On the construction of pullbacks for safe Petri nets

Eric Fabre

To cite this version:

Eric Fabre. On the construction of pullbacks for safe Petri nets. [Research Report] RR-5722, INRIA. 2005, pp.13. inria-00070296

HAL Id: inria-00070296
https://hal.inria.fr/inria-00070296
Submitted on 19 May 2006

INRIA

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

On the construction of pullbacks for safe Petri nets

Eric Fabre

N°5722

October 1st, 2005

Systèmes communicants

ISSN 0249-6399 ISRN INRIA/RR--5722--FR+ENG

Rapport de recherche

On the construction of pullbacks for safe Petri nets

Eric Fabre [*]

Systèmes communicants
Projet Distribcom

Rapport de recherche n ° 5722 — October 1st, 2005 — 13 pages

Abstract: The product of safe Petri nets is a well known operation : it generalizes to concurrent systems the usual synchronous product of automata. In this short note, we consider the definition of pullbacks of safe PNs, another categorical construction. Pullbacks generalize the product to nets which interact both by synchronized transitions and by a shared sub-net.

Key-words: safe Petri net, category theory, Winskel morphism, completeness, pullback, equalizer

(Résumé : tsvp)

---

[*] eric.fabre@irisa.fr

Unité de recherche INRIA Rennes
IRISA, Campus universitaire de Beaulieu, 35042 RENNES Cedex (France)
Téléphone : 02 99 84 71 00 - International : +33 2 99 84 71 00
Télécopie : 02 99 84 71 71 - International : +33 2 99 84 71 71

Construction des pullbacks pour les réseaux de Petri saufs

Résumé : Le produit de réseaux de Petri saufs (éventuellement à labels) est une opération bien connue : on peut la voir comme une généralisation du produit synchrone d'automates à des systèmes concurrents. Dans cette note, on s'intéresse à la construction de pullbacks de réseaux saufs, une autre construction catégorique. Les pullbacks généralisent le produit de réseaux en permettant une interaction non seulement par la synchronisation de transitions, mais aussi par partage de places et de transitions.

Mots-clé : réseau de Petri sauf, théorie des catégories, morphisme de Winskel, complétude, pullback, égaliseur

Contents

1 Introduction

We consider the category *Nets* of safe Petri nets (PN) as defined by Winskel in [2]. Safe Petri nets provide a natural and widespread model for concurrent systems. A product $\times_N$ was defined in [2] for safe PNs, that can be considered as a generalization of the usual synchronous product of automata. In practice, this product is essentially interesting when specialized to labeled nets: roughly speaking, it would then synchronize transitions of two nets as soon as they carry the same label. As a nice property, $\times_N$ is the categorical product in *Nets*. Pushing forward this idea, it can be interesting to derive a notion of pullback for PNs. While the product assumes that nets interact through common events, the pullback goes further and also allows interactions by shared places and transitions.

The notion of pullback has been extensively explored for other models of concurrency (transition graphs, graph grammars, etc.) [6], or for other categories of Petri nets [3] (proposition 11). But the choice of net morphisms plays a crucial role, and apparently the construction of pullbacks in the category *Nets* of [2] is still missing. This category remains of great interest however, because it allows foldings (and so unfoldings!), and already has a product.

Let us mention some contributions to the topic. B. Koenig provides in [8] a definition for specific pullback diagrams. M. Bednarczyk *et al.* prove in [7] that *Nets* is finitely complete, so all pullbacks exist. But the result is obtained in a much more general setting, and is hard to specialize to the case of safe nets (*dixit* one of the authors). Finally, let us stress that [7] mentions in its introduction (p.3) that the existence of a pullback construction for safe Petri nets has been reported... although they have not been able to locate any reference! It is therefore useful that we try to provide a direct definition.

We proceed in several steps. We first consider unlabeled nets. It is a well known fact that the labeling is essentially a decoration that can be reincorporated at no cost in net operations (see [5]), which we do at the end of the paper. Secondly, we recall that a pullback operation can be derived from a product and an equalizer (see [1], chap. V-2, thm. 1, and [6], sec. 5). Since all products exist in *Nets*, we simplify the construction (and proofs) by attempting to build equalizers, which is the heart of the contribution. We finally gather all pieces to give a comprehensive definition of the pullback of labeled Petri nets, first in the general case, then in the specific case where morphisms are partial functions.

2 Notations

Net. We denote Petri nets by $\mathcal{N} = (P, T, \rightarrow, P^0)$, representing respectively places, transitions, initially marked places and the flow relation. For each place $p \in P$, we assume $|p^\bullet \cup {}^\bullet p| \geq 1$, and for each transition $t \in T$, $|t^\bullet| \geq 1$ and $|{}^\bullet t| \geq 1$. For labeled nets, we take $\mathcal{N} = (P, T, \rightarrow, P^0, \lambda, \Lambda)$ where $\lambda : T \rightarrow \Lambda$ is the labeling function.

