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Abstract: An $(n, p, n+f)$-*network* G is a graph (V, E) where the vertex set V is partitioned into four subsets $\mathcal{P}$, $\mathcal{I}$, $\mathcal{O}$ and $\mathcal{S}$ called respectively the *priorities*, the *ordinary inputs*, the *outputs* and the *switches*, satisfying the following constraints: there are p priorities, $n - p$ ordinary inputs and $n + f$ outputs; each priority, each ordinary input and each output is connected to exactly one switch; switches have degree at most 4. An $(n, p, n + f)$-network is a (n, p, f)-*repartitor* if for any disjoint subsets $\mathcal{F}$ and $\mathcal{B}$ of $\mathcal{O}$ with $|\mathcal{F}| = f$ and $|\mathcal{B}| = p$, there exist in G, n edge-disjoint paths, p of them from $\mathcal{P}$ to $\mathcal{B}$ and the $n - p$ others joining $\mathcal{I}$ to $\mathcal{O} \setminus (\mathcal{B} \cup \mathcal{F})$. The problem is to determine the minimum number $R(n, p, f)$ of switches of an (n, p, f)-repartitor and to construct a repartitor with the smallest number of switches.

In this paper, we show how to build general repartitors from $(n, 0, f)$-repartitors also called $(n, n + f)$-*selectors*. We then consrtuct selectors using more powerful networks called *superselectors*. An $(n, 0, n)$-network is an n-*superselector* if for any subsets $\mathcal{I}' \subset \mathcal{I}$ and $\mathcal{O}' \subset \mathcal{O}$ with $|\mathcal{O}'| = |\mathcal{I}'|$, there exist in G, $|\mathcal{O}'|$ edge-disjoint paths joining $\mathcal{I}'$ to $\mathcal{O}'$. We show that the minimum number of switches of an n-superselector $S^+(n)$ is at most $17n+O(log(n))$. We then deduce that $R(n, p, f) \leq \frac{69}{2}n + \frac{35}{2}f - 33p + O(\log(n + f))$ if $p \leq \frac{n-f}{2}$, $R(n, p, f) \leq 18n+34f+O(\log(n+f))$, if $\frac{n-f}{2} \leq p \leq \frac{n+f}{2}$ and $R(n, p, f) \leq \frac{3}{2}n+\frac{35}{2}f+33p+O(\log(n+f))$ if $p \geq \frac{n+f}{2}$. Finally, we give lower bounds for $R(n, 0, f)$ and $S^+(n)$ and show optimal networks for small value of n.

Key-words: network design, on-board network, fault tolerance, vulnerability

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Repartiteurs, sélecteurs et supersélecteurs

Résumé : Un $(n, p, n+f)$-*réseau* G est un graphe (V, E) dont l'ensemble de sommets V est partitionné en quatre sous-ensembles $\mathcal{P}$, $\mathcal{I}$, $\mathcal{O}$ et $\mathcal{S}$ appelés respectivement les *prioritiés*, les *entrées ordinaires*, les *sorties* et les *commutateurs*, satisfaisant les contraintes suivantes : il y a p priorités, $n-p$ entrées ordinaires et $n+f$ sorties; chaque priorité, entrée ordinaire ou sortie est reliée à exactement un commutateur; les commutateurs ont degré au plus 4. Un $(n, p, n+f)$-réseau est un (n, p, f)-*répartiteur* si pour toute paire de sous-ensembles disjoints $\mathcal{F}$ et $\mathcal{B}$ de $\mathcal{O}$ avec $|\mathcal{F}| = f$ et $|\mathcal{B}| = p$, il existe dans G, n chemins arêtes-disjoints, p d'entre eux allant de $\mathcal{P}$ à $\mathcal{B}$ et les $n-p$ autres de $\mathcal{I}$ à $\mathcal{O} \setminus (\mathcal{B} \cup \mathcal{F})$. Le problème consiste à déterminer le nombre minimum $R(n, p, f)$ de commutateurs d'un (n, p, f)-répartiteur et de construire un répartiteur avec le moins possible de commutateurs.

Dans ce rapport, nous montrons comment construire des répartiteurs à partir de répartiteurs particuliers, nommément les $(n, 0, f)$-répartiteurs aussi appelés $(n, n+f)$-*selecteurs*. Nous construisons ensuite des sélecteurs par l'intermédiaire de réseaux plus puissants appelés *supersélecteurs*. Un $(n, 0, n)$-réseau est un n-*supersélecteur* si pour toute paire d'ensembles $\mathcal{I}' \subset \mathcal{I}$ et $\mathcal{O}' \subset \mathcal{O}$ vérifiant $|\mathcal{O}'| = |\mathcal{I}'|$, il existe dans G, $|\mathcal{O}'|$ chemins arête-disjoints reliant $\mathcal{I}'$ à $\mathcal{O}'$. Nous montrons que le nombre minimum de commutateurs d'un n-supersélecteur, $S^+(n)$, est au plus $17n + O(log(n))$. Nous en déduisons ensuite que $R(n, p, f) \leq \frac{69}{2}n + \frac{35}{2}f - 33p + O(\log(n+f))$ si $p \leq \frac{n-f}{2}$, $R(n, p, f) \leq 18n + 34f + O(\log(n+f))$, si $\frac{n-f}{2} \leq p \leq \frac{n+f}{2}$ et $R(n, p, f) \leq \frac{3}{2}n + \frac{35}{2}f + 33p + O(\log(n+f))$ si $p \geq \frac{n+f}{2}$. Enfin, nous donnons des bornes inférieures pour $R(n, 0, f)$ et $S^+(n)$ et exhibons des réseaux optimaux pour de petites valeurs de n.

Mots-clés : conception de réseaux, réseau embarqué, tolérance aux pannes, vulnérabilité

1 Introduction

Modern telecommunications satellites are very complex to design and an important industrial issue is to provide robustness at the lowest possible cost. A key component of telecommunication satellites is an interconnection network which allows to redirect signals received by the satellite to a set of amplifiers where the signals will be retransmitted. In this paper, we consider a certain type of interconnection network as asked by Alcatel Space Industries. The network is made of expensive switches ; so we want to minimize their number subject to the following conditions : Each input and output is adjacent to exactly one link ; each switch is adjacent to exactly four links ; there are n inputs (signals) and $n + f$ outputs (amplifiers) ; among the $n + f$ outputs, f can fail permanently; among the n input signals, p of them called priorities must be connected to the amplifiers providing the best quality of service (that is to some specific outputs) and the other signals should be sent to other amplifiers. Note that the priority signals are given, but the amplifiers providing the quality of service change during the life of the satellite and so the networks should be able to route the signals for any set of f failed outputs and any set of p best quality outputs.

This problem can be formally restated as follows:

Definition 1 An $(n, p, n + f)$-*network* G is a graph (V, E) where the vertex set V is partitioned into four subsets $\mathcal{P}$, $\mathcal{I}$, $\mathcal{O}$ and $\mathcal{S}$ called respectively the *priorities*, the *ordinary inputs*, the *outputs* and the *switches*, satisfying the following constraints:

- there are p priorities, $n - p$ ordinary inputs and $n + f$ outputs;
- each priority, each ordinary input and each output is connected to exactly one switch;
- switches have degree at most 4.

An $(n, p, n + f)$-network is a (n, p, f)-*repartitor* if for any disjoint subsets $\mathcal{F}$ and $\mathcal{B}$ of $\mathcal{O}$ with $|\mathcal{F}| = f$ and $|\mathcal{B}| = p$, there exist in G, n edge-disjoint paths, p of them from $\mathcal{P}$ to $\mathcal{B}$ and the $n - p$ others joining $\mathcal{I}$ to $\mathcal{O} \setminus (\mathcal{B} \cup \mathcal{F})$. The set $\mathcal{F}$ corresponds to set of failures and $\mathcal{B}$ to the set of amplifiers providing the best quality of service. We denote $R(n, p, f)$ the minimum number of switches (i.e. cardinality of $\mathcal{S}$) of a valid (n, p, f)-repartitor. A (n, p, f)-repartitor with $R(n, p, f)$ switches will be called *minimum*.

Problem 1 Determine $R(n, p, f)$ and construct a minimum (or almost minimum) repartitor.

A particular case of repartitors are those with only one type of inputs, i.e. $(n, 0, f)$-repartitors (or (n, n, f)-repartitors) also called $(n, n + f)$-*selectors*. A (p, n)-selector is a network with a set $\mathcal{I}$ of p inputs and a set $\mathcal{O}$ of n outputs, such that for any subset $\mathcal{O}'$ of p outputs there exists a set of p edge-disjoint paths connecting $\mathcal{I}$ to $\mathcal{O}'$.

In [4] and [6], $(n, n + f)$-selectors) with f fixed, were studied. Let us denote $S(p, n)$ the minimum number of switches (i.e. cardinality of $\mathcal{S}$) of a (p, n)-selector. In [4], it is shown that $S(n, n + 2) = R(n, 0, 2) = n$. In [6], the following values for small f are also given : $S(n, n+4) = R(n, 0, 4) = n + \lceil \frac{n}{4} \rceil$; $S(n, n+6) = R(n, 0, 6) = n + \frac{n}{4} + \sqrt{\frac{n}{8}} + O(1)$; $S(n, n+8) = R(n, 0, 8) = n + \frac{n}{3} + \frac{2}{3}\sqrt{\frac{n}{3}} + O(\sqrt[4]{n})$ and $S(n, n + 12) = R(n, 0, 12) = n + \frac{3n}{7} + O(\sqrt{n})$.

It is also proved that $\frac{3n}{2} - O(\frac{n}{f}) \leq S(n, n+f) = R(n, 0, f) \leq \frac{3n}{2} + g(f)$ with g a function depending only on f. However, this function is exponential in f, hence if $log(n) = o(f)$ this upper bound is very bad.

In [5], it is shown that $R(n, p, 0) \leq n - p + \frac{n}{2}\lceil \log_2 p \rceil$. Moreover, some exact values of $R(n, p, f)$ were given when p and f are small.

The main objective of this paper (sectionconstruction) is to prove an upper bound for $R(n, p, f)$ which is linear in n, p and f.

We first show Subsection 2.1 how to construct a repartitor from two selectors and derive the inequality $R(n, p, f) \leq S(p, n+f) + S(n-p, n+f) + n$. We then focus on selectors.

Selectors are an anloguous of (the opposite of) concentrators (see [7]). An *(n,m)-concentrator* is a directed acyclic graph with n distinguished vertices called *inputs* and a disjoint set of m distinguished vertices *outputs* such that for any subset A of m inputs there exists a set of m vertex-disjoint paths connecting A to the outputs. Lots of papers are devoted to study the minimum number of edges of an (n, m)-concentrator.

Here we want to minimize the number of switches of selectors which is very close to minimize the number of edges since every switch has degree at most 4, so the number of edges of a minimum (p, n)-selector is at most $\frac{1}{2}(4S(p, n) + n + p)$.

In order to build selectors, we first build more powerful networks called *superselectors*. An $(n, 0, n)$-network is an *n-superselector* if for any subsets $\mathcal{I}' \subset \mathcal{I}$ and $\mathcal{O}' \subset \mathcal{O}$ with $|\mathcal{O}'| = |\mathcal{I}'|$, there exist in G, $|\mathcal{O}'|$ edge-disjoint paths joining $\mathcal{I}'$ to $\mathcal{O}'$. We will denote $S^+(n)$ the minimum number of switches (i.e. cardinality of $\mathcal{S}$) of a n-superselector. A n-superselector with $S^+(n)$ switches will be called *minimum*.

Superselectors are very powerful since they are like (p, n)-selectors for every $1 \leq p \leq n$. Indeed, it is easy to see that the network obtained from a n-superselector by deleting any set of $n - p$ inputs is a (p, n)-selector.

