_k - \sin \theta_k \mathbb{E}(p_k | \mathcal{F}_k) \leq V'_k,$$

and for $k \geq K$, $\mathbb{E}(V'_{k+1} | \mathcal{F}_k) = V'_k = V_K$. Therefore $(V'_k)_{k \in \mathbb{N}}$ is a supermartingale (for the filtration $\{\mathcal{F}_k\}_{k \in \mathbb{N}}$).

Let $S_0 = 0$, $S_n = \sum_{k=0}^{n-1} q_k$ and $M_n = \max_{0 \leq k \leq n} S_k$. $(S_k)_{k \in \mathbb{N}}$ is a martingale with mean 0.

Lemma 19 *If $k \leq K$ and $M_k \leq x^\gamma$,*

$$V_k \leq S_k.$$

Proof. We prove this result by iteration, by symmetry we can suppose that $0 \leq \theta_k < \pi/2$. Since $X - \mathcal{A}(X) \in \mathcal{C}(\alpha, X/|X|)$, $|q_k| \leq p_k \tan \alpha$. Notice also that if $M_k \leq x^\gamma$ and $k \leq K$ then $\tan \theta_k \leq x^\gamma / (\tan \alpha x^\gamma) \leq 1/\tan \alpha$. It is sufficient to prove that for $\tan \theta \in [0, 1/\tan \alpha]$ and $|q| \leq p \tan \alpha$ that

$$q \cos \theta - p \sin \theta \leq q$$

If $q \geq 0$, there is nothing to prove. If $q \leq 0$ then we have to check that $|q| \leq p \sin \theta / (1 - \cos \theta)$. However since $|q| \leq p \tan \alpha \leq p / \tan \theta \leq p \sin \theta / (1 - \cos \theta)$ since $1/\tan \theta = \cos \theta / \sin \theta \leq \sin \theta / (1 - \cos \theta)$. \square

Lemma 20 *Let $r' < r$, for all $t > 0$, there exists $C_t > 0$ such that*

$$\mathbb{P}(M_n \geq n^\gamma t) \leq C_t n^{1-\gamma r'}.$$

Proof. This lemma is a consequence of Theorem 3.1 Equation (3.3) of Gut [16] (see also Theorem 2 in Baum and Katz [5]). This theorem is stated for a sum of independent variables but it applies to our case also. Indeed, we have the following two key features:

1. $(|q_k|)_{k \in \mathbb{N}}$ is stochastically bounded by an iid sequence $(Z_k)_{k \in \mathbb{N}}$ with $\mathbb{P}(Z_k \geq t | \mathcal{F}_k) \leq C_1 t^{-r}$, thus $\mathbb{E}Z^{r'} < \infty$.
2. $(q_k)_{k \in \mathbb{N}}$ are nearly independent: if $n, m, k \neq l \in \mathbb{N}$, $\mathbb{E}q_k^{2n+1} q_l^m = 0$ and $\mathbb{E}|q_k^{2n} q_l^{2m}| \leq \mathbb{E}Z^{2n} \mathbb{E}Z^{2m}$.

Since the proofs of Gut, Baum and Katz rely only Markov inequality and truncation, their results apply to our case. \square

Lemma 21 *For a memoryless navigation, if for $|X| \geq x_0$, $\mathbb{P}(P(X) \geq c) \geq \epsilon > 0$ then for all $\mu < c\epsilon$, there exist constants C_1 and C_0 such that:*

$$\mathbb{P}(H(X) \geq |X|/\mu) \leq C_1 \exp(-C_0|X|).$$

Proof. Let $\tau(X) = \inf\{k : |X_k| \leq x_0\}$ and $(B_k), k \in \mathbb{N}$ be an iid sequence of Bernoulli variables with $\mathbb{P}(B_1 = 0) = 1 - \epsilon$ and $\mathbb{P}(B_1 = 1) = \epsilon$. We write $1/\mu = 1/\mu' + \eta$, for $\eta > 0$ and $\mu' < c\epsilon$, we have:

$$\begin{aligned} \mathbb{P}(H(X) \geq |X|/\mu) &\leq \mathbb{P}(N(B(0, x_0) \geq \eta|X|) + \mathbb{P}(\tau(X) \geq |X|/\mu') \\ &\leq \mathbb{P}(N(B(0, x_0) \geq \eta|X|) + \mathbb{P}\left(\sum_{k=0}^{\lfloor |X|/\mu' \rfloor - 1} B_k < |X| - x_0\right) \\ &\leq \mathbb{P}(N(B(0, x_0) \geq \eta|X|) + \mathbb{P}\left(\sum_{k=0}^{\lfloor |X|/\mu' \rfloor - 1} B_k < |X|/c\right) \\ &\leq C_1 \exp(-C_0|X|) \end{aligned}$$

where we have the inequality $\mathbb{P}(N(B(0, x_0)) \geq t) \leq \exp(-t \ln(t/(e\pi_d x_0^d)))$ and Hoeffding's inequality: for $t < n\epsilon$, $\mathbb{P}(\sum_{k=0}^{n-1} B_k < t) \leq 2 \exp((t - n\epsilon)^2/(2n))$. \square

We are now in position to conclude the proof of Theorem 7 when Equation (54) holds.

For $l < n$, let $S_{l,n} = \sum_{k=l}^{n-1} q_k$, $S_{l,l} = 0$, $M_n = \max_{0 \leq k \leq n} S_k$ and $m_n = \max_{0 \leq k \leq n} -S_k = |\min_{0 \leq k \leq n} S_k|$. Finally, we define

$$K' = \inf\{k : |U_k| \leq x^\gamma \tan \alpha\}$$

A slight variation of Lemma 19 gives if $k \leq K'$, $m_n \leq x^\gamma$ and $M_n \leq x^\gamma$: $V_k \leq \max_{0 \leq l \leq k} S_{l,k} \leq S_k + m_k$. Hence

$$\mathbb{P}(\max_{0 \leq k < K'} V_k \geq x^\gamma, K' \leq n) \leq \mathbb{P}(M_n \geq x^\gamma/2) + \mathbb{P}(m_n \geq x^\gamma/2). \quad (55)$$

Note also that changing (q_k) into $(-q_k)$ in Lemma 20 gives $\mathbb{P}(m_n \geq n^\gamma t) \leq C_t n^{1-\gamma r'}$. From the isotropy of the navigation, we get:

$$\begin{aligned} \mathbb{P}(\Delta(X) \geq \frac{x^\gamma}{\sqrt{d} \cos \alpha}) &\leq 2d \mathbb{P}(\sup_{k \geq 0} V_k \geq \frac{x^\gamma}{\cos \alpha}) \\ &\leq 2d \mathbb{P}(\max_{0 \leq k < K'} V_k \geq x^\gamma) \end{aligned} \quad (56)$$

$$\begin{aligned} &\leq 2d \mathbb{P}(\max_{0 \leq k < K'} V_k \geq x^\gamma, K' \leq x/\mu) + 2d \mathbb{P}(H(X) \geq x/\mu) \\ &\leq C_1 x^{1-\gamma r'} + C_1 \exp(-C_0 x) \end{aligned} \quad (57)$$

Equation (56) stems from the following fact: if $|U_k| \leq x^\gamma \tan \alpha$ and $|V_k| \leq x^\gamma$ then (since $\mathcal{A}(X) \in B(0, |X|)$) $V_{k+1} \leq \sqrt{U_k^2 + V_k^2} \leq x^\gamma / \cos \alpha$. In Equation (57) we have used Equation (55), Lemma 20 and Lemma 21. If r' is close enough to r we have $\gamma r' - 1 > d$ and this concludes the proof of Theorem 7 when Equation (54) holds.

5.3.2 General Case

The general case is a consequence of the previous case. Indeed there exists α_x such that with $|X| = x > 0$:

$$\mathbb{P}(\mathcal{A}(X) - X \in \mathcal{C}(\alpha_x, -X/x)) \geq \frac{1}{2}, \quad (58)$$

where $\mathcal{C}(\alpha, e_1)$ was defined after Equation (54). We assume that α_x is the minimum angle such that Equation (58) holds.