Morphism. A morphism $\phi : \mathcal{N}_1 \rightarrow \mathcal{N}_2$ between nets $\mathcal{N}_i = (P_i, T_i, \rightarrow_i, P_i^0)$ is a pair (ϕ_P, ϕ_T) where

C1. ϕ_T is a partial function from T_1 to T_2, and ϕ_P a relation between P_1 and P_2,

C2. $P_2^0 = \phi_P(P_1^0)$ and $\forall p_2 \in P_2^0, \exists! p_1 \in P_1^0 : p_1 \xleftrightarrow{\phi_P} p_2$,

C3. if $p_1 \xleftrightarrow{\phi_P} p_2$ then the restrictions $\phi_T : {}^\bullet p_1 \rightarrow {}^\bullet p_2$ and $\phi_T : p_1{}^\bullet \rightarrow p_2{}^\bullet$ are total functions, C4. if $t_2 = \phi_T(t_1)$ then the restrictions $\phi_P^{op} : {}^\bullet t_2 \to {}^\bullet t_1$ and $\phi_P^{op} : t_2{}^\bullet \to t_1{}^\bullet$ are total functions.

where ϕ_P^{op} denotes the opposite relation to ϕ_P. Notice that points 3 and 4 entail that the pair (ϕ_P, ϕ_T) preserves the flow relation (on its domain of definition). Observe also that point 3 implies that if ϕ_P is defined at $p_1 \in P_1$, then ϕ_T is defined at all transitions $t_1 \in T_1$ connected to p_1. In the sequel, we will simply write ϕ for ϕ_P or ϕ_T, and $\phi(X)$ to denote places in relation with at least one place in X. By $Dom(\phi)$, we represent the elements of $\mathcal{N}_1$ (places or transitions) where ϕ is defined, *i.e.* $\phi^{-1}(P_2 \cup T_2)$.

Remark. Notice that condition C2 becomes a consequence of C3 and C4 when one assumes the existence of fake initial transitions $t_{i,0}$ in each $\mathcal{N}_i$, such that $t_{i,0}{}^\bullet = P_i^0$ and $t_{2,0} = \phi(t_{1,0})$. We shall use this trick in the sequel to simplify proofs (focusing on C3, C4 and omitting to check C2).

Safe Petri nets with the above definition of morphisms define the category *Nets* . For labeled nets, we naturally consider label-preserving morphisms, which yields the category λ*Nets* .

Decomposition of the pullback. Let $\mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2$ be nets, and $f_i : \mathcal{N}_i \to \mathcal{N}_0$, $i = 1, 2$ be net morphisms, so $\mathcal{N}_0$ forms a kind of interface between $\mathcal{N}_1$ and $\mathcal{N}_2$, or better a common constraint[1]. We look for a terminal net $\mathcal{N} = (P, T, \to, P^0)$, associated to morphisms $g_i : \mathcal{N} \to \mathcal{N}_i$, $i = 1, 2$, such that (fig. 1):

$$f_1 \circ g_1 \quad = \quad f_2 \circ g_2 \tag{1}$$

By "terminal," we mean the universal property of the pullback: whenever there exists another triple $(\mathcal{N}_3, h_1, h_2)$ satisfying the same commutative diagram, there exists a unique morphism $\psi : \mathcal{N}_3 \to \mathcal{N}$ preserving the commutativity, namely $h_i = g_i \circ \psi$. We denote the pullback by $\mathcal{N}_1 \wedge^{\mathcal{N}_0} \mathcal{N}_2$, or by $\mathcal{N}_1 \wedge \mathcal{N}_2$ for short.

$$\begin{tikzcd}
 & N_3 \arrow[d, dashed, "\psi"] \arrow[ddl, dashed, bend right, "g_1"'] \arrow[ddr, dashed, bend left, "g_2"] & \\
 & N \arrow[dl, bend right, "\pi_1"'] \arrow[dr, bend left, "\pi_2"] & \\
N_1 \arrow[dr, bend right, "f_1"'] & & N_2 \arrow[dl, bend left, "f_2"] \\
 & N_0 &
\end{tikzcd}$$

Figure 1: *Commutative diagram of the pullback* $\mathcal{N} = \mathcal{N}_1 \wedge \mathcal{N}_2$.

It is well known that the pullback operation can be decomposed into a product, followed by an equalization. Consider the product net $\mathcal{N}_1 \times_N \mathcal{N}_2$, and the associated canonical projections $\pi_i : \mathcal{N}_1 \times_N \mathcal{N}_2 \to \mathcal{N}_i$, $i = 1, 2$. In general, $\mathcal{N}_1 \times_N \mathcal{N}_2$ and the π_i do not satisfy the pullback condition, *i.e.* $f_1 \circ \pi_1 \neq f_2 \circ \pi_2$. However, by equalizing them, one gets the desired result. $(\mathcal{N}, e)$ equalizes $f_1 \circ \pi_1$ and $f_2 \circ \pi_2$ iff $(f_1 \circ \pi_1) \circ e = (f_2 \circ \pi_2) \circ e$, and for any other candidate $(\mathcal{N}_3, h)$ there exists a unique $\psi : \mathcal{N}_3 \to \mathcal{N}$ such that $h = e \circ \psi$ (fig. 2). It is straightforward to check that $(\mathcal{N}, \pi_1 \circ e, \pi_2 \circ e)$ yields the desired pullback. For details, we refer the reader to [1], chap. V-2, thm. 1, or to [6], sec. 5 where this construction is also used.