Proposition 1 *For any $p \leq n$, $S(p, n) \leq S^+(n)$.*

Then we give two constructions of superselectors. The first one (Subsection 2.2) uses a technique inspired by Pippenger's construction [8] of superconcentrators. It yields an n-superselector with $17n + O(\log(n))$ switches. However, this construction is based on the existence of *funnels* which is proved in Lemma 2 in a non constructive way. Hence, we give Subsection 2.3 an explicit construction of an n-superselectors with $20n + O(1)$ switches. It is based an a result of Alon and Capalbo [1]. Then, in Subsection 2.4, we give constructions of selectors from superselectors. We show :

If $p \geq n/2$ then $S(p, n) \leq \frac{n}{2} + 17p + O(\log(n))$.
If $p \leq n/2$ then $S(p, n) \leq 17n - 16p + O(\log(n))$.

We then deduce:

If $p \leq \frac{n-f}{2}$, then $R(n, p, f) \leq \frac{69}{2}n + \frac{35}{2}f - 33p + O(\log(n+f))$.
If $\frac{n-f}{2} \leq p \leq \frac{n+f}{2}$ then $R(n, p, f) \leq 18n + 34f + O(\log(n+f))$.
If $p \geq \frac{n+f}{2}$, then $R(n, p, f) \leq \frac{3}{2}n + \frac{35}{2}f + 33p + O(\log(n+f))$.

In the second part of this paper, we study lower bounds to the number of switches in a selector or superselector. In Subsection 3.1, we prove that:

if p is even then $S(p,n) \geq \frac{2^{p/2}-1}{2^{p/2}}n + \Theta(1)$;

if p is odd then $S(p,n) \geq \frac{2^{(p+1)/2}-3}{2^{(p+1)/2}}n + \Theta(1)$.

We conjecture that equality holds. We establish it for $p \leq 6$ in Subsection 3.2. In Subsection 3.3, we give lower bounds for $S^+(n)$ and in Subsection 3.4, we show optimal superselectors for small values of n.

2 Construction of superselectors, selectors and repartitors

2.1 Constructing repartitors with selectors

Lemma 1 $R(n,p,f) \leq S(p,n+f) + S(n-p,n+f) + n + f$

Proof. Let S be a $(p,n+f)$-selector with output-set $\{o_1, o_2, \ldots, o_{n+f}\}$ and S' an $(n-p,n+f)$-selector with output-set $\{o'_1, o'_2, \ldots, o'_{n+f}\}$. Let H be the $(n,p,n+f)$-network constructed from S and S' by replacing each pair $\{o_i, o'_i\}$, $1 \leq i \leq n+f$, by a switch s_i adjacent to an output q_i and the neighbours of o_i and o'_i. See Figure 1. The priorities of H

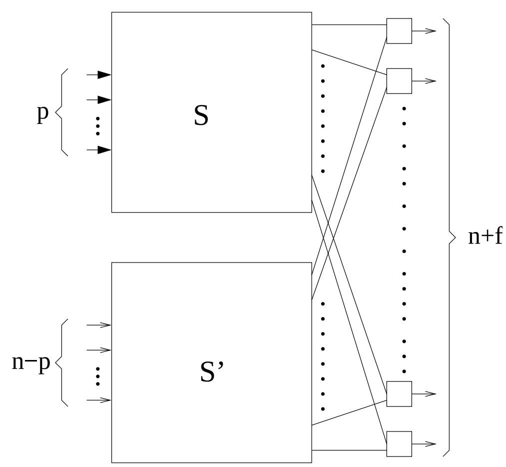

Figure 1: Construction of an (n,p,f)-repartitor from a $(p,n+f)$- and an $(n-p,n+f)$-selector

are the inputs of S and its ordinary inputs the inputs of S'. It is easy to check that H is a $(n,p,0)$-repartitor. Indeed the priorities are routed through S, the ordinary inputs through S' and the switches s_i allow us to select a priority path or an ordinary one. □

From an (n, p, f)-repartitor, one can easily construct a $(p, n+f)$-selector or an $(n-p, n+f)$-selector by removing inputs and outputs. So $S(p, n+f)$ and $S(n-p, n+f)$ are less or equal to $R(n, p, f)$. Hence, if a (n, p, f)-repartitor is constructed from two optimum selectors using the above construction, it has at most $2R(n, p, f)+n+f$ switches. So finding minimum (or almost minimum) selectors will give us fairly small repartitors. Hence in the remaining of the paper, we will focus on selectors.

In order to construct selectors, we first construct more powerful networks: the superselectors.

2.2 Non explicit construction of superselectors

We first give a recursive construction for superselectors. Therefore we need a preliminary lemma due to Pippenger [8].

Let θ be the function from $\mathbb{N}$ into $\mathbb{N}$ defined by $\theta(n) = 4\left\lceil \frac{n}{6} \right\rceil$.

Lemma 2 (Pippenger [8]) *For every n, there is a bipartite graph $Bip_n = (A, B)$ with $|A| = n$ and $|B| = \theta(n)$, in which every vertex of A has outdegree 6, every vertex of B has indegree 9, and, for every $k \leq n/2$ and every set $S \subset$ of k vertices, there exists a matching saturating S.*

Theorem 1

$$S^+(n) \leq 17n + O(\log n)$$

Proof. Let S_n be the $(n, 0, n)$-network defined recursively as follows : If $n \leq 4$, let S_n be the (optimum) superselector defined Subsection 3.4.

If $n \geq 6$, let Bip_n be the bipartite graph defined in Lemma 2 with $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_{\theta(n)}\}$ and $\{i'_j, 1 \leq j \leq \theta(n)\}$ and $\{o'_j, 1 \leq j \leq \theta(n)\}$ be the input and output sets of $S_{\theta(n)}$. Let us construct S_n as follows (See Figure 2) : For $1 \leq l \leq n$, replace the vertex a_l by an path $A_l = (u_l^1, u_l^2, u_l^3)$ incident to the 6 edges of Bip_n incident to a_l (1 incident to u_l^1, 2 incident to u_l^2 and 3 incident to u_l^3) and connect u_l^1 to an input i_l and an output o_l. For $1 \leq l \leq \theta(n)$, replace the vertex b_l by an path $B_l = (v_l^1, v_l^2, v_l^3, v_l^4)$ incident to the 9 edges of Bip_n incident to b_l (3 incident to v_l^1 and 2 to each other switch), identify v_l^4 with i'_l and o'_l. This is possible since i'_l and o'_l are adjacent to the same switch (corresponding to u_l^1) by construction.

Let us prove by induction on n that S_n is an n-superselector.
Let $\mathcal{I}'$ and $\mathcal{O}'$ be sets of inputs and outputs with the same cardinality. Let $I_i = \{j, i_j \in \mathcal{I}'\}$ and $I_o = \{j, o_j \in \mathcal{I}'\}$. For $j \in I_i \cap I_o$, let $P_j = (i_j, a_j^1, o_j)$. Let $J_i = I_i \setminus I_o$ and $J_o = I_o \setminus I_i$. It remains to find paths from the inputs $i_j, j \in J_i$ to outputs $o_j, j \in J_o$.

By Lemma 2, in Bip_n, there is a matching M_i which saturates $\{a_j, j \in J_i\}$ and a matching M_o which saturates $\{a_j, j \in J_o\}$. Note that since J_i and J_o are disjoint $M_i \cap M_o = \emptyset$. For $j \in J_i$, let $g(j)$ be the integer such that $a_j b_{g(j)} \in M_i$ and for $j \in J_o$, let $h(j)$ be the integer such that $a_j b_{h(j)} \in M_o$.

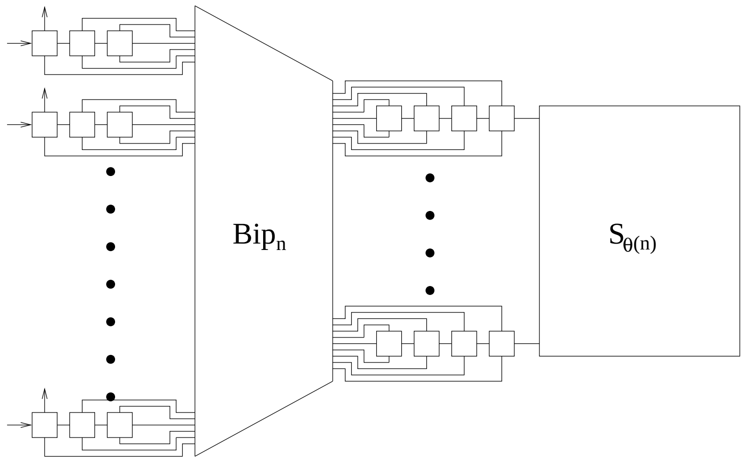

Figure 2: The network S_n

Set $L = g(J_i) \cap h(J_o)$, $L_i = g(J_i) \setminus L$ and $L_o = h(J_o) \setminus L$. Since $S_{\theta(n)}$ is a $\theta(n)$-superselector. There is a set $\mathcal{R}$ of edge-disjoint paths joining $\{v_l^4, l \in L_i\}$ to $\{v_l^4, l \in L_o\}$. For every $l \in L_i$, let R_l be the path of $\mathcal{R}$ joining v_l^4 to a vertex $v_{f(l)}^4$ with $f(l) \in L_o$.

Let $j \in J_i$.

If $g(j) \notin L$, let $j' = h^{-1}[f(g(j))]$. There is a path $Q_j, j \in J_i$ corresponding to $a_j b_{g(j)}$ from i_j to a vertex $v_{g(j)}^4$ and a path Q'_j from $v_{h(j')}^4$ to $o_{j'}$ corresponding to the edge $a_{j'} b_{h(j')}$. Hence $P_j = Q_J R_j Q'_j$ route i_j to $o_{j'}$.

If $g(j) \in L$, let $j' \in J_o$ such that $h(j') = g(j)$. By construction of S_n, there is a path $Q_j, j \in J_i$ corresponding to $a_j b_{g(j)}$ from i_j to a vertex x of $B_{g(j)}$ and a path Q'_j from x to $o_{j'}$ corresponding to the edge $a_{j'} b_{h(j')}$. Hence $P_j = Q_J Q'_j$ route i_j to $o_{j'}$.

Obviously the P_j, $j \in I_i$, are edge disjoint since a path A_j or B_l intersects at most one of them. So they are the desired paths.