Lemma 22

$$\alpha = \sup_{x > 0} \alpha_x < \frac{\pi}{2}.$$

Proof. Let $X = -x e_1$, $x > 0$ and \mathcal{A}_{e_1} the associated directed navigation of \mathcal{A} , $\mathcal{A}(X) - X$ converges weakly to $\mathcal{A}_{e_1}(0)$. The directed navigation has non-negative progress and there

exists $\beta \in [0, \pi/2)$ such that $\mathbb{P}(\mathcal{A}_{e_1}(0) \in \mathcal{C}(\beta, e_1)) \geq 3/4$. Hence for $x \geq x_1$ large enough: $\alpha_x \leq \beta < \pi/2$.

For $x \leq x_0$, $\mathbb{P}(N^0(B(0, x)) > 1) \leq 1/2$ hence $\alpha_x = 0$.

It remains to treat the case $x_0 < x < x_1$. Let $L(x, \beta) = X + B(0, x) \cap \mathcal{C}^c(\beta, e_1)$, it appears easily that $|L(x, \beta)| \leq |L(x_1, \beta)| \leq C_1 \beta x_1^d$ and

$$\begin{aligned} \mathbb{P}(\mathcal{A}(X) - X \notin \mathcal{C}(\beta, e_1)) &\leq \mathbb{P}(N(L(x, \beta) \cap B(0, t)) > 0) \\ &\leq C_1 \beta x_1^d. \end{aligned}$$

Therefore for β large enough $\mathbb{P}(\mathcal{A}(X) - X \in \mathcal{C}(\beta, e_1)) \geq \frac{1}{2}$ and this concludes the proof. \square

Now we define $\theta = \inf\{k \geq 1 : X_k - X_{k-1} \in \mathcal{C}(\alpha, e_1)\}$ where α was defined in Lemma 22. From Lemma 22, θ is dominated by a geometric variable with parameter $1/2$. We then consider:

$$\tilde{\mathcal{A}}(X) = \mathcal{A}^\theta(X).$$

$\tilde{\mathcal{A}}$ is an isotropic navigation with non-negative progress moreover it satisfies Equation (54) holds. We will denote by $\tilde{\cdot}$ a variable defined in the previous paragraph for $\tilde{\mathcal{A}}$. For example

$$\tilde{K} = \inf\{k : \tilde{V}_k < 0 \text{ or } |\tilde{U}_k| \leq x^\gamma \tan \alpha\}.$$

Let $r' < r'' < r$. A rough bound and a use of Markov inequality gives:

$$\begin{aligned} \mathbb{P}(|\tilde{\mathcal{A}}(X) - X| \geq t) &\leq \mathbb{P}(\theta > n) + \mathbb{P}\left(\sum_{k=0}^{n-1} |X_{k+1} - X_k| \geq t\right) \\ &\leq 2^{-n} + \sum_{k=0}^{n-1} \mathbb{P}(|X_{k+1} - X_k| \geq t/n) \\ &\leq 2^{-n} + C_1 n^{r+1} t^{-r} \\ &\leq C_1 t^{-r''}, \end{aligned}$$

by picking $n = \lfloor c \ln t / \ln 2 \rfloor$. We deduce that for all $r' < r$:

$$\sup_{X \in \mathbb{R}^d} \mathbb{E}|\tilde{\mathcal{A}}(X) - X|^{r'} < \infty.$$

We can thus apply Theorem 7 to $\tilde{\mathcal{A}}$ and we get:

$$\mathbb{P}(\tilde{\Delta}(X) \geq |X|^\gamma) \leq C_1 |X|^{-d-\eta},$$

this last inequality does not lead directly to the desired result. We circumvent this difficulty by introducing a new variable:

$$\bar{V}_k = \max_{\theta_k \leq l < \theta_{k+1}} V_l - V_{\theta_k} = \max_{\theta_k \leq l < \theta_{k+1}} V_l - \tilde{V}_k,$$

where $\theta_0 = 0$, $\theta_1 = 1$ and $\theta_{k+1} = \inf\{l > \theta_k : X_l - X_{l-1} \in \mathcal{C}(\alpha, e_1)\}$. We have:

Lemma 23 *Let $r'' < r'$, for all $t > 0$ there exists a constant C_t such that:*

$$\mathbb{P}(\max_{0 \leq k < n} \bar{V}_k \geq tn^\gamma) \leq C_t n^{1-\gamma r''}$$

Proof. The proof uses always the same type of rough stochastic bounds. Using Hölder inequality, we have:

$$\begin{aligned} \mathbb{E}|\bar{V}_k|^{r''} &\leq \sum_{n=1}^{\infty} \mathbb{E} \mathbb{1}(\theta_k \geq n) |V_{\theta_k+n} - V_{\theta_k+n-1}|^{r''} \\ &\leq \sum_{n=1}^{\infty} (\mathbb{P}(\theta_k \geq n))^{1-r''/r'} (\mathbb{E}|V_{\theta_k+n} - V_{\theta_k+n-1}|^{r'})^{r''/r'} \\ &\leq \sum_{n=1}^{\infty} 2^{-(n-1)(1-r''/r')} (\mathbb{E}|X_{\theta_k+n} - X_{\theta_k+n-1}|^{r'})^{r''/r'} \\ &\leq C_1, \end{aligned}$$

indeed, for $n \geq 1$, $|X_{\theta_k+n} - X_{\theta_k+n-1}|^{r'}$ is given X_{θ_k+n-1} independent of θ_k . We thus have uniformly in X : $\mathbb{P}(\bar{V}_k \geq t | \mathcal{F}_k) \leq \mathbb{P}(Z_k \geq t)$ with $(Z_k)_{k \in \mathbb{N}}$ is an iid sequence and

$$\mathbb{P}(Z_k \geq t | \mathcal{F}_k) = 1 \wedge C_1 t^{-r''}.$$

The final step follows from elementary inequalities:

$$\begin{aligned} \mathbb{P}(\max_{0 \leq k < n} \bar{V}_k \geq tn^\gamma) &\leq \mathbb{P}(\max_{0 \leq k < n} Z_k \geq tn^\gamma) \\ &\leq 1 - \prod_{k=0}^{n-1} \mathbb{P}(Z_k \geq tn^\gamma) \\ &\leq 1 - (1 - C_1 t^{-r''} n^{-\gamma r''})^n \\ &\leq 1 - \exp(-C_1 t^{-r''} n^{1-\gamma r''}) \\ &\leq C_1 t^{-r''} n^{1-\gamma r''}. \end{aligned}$$

for n large enough. We then relax the assumption on n by increasing C_1 . □

The end of the proof is as in the previous paragraph:

$$\begin{aligned} \mathbb{P}(\Delta(X) \geq 2x^\gamma) &\leq \mathbb{P}(\tilde{\Delta}(X) \geq x^\gamma) + \mathbb{P}(\max_{0 \leq k < x/\mu} \bar{V}_k \geq x^\gamma) + \mathbb{P}(\tilde{H}(X) \geq x/\mu) \\ &\leq C_1 x^{1-\gamma r''} \end{aligned}$$

5.4 Isotropic Regenerative Navigation

With Theorem 7, we have treated so far the case of memoryless navigation. For isotropic regenerative navigation a similar result holds. Let \mathcal{A} be a regenerative navigation with

regenerative time θ : \mathcal{A}^θ is a memoryless navigation. We define:

$$L^\theta(X) = \max_{0 \leq k < \theta} |X_k - X|.$$

We have the following corollary of Theorem 7.

Corollary 4 *Let $\gamma \in (1/2, 1)$, if the navigation is isotropic, regenerative, with non-negative progress and*

- $\sup_{X \in \mathbb{R}^d} \mathbb{E} L^\theta(X)^r < \infty$ with $r > (d+1)/\gamma$
- For $|X| \geq x_0$, $\mathbb{P}(P(X) \geq c) \geq \epsilon$ with $x_0, c, \epsilon > 0$.

Then for some $\eta > 0$, there exists C_1 such that

$$\mathbb{P}(\Delta(X) \geq |X|^\gamma) \leq C_1 |X|^{-d-\eta}.$$

and \mathcal{T}_0 is f -straight with $f(x) = |x|^{1-\gamma}$.