---

[1] The term "interface" suggests that all interactions are captured by the intermediary net $\mathcal{N}_0$, which is not the case. The two nets $\mathcal{N}_1, \mathcal{N}_2$ may still have interactions outside $\mathcal{N}_0$ by means of synchronized transitions, as it will be obvious in the pullback definition given at the end of these notes.

$$\begin{tikzcd}
N_3 \arrow[d, dashed, "\psi"'] \arrow[dr, dashed, "h"] \arrow[drr, dashed, bend left] & & \\
N \arrow[r, "e"] \arrow[rr, bend right] & N_1 \times N_2 \arrow[r, bend left, "f_1 \circ \pi_1"] \arrow[r, bend right, "f_2 \circ \pi_2"'] & N_0
\end{tikzcd}$$

Figure 2: *Equalizing* $f_1 \circ \pi_1$ *and* $f_2 \circ \pi_2$.

3 Equalizer in *Nets*

Consider two nets $\mathcal{N}_i = (P_i, T_i, \rightarrow_i, P_i^0), i = 1, 2$ related by two morphisms $f, g : \mathcal{N}_1 \rightarrow \mathcal{N}_2$. We want to build the equalizer $(\mathcal{N}, e)$ of f and g, *i.e.* a net $\mathcal{N}$ and a morphism $e : \mathcal{N} \rightarrow \mathcal{N}_1$ satisfying $f \circ e = g \circ e$, and such that for any other candidate pair $(\mathcal{N}_3, h)$ there exists a unique morphism $\psi : \mathcal{N}_3 \rightarrow \mathcal{N}$ satisfying $h = e \circ \psi$ (fig. 3).

$$\begin{tikzcd}
N_3 \arrow[d, dashed, "\psi"'] \arrow[dr, dashed, "h"] \arrow[drr, dashed, bend left] & & \\
N \arrow[r, "e"] \arrow[rr, bend right] & N_1 \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & N_2
\end{tikzcd}$$

Figure 3: *A pair* $(\mathcal{N}, e)$ *equalizing* f *and* g.

3.1 Equalizer and coequalizer in *Sets*

We recall here two classical results that will be instrumental in the sequel.

3.1.1 Equalizer

We consider the category of sets with *partial* functions as morphisms (or equivalently pointed set with total functions). Let T_1, T_2 be two sets related by partial functions $f, g : T_1 \rightarrow T_2$. The equalizer of f and g is the pair (T, e) where

$$T = \{t_1 \in T_1 : f(t_1) = g(t_1) \text{ or both } f \text{ and } g \text{ are undefined at } t_1\} \tag{2}$$

and e is the canonical injection of T into T_1 (we'll use the shorthand $t_1 \in T$ instead of $t \in T, t_1 = e(t)$). In the setting of pointed sets, where functions point to the special value ϵ of a set to mean "undefined," (2) takes the simplest form $f(t_1) = g(t_1)$.

Given another candidate pair (T_3, h), the unique morphism (partial function) $\psi : T_3 \rightarrow T$ is obtained by $\psi = e^{-1} \circ h$ (it is easy to check that $Im(h) \subseteq T$).

3.1.2 Coequalizer

We now consider the category of sets with *total* functions. The coequalizer diagram corresponds to fig. 3 with all arrows reversed. Let S_2, S_1 be two sets related by total functions $F, G : S_2 \rightarrow S_1$, and denote by (S, E) the coequalizer of F and G. The construction is a bit more complex.

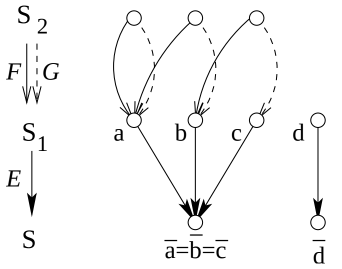

Figure 4: *Coequalizing the total functions F and G.*

Define the relation R on elements of S_1 by

$$p_1 \, R \, p_1' \quad \Leftrightarrow \quad \exists p_2 \in S_2, \; \{p_1, p_1'\} = \{F(p_2), G(p_2)\} \tag{3}$$

and consider the equivalence relation $\equiv$ generated by R. We denote by $[p_1]$ the class of p_1 for $\equiv$. Then

$$S \quad = \quad \{[p_1] : p_1 \in S_1\} \tag{4}$$

and the function $E : S_1 \rightarrow S$ is simply the quotient operation, *i.e.* $E(p_1) = [p_1]$.

Given another candidate pair (S_3, H), the unique morphism (total function) $\Psi : S_3 \rightarrow S$ is obtained by $\Psi = H \circ E^{-1}$, or in other words by $\forall [p_1] \in S, \Psi([p_1]) = H(p_1)$. Indeed, it is easy to check that H is necessarily class invariant.

3.2 Candidate equalizer in *Nets*

Let $(\mathcal{N}, e)$ denote the desired equalizer, with $\mathcal{N} = (P, T, \rightarrow, P^0)$ and $e : \mathcal{N} \rightarrow \mathcal{N}_1$.

Transitions. On transition sets, $f, g : T_1 \rightarrow T_2$ are partial function, so we adopt definition (2) for T and e on T.

Places. On place sets, the definition is a bit more complex. The morphism definition in *Nets* actually states in C4 that $\phi^{op} : {}^\bullet t_2 \rightarrow {}^\bullet t_1$ and $\phi^{op} : t_2{}^\bullet \rightarrow t_1{}^\bullet$ are total functions, for $t_2 = \phi(t_1)$, which orients us to co-equalizers in *Sets*. So let t be a transition of T, with $t_1 = e(t) \in T_1$.