Now $|S_n| \leq 3n + 4\theta(n) + |S_{\theta(n)}|$. For $n \geq 4$, define $\theta^0(n) = n$ and $\theta^{t+1}(n) = \theta(\theta^t(n))$. Pick t the smallest integer such that $\theta^{t+1}(n) = 4$. Since $\frac{2}{3} < 1$, then $t = O(\log n)$ since $\frac{2}{3} < 1$. Now applying the above equation $t + 1$ times, we get

$$|S_n| \quad \leq \quad 3n + 7(\theta^1(n) + \cdots + \theta^t(n)) + 16 + S^+(4)$$

It is easy to show by induction on t that $\theta^t(n) \leq \left(\frac{2}{3}\right)^t n + 8$. This implies that

$$|S_n| \leq 17n + 56(t+1) + O(1) \leq 17n + O(\log(n))$$

□

2.3 Explicit construction for superselectors

In order to construct explicitely an n-superselector, we need the following Lemma due to Alon and Capalbo [1]:

Lemma 3 (Alon and Capalbo [1]) *There exists an explicit* 10*-regular bipartite graph* $H_n = (A, B)$ *with* $A = \{a_1, a_2, \ldots, a_n\}$*,* $B = \{b_1, b_2, \ldots, b_n\}$ *and* $n = k \times 2^{l+1}$ *with* $k = 63 \times 64 \times 65$ *such that for any subsets* A_1 *and* A_2 *of* A *of the same cardinality* $|A_1| = |A_2|$*, there exists two matchings* M_1 *and* M_2 *such that:*

(i) M_1 *saturates* A_1 *and* M_2 *saturates* A_2*;*

(ii) Let i *be any integer in* $\{1, 2, \ldots, n/2\}$*. Then if* M_1 *covers* b_i *and* $b_{i+n/2}$ *then* M_2 *covers at least one vertex of* $\{b_i, b_{i+n/2}\}$*. Similarly, if* M_2 *covers* b_i *and* $b_{i+n/2}$ *then* M_1 *covers at least one vertex of* $\{b_i, b_{i+n/2}\}$*.*

Using this bipartite graph and a method similar to Alon and Capalbo's one for superconcentrators, we construct explicitely superselector recursively :

Theorem 2 *There exists an explicit* n*-superselector* $20n + O(1)$ *switches.*

Proof. Let N_n be the network constructed recursively as follows. Take the bipartite graph H_n constructed in Lemma 3 and $N_{n/2}$ with input set $\{i'_1, i'_2, \ldots, i'_{n/2}\}$ and output set and $\{o'_1, o'_2, \ldots, o'_{n/2}\}$.
For $1 \leq j \leq n$, replace the vertices a_j by a path $A_j = (u_j^1, u_j^2, u_j^3, u_j^4, u_j^5)$ incident to the 10 edges of H_n incident to a_j (one incident to u_j^1, two incident to each of u_j^2, u_j^3, u_j^4 and three incident to u_j^5) and connect u_j^1 to an input i_j and an output o_j.
For $1 \leq l \leq n/2$, replace the two vertices b_l and $b_{l+n/2}$ by a path $B_l = (v_l^1, v_l^2, \ldots, v_l^{10})$ incident to the 20 edges of H_n incident to b_l or $b_{l+n/2}$, (Each switch is incident to one edge of incident to b_l and one edge incident to $b_{l+n/2}$). Finally, for $1 \leq l \leq n/2$, remove i'_l and o'_l and connect their common neighbour s'_l in N' with v_l^1 and v_l^{10}. The network N_n is depicted Figure 3.

Let us prove by induction that N_n is an n-superselector.
Let $\mathcal{I}' \subset \mathcal{I}$ and $\mathcal{O}' \subset \mathcal{O}$ be two sets of the same cardinality k. Set $I_1 = \{j, i_j \in \mathcal{I}'\}$, $I_2 = \{j, i_j \in \mathcal{I}'\}$, $I = I_1 \cap I_2$, $J_1 = I_1 \setminus I$ and $J_2 = I_2 \setminus I$. If $j \in I$, route i_j to o_j via $P_j = (i_j, v_j^1, o^j)$. It remains to find paths from $\{i_j, j \in J_1\}$ to $\{o_j, j \in J_2\}$.
Let $A_1 = \{a_j, j \in J_1\}$ and $A_2 = \{a_j, j \in J_2\}$, and M_1 and M_2 be the two matchings constructed as in Lemma 3. For $j \in J_i$, let $g(j)$ be the integer such that an edge e among $a_j b_{g(j)}$ and $a_j b_{g(j)+n/2}$ is in M_1 and let $u'_j v'_j$ the edge of N_n corresponding to the edge e. By construction $u'_j \in V(A_j)$ and $v'_j \in V(B_{g(j)})$. Let Q'_j be the path from i_j to v'_j in $A_j \cup e$. For $j \in J_o$, let $h(j)$ be the integer such that an edge e among $a_j b_{h(j)}$ and $a_j b_{h(j)+n/2}$ is in M_2 and let $u''_j v''_j$ the edge of N_n corresponding to the edge e and Q''_j the path from v''_j to o_j. For $0 \leq a, b \leq 2$, $L_{a,b}$ be set set of integers l such that $|g^{-1}(l)| = a$ and $|h^{-1}(l)| = b$. By the property (ii) of M_1 and M_2, $L_{2,0}$ and $L_{0,2}$ are empty.

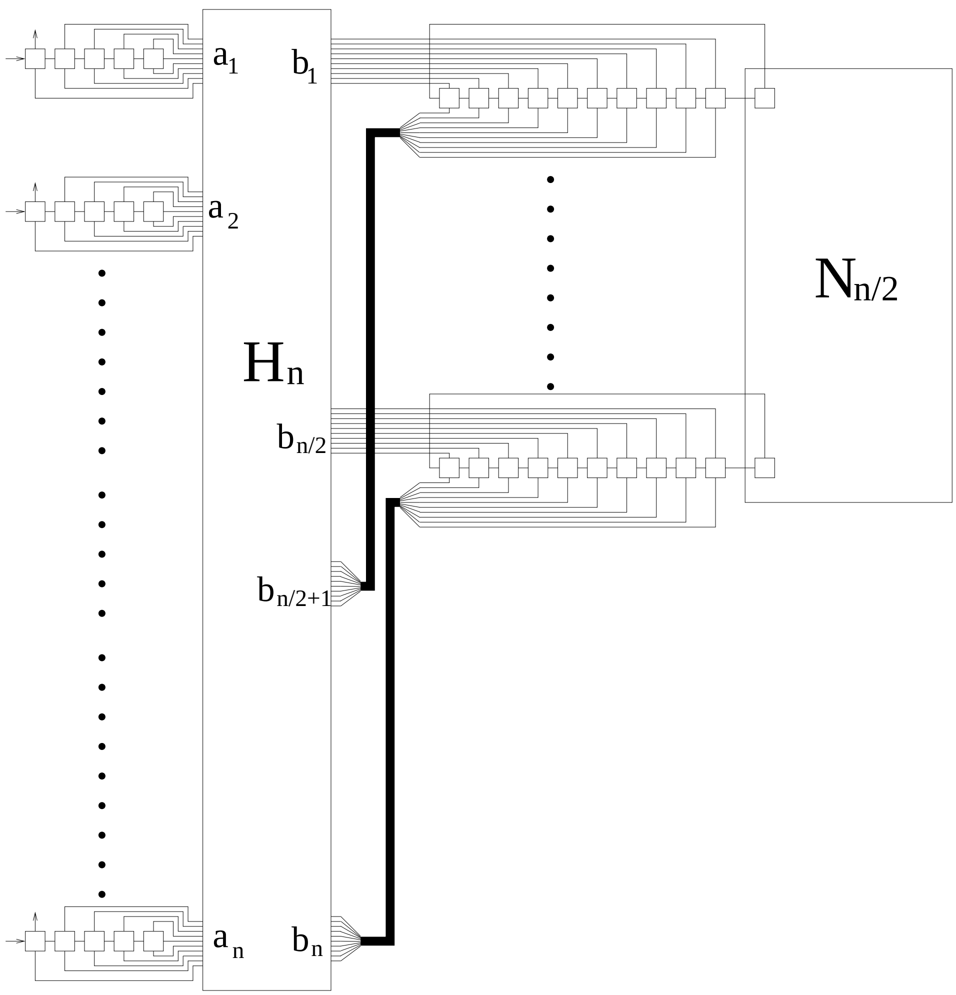

Figure 3: The network N_n.

If $l \in L_{2,2}$, set $\{j,j'\} = g^{-1}(l)$ and $\{k,k'\} = h^{-1}(l)$. In the cycle induced by $V(B_l) \cup s'_l$ there exist two edge-disjoint paths joining $\{v'_j, v'_{j'}\}$ to $\{v''_k, v''_{k'}\}$. Free to exchange j and j' we may assume that a path Q_j goes from v'_j to v''_k and another $Q_{j'}$ goes from $v'_{j'}$ to $v''_{k'}$. Then route i_j to o_k via $P_j = Q'_j Q_j Q''_k$ and $i_{j'}$ to $o_{k'}$ via $P_{j'} = Q'_{j'} Q_{j'} Q''_{k'}$.
If $l \in L_{1,1}$, set $\{j\} = g^{-1}(l)$ and $\{k\} = h^{-1}(l)$. There is a subpath Q_j of B_l which goes from v'_j to v''_k. Then route i_j to o_k via $P_j = Q'_j Q_j Q''_k$.
If $l \in L_{2,1}$, set $\{j,j'\} = g^{-1}(l)$ and $\{k\} = h^{-1}(l)$. In the cycle induced by $V(B_l) \cup s'_l$ there exist two edge-disjoint paths Q_{j_l} from v'_{j_l} to s'_l and the other $Q_{j'_l}$ from v'_{j_l} to v''_k. Route $i_{j'_l}$ to o_k via $P_{j'_l} = Q'_{j'_l} Q_{j'_l} Q''_k$.
If $l \in L_{1,0}$, set $\{j\} = g^{-1}(l)$. There is a subpath Q_j of B_l which goes from v'_j to s'_l.
If $l \in L_{1,2}$, set $\{j\} = g^{-1}(l)$ and $\{k,k'\} = h^{-1}(l)$. In the cycle induced by $V(B_l) \cup s'_l$ there exist two edge-disjoint paths Q_{k_l} from s'_l to v''_{k_l} and the other $Q_{k'_l}$ from v'_j to $v''_{k'_l}$. Route i_j to $o_{k'_l}$ via $P_j = Q'_j Q_{k_l} Q''_{k'_l}$.
If $l \in L_{0,1}$, set $\{k\} = h^{-1}(l)$. There is a subpath Q_k of B_l which goes from s'_l to v''_k.

Now, since $N_{n/2}$ is an $(n/2)$-superselector, in $N_{n/2}$, there is a set $\mathcal{R}$ edge-disjoint paths joining $\{s'_l, l \in L_{2,1} \cup L_{1,0}\}$ to $\{s'_l, l \in L_{1,2} \cup L_{0,1}\}$. For $l \in L_{2,1} \cup L_{1,0}$, let R_l be the path of $\mathcal{R}$ with end s'_l and let $s'_{f(l)}$ be its other end.

For $j \in \{j_l, l \in L_{2,1}\} \cup g^{-1}(L_{1,0})$, let $l = f(g(j))$, and $k = k_l$ if $l \in L_{1,2}$ and $k = h^{-1}(l)$ otherwise. Then route i_j to o_k via $P_j = Q'_j Q_j R_{g(j)} Q''_k$.

Oviously N_n has $|N_{n/2}| + 10n$ switches. So by induction, one can explicitly construct an n-superselector with at most $20n + O(1)$ switches. □

2.4 Applications to selectors and repartitors

Theorem 1 and Proposition 1 yield $S(p,n) \leq 17n + O(\log(n))$. However, better upper bounds may be achieved by constructing a (p,n)-selector from smaller superselectors.

Theorem 3

If $p \geq n/2$ *then* $S(p,n) \leq \frac{n}{2} + 17p + O(\log(n))$.
If $p \leq n/2$ *then* $S(p,n) \leq 17n - 16p + O(\log(n))$.

Proof. Suppose first that $p \geq n/2$. Let $G(n,p)$ be the $(p,0,n)$-network constructed as follows from S_p be the p-superselector constructed in Subsection 2.2 with input set $\{i'_1, \ldots, i'_p\}$ and output set $\{o'_1, \ldots, o'_p\}$. For $1 \leq j \leq \lceil n/2 \rceil$, create a switch t_j connected to an input i_j and two outputs o_j and o_{n+1-j} (except eventually $t_{\lceil n/2 \rceil}$ which is adjacent to a unique output o_j if n is odd), remove i'_j and o'_j and connect t_j to their common neighbour v_j. For $\lceil n/2 \rceil + 1 \leq j \leq p$, let i'_j be the input i_j of G and remove o'_j. See Figure 4 left.