Proof. The proof as already being done in §5.3.2. Indeed, let e_1, e_2 in S^{d-1} with $\langle e_1, e_2 \rangle = 0$ and $X = xe_1$, $x > 0$. We define $V_k = \langle X_k, e_2 \rangle$ and $\tilde{\mathcal{A}}(X) = \mathcal{A}^\theta(X)$. We may apply Theorem 7 to $\tilde{\mathcal{A}}$ and as in §5.3.2, let $\bar{V}_k = \max_{\theta_k \leq l < \theta_{k+1}} V_l - V_{\theta_k} \leq L^\theta(X_{\theta_k})$. Lemma 23 holds and we conclude similarly. \square

The next corollary is a consequence of Corollary 3.

Corollary 5 *If the navigation is isotropic, regenerative, with non-negative progress and*

- $\sup_{X \in \mathbb{R}^d} \mathbb{E} L^\theta(X)^r < \infty$ for all $r \geq 1$.
- For $|X| \geq x_0$, $\mathbb{P}(P(X) \geq c) \geq \epsilon$ with $x_0, c, \epsilon > 0$.

Then for all $\epsilon > 0$ and $n \in \mathbb{N}$, there exists C_1 such that

$$\mathbb{P}(\Delta(X) \geq |X|^{1/2+\epsilon}) \leq C_1 |X|^{-n},$$

and \mathcal{T}_0 is f -straight with $f(x) = |x|^{-1/2+\epsilon}$.

A simple way to bound $L^\theta(X)$ is to note that for $r \in \mathbb{N}$:

$$L^\theta(X)^r \leq \left(\sum_{k=1}^{\theta} |X_k - X_{k-1}| \right)^r \leq \theta^{r-1} \sum_{k=1}^{\theta} |X_k - X_{k-1}|^r.$$

As an application, for the navigation on the Small World graph with connection function $f(r) \sim cr^{-\beta}$, Using a couple of times Holder inequality, we obtain:

Corollary 6 *For $\beta > d$, there exists $C \geq 1$ such that if $\gamma\beta > (\gamma + C)d + C$, then for some $\eta > 0$, there exists C_1 such that*

$$\mathbb{P}(\Delta(X) \geq |X|^\gamma) \leq C_1 |X|^{-d-\eta}.$$

and \mathcal{T}_0 is f -straight with $f(x) = |x|^{\gamma-1}$.

Note that a bound for the constant C could be explicitly computed. We only point out that for a Small World navigation on a PWIT, $C = 1$.

Proof. For $\beta > d$, the Small World navigation is isotropic regenerative with non-negative progress. Moreover we have $\mathbb{P}(|X - \mathcal{A}(X)| \geq t) \leq C_1 c t^{d-\beta}$, similarly the tail of θ is bounded by a constant times $t^{2+d-\beta}$. We then use Equation (5.4) and Holder inequality. \square

6 Shape of the Navigation Tree

6.1 Shape of Memoryless Navigation

Another interesting feature is the set of points at tree-distance less than k from the origin $\mathcal{T}_0(k) = \{X \in N : \mathcal{A}^k(X) = 0\}$. The set of assumptions under consideration is:

$$(A6.2) \left\{ \begin{array}{l} (i) \quad \mathcal{A} \text{ is a memoryless navigation with non-negative progress.} \\ (ii) \quad \sup_{X \in \mathbb{R}^d} \mathbb{E}P(X)^r < \infty \text{ for some } r > d + 2 \\ (iii) \quad F_X \text{ converges weakly to } F \text{ with } \mu = \int rF(dr) > 0. \end{array} \right.$$

Theorem 8 *Under the foregoing Assumption (A6.2), for all $\epsilon > 0$ there exists a.s. K such that if $k \geq K$:*

$$N \cap B(0, (1 - \epsilon)k\mu) \subset \mathcal{T}_0(k) \subset B(0, (1 + \epsilon)k\mu). \quad (59)$$

Moreover a.s. and in L^1 :

$$\frac{|\mathcal{T}_0(k)|}{\pi_d k^d} \rightarrow \mu^d, \quad (60)$$

In other words, the navigation tree generated by a PPP inside a ball grows linearly with the number of points. The main aim of this section is to prove Theorem 8, and in particular the fact that G_k/k^2 a.s. tends to a constant when k tends to ∞ . In the literature, this constant is known as the *volume growth*. The intuition behind Theorem 8 is as follows: from Proposition 7, a point k hops away from the origin is asymptotically at Euclidean distance $D_k \sim k\mu^d$ from the origin. The ball of radius D_k contains $\pi_d D_k^d$ points in N asymptotically. In order to prove Theorem 8, we need an estimate of the tail of the fluctuations of $H(X)$ around its mean. The proof of the next theorem is the heart of the proof of Theorem 8.

Theorem 9 *Under the foregoing Assumption (A6.2), let $r' < r$, for all $\nu < \mu$, there exists a positive constant C_1 :*

$$\text{if } |X| < n\nu - 1 \text{ and } n \geq 1, \quad \mathbb{P}(H(X) > n) \leq C_1 n(n\nu - |X|)^{-r'}.$$

Similarly for $\nu > \mu$:

$$\text{if } |X| > 1 + n\nu \text{ and } n \geq 1, \quad \mathbb{P}(H(X) < n) \leq C_1 n(|X| - n\nu)^{-r'}.$$

In particular if $\nu > \mu$, consider $n = \lfloor x/(2\nu) \rfloor$, we obtain:

$$P(H(X) > \frac{x}{\nu}) \leq C_1 |X|^{1-r'}, \quad (61)$$

(and similarly for $\nu < \mu$).

We first prove Theorem 8.

Proof. We define $G_k = |\mathcal{T}_0(k)| = \sum_{X \in \mathcal{N}} \mathbf{1}(H(X) \leq k)$, G_k is the size of the ball of center 0 and radius k for the graph-distance in \mathcal{T}_0 . We begin by the proof of Equation (60), let $\epsilon \in (0, 1)$, we write

$$\begin{aligned} |G_k - N(B(0, \mu k))| &\leq \sum_{X \in \mathcal{N}} \mathbf{1}(X \notin B(0, \mu k) \cap H(X) \leq k) \\ &\quad + \sum_{X \in \mathcal{N}} \mathbf{1}(X \in B(0, \mu k) \cap H(X) > k) \\ &\leq \sum_{X \in \mathcal{N}} \mathbf{1}(X \notin B(0, (1+\epsilon)\mu k) \cap H(X) \leq k) \\ &\quad + N(B(0, (1+\epsilon)\mu k) \setminus B(0, (1-\epsilon)\mu k)) \\ &\quad + \sum_{X \in \mathcal{N}} \mathbf{1}(X \in B(0, (1-\epsilon)\mu k) \cap H(X) > k) \\ &\leq I_k + J_k + L_k. \end{aligned}$$

From Slyvniak-Campbell's formula and using Equation (61) for $\nu = \mu(1 + \epsilon/2)$:

$$\begin{aligned} \mathbb{E}I_k &= \int_{\mathbb{R}^d \setminus B(0, (1+\epsilon)\mu k)} \mathbb{P}(H(X) \leq k) dX \\ &\leq \omega_{d-1} \int_{(1+\epsilon)\mu k}^{\infty} C_1 x^{1-r'} x^{d-1} dx \\ &\leq C_1 k^{d-r'+1}. \end{aligned}$$

From the Borel Cantelli Lemma, we obtain that almost surely $I_k = 0$ for k large enough. Similarly, let $\nu = (1 - \epsilon/2)\mu$, we get:

$$\begin{aligned} \mathbb{E}L_k &= \int_{B(0, (1-\epsilon)\mu k)} \mathbb{P}(H(X) \geq k) dX \\ &\leq \omega_{d-1} \int_0^{(1-\epsilon)\mu k} C_1 k (k\nu - x)^{-r'} x^{d-1} dx \\ &\leq \frac{\omega_{d-1} C_1 2^{r'}}{(k\epsilon\mu)^{r'-1}} \int_0^{(1-\epsilon)\mu k} x^{d-1} dx \\ &\leq C_1 \epsilon^{1-r'} k^{d-r'+1}. \end{aligned}$$

We deduce that almost surely $L_k = 0$ for k large enough.