Assume first that f, g are defined at t_1, and $f(t_1) = g(t_1) = t_2 \in T_2$. We take for e^{op} in ${}^\bullet t_1$ the coequalizer of $f^{op}, g^{op} : {}^\bullet t_2 \rightarrow {}^\bullet t_1$. Eq. (3) thus defines $R^{{}^\bullet t_1}$, the equivalence relation $\equiv^{{}^\bullet t_1}$ and place classes $[p_1]^{{}^\bullet t_1}$. And similarly in the post-set of t_1.

When f, g are both undefined at t_1, we take for e^{op} in ${}^\bullet t_1$ (or $t_1{}^\bullet$) the coequalizer of functions f^{op}, g^{op} from the empty set. So e^{op} is simply the identity.

In summary, the place set P of $\mathcal{N}$ is a subset of 2^{P_1} given by

$$P \quad = \quad \{[p_1]^{{}^\bullet t_1} : t_1 \in T, \; p_1 \in {}^\bullet t_1\} \cup \{[p_1]^{t_1{}^\bullet} : t_1 \in T, \; p_1 \in t_1{}^\bullet\} \tag{5}$$

and the relation e on places is simply given by $p \overset{e}{\longleftrightarrow} p_1$ iff $p_1 \in p$. Observe that a place $p_1 \in P_1$ not connected to a transition of T has no counterpart in P.

Lemma 1 *Let $t_1, t_1' \in T$. Assume $p_1, p_1' \subseteq t_1'{}^\bullet \cap {}^\bullet t_1$, then*

$$p_1 \equiv^{t_1'{}^\bullet} p_1' \quad \Longleftrightarrow \quad p_1 \equiv^{{}^\bullet t_1} p_1' \tag{6}$$

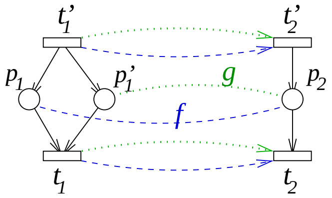

Figure 5: *Identity of equivalence classes.*

Proof. Assume $p_1 \neq p'_1$ and $p_1 \, R^{t'_1{}^\bullet} \, p'_1$. This means f, g are defined at t_1, $f(t'_1) = t'_2 = g(t'_1)$, and for example[2] $\exists p_2 \in t'_2{}^\bullet \; : \; p_1 \xleftrightarrow{f} p_2 \xleftrightarrow{g} p'_1$. Let $t_2 = f(t_1) = g(t_1)$, by C4 on f or g, one has $p_2 \in {}^\bullet t_2$, whence $p_1 \, R^{{}^\bullet t_1} \, p'_1$. This proves $[p_1]^{t'_1{}^\bullet} \subseteq [p_1]^{{}^\bullet t_1}$. One can show in the same way the reverse inclusion, which proves the lemma. □

Naturally, the lemma holds also for the other arrow orientations, *i.e.* for $p_1, p'_1 \subseteq t'_1{}^\bullet \cap t_1{}^\bullet$ and for $p_1, p'_1 \subseteq {}^\bullet t'_1 \cap {}^\bullet t_1$.

Initial places. In eq. (5), we assume the existence of (fake) transitions $t_{i,0}$ with $t_{i,0}{}^\bullet = P_i^0$ and $f(t_{1,0}) = g(t_{1,0}) = t_{2,0}$. So initial places in P are given by

$$P^0 \quad = \quad \{[p_1]^{t_{1,0}{}^\bullet} : p_1 \in P_1^0\} \tag{7}$$

For $p_1 \in P_1$ and $t_1 \in T_1$, notice that the equivalence class $[p_1]^{{}^\bullet t_1}$ (or equivalently $[p_1]^{t_1{}^\bullet}$) may both contain marked places of P_1^0 and unmarked places of $P_1 \setminus P_1^0$. Such a class is not taken as an initial place of $\mathcal{N}$. See the example of p' in fig. 6.

Conversely, assume an equivalence class $[p_1]^{{}^\bullet t_1}$ (for example) satisfies $[p_1]^{{}^\bullet t_1} \subseteq P_1^0$. By lemma 1, $[p_1]^{{}^\bullet t_1} = [p_1]^{t_1^0{}^\bullet}$ which corresponds to an initial place of $\mathcal{N}$. We could thus take as an alternate definition:

$$P^0 \quad = \quad \{p \in P \; : \; e(p) \subseteq P_1^0\} \tag{8}$$

Flow relation. It is obviously defined by $p \to t$ when $e(t) = t_1$ and $p = [p_1]^{{}^\bullet t_1}$ for some $p_1 \in {}^\bullet t_1$. But, using lemma 1, we can derive the simpler criterion:

$$p \to t \quad \Longleftrightarrow \quad e(p) \subseteq {}^\bullet e(t) \;\; \text{in} \;\; \mathcal{N}_1 \tag{9}$$

We proceed symmetrically for $t \to p$.