Let us prove that $G(n,p)$ is a (p,n)-selector. Let $\mathcal{I}'$ and $\mathcal{O}'$ be set of inputs and outputs of $G(n,p)$ with the same cardinality. Set $I = \{j, \{o_j, o_{n+1-j}\} \subset \mathcal{O}'\}$, $I_1 = \{j, \{o_j, o_{n+1-j}\} \cap \mathcal{O}' = \{o_j\}\}$, $I_2 = \{j, \{o_j, o_{n+1-j}\} \cap \mathcal{O}' = \{o_{n+1-j}\}\}$, and $J = \{j, \{o_j, o_{n+1-j}\} \cap \mathcal{O}' = \emptyset\} \cup \{\lceil n/2 \rceil + 1, \ldots, p\}$. If $j \in I_1$ then route i_j to o_j via t_j, and $j \in I_2 \cup I$ then route i_j to

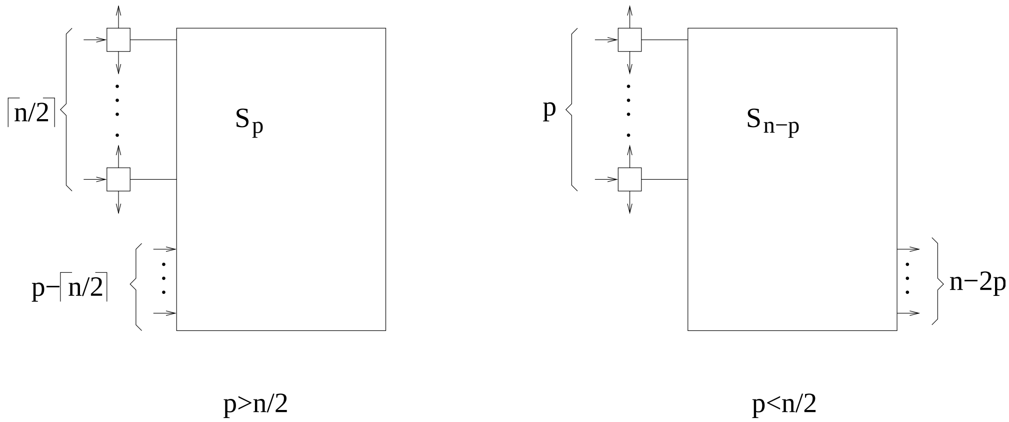

Figure 4: Construction of a (p,n)-selector from S_p or S_{n-p}

o_{n+1-j} via t_j. It remains to route the inputs i_j, $j \in J$ to outputs o_j, $j \in I$. This is possible through S_p because it is a p-superselector and t_j is adjacent to the common neighbour of i'_j and o'_j and then may be identified to each of these two.

Suppose now that $p \geq n/2$. Let $G(n,p)$ be the $(p,0,n)$-network constructed as follows from the $(n-p)$-superselector S_{n-p} constructed in Subsection 2.2 with input set $\{i'_1,\dots,i'_p\}$ and output set $\{o'_1,\dots,o'_p\}$. For $1 \leq j \leq p$, create a switch t_j connected to an input i_j and two outputs o_j and o_{n+1-j}, remove i'_j and o'_j and connect t_j to their common neighbour v_j. For $2p+1 \leq j \leq n-p$, remove i'_j and set $o_j = o'_j$. See Figure 4 right.

Let us prove that $G(n,p)$ is a (p,n)-selector. Let $\mathcal{I}'$ and $\mathcal{O}'$ be set of inputs and outputs of $G(n,p)$ with the same cardinality. Set $I = \{j, \{o_j, o_{n+1-j}\} \subset \mathcal{O}'\} \cup \{2p+1,\dots,n-p\}$, $I_1 = \{j, \{o_j, o_{n+1-j}\} \cap \mathcal{O}' = \{o_j\}\}$, $I_2 = \{j, \{o_j, o_{n+1-j}\} \cap \mathcal{O}' = \{o_{n+1-j}\}\}$, and $J = \{j, \{o_j, o_{n+1-j}\} \cap \mathcal{O}' = \emptyset\}$. If $j \in I_1$ then route i_j to o_j via t_j, and $j \in I_2 \cup I$ then route i_j to o_{n+1-j} via t_j. It remains to route the inputs i_j, $j \in J$ to outputs o_j, $j \in I$. This is possible through S_{n-p} because it is a $(n-p)$-superselector and t_j is adjacent to the common neighbour of i'_j and o'_j and then may be identified to each of these two. □

From Proposition 1 and Theorem 3, we derive the following upper bounds of $R(n,p,f)$.

Corollary 1

If	$p \leq \frac{n-f}{2}$,	*then*	$R(n,p,f) \leq \frac{69}{2}n + \frac{35}{2}f - 33p + O(\log(n+f))$.
If	$\frac{n-f}{2} \leq p \leq \frac{n+f}{2}$	*then*	$R(n,p,f) \leq 18n + 34f + O(\log(n+f))$.
If	$p \geq \frac{n+f}{2}$,	*then*	$R(n,p,f) \leq \frac{3}{2}n + \frac{35}{2}f + 33p + O(\log(n+f))$.

Analogously, to Theorem 3, replacing S_n by N_n in the construction one can explicitly construct (p,n)-selectors with $20n + 19p$ switches if $p \geq n/2$ and $n/2 + 20p$ switches if

$p \geq n/2$. We then can derive (n,p,f)-repartitor with $20n+20f-19p$ if $p \leq \frac{n-f}{2}$, $21n+40f$ switches if $\frac{n-f}{2} \leq p \leq \frac{n+f}{2}$, and $n+20f+19p$ if $p \geq \frac{n+f}{2}$.

3 Some lower bounds and minimum networks

3.1 Lower bounds for selectors

Let W be a set of vertices of a network. We denote by $in(W)$ (resp. $out(W)$, $sw(W)$) the number of inputs (resp. outputs, switches) in W. An edge connecting W and $\overline{W} = V \setminus W$ is said to be W-*cutting*. The set of W-cutting edges is denoted by $\Delta(W)$ and its cardinality is denoted by $deg(W)$.

Proposition 2 *A $(p,0,n)$-network is a (p,n)-selector if and only if for every subset W:*

$$deg(W) \geq \min\{p, out(W)\} - in(W)$$

Proof. Let $\mathcal{O}'$ be a fixed set of p outputs and let $out'(W) = |W \cap \mathcal{O}'|$. A variant of the Ford-Fulkerson Theorem states that the problem is feasible if and only if

$$\forall\ W \subset V : deg(W) \geq demand(W) = out'(W) - in(W).$$

It follows that a (p,n)-network is (p,n)-selector if and only if:

$$\forall\ W \subset V : deg(W) \geq \max\{out'(W) | \mathcal{O}' \text{ set of } p \text{ outputs}\} - in(W).$$

Now $\max\{out'(W)|\mathcal{O}' \text{ set of } p \text{ outputs}\}$ is the maximum number of outputs of W in $\mathcal{O}'$. This maximum is attained either by choosing all the outputs in W to be in $\mathcal{O}'$ if $out(W) \leq p$, or by choosing p outputs in W to be in $\mathcal{O}'$ if $out(W) \geq p$. Hence, $\max\{out'(W)|\mathcal{O}' \text{ set of } p \text{ outputs}\} = \min\{p, out(W)\}$. □

Let $\mathcal{S}_0$ (resp. $\mathcal{S}_1$, $\mathcal{S}_2$) be the set of switches adjacent to no output (resp. one output, two outputs) and s_0 (resp. s_1, s_2) its cardinality.

Let $\mathcal{S}_0^0$ (resp. $\mathcal{S}_0^1$, $\mathcal{S}_0^2$) be the set of switches of $\mathcal{S}_0$ adjacent to no vertex (resp. one vertex, two vertices) of $\mathcal{S}_2$ and s_0^0 (resp. s_0^1, s_0^2) its cardinality.

Let $\mathcal{S}_1^0$ (resp. $\mathcal{S}_1^1$) be the set of switches of $\mathcal{S}_1$ adjacent to no vertex (resp. one vertex) of $\mathcal{S}_2$ and s_1^0 (resp. s_1^1) its cardinality.

Let us define the sets $\mathcal{U}_i$ and $\mathcal{T}_i$ inductively by: $\mathcal{U}_0 = \mathcal{S}_1^1$ and $\mathcal{T}_0 = \mathcal{S}_1^0$. $\mathcal{U}_{i+1}$ (resp. $\mathcal{T}_{i+1}$) is the set of switches of $\mathcal{T}_i$ having exactly one (resp. no) neighbour in $\mathcal{U}_i$. The cardinality of $\mathcal{U}_i$ is denoted u_i.

Let us denote by k_1^0 (resp. k_1^1, k_2) the number of inputs adjacent to $\mathcal{S}_1^0$ (resp. $\mathcal{S}_1^1$, $\mathcal{S}_2$).

From Proposition 2, we prove the following assertions:

Proposition 3 *1. If $p \geq 2$, a switch is adjacent to at most two outputs.*

2. *If $p \geq 3$ then two switches of $\mathcal{S}_2$ are not adjacent.*

3. *If $p \geq 4$, a switch of $\mathcal{S}_1$ is adjacent to at most one switch of $\mathcal{S}_2$.*

4. *If $p \geq 5$, a switch of $\mathcal{S}_0$ is adjacent to at most two switches of $\mathcal{S}_2$.*

5. *If $p \geq 2i+4$, then $(\mathcal{U}_i; \mathcal{T}_i)$ is a partition of $\mathcal{T}_{i-1}$.*

6. *If $p \geq 2i+5$, then any two elements of $\mathcal{U}_i$ are not adjacent.*

7. *If $p \geq i+6$, then any element of $\mathcal{U}_i$ is not adjacent to any element of $\mathcal{S}_0^2$.*

Proof. 1. Suppose that a switch S is adjacent to three outputs. Let W be the set consisting in S and its three adjacent outputs. Then $deg(W) = 1$, $out(W) = 3$ and $in(W) = 0$. So if $p \geq 3$ then W contradicts Proposition 2.

2. Suppose that two switches S and S' of $\mathcal{S}_2$ are adjacent. Let W be the set consisting in S, S' and their four adjacent outputs. Then $deg(W) = 2$, $out(W) = 4$ and $in(W) = 0$. So if $p \geq 3$ then W contradicts Proposition 2.

3. Suppose that a switch S_1 of $\mathcal{S}_1$ is adjacent to two switches of $\mathcal{S}_2$, S_2 and S_2'. Let W be the set consisting in S_1, S_2, S_2' and their five adjacent outputs. Then $deg(W) = 3$, $out(W) = 5$ and $in(W) = 0$. So if $p \geq 4$ then W contradicts Proposition 2.

4. Suppose that a switch S_0 of $\mathcal{S}_0$ is adjacent to three switches of $\mathcal{S}_2$, S_2, S_2' and S_2''. Let W be the set consisting in S_0, S_2, S_2', S_2'' and their six adjacent outputs. Then $deg(W) = 4$, $out(W) = 6$ and $in(W) = 0$. So if $p \geq 5$ then W contradicts Proposition 2.

5. Suppose that a switch T of $\mathcal{T}_{i-1}$ is adjacent to two switches U_{i-1} and U'_{i-1} of $\mathcal{U}_{i-1}$. For $j = 2$ to i, let U_{i-j} (resp. U'_{i-j}) be the swich of $\mathcal{U}_{i-j}$ adjacent to U_{i-j+1} (resp. U'_{i-j+1}). Let S_2 (resp. S_2') be the switch of $\mathcal{S}_2$ adjacent to U_0 (resp. U_0'). Let W be the set consisting in T, U_j, $0 \leq j \leq i-1$, U_j', $0 \leq j \leq i-1$, S_2, S_2' and their adjacent outputs. Then $deg(W) = 2i+3$, $out(W) = 2i+5$ and $in(W) = 0$. So, if $p \geq 2i+4$ then W contradicts Proposition 2.