The ergodic properties of the PPP imply that

$$\frac{J_k}{k^d} = \frac{N(B(0, (1+\epsilon)\mu k) \setminus B(0, (1-\epsilon)\mu k))}{k^d}$$

converges almost surely and in mean toward $2d\pi_d(\mu\epsilon)^{d-1}$ (notice that $N(B(0, (1+\epsilon)\mu k) \setminus B(0, (1-\epsilon)\mu k))$ is not an increasing sequence of convex sets, to prove this convergence, we need to use the independency properties of the PPP). We thus have proved that for all $\epsilon > 0$, almost surely,

$$\limsup_k \frac{|G_k - N(B(0, \mu k))|}{k^d} \leq 2d\pi_d(\mu\epsilon)^{d-1}.$$

Hence, almost surely,

$$\lim_k \frac{G_k}{k^d} = \lim_k \frac{N(B(0, \mu k))}{k^d} = \pi_d \mu^d.$$

The proof for the L^1 convergence is a consequence of Scheffe's Lemma.

Equation (59) holds since we have seen that a.s. for k large enough I_k and L_k are both equal to 0. I_k is the cardinal of $\mathcal{T}(k) \cap B(0, p+\epsilon)^c$ and L_k is the cardinal of $\mathcal{T}(k)^c \cap B(0, p-\epsilon)$. \square

6.2 Proof of Theorem 9

We now turn to the proof of Theorem 9.

As usual let $X_k = \mathcal{A}^k(X)$ and $H(x) = \inf\{k : X_k = 0\}$. For $1 \leq k \leq H(x)$, we define the progress: $P_k(X) = |X_{k-1}| - |X_k|$ and for $k \geq H(x)$, $P_k(X) = 0$. We fix $r' < r'' < r$.

Case $\nu < \mu$.

There exists $\nu' > 0$ such that $\nu' < \nu < \mu$ and $|X| < \nu'n - 1$. Since $(P(X))$ is uniformly integrable, there exists x_0 such that:

$$\text{if } |X| \geq x_0 \text{ then } \mathbb{E}P(X) \geq \nu. \quad (62)$$

Let $l < n$ we have:

$$\begin{aligned} \mathbb{P}(H(X) > n) &= \mathbb{P}\left(\sum_{k=0}^{n-1} P_k < |X|\right) \\ &\leq \mathbb{P}N(B(0, x_0) > l) + \mathbb{P}\left(\sum_{k=0}^{n-l-1} P_k < |X| - x_0\right) \\ &\leq \exp\left(-l \ln \frac{l}{e\pi^d x_0^d}\right) + \mathbb{P}\left(\sum_{k=0}^{n-l-1} P_k - \mathbb{E}(P_k | \mathcal{F}_k) < |X| - x_0 + \sum_{k=0}^{n-l-1} \mathbb{E}(P_k | \mathcal{F}_k)\right) \\ &\leq \exp\left(-l \ln \frac{l}{e\pi^d x_0^d}\right) + \mathbb{P}\left(\sum_{k=0}^{n-l-1} P_k - \mathbb{E}(P_k | X_k) < |X| - (n-l)\nu\right), \end{aligned} \quad (63)$$

where in Equation (63) we have used Equation(62) together with Assumption (i): $\mathbb{E}(P_k|\mathcal{F}_k) = \mathbb{E}(P_k|X_k)$ and

$$\left\{ \sum_{k=0}^{n-l-1} P_k < |X| - x_0 \right\} \subset \bigcup_{0 \leq k \leq n-l-1} \left\{ |X_k| \geq x_0 \right\} \subset \bigcup_{0 \leq k \leq n-l-1} \left\{ \mathbb{E}(P_k|X_k) \geq \nu \right\}.$$

We define $Y_k = P_k - \mathbb{E}(P(X_k)|X_k)$, we notice by Assumption (ii):

$$\mathbb{E}Y_k = 0 \quad \text{and} \quad \mathbb{E}|Y_k|^r \leq C_1.$$

The sequence $(Y_k)_{k \in \mathbb{N}}$ is not independent however, it is nearly independent:

1. $(|Y_k|)_{k \in \mathbb{N}}$ is stochastically dominated by an iid sequence $(Z_k)_{k \in \mathbb{N}}$ with $\mathbb{E}Z_k^{r''} < \infty$.
2. If $p, q \in \mathbb{N}$ and $k \neq l$ then $\mathbb{E}Y_k^{2p+1}Y_l^q = 0$ and $\mathbb{E}Y_k^{2p}Y_l^{2q} \leq \mathbb{E}Z_k^{2p}\mathbb{E}Z_l^{2q}$.

We can thus apply Lemma 26 which is stated for iid variables but still holds since it is based only on truncation and a systematic use of Markov inequalities. We obtain if $\mu \geq 1$ and $|X| < m\nu - t_0$, $t_0 > 0$:

$$\mathbb{P}\left(\sum_{k=0}^{m-1} P_k - \mathbb{E}(P_k|X_k) < |X| - m\nu\right) \leq C_1 m(m\nu - |X|)^{-r'},$$

Hence, using this last inequality in Equation (63), and considering $l = \lfloor (\nu'/\nu - 1)n \rfloor$ we get, (since $(n-l)\nu \geq n\nu' > |X|$)

$$\mathbb{P}(H(X) > n) \leq \exp(-C_0 n) + C_1 n(n\nu' - |X|)^{-r'}, \quad (64)$$

then since $n \geq (n\nu' - |X|)/\nu'$, we obtain our result (with ν' instead on ν).

Case $\nu > \mu$. This case is slightly simpler, there exists x_1 such that:

$$\text{if } |X| \geq x_1 \text{ then } \mathbb{E}P(X) \leq \nu. \quad (65)$$

Following the same computation as in the case $\nu < \mu$

$$\begin{aligned} \mathbb{P}(H(X) \leq n) &= \mathbb{P}\left(\sum_{k=0}^{n-1} P_k = |X|\right) \\ &\leq \mathbb{P}\left(\sum_{k=0}^{n-1} P_k > |X| - x_1\right) \\ &\leq \mathbb{P}\left(\sum_{k=0}^{n-1} P_k - \mathbb{E}(P_k|\mathcal{F}_k) > |X| - x_1 + \sum_{k=0}^{n-1} \mathbb{E}(P_k|\mathcal{F}_k)\right) \\ &\leq \mathbb{P}\left(\sum_{k=0}^{n-1} P_k - \mathbb{E}(P_k|X_k) > |X| - n\nu\right) \\ &\leq C_1 n(|X| - n\nu)^{-r'}, \end{aligned}$$

where we have used the same argument and Lemma 26.

6.3 Shape of Regenerative Navigation

We extend Theorems 59 and 9 to regenerative navigation. Let \mathcal{A} be a regenerative navigation and θ its associated regenerative time. We define $P^\theta(X) = |X| - |X_\theta| = |X| - |\mathcal{A}^\theta(X)|$, the assumptions is as follows

$$(A6.3) \begin{cases} (i) & \mathcal{A} \text{ is a regenerative navigation with non-negative progress.} \\ (ii) & \sup_{X \in \mathbb{R}^d} \mathbb{E}P^\theta(X)^r < \infty \text{ and } \mathbb{E}\theta^r < \infty \text{ for some } r > d + 2 \\ (iii) & F_X \text{ converges weakly to } F \text{ with } \int rF(dr) > 0. \end{cases}$$

We denote by $\bar{\theta} = \lim_{|X| \rightarrow \infty} \mathbb{E}\theta(X)$ and $\mu = 1/\bar{\theta} \int rF(dr) > 0$. From Proposition 7 and Lemma 9, as $|X|$ tends to infinity a.s. $H(X)/|X| \rightarrow \mu$. Not surprisingly, we obtain the next two results as corollaries of Theorems 8 and 9.

Corollary 7 *Under the foregoing Assumption (A6.3), let $r' < r$, for all $\nu < \mu$, there exists a positive constant C_1 :*

$$\text{if } |X| < n\nu - 1 \text{ and } n \geq 1, \quad \mathbb{P}(H(X) > n) \leq C_1 n (n\nu - |X|)^{-r'}.$$

Similarly for $\nu > \mu$:

$$\text{if } |X| > 1 + n\nu \text{ and } n \geq 1, \quad \mathbb{P}(H(X) < n) \leq C_1 n (|X| - n\nu)^{-r'}.$$

Corollary 8 *Under the foregoing Assumption (A6.3), the conclusions of Theorem 8 hold for \mathcal{A} .*

Corollary 8 follows from Corollary 7 exactly as Theorem 9 implies Theorem 8. We now prove corollary 7.