Example. Fig. 6 illustrates this construction. Observe that $p_1 R^{t'_1{}^\bullet} p'_1$ and $p_1 R^{{}^\bullet t_1} p''_1$, which results in two classes/places in $\mathcal{N}$, both related to p_1 by e. These places must indeed be distinguished: by merging places p' and p in $\mathcal{N}$, *i.e.* by aggregating classes sharing one or more places of P_1, the resulting e wouldn't be a morphism (C3 violated).

3.3 Coherence of the definition

$e : \mathcal{N} \to \mathcal{N}_1$ **is a net morphism.** C1 holds by definition, and with the trick of fake initial transitions, C2 is a consequence of C3 and C4, which we only need to examine.

---

[2] The other possibility is $p_1 \xleftrightarrow{g} p_2 \xleftrightarrow{f} p'_1$, but this doesn't affect the proof.

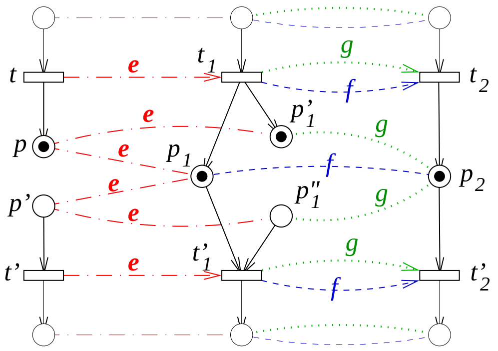

Figure 6: *The equalizer* $(\mathcal{N}, e)$ *(left) for nets* $\mathcal{N}_1$ *(center) and* $\mathcal{N}_2$ *(right) related by two morphisms* f, g*. Notice that* t', t'_1, t'_2 *could be the "fake" initial transitions, provided their input places would be removed.*

C4 obviously holds by construction of places of P: if $t_1 = e(t)$, then $e^{op} : {}^\bullet t_1 \to {}^\bullet t$ defined by $e^{op}(p_1) = [p_1]^{{}^\bullet t_1}$ is a total function. Similarly for $e^{op} : t_1{}^\bullet \to t^\bullet$.

For C3, consider $p \to t$ in $\mathcal{N}$, such that $p \xleftrightarrow{e} p_1$ and $e(t) = t_1$. We want to check that $p_1 \to_1 t_1$ in $\mathcal{N}_1$. By definition of the flow in $\mathcal{N}$, one has $p \to t$ iff $e(p) \subseteq {}^\bullet e(t) = {}^\bullet t_1$, and $p \xleftrightarrow{e} p_1$ iff $p_1 \in p$, so $p_1 \to_1 t_1$ holds. The same reasoning proves that $e : {}^\bullet p \to {}^\bullet p_1$ is also a total function.

$\mathcal{N}$ is a safe net. By a standard argument: since $e : \mathcal{N} \to \mathcal{N}_1$ is a net morphism, it maps runs of $\mathcal{N}$ to runs of $\mathcal{N}_1$. So if $\mathcal{N}$ is not safe, one of its run fills some place with more than one token, which reveals by e a non safe run in $\mathcal{N}_1$, because e is a total function on T.

$(\mathcal{N}, e)$ satisfies the commutative diagram. This is true by construction for the partial functions on transitions. It also holds locally for relations on places, *i.e.* around triples of transitions (t, t_1, t_2) with $t_1 = e(t), t_2 = f(t_1) = g(t_1))$. This allows to reach completely the place relations e, f, g.

3.4 Universal property

Assume the pair $(\mathcal{N}_3, h)$ satisfies $f \circ h = g \circ h$, with $N_3 = (P_3, T_3, \to_3, P_3^0)$ and $h : \mathcal{N}_3 \to \mathcal{N}_1$. We look for a (unique) $\psi : \mathcal{N}_3 \to \mathcal{N}$ satisfying $h = e \circ \psi$.

Definition of ψ. On transitions, ψ is uniquely given by $\psi = e^{-1} \circ h$, as it was seen in section 3.1.1.

For places, consider a triple $(t_3, t, t_1) \in T_3 \times T \times T_1$ of related transitions: $\psi(t_3) = t$ and $h(t_3) = t_1 = e(t)$. We say that such a triple (t_3, t, t_1) *forms a triangle.* From section 3.1.2, we know that $\psi^{op} : {}^\bullet t \to {}^\bullet t_3$ is uniquely defined from $h^{op} : {}^\bullet t_1 \to {}^\bullet t_3$ by

$$\forall p_1 \in {}^\bullet t_1, \quad \psi^{op}([p1]^{{}^\bullet t_1}) \;=\; h^{op}(p_1) \cap {}^\bullet t_3 \tag{10}$$

(recall that h^{op} is necessarily class invariant on ${}^\bullet t_1$). We proceed similarly to define $\psi^{op} : t^\bullet \to t_3{}^\bullet$.

ψ satisfies the commutative diagram. By construction of ψ, $h = e \circ \psi$ is obvious on transitions, and locally on places (*i.e.* around triangles of transitions). To show that the relation holds globally on places, consider $p_3 \in P_3$. By assumption, p_3 is connected to at least one transition t_3 in $\mathcal{N}_3$. If h is defined at p_3 and $p_3 \xleftrightarrow{h} p_1$, then h is also defined at t_3 (by C3), $h(t_3) = t_1 \in T$ and p_1 is connected to t_1. We then use $h = e \circ \psi$ around the triangle (t_3, t, t_1), where $t = \psi(t_3)$.