6. Suppose that two switches U_i and U_i' of $\mathcal{U}_i$ are adjacent. For $j = 1$ to i, let U_{i-j} (resp. U'_{i-j}) be the swich of $\mathcal{U}_{i-j}$ adjacent to U_{i-j+1} (resp. U'_{i-j+1}). Let S_2 (resp. S_2') be the switch of $\mathcal{S}_2$ adjacent to U_0 (resp. U_0'). Let W be the set consisting in U_j, $0 \leq j \leq i$, U_j', $0 \leq j \leq i$, S_2, S_2' and their adjacent outputs. Then $deg(W) = 2i+4$, $out(W) = 2i+6$ and $in(W) = 0$. So, if $p \geq 2i+5$ then W contradicts Proposition 2.

7. Suppose that a switch S_0 of $\mathcal{S}_0^2$ and a switch U_i of $\mathcal{U}_i$ are adjacent. For $j = 1$ to i, let U_{i-j} be the swich of $\mathcal{U}_{i-j}$ adjacent to U_{i-j+1}. Let S_2 be the switch of $\mathcal{S}_2$ adjacent to U_0 and S_2' and S_2'' be the two vertices of $\mathcal{S}_2$ adjacent to S_0. Let W be the set consisting in U_j, $0 \leq j \leq i$, S_2, S_0, S_2' and S_2'' and their adjacent outputs. Then $deg(W) = i+5$, $out(W) = i+7$ and $in(W) = 0$. So, if $p \geq i+6$ then W contradicts Proposition 2.

□

From this Proposition, we deduce the following equations:

Corollary 2
If $p \geq 3$,

$$n = 2s_2 + s_1 \tag{1}$$

$$2s_2 \leq 3s_1 + 4s_0 + p \tag{2}$$

If $p \geq 4$,

$$2s_2 \leq s_1 + 4s_0 + p \tag{3}$$

If $p \geq 5$,

$$2s_2 = s_1^1 + 2s_0^2 + s_0^1 + k_2 \tag{4}$$

If $p \geq 2i + 5$,

$$2s_1^1 + \sum_{j=1}^{i} u_j \leq 3t_i + 3s_0^1 + 4s_0^0 + k_1^1 + k_1^0 \tag{5}$$

If $p \geq 2i + 6$,

$$2s_1^1 + \sum_{j=1}^{i} u_j \leq u_{i+1} + 3s_0^1 + 4s_0^0 + k_1^1 + k_1^0 \tag{6}$$

Proof. Eq. (1) is obtained by counting the number e of edges in $\Delta(\mathcal{O})$. Every output is adjacent to one edge so $e = n$. By Proposition 3, a switch is adjacent to at most two outputs. Then $n = 2s_2 + s_1$.

Eq. (2) and (3) are obtained by counting e_0 the number of edges between $\mathcal{S}_2$ and $\mathcal{S}_1 \cup \mathcal{S}_0 \cup \mathcal{I}$. By Proposition 3-2, a switch in $\mathcal{S}_2$ is adjacent to exactly two vertices of $\mathcal{S}_1 \cup \mathcal{S}_0 \cup \mathcal{I}$. So $e_0 = 2s_2$. A switch of $\mathcal{S}_0$ is adjacent to at most 4 switches in $\mathcal{S}_2$ and a switch of $\mathcal{S}_1$ is adjacent to at most 3 switches in $\mathcal{S}_2$ or at most one switch in $\mathcal{S}_2$ if $p \geq 4$ by Proposition 3-3. So $e_0 \leq 3s_1 + 4s_0 + p$ or if $p \geq 4$, $e_0 \leq s_1 + 4s_0 + p$.

Eq. (4) is also obtained by counting e_0 : if $p \geq 5$, by Proposition 3-3, $\mathcal{S}_0$ is partitionned in $(\mathcal{S}_0^2, \mathcal{S}_0^1, \mathcal{S}_0^0)$.

Eq. (5) and (6) are obtained by counting the number e_i of edges between $\bigcup_{j=0}^{i} \mathcal{U}_j$ and $\mathcal{I} \cup \mathcal{S}_0 \cup \mathcal{T}_i$. There are $2s_1^1$ edges between $\mathcal{S}_1^1$ and $\mathcal{S}_0 \cup \mathcal{S}_1$. And by definition of U_1 and Proposition 3-5, there are u_1 edges between $\mathcal{S}_1^1$ and $\mathcal{U}_1$ and no between $\mathcal{S}_1^1$ and $\bigcup_{j=2}^{i} \mathcal{U}_j$. So there are $2s_1^1 - u_1$ edges between $\mathcal{S}_1^1$ and $\mathcal{T}_i$. Analogously, for every $1 \leq j < i$, there are $2u_j - u_{j+1}$ edges between $\mathcal{U}_j$ and $\mathcal{I} \cup \mathcal{S}_0 \cup \mathcal{T}_i$. If $p \geq 2i+5$, then by Proposition 3-6, there at $2u_i$ edges between $\mathcal{U}_i$ and $\mathcal{I} \cup \mathcal{S}_0 \cup \mathcal{T}_i$. Because there are no edges between $\bigcup_{j=0}^{i} \mathcal{U}_j$ and $\mathcal{S}_0^2$, it follows that $e_i = 2s_1^1 + \sum_{j=1}^{i} u_j$. Moreover, switches of $\mathcal{T}_i \cup \mathcal{S}_0^1$ are adjacent to at most 3 switches not in $\mathcal{S}_2$, switches of $\mathcal{S}_0^0$ have degree at most 4 and $k_1^1 + k_0^1$ inputs are adjacent to switches in $\mathcal{S}_1$. So $e_i \leq 3t_i + 3s_0^1 + 4s_0^0 + k_1^1 + k_1^0$. If $p \geq 2i + 6$, by Proposition 3-4 the only switches of $\mathcal{T}_i$ which are adjacent to vertices of $\bigcup_{j=0}^{i} \mathcal{U}_j$ are those of $\mathcal{U}_{i+1}$ which are adjacent to exactly one by definition. So $e_i \leq u_{i+1} + 3s_0^1 + 4s_0^0 + k_1^1 + k_1^0$.

□

Theorem 4

1. If $p \geq 2p'-1$, $\quad S(p,n) \geq \dfrac{2^{p'+1}-3}{2^{p'+1}}n - \dfrac{2^{p'}-3}{2^{p'+1}}p$

2. If $p \geq 2p'$, $\quad S(p,n) \geq \dfrac{2^{p'}-1}{2^{p'}}n - \dfrac{2^{p'-1}-1}{2^{p'}}p.$

Proof. Since a minimum (p,n)-selector must be connected, it follows that $S(p,n) \geq 1/2(p+n-2)$, hence $S(1,n) \geq \lfloor \frac{n}{2} \rfloor$ and $S(2,n) \geq \lceil \frac{n}{2} \rceil$.

If $p \geq 3$, Eq. (1) + 1/5 Eq. (2) gives $8/5s_2+8/5s_1+4/5s_0 \geq n-\frac{3}{5}$. Thus $S(3,n) \geq \frac{5n}{8}-\frac{3}{8}$.

If $p \geq 4$, Eq. (1) + $\frac{1}{3}$ Eq. (3) gives $4/3s_0+4/3s_1+4/3s_2 \geq n-4/3$. Thus $S(4,n) \geq \frac{3n}{4}-1$.

Suppose now that $p \geq 5$.

1) Set $l = p'-3$.

Eq. (1) + $\dfrac{1}{2^{l+4}-3}\left\{(2^{l+3}-3)\text{Eq. (4)} + \sum_{i=0}^{l-1} 2^{l+1-i}\text{Eq. (6)}[i] + \text{Eq. (5)}[l]\right\}$ yields:

$$\begin{aligned} n \quad \leq \quad & \frac{2^{l+4}}{2^{l+4}-3}\left(s_2+s_1^1+\sum_{i=1}^{l}u_i+t_l\right)+\frac{2^{l+4}-6}{2^{l+4}-3}s_0^2+\frac{7\times 2^{l+1}-12}{2^{l+4}-3}s_0^1+\frac{2^{l+4}-12}{2^{l+4}-3}s_0^0 \\ & +\frac{2^{l+3}-3}{2^{l+4}-3}k_2+\frac{2^{l+2}-4}{2^{l+4}-3}k_1^1+\frac{2^{l+1}-4}{2^{l+4}-3}k_1^0 \end{aligned}$$

Thus $n \leq \frac{2^{l+4}}{2^{l+4}-3}s + \frac{2^{l+3}-3}{2^{l+4}-3}p$.

2) Set $l = p'-3$.

Eq. (1) + $\dfrac{1}{2^{l+3}-1}\left\{(2^{l+2}-1)\text{Eq. (4)} + \sum_{i=0}^{l} 2^{l-i}\text{Eq. (6)}[i]\right\}$ yields:

$$\begin{aligned} n \quad \leq \quad & \frac{2^{l+3}}{2^{l+3}-1}\left(s_2+s_1^1+\sum_{i=1}^{l+1}u_i\right)+t_{l+1}+\frac{2^{l+3}-2}{2^{l+3}-1}s_0^2+\frac{7\times 2^{l}-2}{2^{l+3}-1}s_0^1+\frac{2^{l+3}-4}{2^{l+3}-1}s_0^0 \\ & +\frac{2^{l+2}-1}{2^{l+3}-1}k_2+\frac{2^{l+1}-1}{2^{l+3}-1}k_1^1+\frac{2^{l}-1}{2^{l+3}-1}k_1^0 \end{aligned}$$

Thus $n \leq \frac{2^{l+3}}{2^{l+3}-1}s + \frac{2^{l+2}-1}{2^{l+3}-1}p$.

□

We conjecture that the inequalities obtained in the above corollary are tight:

Conjecture 1 *Let p be a fixed non-negative integer,*

if p is even then $S(p,n) = \dfrac{2^{p/2}-1}{2^{p/2}}n + \Theta(1)$*;*

if p is odd then $S(p,n) = \dfrac{2^{(p+3)/2}-3}{2^{(p+3)/2}}n + \Theta(1)$*.*

3.2 Minimum (p, n)-selectors for p fixed

We now show that Conjecture 1 holds for $p \leq 6$.

Therefore we prove a reinforcement of Proposition 2, which allows us to check the cut criterion only for a certain kind of subsets called *suitable*. A subset is *suitable* if it is connected, with no input and containing all the outputs adjacent to its switches.

Proposition 4 *A (p, n)-network is a (p, n)-selector if and only if $deg(W) \geq \min\{p, out(W)\}$ for any suitable subset W.*

Proof. Suppose that $deg(W) \geq \min\{p, out(W)\}$ for any suitable subset W.

Let us first prove that for any subset X connected with no input then $deg(X) \geq \min\{p, out(X)\}$. Let W be the set obtained from X by adding all the outputs adjacent to a switch of X. Then $deg(W) \leq deg(X)$ and $out(W) \geq out(X)$. So $deg(X) \geq deg(W) \geq \min\{p, out(W)\} \geq \min\{p, out(X)\}$.

Let us prove that for any subset Y with no input then $deg(Y) \geq \min\{p, out(Y)\}$, by induction on the number c of connected component. The result is true if $c = 1$. Suppose now that it is true for c, and suppose Y has $c+1$ connected components. Let C be one of these and $X = Y \setminus C$. We have $deg(Y) = deg(C) + deg(X) \geq \min\{p, out(C)\} + \min\{p, out(X)\}$. Since $out(Y) = out(C) + out(X)$, we obtain $deg(Y) \geq \min\{p, out(Y)\}$.