Proof. Let (θ_k) denote the regenerative sequence, $\tilde{\mathcal{A}}(X) = \mathcal{A}^\theta(X)$ and $H^\theta(X) = \inf\{k : X_{\theta_k} = 0\} = \sup\{k : \tilde{\mathcal{A}}^k(X) = 0\}$.

We assume first that $|X| < n\nu - 1$ and $\nu < \mu$. We may find $0 < \delta < \bar{\theta}$ such that $\nu' = \nu\bar{\theta}/\delta < \mu$ and $|X| < \nu'n - 1$. We get

$$\begin{aligned} \mathbb{P}(H(X) > n) &\leq \mathbb{P}(H^\theta(X) > \frac{n}{\delta}) + \mathbb{P}(\theta_{\lfloor \frac{n}{\delta} \rfloor} < n) \\ &\leq \mathbb{P}(H^\theta(X) > \frac{n}{\delta}) + \mathbb{P}\left(\sum_{k=0}^{\lfloor \frac{n}{\delta} \rfloor - 1} \theta_{k+1} - \theta_k < n\right) \\ &\leq \mathbb{P}(H^\theta(X) > \frac{n}{\delta}) + \mathbb{P}\left(\left|\sum_{k=0}^{\lfloor \frac{n}{\delta} \rfloor - 1} (\theta_{k+1} - \theta_k - \bar{\theta})\right| > n(1 - \bar{\theta}/\delta) - 1\right) \end{aligned} \quad (66)$$

We may apply Theorem 9 to $\tilde{\mathcal{A}}$ and $\tilde{\nu} = \nu'\bar{\theta} < \mu\bar{\theta}$ since we have $|X| < n\nu' - 1 < (\frac{n}{\delta})(\nu'\bar{\theta}) - 1$. The first term in the latter inequality (66) is thus bounded by $C_1 n / \delta (n\nu'\bar{\theta}/\delta - |X|)^{-r'} = C_1 n / \delta (n\nu - |X|)^{-r'}$. We can also apply Lemma 26 to the sequence of iid variables $Y_k =$

$\theta_{k+1} - \theta_k - \bar{\theta}$. Thus we may upper bound the second term in Equation (66) by $C_1(1 - \bar{\theta}/\delta - 1/n)^{-r'} n^{1-r'}$ for n large enough to guarantee $1 - \bar{\theta}/\delta - 1/n > t_0$ with $0 < t_0 < 1 - \bar{\theta}/\delta$. Finally we obtain (since $n \geq (n\nu - |X|)/\nu$) for n large enough:

$$\mathbb{P}(H(X) > n) \leq C_1 n (n\nu - |X|)^{-r'}.$$

By increasing suitably C_1 we obtain the result for all $n \geq 1$. \square

As an example, for the Small World model, we easily get:

Corollary 9 - For $\beta > d$, there exists $C \geq 1$ such that if $\beta > (C + 1)d + 2C$, then the conclusions of Corollaries 7 and 8 hold true with μ computed in Theorem 5.

- For $\beta = d$, let $\tilde{\mu}$ be as in Theorem 6, for all $\epsilon > 0$ there exists a.s. K such that if $k \geq K$:

$$N \cap B(0, \exp((1 - \epsilon)k\tilde{\mu})) \subset \mathcal{T}_0(k) \subset B(0, \exp((1 + \epsilon)k\tilde{\mu})).$$

Moreover a.s. and in L^1 :

$$\frac{\ln |\mathcal{T}_0(k)|}{k} \rightarrow d\tilde{\mu}.$$

- For $d - 2 < \beta < d$, let $\alpha = 1 - (d - \beta)/2$, for all $\epsilon > 0$ there exists a.s. K such that if $k \geq K$:

$$N \cap B(0, \exp(\alpha^{(1-\epsilon)k})) \subset \mathcal{T}_0(k) \subset B(0, \exp(\alpha^{(1+\epsilon)k})).$$

Moreover a.s. and in L^1 :

$$\frac{\ln \ln |\mathcal{T}_0(k)|}{k} \rightarrow \ln \alpha.$$

7 Appendix

7.1 Further examples of navigation

7.1.1 Ray Navigation

This navigation is built up artificially from the directed navigation introduced by Ferrari et al. in [11] to obtain their Poisson forest. The main interest of this navigation is that its mathematical analysis is fairly simple, indeed this navigation is memoryless for a PPP.

For $X \in \mathbb{R}^d \setminus \{0\}$ and $t \in \mathbb{R}_+$, let $\mathcal{C}(X) = \{Y \in \mathbb{R}^d : |Y| < |X|, 0 \leq \langle X, Y \rangle \leq |X||Y| \cos(2\pi \wedge |X|^{-1})\}$. $\mathcal{C}(X)$ is a cone intersected with $B(0, X)$ tuned to guarantee that for $|X| \geq 1$, $\overline{\mathcal{C}(X)} \cap \partial B(0, |X|)$ is a calotte on S^{d-1} with $(d - 1)$ -Lebesgue measure the volume of the unit ball in \mathbb{R}^{d-1} (i.e. $\pi^{d/2}/\Gamma(d/2 + 1)$). If $0 \in N$, the ray navigation from 0 to X is defined as (see Figure 3):

$$\mathcal{A}(X) = |Y| \quad \text{if } |Y| < |X| \text{ and } \overline{\mathcal{C}(X) \cap B(0, |X| - |Y|)}^c \cap N = \emptyset.$$

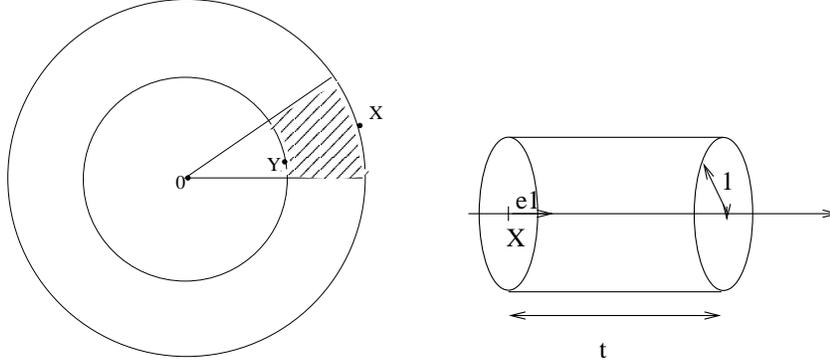


Figure 3: *Left:* $\mathcal{A}(X) = Y$ if the dashed region has no point (dim. 2). *Right:* $R(X,t)$ in dim. 3.

Let $R(X, t)$ denote the open cylinder of height $t > 0$ with direction e_1 generated by a $(d - 1)$ -dimensional ball of center X and radius 1 orthogonal to e_1 (see Figure 3). The directed navigation introduced by Ferrari et al. is:

$$\mathcal{A}_{e_1}(X) = |Y| \quad \text{if } \langle Y - X, e_1 \rangle > 0 \text{ and } R(X, \langle Y - X, e_1 \rangle) \cap N = \emptyset.$$

7.1.2 Hierarchical Navigation

In view of applications, it is interesting to consider more sophisticated navigation algorithms, for example the closest point between the ancestor given by a radial navigation and a small world navigation.

A more appealing model is as follows. We divide our locally finite point set N into point sets N_1 and N_2 . If $X \in N_1$ then a navigation \mathcal{A}_1 is performed on the point set N and if $X \in N_2$ then a navigation \mathcal{A}_2 is performed.