ψ is a net morphism. It obviously satisfies C1, and C4 is imposed by the construction of ψ on places. So only C3 has to be checked, which is the difficult part of the proof.

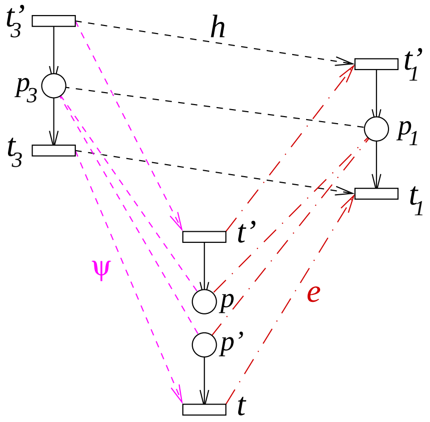

Figure 7: *Proof that ψ satisfies C3.*

For C3, consider a pair of places $(p_3, p) \in P_3 \times P$ related by ψ (*i.e.* $p_3 \xleftrightarrow{\psi} p$) and assume $p_3 \rightarrow t_3$ in $\mathcal{N}_3$. We want to show that ψ is defined at t_3, and $\psi(t_3) \in p^\bullet$ in $\mathcal{N}$. By definition of ψ on places, there exists a triangle $(t'_3, t', t'_1) \in T_3 \times T \times T_1$ such that for example[3] $t'_3 \rightarrow_3 p_3$, $t'_1 \rightarrow_1 p_1$, $t \rightarrow p$ and $p = [p_1]^{t_1'^\bullet}$ (see Fig. 7).

h is defined at p_3, thus also at t_3 by C3. Since $f \circ h = g \circ h$, one has $t_1 = h(t_3) \in T$. So there exists $t \in T$ with $e(t) = t_1$ and thus we already know that ψ is defined at t_3: $\psi(t_3) = t$. In other words, $(t_3, t, t_1) \in T_3 \times T \times T_1$ forms another triangle. Since e is a morphism, let p' be the image of p_1 by $e^{op} : {}^\bullet t_1 \rightarrow {}^\bullet t$, so $p' = [p_1]^{{}^\bullet t_1}$. By definition of ψ in the presets of the triangle (t_3, t, t_1), see (10), one has $p_3 \xleftrightarrow{\psi} p'$. To conclude the proof, we thus have to show that $p = p'$. We essentially use the fact that h is a morphism satisfying $f \circ h = g \circ h$.

Let p'_1 be a place of $t_1'^\bullet$ such that $p_1 \equiv^{t_1'^\bullet} p'_1$. We know that $p_3 \xleftrightarrow{h} p'_1$, because $h^{op} : t_1'^\bullet \rightarrow t_3'^\bullet$ is class invariant (a consequence of $f \circ h = g \circ h$). From $p_3 \rightarrow_3 t_3$ in $\mathcal{N}_3$ and $p_3 \xleftrightarrow{h} p'_1$, we derive by C3 that $p'_1 \rightarrow_1 t_1 = h(t')$. We are now exactly in the situation of lemma 1, so $p_1 \equiv^{{}^\bullet t_1} p'_1$. We have thus proved that $[p_1]^{t_1'^\bullet}$ and $[p_1]^{{}^\bullet t_1}$ are identical, or in other words $p = \bar{p}$.

4 Synthesis

We now reassemble all elements to provide a definition for pullbacks of safe labeled nets. The next section recalls the product definition (assuming a simple synchronization algebra), that we combine to the equalizer to obtain the pullback.

---

[3] Equivalently, we could have assumed that the related places lie in the presets (instead of post-sets) of a transition triangle.

4.1 Product

Let $\mathcal{N}_i = (P_i, T_i, \rightarrow_i, P_i^0, \Lambda_i, \lambda_i), i = 1, 2$ be two labeled nets. For net products, we assume a simple synchronization algebra : two transitions carrying the same label have to synchronize, while transitions carrying a private label remain private. Private labels are those in $(\Lambda_1 \setminus \Lambda_2) \cup (\Lambda_2 \setminus \Lambda_1)$. The product $\bar{\mathcal{N}} = \mathcal{N}_1 \times_N \mathcal{N}_2$ and the associated projections $\pi_i : \bar{\mathcal{N}} \rightarrow \mathcal{N}_i$ are defined as follows[4] :

1. $\bar{P} = \{(p_1, \star) : p_1 \in P_1\} \cup \{(\star, p_2) : p_2 \in P_2\}$: disjoint union of places, $\pi_i(p_1, p_2) = p_i$ if $p_i \neq \star$ and is undefined otherwise,
2. $\bar{P}^0 = \pi_1^{-1}(P_1^0) \cup \pi_2^{-1}(P_2^0)$,
3. the transition set $\bar{T}$ is given by

$$\begin{array}{rcl} \bar{T} & = & \{(t_1, \star) \ : \ t_1 \in T_1, \ \lambda_1(t_1) \in \Lambda_1 \setminus \Lambda_2\} \\ & \cup & \{(\star, t_2) \ : \ t_2 \in T_2, \ \lambda_2(t_2) \in \Lambda_2 \setminus \Lambda_1\} \\ & \cup & \{(t_1, t_2) \in T_1 \times T_2 \ : \ \lambda_1(t_1) = \lambda_2(t_2) \in \Lambda_1 \cap \Lambda_2\} \end{array}$$

   $\pi_i(t_1, t_2) = t_i$ if $t_i \neq \star$ and is undefined otherwise,
4. the flow $\rightarrow$ is defined by ${}^\bullet t = {}^\bullet \pi_1(t) \cup {}^\bullet \pi_2(t)$ and $t^\bullet = \pi_1(t)^\bullet \cup \pi_2(t)^\bullet$, assuming ${}^\bullet \pi_i(t) = \pi_i(t)^\bullet = \emptyset$ if π_i is undefined on t,
5. $\bar{\Lambda} = \Lambda_1 \cup \Lambda_2$ and $\bar{\lambda}$ is the unique labeling preserved by the π_i.