Let us now prove that for any subset Z, $deg(Z) \geq \min\{p, out(Z)\} - in(Z)$. Let Y be the set obtained from Z by removing all the inputs. We have $deg(Y) \geq deg(Z) - in(Z)$, and $out(Y) = out(Z)$. Now $deg(Y) \geq \min\{p, out(Y)\}$, so $deg(Z) \geq \min\{p, out(Z)\} - in(Z)$. □

Theorem 5

$$\begin{aligned} S(1,n) &= \left\lfloor \frac{n}{2} \right\rfloor \\ S(2,n) &= \left\lceil \frac{n}{2} \right\rceil \end{aligned} \tag{7}$$

Proof. Let P_i be the network consisting of a path $(v_1, v_2, \ldots, v_i)$ of switches and $2i$ outputs $1 \leq o_j \leq 2i$ such that for every $1 \leq j \leq i$, then v_j is adjacent to o_j and o_{i+j}. Let $S_{1,2i+1}$ (resp. $S_{2,2i}$) be the network obtained from P_i by adding an input adjacent to v_1 and an output (resp. an input) adjacent to v_i see Figure 5.

Let W be a suitable subset of $S_{1,2i+1}$. And let j be the smallest integer such that $v_j \in W$. Then v_i is adjacent to an element in $\overline{W}$. Thus $deg(W) \geq 1$. By Proposition 4, it follows that $S_{1,2i+1}$ is a $(1, 2i + 1)$-selector.

Analogously considering j and j' the smallest and largest integer such that v_j is in a suitable subset W of $S_{2,2i}$, we obtain that $deg(W) \geq 2$ for any suitable subset of $S_{2,2i}$. Hence $S_{2,2i}$ is a $(2, 2i)$-selector by Proposition 4.

The network $S_{1,2i}$ (resp. $S_{2,2i-1}$) obtained from $S_{1,2i+1}$ (resp. $S_{2,2i}$) by removing an output is obviously a $(1, 2i)$-selector (resp. $(2, 2i - 1)$-selector). □

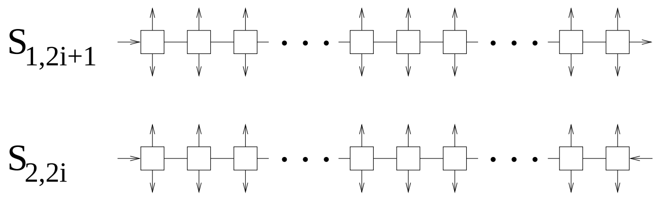

Figure 5: Minimum $(1, 2i+1)$- and $(2, 2i)$-selectors.

Theorem 6

$$S(3,n) \quad = \quad \left\lceil \frac{5n}{8} \right\rceil + \Theta(1)$$

Proof. Let $S_{3,8i+5}$ be the network depicted Figure 6. Let W be a suitable subset of $S_{3,8i+5}$.

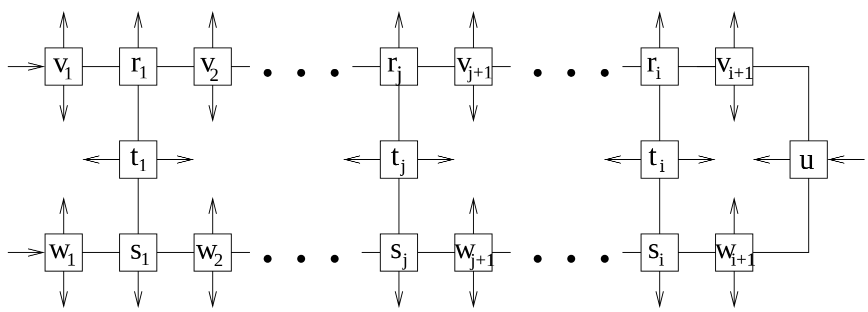

Figure 6: Minimum $(3, n)$-selector

If W contains a unique switch then $deg(W) \geq 2 \geq out(W)$. Set $u = r_{i+1} = s_{i+1}$ and let $R = \{r_j, 1 \leq j \leq i+1\}$ and $S = \{r_j, 1 \leq j \leq i+1\}$. Suppose now that W contains at least two switches. Then because W is connected, it contains an element of $R \cup S$. By symmetry, we may assume that $W \cap R$ is not empty. Let j and j' be the smallest and largest integer such that $r_j \in W$. Then if $v_j \in \overline{W}$ then $v_j r_j \in \Delta(W)$ otherwise $v_j r_{j-1}$ (with r_0 being the input adjacent to v_1) is in $\Delta(W)$. Analogously, if $v_{j'+1} \in \overline{W}$ then $v_{j'+1} r_{j'} \in \Delta(W)$ otherwise $v_{j'+1} r_{j'+2}$ (with r_{i+2} being the input adjacent to u) is in $\Delta(W)$.

Suppose first that $W \cap S \neq \emptyset$. Let j'' be the smallest integer such that $s_{j''} \in W$. There is a cutting edge which is incident to $w_{j''}$. Hence $deg(W) \geq 3$.

Suppose now that $W \cap S = \emptyset$. Then $j \leq i$ and $r_j t_j$ or $t_j s_j$ is in $\Delta(W)$. Again $deg(W) \geq 3$.

Thus by Proposition 4, $S_{3,8i+5}$ is a $(3, 8i+5)$-selector. Obviously, for $1 \leq j \leq 7$, $S_{3,8i+5-j}$ obtained from $S_{3,8i+5}$ by removing j outputs is a $(3, 8i+5-j)$-selector. $\square$

Theorem 7

$$S(4,n) \quad = \quad \frac{3}{4}n + \Theta(1)$$

Proof. Let $S_{4,4i}$ be the network depicted Figure 7.

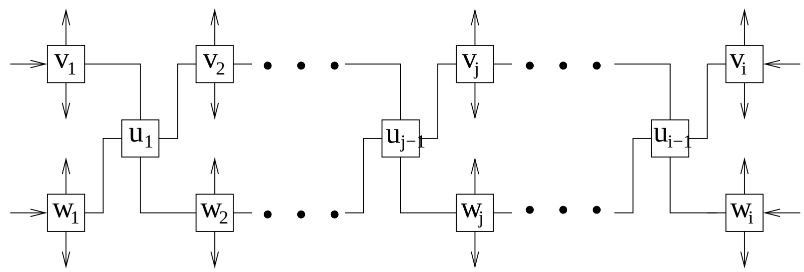

Figure 7: Minimum $(4,n)$-selector

Let W be a suitable subset of $S_{4,4i}$. If W contains a unique switch then $deg(W) \geq 2 \geq out(W)$. Suppose now that it contains at least two switches. Then because W is connected, it contains at least one of the u_j. Let j and j' be the smallest and largest integer such that $u_j, u_{j'} \in W$. Then one of the two edges $v_j u_j$ and $v_j u_{j-1}$ is in $\Delta(W)$ since $u_{j-1} \notin W$ (with u_0 the input adjacent to v_1). Analogously w_j, $v_{j'+1}$ and $w_{j'+1}$ are incident to a W-cutting edge. Hence $deg(W) \geq 4$. Therefore, by Proposition 4, $S_{4,4i}$ is a $(4,4i)$-selector.

Obviously, for $1 \leq j \leq 3$, the network $S_{4,4i-j}$, obtained from $S_{4,4i}$ by removing j outputs is a $(4, 4i-j)$-selector. □

Theorem 8

$$S(5,n) \quad = \quad \frac{13}{16}n + \Theta(1) \tag{8}$$

Proof. It is simple matter to check that the network $S_{5,16i}$ depicted Figure 8 is a $(5,16i)$-selector. For $1 \leq j \leq 15$, the network $S_{5,16i-j}$, obtained from $S_{5,16i}$ by removing j outputs is a $(5, 16i-j)$-selector. □

Theorem 9

$$S(6,n) \quad = \quad \frac{7}{8}n + \Theta(1)$$

Proof. Let $S_{6,8i}$ be the network depicted Figure 9. Set $P_A = (a_1, a_2, \ldots, a_i)$, $P_B = (b_1, b_2, \ldots, b_i)$, $P_C = (c_1, c_2, \ldots, c_i)$ and $P_D = (d_1, d_2, \ldots, d_i)$. Let W be a suitable set of $S_{6,8i}$.

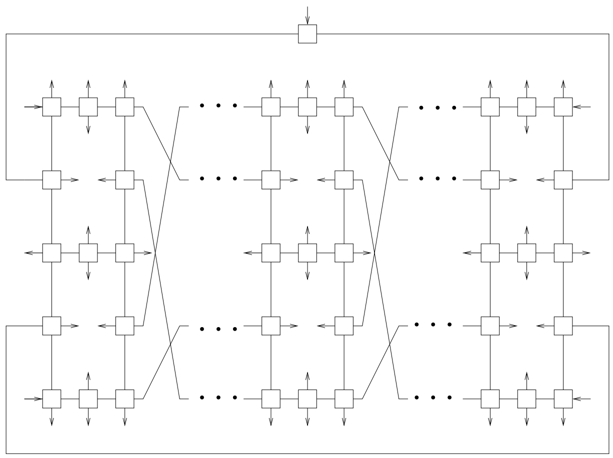

Figure 8: Minimum $(5, n)$-selector

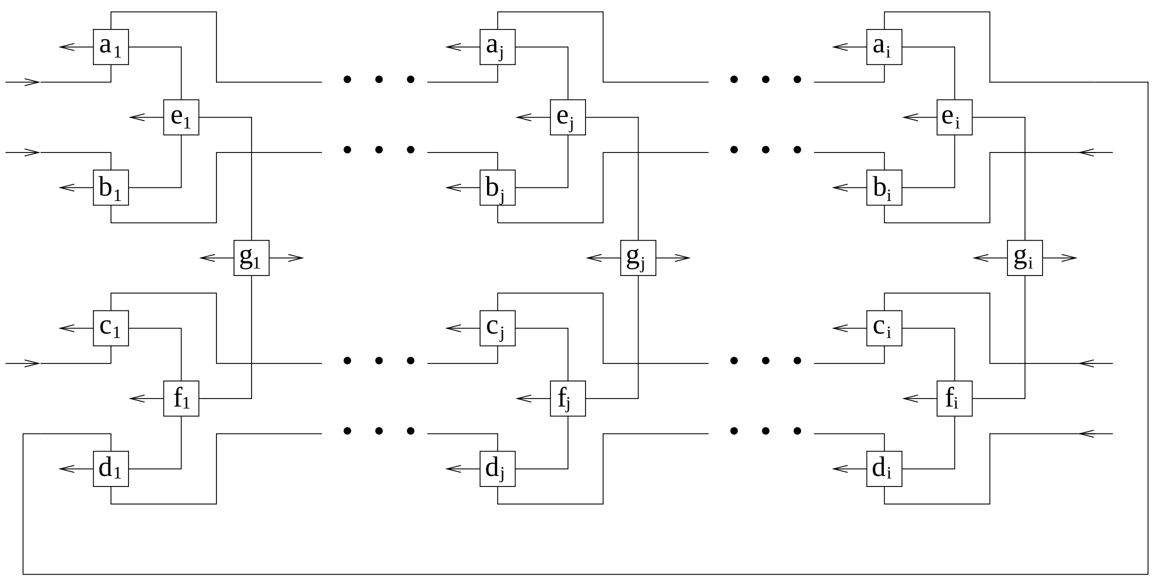

Figure 9: Minimum $(6, n)$-selector

Assume first that W has $sw < 6$ switches. Since there is no cycle of length less than 6 and the distance between to switches of G is at least 6, then W is a tree containing at most one element of G. Thus, $deg(W) \geq 2sw + 2 - out(W)$ and $out(W) \leq sw + 1$. Thus, $deg(W) \geq out(W)$.

Suppose now that W has at least 6 switches. Let us prove that $deg(W) \geq 6$.