We consider the following example, in a network there are two types of vertices, N_1 and N_2 . Vertices in N_1 are highly connected whereas vertices in N_2 are poorly connected. The intensity of N_1 is much smaller than the intensity of N_2 . Let $G_D = (N_1 \cup N_2, E_D)$ be the Delaunay graph on $N_1 \cup N_2$ and $G_{SW} = (N_1, E_{SW})$ denote a Small World graph on N_1 . In a wireless communication scenario, E_D could be a wireless link (short) and E_{SW} a wired link (long), N_2 is the set of wireless users in an ad-hoc network and N_1 the set of entry points to a wired networks. A maximal progress navigation is performed on the graph $G = (N, E_D \cup E_{SW})$. A hierarchical structure naturally appears, a navigation from $X \in N_2$ to $0 \in N_2$ will probably start by short links on G_D until it finally reaches a point in N_1 then long links on G_{SW} are followed until the path gets close of the destination. Then the path ends with a sequence of short links on G_D to its destination.

It is of course possible to combine more general navigation schemes by dividing N into k point sets.

7.2 Collection of technical Lemmas

Lemma 24 *Let f be a measurable non-negative function and $\lim_{x \rightarrow +\infty} f(x) = 0$. There exists a measurable positive non-decreasing function $g(x)$ with $\lim_{x \rightarrow \infty} g(x) = \infty$ such that:*

$$\lim_{x \rightarrow +\infty} \int_x^{x+g(x)} f(y) dy = 0.$$

Proof. Let $F(x) = \sup_{y \geq x} f(y)$, F satisfies the same hypothesis than f and F is non-increasing. If $F(x) = 0$ for x large enough any function g will work. Otherwise $F(x) > 0$ and the function $g(x) = 1/\sqrt{F(x)}$ trivially satisfies all the requirements. Indeed:

$$\int_x^{x+g(x)} f(y) dy \leq g(x)F(x) \leq \sqrt{F(x)}.$$

□

Lemma 25 *Let $(X_n), n \in \mathbb{N}$, be a sequence of real valued random variables adapted to a filtration $\{\mathcal{F}_n\}, n \in \mathbb{N}$. Assume that $X_{n+1} | \mathcal{F}_n \stackrel{st}{\leq} Y$, where Y is a random variable. Then for all stopping time N :*

$$\sum_{k=1}^N X_k \stackrel{st}{\leq} \sum_{k=1}^N Y_k,$$

where $(Y_k), k \in \mathbb{N}$, is a iid sequence with the distribution of Y .

Note that the sequence $(Y_k)_{k \in \mathbb{N}}$ is not necessarily independent of N .

Proof. For all n in \mathbb{N} , we check easily that

$$\sum_{k=1}^n X_k \stackrel{st}{\leq} \sum_{k=1}^n Y_k.$$

We assume first that a.s. $N \leq n$. Notice $\mathbf{1}(N \geq (k+1))$ is \mathcal{F}_k -measurable, thus $X_{k+1} \mathbf{1}(N \geq k+1) \stackrel{st}{\leq} Y_k \mathbf{1}(N \geq k+1)$. Since a.s. $\sum_{k=1}^N X_k = \sum_{k=1}^n \mathbf{1}(N \geq k) X_k$, we deduce:

$$\sum_{k=1}^N X_k \stackrel{st}{\leq} \sum_{k=1}^N Y_k.$$

For the general case, we consider $N_n = N \wedge n$ and we let n tends to infinity. □

Lemma 26 *Let $(X_k), k \in \mathbb{N}$, be a sequence of iid real valued random variable. We assume that $\mathbb{E}X_1 = 0$ and $\mathbb{E}|X_1|^r < \infty$ for some $r > 1$. Then for all $1 < r' < r$, and $t_0 \geq 0$ there exists C_1 such that, for all $t > t_0$ and $n \geq 1$:*

$$\mathbb{P}(|\sum_{k=1}^n X_k| \geq tn) \leq C_1 t^{-r'} n^{1-r'}.$$

Proof. A proof of this lemma follows step by step the proof of Theorem 4 of Baum and Katz [5]. As it is pointed there, we can suppose that the distribution of X_1 is symmetric. Let $X_k^{[tn]} = X_k \mathbb{1}(X_k \leq tn)$, we write:

$$\begin{aligned} n^{r'-1} \mathbb{P}\left(\left|\sum_{k=1}^n X_k\right| \geq tn\right) &\leq n^{r'-1} n \mathbb{P}(|X_k| \geq tn) + n^{r'-1} \mathbb{P}\left(\left|\sum_{k=1}^n X_k^{[tn]}\right| \geq tn\right) \\ &\leq t^{-r'} \mathbb{E}|X_k|^{r'} + n^{r'-1} \mathbb{P}\left(\left|\sum_{k=1}^n X_k^{[tn]}\right| \geq tn\right), \end{aligned}$$

where we have used Markov inequality. We bound the second term exactly as in the proof of Theorem 4 of [5], let p be an even integer strictly larger than $2r' - 1$, from Markov inequality

$$n^{r'-1} \mathbb{P}\left(\left|\sum_{k=1}^n X_k^{[tn]}\right| \geq tn\right) \leq t^{-p} n^{r'-p-1} (n \mathbb{E}X_1^{[tn]^p} + n(n-1) \mathbb{E}X_1^{[tn]^{p-2}} \mathbb{E}X_1^{[tn]^2} + \dots)$$

Let $\{2i_1, \dots, 2i_m\}$ be a partition of r into positive even integers, the corresponding term in the preceding expansion is bounded by $t^{-p} n^{r'+m-p-1} \mathbb{E}X_1^{[tn]^{i_1}} \dots \mathbb{E}X_1^{[tn]^{i_m}}$. Note that $\mathbb{E}X_1^{[tn]^{2i}} \leq C_1$ if $2i \leq r$ and $\mathbb{E}X_1^{[tn]^{2i}} \leq C_1 t^{2i-r'} n^{2i-r'}$ if $2i > r'$. Hence if $W = \{j : 2i_j > r'\}$ we bound the term by $C_1 t^{-p} n^{r'+m-p-1} \prod_{j \in W} t^{2i_j-r'} n^{2i_j-r'} \leq C_1 t^{-|W|r'-p+\sum_{j \in W} 2i_j} f(n)$. It is proved in Theorem 4 of [5] that $f(n) \in \ell^0$, it remains to check the exponent in t : $\alpha = -|W|r' - p + \sum_{j \in W} 2i_j$. If $W = \emptyset$ then it reduces to $\alpha = -p \leq -r'$ and if $|W| \geq 1$ since $\sum_{j \in W} 2i_j \leq 2r$ we obtain $\alpha \leq -r'$ and this concludes the proof. \square

7.3 Some Results in Renewal Theory

Lemma 27 *Let $S_0 = 0$ and $S_n = \sum_{k=0}^{n-1} U_k$ where (U_n) is an i.i.d. sequence of positive reals with common distribution F . We assume that for some $0 < \alpha < 1$ and $c > 0$, as t goes to infinity $\overline{F}(t) \sim c/t^\alpha$. Define $\tau(x) = \inf\{n : S_n \geq x\}$, as x tends to $+\infty$ we have:*

$$\frac{\tau(x)}{x^\alpha} \Rightarrow \frac{1}{\Gamma(1-\alpha)c} \chi_\alpha^{-1/\alpha},$$

where χ_α is an α -stable random variable.

Proof. This lemma is a restatement of Equation (XI.5, 5.6) in [10]. \square

This lemma is a corollary of Rogozin's Relative Stability Theorem (Theorem 8.8.1. of [6]).

Lemma 28 *Let $S_0 = 0$ and $S_n = \sum_{k=0}^{n-1} U_k$ where (U_n) is an i.i.d. sequence with common distribution F . We assume that for some $c > 0$, as t goes to infinity $\overline{F}(t) \sim c/t$. Define $\tau(x) = \inf\{n : S_n \geq x\}$, as x tends to $+\infty$, a.s. we have:*

$$\lim_{x \rightarrow +\infty} \frac{\tau(x) \ln x}{x} = \frac{1}{c}.$$

The next lemma is a direct consequence of Rogozin's Relative Stability Theorem (Theorem 8.8.1. of [6]) and the stability of positive stable laws (see §8.3.5 in [6]).