4.2 Pullback

Assume the $f_i : \mathcal{N}_i \rightarrow \mathcal{N}_0$ are morphisms of labeled nets. The pullback $\mathcal{N} = \mathcal{N}_1 \wedge \mathcal{N}_2$ is defined as follows, by combining the definitions of product and equalizer (section 2).

Transitions. We distinguish "shared" transitions in $\mathcal{N}_1$ and $\mathcal{N}_2$, *i.e.* those having an image in $\mathcal{N}_0$, from "private" ones, the others. For private transitions, the definition of the pullback mimics the definition of the product. For shared transitions, only pairs that match through the f_i are preserved.

$$\begin{array}{rcll} T_s & = & \{(t_1, t_2) \in T_1 \times T_2 \ : \ t_i \in Dom(f_i), \ f_1(t_1) = f_2(t_2)\} & (11) \\ T_p & = & \{(t_1, t_2) \in T_1 \times T_2 \ : \ t_i \notin Dom(f_i), \ \lambda_1(t_1) = \lambda_2(t_2)\} & \\ & & \cup \{(t_1, \star) \ : \ t_1 \in T_1, \ t_1 \notin Dom(f_1), \ \lambda_1(t_1) \in \Lambda_1 \setminus \Lambda_2\} & \\ & & \cup \{(\star, t_2) \ : \ t_2 \in T_2, \ t_2 \notin Dom(f_2), \ \lambda_2(t_2) \in \Lambda_2 \setminus \Lambda_1\} & (12) \\ T & = & T_s \cup T_p & (13) \end{array}$$

Notice that the label condition doesn't appear in (11) : it comes as a consequence of $f_1(t_1) = f_2(t_2)$, since morphisms preserve labels.

---

[4] Remark : if ones wishes to use the trick of fake initial transitions t_i^0 to define initial markings P_i^0 by $P_i^0 = t_i^{0\bullet}$, one has to assume that each Λ_i contains a special label ϵ^0 reserved to the transition t_i^0.

Places. Places are obtained by inspecting transitions selected in T.

Consider first a private transition $(t_1,t_2) \in T_p$, where one (at most) of the t_i can be $\star$. Assume $p_i \rightarrow_i t_i$ (or equivalently $t_i \rightarrow_i p_i$) in $\mathcal{N}_i$, with $t_i \neq \star$. Observe that necessarily $p_i \notin Dom(f_i)$, otherwise f_i would be defined at t_i. Such a place p_i induces a singleton equivalence class in P, either $(p_1,\star)$, or $(\star,p_2)$. We denote by P_p all such "private" places.

Consider now a pair of shared transitions $(t_1,t_2) \in T_s$, where $f_1(t_1) = t_0 = f_2(t_2)$. Consider for example a place $p_1 \in {}^\bullet t_1$ (or equivalently $p_1 \in t^\bullet{}_1$, and symm. for a place $p_2 \in {}^\bullet t_2{}^\bullet$).

a. If $p_1 \notin Dom(f_1)$, then $[(p_1,\star)]^{\bullet(t_1,t_2)}$ is reduced to $(p_1,\star)$, which yields another private place in P_p.

b. If $p_1 \in Dom(f_1)$, let $p_0 \in P_0 \cap {}^\bullet t_0$ satisfy $p_1 \overset{f_1}{\longleftrightarrow} p_0$. By C4 applied to f_2, there exists $p_2 \in {}^\bullet t_2$ such that $p_2 \overset{f_2}{\longleftrightarrow} p_0$, so $(p_1,\star)\, R^{\bullet(t_1,t_2)}\, (\star,p_2)$ in the product $\mathcal{N}_1 \times_N \mathcal{N}_2$. The resulting equivalence class $[(p_1,\star)]^{\bullet(t_1,t_2)}$, takes the form (Q_1,Q_2), with $\emptyset \neq Q_i \subseteq P_i$, and yields a "shared" place in the pullback.

In summary :

$$
\begin{aligned}
P_p &= \{\, (p_1,\star) \;:\; p_1 \in P_1,\ p_1 \notin Dom(f_1),\ \exists (t_1,\cdot) \in T,\ p_1 \in {}^\bullet t_1{}^\bullet \,\} \\
&\quad \cup \{\, (\star,p_2) \;:\; p_2 \in P_2,\ p_2 \notin Dom(f_2),\ \exists (\cdot,t_2) \in T,\ p_2 \in {}^\bullet t_2{}^\bullet \,\} && (14)\\
P_s &= \{\, (Q_1,Q_2) \;:\; Q_i \subseteq P_i,\ Q_i \subseteq Dom(f_i), \\
&\quad\quad \exists (t_1,t_2) \in T_s,\ Q_1 \uplus Q_2 \text{ equiv. class of } \equiv^{\bullet(t_1,t_2)} \text{ or of } \equiv^{(t_1,t_2)\bullet} \,\} && (15)\\
P &= P_p \cup P_s && (16)
\end{aligned}
$$

In (14), the dot in $(t_1,\cdot)$ stands for either t_2 or $\star$, and symm. for the second line.