Let us consider the paths $P_1 = P_A P_D$, $P_2 = P_B$ and $P_3 = P_C$. For $1 \leq l \leq 3$, if there is a vertex on P_l then P_l contains at least two cutting edges. In particular, if there is a vertex on each path then $deg(W) \geq 6$.

Suppose now that W intersects two paths P_l. Let T_j be the network induced by $\{a_j, b_j, c_j, d_j, e_j, f_j, g_j\}$. Then each T_j containing a vertex of W contains a W-cutting edge. If there at least two such trees, then $deg(W) \geq 6$ because there are at least four cutting edges on the paths. If there only one, it is easy that there are six cutting edges because there are 4 on P_1, (two on P_A and two on P_D). So $deg(W) \geq 6$.

At last, assume that W intersects one of the P_l. For any tree T_j, if $|W \cap T_j| = 1$, there is one cutting edge in T_j, if $|W \cap T_j| \in \{2, 3\}$ there are two cutting edges and if $|W \cap T_j| = 4$, there are three cutting edges. It follows easily that $deg(W) \geq 6$. $\square$

3.3 Lower bounds for superselectors

Proposition 5 *A minimum n-superselector is connected.*

Corollary 3 $S^+(n) \geq n - 1$.

Proof. Since the switches have degree at most four and inputs and outputs degree one, denoting by e the number of edges connecting two switches, we obtain : $4sw \geq 2e + 2n$. Since a minimum superselector is connected $e \geq sw - 1$. It follows $2sw \geq 2n - 2$. $\square$

Proposition 6 *Let $G = (V, E)$ be an n-selector and W a subset of vertices such that $out(W) + in(W) \leq n$. Then $deg(W) \geq \max\{out(W), in(W)\} \geq \frac{out(W)+in(W)}{2}$.*

Proof. Let W be a set such that $out(W) + in(W) \leq n$. Set $\mathcal{O}' = out(W)$ and $\mathcal{I}'$ be a set of $|\mathcal{O}'|$ inputs in $\mathcal{I} \setminus in(W)$. Such a set $\mathcal{I}'$ exists since $out(W) + in(W) \leq n$. A variant of the Ford-Fulkerson Theorem states that the problem is feasible only if

$$deg(W) \geq demand(W) = \mathcal{O}' \cap W - \mathcal{I}' \cap W = out(W)$$

Analgously, setting $\mathcal{I}' = in(W)$ and $\mathcal{O}'$ be a set of $|\mathcal{I}'|$ outputs in $\mathcal{O} \setminus out(W)$, we get $deg(W) \geq in(W)$. $\square$

Proposition 7 *If $n \geq 3$ then a switch is adjacent to at most two elements of $\mathcal{I} \cup \mathcal{O}$.*

Proof. Suppose a vertex S is adjacent to at least three outputs or inputs. Then W the union of S and its adjacent inputs and outputs. Then $deg(W) = 1$ and $in(W)+out(W) = 3$. So if $n \geq 3$ then W contradicts Proposition 6. □

Since there are n inputs and n outputs in an n-superselector we get the following :

Corollary 4 *If $n \geq 3$ then $S^+(n) \geq n$.*

Definition 2 Let $\mathcal{V}_0$ (resp. $\mathcal{V}_1$, $\mathcal{V}_2$) be the set of switches adjacent to no (resp. one, two) elements of $\mathcal{I} \cup \mathcal{O}$ and v_0 (resp. v_1, v_2) its cardinality. By Proposition 7, $(\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2)$ is a partition of $\mathcal{V}$ thus $sw = v_0 + v_1 + v_2$.

Since there are n outputs and n inputs then

$$2n = 2v_2 + v_1 \tag{9}$$

Assume now that $n \geq 6$.

Proposition 8 *If $n \geq 6$, an element of $\mathcal{V}_2$ is adjacent to at most one element of $\mathcal{V}_2$.*

Proof. Suppose that a swicth V_2 of $\mathcal{V}_2$ is adjacent to two switches V_2' and V_2'' of $\mathcal{V}_2$. Let W be the union of V_2, V_2' V_2'' and their adjacent inputs and outputs. Then $deg(W) = 2$ and $out(W) + in(W) = 6$. Hence if $n \geq 6$, W contradicts Proposition 6. □

Corollary 5 *If $n \geq 6$,*

$$v_2 \leq 3v_1 + 4v_0 \tag{10}$$

Proof. Let us count the number e_2 of edges between $\mathcal{V}_2$ and $\mathcal{V}_1 \cup \mathcal{V}_0$. By Proposition 8, every switch of $\mathcal{V}_2$ is adjacent to at least one switch in $\mathcal{V}_1 \cup \mathcal{V}_0$. So $e_2 \geq v_2$. A switch of $\mathcal{V}_1$ (resp. $\mathcal{V}_0$) is adjacent to at most 3 (resp. at most 4) switches, so $e_2 \leq 3v_1 + 4v_0$. □

Corollary 6 *If $n \geq 6$,*

$$S^+(n) \geq \frac{8}{7}n$$

Proof. Eq. (9) + 1/4 Eq. (10) yields: $2n \leq 7/4v_2 + 7/4v_1 + v_0$. Thus $sw \geq 8/7n$. □

Suppose now that $n \geq 7$.

Proposition 9 *If $n \geq 7$, an element of $\mathcal{V}_1$ is adjacent to at most two elements of $\mathcal{V}_2$.*

Proof. Suppose that a switch V_1 of $\mathcal{V}_1$ is adjacent to three switches V_2, V_2 and V_2'' of $\mathcal{V}_2$. Let W be the union of V_1, V_2, V_2' V_2'' and their adjacent inputs and outputs. Then $deg(W) = 3$ and $out(W) + in(W) = 7$. Hence if $n \geq 7$, W contradicts Proposition 6. $\square$

Let $\mathcal{V}_2'$ (resp. $\mathcal{V}_2''$ be the set of switches of $\mathcal{V}_2$ which are adjacent to one (resp. no) switch of $\mathcal{V}_2$, and v_2' (resp. v_2'') its cardinality. By Proposition 8, $\mathcal{V}_2' \cup \mathcal{V}_2'' = \mathcal{V}_2$. So $v_2 = v_2' + v_2''$.

For $i = 0, 1, 2$, let $\mathcal{V}_1^i$ be the set of switches of $\mathcal{V}_1$ which are adjacent to i switch of $\mathcal{V}_2$ and v_1^i its cardinality. By Proposition 9, $\mathcal{V}_1^2 \cup \mathcal{V}_1^1 \cup \mathcal{V}_1^0 = \mathcal{V}_1$, so $v_1 = v_1^2 + v_1^1 + v_1^0$.

Hence Eq. (9) and (10) become

$$2n = 2v_2' + 2v_2'' + v_1^2 + v_1^1 + v_1^0 \tag{11}$$

Corollary 7 *If $n \geq 7$,*

$$v_2' + 2v_2'' \leq 2v_1^2 + v_1^1 + 4v_0 \tag{12}$$

Proof. By counting e_2 the number of edges between $\mathcal{V}_2$ and $\mathcal{V}_1 \cup \mathcal{V}_0$. The result follows from the definitions of $\mathcal{V}_2'$, $\mathcal{V}_2''$ and $\mathcal{V}_1^i$, $0 \leq i \leq 2$ and the fact that a switch of $\mathcal{V}_0$ has degree at most 4. $\square$

Proposition 10 *If $n \geq 7$, a switch of $\mathcal{V}_2'$ is not adjacent to a switch of $\mathcal{V}_1^2$.*

Proof. Suppose that a switch V_2' of $\mathcal{V}_2'$ is adjacent to a switch V_1^2 of $\mathcal{V}_1^2$. Let W_2' be the switch of $\mathcal{V}_2'$ adjacent to V_2' and T_2' the switch of $\mathcal{V}_2$ distinct from V_2' adjacent to V_1^2. Let W be the union of V_2', W_2', T_2' V_1^2 and their adjacent inputs and outputs. Then $deg(W) = 3$ and $out(W) + in(W) = 7$. Hence if $n \geq 7$, W contradicts Proposition 6. $\square$

Corollary 8 *If $n \geq 7$,*

$$v_2' \leq v_1^1 + 4v_0 \tag{13}$$

Corollary 9 *If $n \geq 7$,*

$$S^+(n) \geq \frac{5}{4}n$$

Proof. Eq. (11) + 1/5 Eq. (12) + 1/5 Eq. (13) yields: $2n \leq 8/5(v_2' + v_2'' + v_0 + 7/5(v_1^2 + v_1^1) + v_1^0$. Thus $sw \geq 5/4n$. $\square$

Suppose now that $n \geq 10$.

Proposition 11 *If $n \geq 10$ then an element of $\mathcal{V}_0$ is adjacent to at most two elements of $\mathcal{V}_2'$.*

Proof. Suppose that a switch V_0 of $\mathcal{V}_0$ is adjacent to three switches V_2, W_2, T_2 of $\mathcal{V}_2'$. Let V_2', (resp. W_2') be the neighbour of V_2 (resp. W_2) in $\mathcal{V}_2'$. Let W be the set consisting in V_0, V_2, V_2', W_2, W_2', T_2' and their adjacent inputs and outputs. $deg(W) = 4$ and $in(W) + out(W) = 10$. So if $n \geq 10$, then W contradicts Proposition 6. $\square$

Corollary 10 *If* $n \geq 10$

$$v'_2 \leq v_1^1 + 2v_0 \tag{14}$$

Corollary 11 *If* $n \geq 10$*,*

$$S^+(n) \geq \frac{4}{3}n$$

Proof. Eq. (11) + 1/4 Eq. (12) + 1/4 Eq. (14) yields: $2n \leq 3/2(v'_2 + v''_2 + v_0 + v_1^2 + v_1^1) + v_1^0$. Thus $sw \geq 4/3n$. □

For $i = 0, 1, 2$, let $\mathcal{V}_0^i$ be the set of switches of $\mathcal{V}_0$ which are adjacent to i switch of $\mathcal{V}'_2$ and v_0^i its cardinality. By Proposition 11, $\mathcal{V}_0^2 \cup \mathcal{V}_0^1 \cup \mathcal{V}_0^0 = \mathcal{V}_0$, so $v_0 = v_0^2 + v_0^1 + v_0^0$.

Proposition 12 *If* $n \geq 10$*, two elements of* $\mathcal{V}_1^2$ *are not adjacent.*

Proof. Suppose that a switch S of $\mathcal{V}_1^2$ is adjacent to a switch S' of $\mathcal{V}_1^2$. Let V_2 and W_2 be the two neighbours of S in $\mathcal{V}_2$ and V'_2 and W'_2 be the two neighbours of S' in $\mathcal{V}_2$. Let W be the set consisting in S, V_2, V'_2, S', V'_2, W'_2 and their adjacent inputs and outputs. $deg(W) = 4$ and $in(W) + out(W) = 10$. So if $n \geq 10$, then W contradicts Proposition 6. □

3.4 Minimum n-superselectors

Proposition 13

$$\begin{aligned} i) &\quad S^+(1) = 0 \\ ii) &\quad S^+(2) = 1 \end{aligned}$$

Proof.

i) Obvious.

ii) The network S_2 consisting of a switch adjacent to two inputs and two outputs is a 2-superselector. Thus $S^+(2) \leq 1$. And by Corollary 3 $S^+(2) \geq 1$.

□

The following lemma immediatly follows from the definition of superselector:

Lemma 4 *Let* G *be an* n*-superselector with a switch* s *adjacent to an input* i*, an ouput* o *and two switches* s_1 *and* s_2*. Then the network* G' *obtained by removing* s*,* i*,* o *and adding the link* s_1s_2 *is an* $(n-1)$*-superselector.*

Proposition 14 *If* $3 \leq n \leq 5$ *then* $S^+(n) = n$*.*

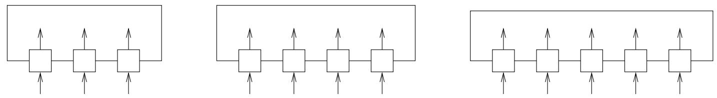

Figure 10: Minimum n-superselector for $3 \leq n \leq 5$

Proof. By Corollary 4, $S^+(n) \geq n$, if $n \geq 3$.