Lemma 29 *Let (X_k) be an i.i.d. sequence of non-negative reals with distribution F and define $S_n = \sum_{k=1}^n X_k$. Assume that*

$$C_0 t^{-\alpha} \leq \overline{F}(t) \leq C_1 t^{-\alpha}$$

for some positive C_0, C_1 . If $\alpha = 1$, there exists a function $C_0 \leq C(n) \leq C_1$ such that a.s. :

$$\frac{S_n}{C(n)n \ln n} \rightarrow 1.$$

If $0 < \alpha < 1$ and χ_α denotes an α -stable random variable, there exists a function $C_0 \leq C(n) \leq C_1$

$$\frac{S_n}{C(n)n^{1/\alpha}} \Rightarrow \chi_\alpha.$$

Lemma 30 *Let $S_0 = 0$ and $S_n(u) = \sum_{k=0}^{n-1} \mathbf{1}(U_n < u) U_n$ where (U_n) is an i.i.d. sequence of positive reals with common distribution F . We assume that for some $0 < \alpha < 1$ and $c > 0$, as t goes to infinity $\overline{F}(t) \sim c/t^\alpha$. Let $\tau_\epsilon(x) = \inf\{n : S_n(x\epsilon(x)) \geq x\}$, with $\lim_{x \rightarrow \infty} \epsilon(x) = 0$ and $\lim_{x \rightarrow \infty} x\epsilon(x) = \infty$. Then as x tends to infinity, a.s.:*

$$\lim_x \frac{\mathbb{E}\tau_\epsilon(x)}{x^\alpha \epsilon(x)^{\alpha-1}} = \frac{1-\alpha}{c}.$$

Similarly, if $\overline{F}(t) \sim c/t$, a.s.:

$$\lim_x \frac{\mathbb{E}\tau_\epsilon(x) \ln(x\epsilon(x))}{x} = \frac{1}{c}.$$

Proof. Notice that $m(x) = \mathbb{E}U_n \mathbf{1}(U_n < x\epsilon(x)) \sim_x x^{1-\alpha} \epsilon(x)^{1-\alpha} c / (1-\alpha)$. We have $x \leq S_{\tau_\epsilon(x)} \leq x + U_{\tau_\epsilon(x)}$, thus from Wald equality, we have: $x \leq m(x) \mathbb{E}\tau_\epsilon(x) \leq x + \mathbb{E}U_{\tau_\epsilon(x)} \mathbf{1}(U_{\tau_\epsilon(x)} < x\epsilon(x)) \leq x(1+\epsilon(x))$. Hence for x large enough: $x^\alpha \epsilon(x)^{\alpha-1} (1-\alpha)/c \leq \mathbb{E}\tau_\epsilon(x) \leq x^\alpha \epsilon(x)^{\alpha-1} (1+\epsilon(x))(1-\alpha)/c$. For $\alpha = 1$, the proof is identical. \square

Remark 6 *Assume $0 < \alpha < 1$ and let $\epsilon_k(x) = |x|^{-1/k}$ so that $x\epsilon_k(x) = |x|^{(k-1)/k}$. For $k > 1$, we have $m_k(x) = \mathbb{E}U_n \mathbf{1}(U_n < |x|^{(k-1)/k}) \sim_x x^{(1-\alpha)(k-1)/k} c / (1-\alpha) = l_k(x)$ uniformly in k : $\lim_x \sup_{k>1} m_k/l_k(x) = 1$. As in the proof of Lemma 30, for x large enough, for all k we have $\mathbb{E}\tau_{\epsilon_k}(x) \leq x^{1/k+\alpha(k-1)/k} (1+|x|^{-1/k})(1-\alpha)/c$. Therefore:*

$$\limsup_x \sup_k \frac{\mathbb{E}\tau_{\epsilon_k}(x)}{x^{1/k+\alpha(k-1)/k}} \leq 2 \frac{1-\alpha}{c}. \quad (67)$$

7.4 Tail Inequality in the GI/GI/ ∞ Queue

Let $\{\sigma_n, \tau_n\}, n \in \mathbb{Z}$, be an i.i.d. sequence of $\mathbb{R}_+ \times \mathbb{R}_+$ -valued random variables representing the service times and inter-arrival times in a GI/GI/ ∞ queue. The random variables (σ_n) and (τ_n) are independent. We set $T_0 = 0$ as the arrival time of customer 0; for $n \geq 1$, $T_n = \sum_{k=0}^{n-1} \tau_k$ is the arrival time of the n^{th} customer. Let $Y \in \mathbb{R}_+$ be a non-negative initial condition, independent of the $\{\sigma_n, \tau_n\}$ sequence. We set $W_0^{[Y]} = Y$, and for $n \geq 1$, we define

$$\begin{aligned} W_n^{[Y]} &= \max \left(Y - \sum_{k=0}^{n-1} \tau_k, \max_{1 \leq i \leq n} \sigma_{i-1} - \sum_{k=i}^{n-1} \tau_k \right)^+ \\ &= \max \left(Y - (T_n - T_0), \max_{1 \leq i \leq n} \sigma_{i-1} - (T_n - T_i) \right)^+ \\ &= \max \left(W_{n-1}^{[Y]} - \tau_{n-1}, \sigma_{n-1} \right). \end{aligned}$$

(where by convention $\sum_{k=n}^{n-1} \cdot = 0$). The random variable $W_n^{[Y]}$ is the largest residual service time just after the arrival of the n^{th} customer in the GI/GI/ ∞ queue with initial condition Y .

Let \mathcal{F}_n be the σ -field generated by the random variables Y and $\{(\sigma_k, \tau_k), k = 0, \dots, n-1\}$. Consider the $\{\mathcal{F}_n\}$ -stopping time:

$$\theta(Y) = \theta = \inf\{n \geq 1 : W_n^{[Y]} = 0\}. \quad (68)$$

θ is the time needed to empty all queues.

7.4.1 Light Tail Case

The following additional assumptions are made:

- (i) There exist a constant $s > 0$ such that $\mathbb{E} \exp(s\sigma_1) < \infty$ and $\mathbb{E} \exp(sY) < \infty$.
- (ii) $\mathbb{P}(\tau_1 > 0) > 0$.
- (ii) $\mathbb{P}(\sigma_1 = 0) > 0$.

Lemma 31 *Let θ be the stopping time defined in Equation (68). Under the foregoing probabilistic assumptions on Y , (τ_n) and (σ_n) , there exists $s > 0$ such that:*

$$\mathbb{E} \exp(s\theta) < \infty.$$

Proof. The Loynes' sequence $\{M_n\}$ of this GI/GI/ ∞ queue is defined by $M_0 = 0$ and

$$M_n = \max_{-n+1 \leq i \leq 0} \left(\sigma_{i-1} - \sum_{k=i}^{-1} \tau_k \right)^+, \quad n \geq 1.$$

This sequence is non-decreasing in n and it a.s. converges to

$$M = \sup_{i \leq 0} (\sigma_{i-1} - \sum_{k=i}^{-1} \tau_k)^+. \quad (69)$$

The random variable M is a.s. finite. Indeed, we can easily obtain a stronger assertion. Let $s > 0$ such that $\mathbb{E} \exp(s\sigma_1) < \infty$ (such s exists due to Assumption (i)), then:

$$\begin{aligned} \mathbb{E} \exp(sM) &= \mathbb{E} \exp(s \sup_{i \leq 0} (\sigma_{i-1} - \sum_{k=i}^{-1} \tau_k)^+) \\ &\leq 1 + \sum_{i \leq 0} \mathbb{E} \exp(s(\sigma_{i-1} - \sum_{k=i}^{-1} \tau_k)) \\ &\leq 1 + \mathbb{E} \exp(s\sigma_1) \sum_{i \leq 0} \mathbb{E} \exp(-s|i|\tau_1) < \infty. \end{aligned}$$

Now, we define :

$$\nu(Y) = \nu = \inf\{n \geq 2 : Y - \sum_{k=0}^{n-1} \tau_k < 0\}. \quad (70)$$

From time ν on, the initial workload does not count anymore, i.e. for $n \geq \nu$ $W_n^{[Y]} = \max_{2 \leq i \leq n} (\sigma_{i-1} - T_n + T_{i-1})^+$. Note that ν has the same distribution as

$$\nu' = \max\{n \leq -1 : Y - \sum_{k=n}^{-1} \tau_k < 0\}.$$

More generally,

$$\left(\sum_{k=0}^{\nu-1} \tau_k, \sum_{k=1}^{\nu-1} \tau_k, \dots, \tau_{\nu-1} \right) \stackrel{\mathcal{L}}{=} \left(\sum_{k=-\nu'}^{-1} \tau_k, \sum_{k=-\nu'+1}^{-1} \tau_k, \dots, \tau_{-1} \right),$$

which implies that $M_{\nu'}$ and $W_{\nu'}^{[Y]}$ have the same distribution. Since $M_{\nu'} \leq M$, we have

$$W_{\nu'}^{[Y]} \leq_{st} M. \quad (71)$$

Note that this bound is uniform in Y .