Initial places. By abuse of notation, let us identify a private place like $(p_1,\star)$ to $(Q_1,Q_2) = (\{p_1\},\emptyset)$, and $(\star,p_2)$ to $(Q_1,Q_2) = (\emptyset,\{p_2\})$. So (Q_1,Q_2) denotes a general place in P.

$$
P^0 = \{(Q_1,Q_2) \in P : Q_1 \subseteq P_1^0,\ Q_2 \subseteq P_2^0\} \tag{17}
$$

Flow. Let $(Q_1,Q_2) \in P$ and $(t_1,t_2) \in T$ (where one of the t_i can be $\star$). Then

$$
\begin{aligned}
(Q_1,Q_2) \rightarrow (t_1,t_2) &\iff Q_1 \subseteq {}^\bullet t_1 \text{ in } \mathcal{N}_1,\quad Q_2 \subseteq {}^\bullet t_2 \text{ in } \mathcal{N}_2 && (18)\\
(t_1,t_2) \rightarrow (Q_1,Q_2) &\iff Q_1 \subseteq t_1{}^\bullet \text{ in } \mathcal{N}_1,\quad Q_2 \subseteq t_2{}^\bullet \text{ in } \mathcal{N}_2 && (19)
\end{aligned}
$$

with the convention that $\emptyset \subseteq {}^\bullet\star$ and $\emptyset \subseteq \star^\bullet$ hold.

Morphisms g_i. Let (t_1,t_2) be a transition of T, one has $g_i(t_1,t_2) = t_i$ if $t_i \neq \star$, and is undefined otherwise. Let (Q_1,Q_2) be a general place in P, one has $(Q_1,Q_2) \overset{g_i}{\longleftrightarrow} p_i$ iff $p_i \in Q_i$.

4.3 Special case

We examine here the special case where morphisms $f_i : \mathcal{N}_i \rightarrow \mathcal{N}_0$ are partial functions not only on transitions, but also on places (instead of being relations on places). The definition changes only for P_s in (15) : when place duplications are forbidden, equivalence classes of shared places are reduced to two elements only.

$$\begin{aligned} P_s \;=\; \{\, (p_1, p_2) \;:\; & p_i \in P_i \cap Dom(f_i),\; f_1(p_1) = f_2(p_2) = p_0, \\ & \exists (t_1, t_2) \in T_s,\; f_1(t_1) = f_2(t_2) = t_0,\; p_0 \in {}^\bullet t_0{}^\bullet \,\} \end{aligned} \tag{20}$$

This definition coincides with the proposition of [8] (and also to an early version of the present notes), apart from the extra condition that places created in (14) and (20) be connected to at least one transition of the pullback.

Acknowledgment : special thanks to Marek Bednarczyk for fruitful discussions.

References

[1] S. Mac Lane, Categories for the Working Mathematician, Springer-Verlag, 1971.

[2] G. Winskel, A new Definition of Morphism on Petri Nets, LNCS 166, pp. 140-149, 1984.

[3] M. Nielsen, G. Winskel, Petri Nets and Bisimulation, BRICS report no. RS-95-4, Jan. 1995.

[4] G. Winskel, Categories of models for concurrency, Seminar on Concurrency, Carnegie-Mellon Univ. (July 1984), LNCS 197, pp. 246-267, 1985.

[5] G. Winskel, Event structure semantics of CCS and related languages , LNCS 140, 1982, also as report PB-159, Aarhus Univ., Denmark, April 1983.

[6] M.A. Bernarczyk, L. Bernardinello, B. Caillaud, Modular System Development with Pullbacks, ICATPN 2003, LNCS 2679, pp. 140-160, 2003.

[7] M.A. Bernarczyk, A. Borzyszkowski, R. Somla, Finite Completeness of Categories of Petri Nets, Fundamenta Informaticae, vol. 43, no. 1-4, pp. 21-48, 2000.

[8] B. Koenig, Parallel Composition and Unfolding of Petri Nets (Including Some Examples), private communication, 2005.

Unité de recherche INRIA Lorraine, Technopôle de Nancy-Brabois, Campus scientifique,
615 rue du Jardin Botanique, BP 101, 54600 VILLERS LÈS NANCY
Unité de recherche INRIA Rennes, Irisa, Campus universitaire de Beaulieu, 35042 RENNES Cedex
Unité de recherche INRIA Rhône-Alpes, 655, avenue de l'Europe, 38330 MONTBONNOT ST MARTIN
Unité de recherche INRIA Rocquencourt, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex
Unité de recherche INRIA Sophia-Antipolis, 2004 route des Lucioles, BP 93, 06902 SOPHIA-ANTIPOLIS Cedex

Éditeur
INRIA, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)
http://www.inria.fr
ISSN 0249-6399