For $3 \leq n \leq 5$, let S_n be the network consisting of a cycle of n switches s_j, $1 \leq j \leq n$ each of which is adjacent to an input i_j and an output o_j. (see Figure 10). Let us prove that S_5 is a 5-superselector. Let $\mathcal{I}'$ and $\mathcal{O}'$ be subset of $\mathcal{I}$, and $\mathcal{O}$ respectively such that $|\mathcal{I}'| = |\mathcal{O}'| = k \leq 5$. If the input i_j and the output o_j are in $\mathcal{I}'$ and $\mathcal{O}'$ then route i_j to o_j directly through s_j. It remains at most two inputs to route and one can do it using the cycle.

By Lemma 4, S_4 and S_3 are 4- and 3-superselectors. □

Proposition 15 $S^+(6) = 7$

Proof. By Corollary 6, $S^+(6) \geq 48/7$ so $S^+(6) \geq 7$. Now it is simple matter to check that the network depicted Figure 11 is a 6-superselector. □

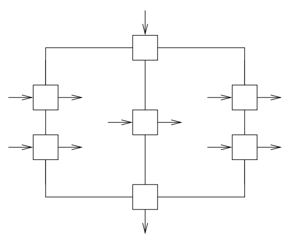

Figure 11: Minimum 6-superselector

Proposition 16 $S^+(7) = 9$ *and* $S^+(8) = 10$*.*

Proof. By Corollary 9, $S^+(7) \geq 35/4$, so $S^+(7) \geq 9$ and $S^+(8) \geq 10$.

Let us prove that S_8, the network depicted Figure 12 right, is a 8-superselector. For $1 \leq j \leq 8$, set $P_j = (i_j, s_j, o_j)$ and $Q_j = (i_j, s_j, s_{j+4}, o_{j+4})$. ($j + 4$ must be understood modulo 8). Let $\mathcal{I}'$ and $\mathcal{O}'$ be sets of k inputs and k outputs of S_8. If $i_j \in \mathcal{I}'$ and $o_j \in \mathcal{O}'$ then route i_j to o_j through P_j. Let $\mathcal{I}'' = \{i_j \in \mathcal{I}', o_j \notin \mathcal{O}'\}$ and $\mathcal{O}'' = \{o_j \in \mathcal{O}', i_j \notin \mathcal{I}'\}$. It remains to route inputs of $\mathcal{I}''$ to outputs of $\mathcal{O}''$ and obviously $|\mathcal{I}''| = |\mathcal{O}''| \leq 4$. If $i_j \in \mathcal{I}''$

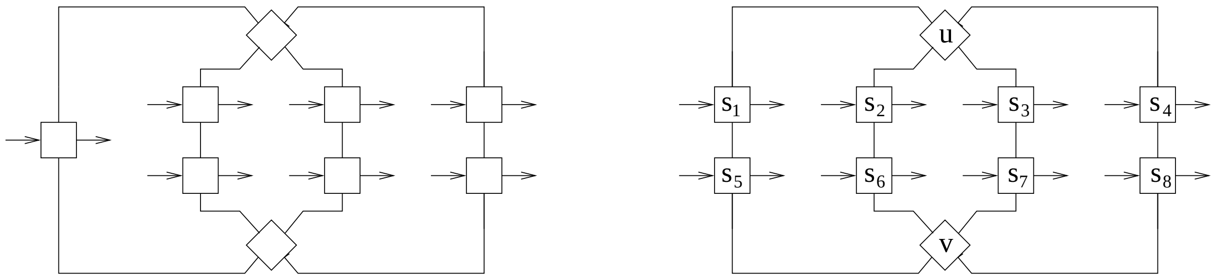

Figure 12: Minimum 7- and 8-superselector

and $o_{j+4} \in \mathcal{O}''$ then route i_j to o_{j+4} through Q_j. Let $\mathcal{I}^3 = \{i_j \in \mathcal{I}', o_j \notin \mathcal{O}'$ and $o_{j+4} \notin \mathcal{O}'\}$ and $\mathcal{O}'' = \{o_j \in \mathcal{O}', i_j \notin \mathcal{I}'\}$ and $i_{j+4} \notin \mathcal{I}'\}$. It remains to route inputs of $\mathcal{I}^3$ to outputs of $\mathcal{O}^3$ and obviously $|\mathcal{I}^3| = |\mathcal{O}^3| \leq 4$. Hence this can be done through u and v.

By Proposition 4, the network depicted Figure 12 left is a 7-superselector. □

Proposition 17

$$S^+(9) = 12$$

Proof. By Corollary 9, $S^+(9) \geq 45/4$, so $S^+(9) \geq 12$.

Let us prove that S_9, the network depicted Figure 13, is a 9-superselector. For $1 \leq j \leq 6$,

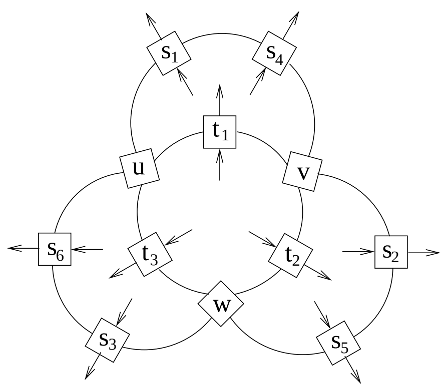

Figure 13: Minimum 9-superselector

set $P_j = (i_j, s_j, o_j)$, $Q_j = (i_j, s_j, s_{j+3}, o_{j+3})$. ($j+3$ must be understood modulo 6) and for $7 \leq j \leq 9$, set $P_j = (i_j, t_{j-6}, o_j)$.

Let $\mathcal{I}'$ and $\mathcal{O}'$ be sets of k inputs and k outputs of S_9. If $i_j \in \mathcal{I}'$ and $o_j \in \mathcal{O}'$ then route i_j to o_j through P_j. It remains then at most 4 inputs to route to four outputs. If $i_j \in \mathcal{I}'$, $o_j \notin \mathcal{O}'$, $i_{j+3} \notin \mathcal{I}'$ and $o_{j+3[\text{mod } 6]} \in \mathcal{O}'$ then route i_j to o_{j+3} through Q_j. It is now easy to see that we may route the remaining inputs to the remaining outputs through u, v and w. □

4 Conclusion and open problems

In Section 2.4, we obtain linear upper bounds for $R(n, p, f)$. However these bounds are far to be satisfactory because they seems far to be optimal and also because their variations are "opposite" to the variations of $R(n, p, f)$. Indeed, according to the value of $R(n, p, f)$ for small n, p and f and the fact that $R(n, p, f) = R(n, n-p, f)$, we conjecture that as a function of p, $R(n, p, f)$ is unimodal and maximum for $p = n/2$.

Conjecture 2
If $p \leq n/2$ then $R(n, p-1, 0) \leq R(n, p, 0)$.
If $p \geq n/2$ then $R(n, p, 0) \leq R(n, p+1, 0)$.

But our upper bound is minimum for $p = n/2$ which is a bit bizarre. Note that Conjecture 2 implies $R(n, p, f) \leq 18n + 34f + O(log((n+f))$ for every p.

It seems that the cornerstone of the problem is to find the smallest constant α such that $R(n, \lceil n/2 \rceil, 0) \leq \alpha n$. If Conjecture 2 is true then $R(n, p, 0) \leq \alpha n$ for every p. Moreover, even if Conjecture 2 turns out to be false, if would imply that $R(n, p, 0) \leq 2\alpha n$ for every p. Indeed, if $p \leq n/2$, one can construct an $(n, p, 0)$-repartitor from an $(n, \lceil n/2 \rceil, 0)$-repartitor and an $(\lfloor n/2 \rfloor, p, 0)$-repartitor. (See Figure 14). Hence, $R(n, p, 0) \leq R(\lfloor n/2 \rfloor, p, 0) + R(n, \lceil n/2 \rceil, 0)$.

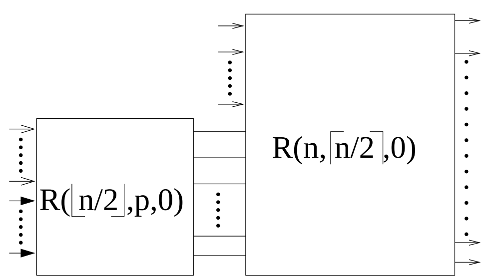

Figure 14: Construction of an $(n, p, 0)$-repartitor from an $(n, \lceil n/2 \rceil, 0)$-repartitor and an $(\lfloor n/2 \rfloor, p, 0)$-repartitor.

Applying this formula recursively with the equality $R(n, p, 0) = R(n, n-p, 0)$, we obtain :

$$R(n, p, 0) \leq \sum_{i=1}^{\log(n)} R(\left\lfloor \frac{n}{2^i} \right\rfloor, \left\lceil \frac{n}{2^{i+1}} \right\rceil, 0)$$

A good manner to improve the upper bounds on $R(n, p, f)$ would be to improve the upper bounds for $S(p, n)$ obtained in Section 2.4. Indeed these bounds are far to be tight and a lot greater than the lower bounds given in Section 3.2 or [6]. A problem is to close the gap between the lower bounds and the upper bounds.

It would also be interesting to get some results on the shape of $S(p, n)$. We conjecture that, for any fixed integer n, $S(p, n)$ increases until an integer $\alpha_n \geq n/2$ and then decreases. In particular, we conjecture :

Conjecture 3 If $p \leq n/2$ then $S(p-1, n) \leq S(p, n)$.

If true, this conjecture would imply $S(p, n) \leq S(n/2, n) \leq 9n + O(log(n))$ if $p \leq n/2$ which is a better upper bound as ours.

One can easily prove a weaker statement than Conjecture 3 :

Proposition 18

$$S(p-1, n-1) \leq S(p, n)$$

Proof. The network obtained from a (p, n)-selector by removing an input and an output is trivially a $(p-1, n-1)$-selector. □

A way to improve the upper bounds on $S(p, n)$ and so those on $R(n, p, f)$ is to improve the bounds on $S^+(n)$. It would also be interesting to close the gap $\frac{4}{3}n \leq S^+(n) \leq 17n + O(log(n))$.

References

[1] N. Alon and M. Capalbo, Smaller explicit superconcentrators. *Internet Math.* **1** (2) (2003), 151–163.

[2] L. A. Bassalygo, Asymptotically optimal switching circuits, *Problemy Pederachi Informatsii*, **17** (1981), 81–88.

[3] B. Beauquier, and E. Darrot. Arbitrary size Waksman networks and their vulnerability. *Parallel Processing Letters*, **12** (3-4) (2002), 287–296.

[4] J.-C. Bermond, E. Darrot, and O. Delmas. Design of fault tolerant on-board networks in satellite. *Networks*, **40** (4) (2002), 202–207.

[5] J.-C. Bermond, F. Havet, and D. Tóth. Fault tolerant on-board networks with priorities. *Networks*, submitted. INRIA research report RR-5363.

[6] J.-C. Bermond, S. Pérennes, and D. Tóth. On the design of fault tolerant flow networks, part I. Manuscript.

[7] F. R. K. Chung, On concentrators, superconcentrators, generalizers, and nonblocking networks, *Bell System Tech. J.*, **58** (1979), 1765–1777.

[8] N. Pippenger, Superconcentrators, *SIAM J. Comput.,* **6** (1977), 298–304.

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