From Assumption (ii) we may find $c > 0$ and $\epsilon > 0$ such that $\mathbb{P}(\tau_1 \geq c) \geq \epsilon$. Let $B_k = c\mathbf{1}(\tau_k \geq c)$. Using the independency between τ and Y and Hoeffding's inequality:

$$\mathbb{P}(\nu > n) \leq \mathbb{P}\left(\sum_{k=0}^{n-1} B_k < Y\right) \leq \mathbb{E} \exp\left(-\frac{(c\epsilon n - Y)^2}{2c^2 n}\right).$$

By assumption (i), Y is such that $\mathbb{P}(Y > t) \leq C_1 \exp(-C_0 t)$, for some positive constants C_0, C_1 , hence:

$$\mathbb{P}(\nu > n) \leq \mathbb{P}(Y > nt_0) + \exp\left(-\frac{(c\epsilon n - nt_0)^2}{2c^2 n}\right) \leq C_1 \exp(-C_0 n). \quad (72)$$

for some positive constants C_0, C_1 , uniformly on the initial conditions Y . Hence we may found some $s > 0$ such that $\mathbb{E} \exp(s\nu) < \infty$.

The sequence $\{W_n^{[Y]}\}$ is a $\{\mathcal{F}_n\}$ -Markov chain and the random variables

$$\nu_{n+1} = \nu_n + \nu(W_{\nu_n}^{[Y]}), \quad n \geq 1,$$

with $\nu(W)$ defined in (70) and with $\nu_1 = \nu = \nu(Y)$, are $\{\mathcal{F}_n\}$ -stopping times. Using what precedes, one gets by induction that each ν_n is a.s. finite and that for all n ,

$$\mathbb{P}(\nu_{n+1} - \nu_n > m | \mathcal{F}_{\nu_n}) = \mathbb{P}(\nu_{n+1} - \nu_n > m | W_{\nu_n}^{[Y]}) \leq C_1 \exp(-C_0 m), \quad \forall m \quad (73)$$

$$\mathbb{P}(W_{\nu_{n+1}}^{[Y]} > x | \mathcal{F}_{\nu_n}) = \mathbb{P}(W_{\nu_{n+1}}^{[Y]} > x | W_{\nu_n}^{[Y]}) \leq \mathbb{P}(M > x), \quad \forall x. \quad (74)$$

Using (73) and a Chernoff type bound, one gets

$$\mathbb{P}(\nu_n > \alpha n) \leq C_1 \exp(-C_0 n), \quad (75)$$

for some positive constants α, C_0, C_1 .

We now turn back to the stopping time θ . First we prove that $\mathbb{P}(M = 0) = p_0 > 0$. M is the stationary solution the Markov Chain. Let M_n be this stationary sequence, $M_1 \stackrel{\mathcal{L}}{=} M_2 = \max(M_1 - \tau_1, \sigma_1)$, $\mathbb{P}(M_1 = 0) = \mathbb{P}(M_1 \leq \tau) \mathbb{P}(\sigma_1 = 0)$. Then assume that $M > 0$ a.s.. By assumption (iii) $\mathbb{P}(\sigma_1 = 0) > 0$ then $M_1 > \tau_1$ a.s.. The independence of M_1 and τ_1 implies that $M > c$ almost surely. Notice that $\mathbb{P}(M < c) \geq \mathbb{P}(M_1 - \tau_1 < 2c) \mathbb{P}(\sigma_1 = 0) \geq \mathbb{P}(c < M \leq 2c) \mathbb{P}(\tau_1 > c) \mathbb{P}(\sigma_1 = 0)$, hence $M > c$ a.s. implies $M > 2c$ a.s.. By iteration, we get for all n $M > nc$ a.s. and this contradicts the finiteness of M .

Equation (73) implies that $\mathbb{P}(W_{\nu_{n+1}}^{[Y]} = 0 | \mathcal{F}_{\nu_n}) \geq \mathbb{P}(M = 0) = p_0$. In the same vein of what precedes when using (75) and (74), one gets that:

$$P(\theta > n) \leq (1 - p_0)^{\lfloor n/\alpha \rfloor} + P(\nu_{\lfloor n/\alpha \rfloor} > n) \leq C_1 e^{-C_0 n},$$

for some positive constants C_0, C_1 . □

7.4.2 Heavy Tail Case

In this paragraph, the probabilistic assumptions are made:

- (i) There exist a constant $\alpha > 1$ such that : $\mathbb{P}(\sigma_1 > t) \leq C_1 t^{-\alpha}$ and $\mathbb{P}(Y > t) \leq C_1 t^{1-\alpha}$.
- (ii) $\mathbb{P}(\tau_1 > 0) > 0$.
- (ii) $\mathbb{P}(\sigma_1 = 0) > 0$.

Lemma 32 *Let θ be the stopping time defined in Equation (68). Under the foregoing probabilistic assumptions on Y , (τ_n) and (σ_n) , θ is a.s. finite. Moreover if $\alpha > 2$, there exists $C_1 > 0$ such that:*

$$\mathbb{E}\theta < \infty \quad \text{and} \quad \mathbb{P}(\theta > t) \leq C_1 t^{2-\alpha}.$$

Proof. The proof follows the proof of Lemma 31 in almost all aspects. We only outline the proof. The tail of the stationary solution M is bounded differently. Let $a > 0$, we notice that:

$$\mathbb{P}(\sup_{i \geq 0} (\sigma_i - ia) \geq t) = 1 - \prod_{i \geq 0} (1 - \mathbb{P}(\sigma_1 \geq t + ia)) \leq C_1 t^{1-\alpha}.$$

From Assumption (ii) we may find $c > 0$ and $\epsilon > 0$ such that $\mathbb{P}(\tau_1 \geq c) \geq \epsilon$. Let $B_k = c\mathbf{1}(\tau_k)$. Fix $0 < a < c\epsilon$ and let $T = \inf\{n : \forall n \geq i, \sum_{k=1}^i \tau_k \geq ia\}$, it follows from Hoeffding's Inequality that $\mathbb{P}(T \geq n) \leq C_0 \exp(-C_1 n)$. Then we have:

$$\mathbb{P}(M \geq t) \leq \mathbb{P}(T \geq n) + \mathbb{P}(\max(\sigma_1, \dots, \sigma_n) \geq t) + \mathbb{P}(\sup_{i \geq 0} (\sigma_i - ia) \geq t) \quad (76)$$

$$\leq C_1 t^{1-\alpha}, \quad (77)$$

(we pick $n = \lfloor t^\gamma \rfloor$, $0 < \gamma \leq 1$). We obtain similarly that:

$$\mathbb{P}(\nu_{n+1} - \nu_n > m | \mathcal{F}_{\nu_n}) \leq C_1 m^{1-\alpha}.$$

where ν was defined as in the proof of Lemma 31 by Equation (70). In particular, since $\alpha > 2$, $\mathbb{E}(\nu_{n+1} - \nu_n | \mathcal{F}_{\nu_n}) \leq \lambda$, for some $\lambda > 0$. The relation $\mathbb{P}(W_{\nu_{n+1}}^{[Y]} = 0 | \mathcal{F}_{\nu_n}) \geq \mathbb{P}(M = 0) = p_0$ still holds. Then, from Doob formula we get $\mathbb{E}(\theta)p_0/\lambda \leq 1$. The statement on the tail of θ follows from a moderate deviation result of Baum and Katz (Theorem 4 in [5]): $\mathbb{P}(\nu_{n+1} - \nu_n > m | \mathcal{F}_{\nu_n}) \leq C_1 m^{1-\alpha}$ implies that $\mathbb{P}(\nu_n \geq \epsilon n) \leq C_1 n^{2-\alpha}$ for $\epsilon < \lambda$. The proof is then parallel to the proof of Lemma 31. \square

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