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Abstract: In a WDM network, routing a request consists in assigning it a route in the physical network and a wavelength. If each request uses at most $1/C$ of the bandwidth of the wavelength, we will say that the grooming factor is C. That means that on a given edge of the network we can groom (group) at most C requests on the same wavelength. With this constraint the objective can be either to minimize the number of wavelengths (related to the transmission cost) or minimize the number of Add Drop Multiplexers (shortly ADM) used in the network (related to the cost of the nodes). We consider here the case where the network is a path on N nodes, P_N. Thus the routing is unique. For a given grooming factor C minimizing the number of wavelengths is an easy problem, well known and related to the load problem. But minimizing the number of ADM's is NP-complete for a general set of requests and no results are known. Here we show how to model the problem as a graph partition problem and using tools of design theory we completely solve the case where $C = 2$ and where we have a static uniform all-to-all traffic (one request for each pair of vertices).

Key-words: Traffic grooming, graph, design theory, WDM

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Groupage de trafic sur le chemin

Résumé : Dans un réseau WDM, le routage d'une requête correspond à l'attribution d'un chemin dans le réseau physique et d'une longueur d'onde. Si chaque requête utilise au plus une fraction $1/C$ de la bande passante offerte par une longueur d'onde, alors il est possible de partager cette bande passante entre C requêtes. Le paramètre C est alors appelé facteur de groupage. Avec cette contrainte, l'objectif peut être soit de minimiser le nombre de longueurs d'ondes (coût de transmission) ou le nombre total de multiplexeurs à insertion/extraction (ADM) utilisés dans le réseau (coût de nœuds).

Ici, nous étudions le cas où le réseau physique est le chemin à N nœuds P_N. Le routage est alors unique et le problème de minimiser le nombre de longueurs d'onde, pour un facteur de groupage donné C, est un problème facile à résoudre. Par contre le problème de minimiser le nombre d'ADMs a été montré NP-complet pour les instances générales et aucun autre résultat n'a été proposé à ce jour.

Nous modélisons le problème de la minimisation du nombre d'ADMs comme un problème de partition des arêtes d'un graphe. Puis, en utilisant des outils de la théorie des designs, nous résolvons le cas de l'instance all-to-all lorsque le facteur de groupage est $C = 2$.

Mots-clés : Groupage de trafic, graphes, théorie des configurations, WDM

Contents

1 Introduction

Traffic grooming is the generic term for packing low rate signals into higher speed streams (see the surveys [15, 24, 26]). By using traffic grooming, one can bypass the electronics in the nodes for which there is no traffic sourced or destinated to it. Typically, in a WDM network, instead of having one SONET Add Drop Multiplexer (shortly ADM) on every wavelength at every node, it may be possible to have ADMs only for the wavelength used at that node (the other wavelengths being optically routed without electronic switching).

In the past many papers on WDM networks had for objective to minimize the transmission cost and in particular the number of wavelengths to be used [10, 1, 13]; recent research has focused on reducing the total number of ADMs used in the network, trying to minimize it.

Here, we consider the particular case of paths (the routing is unique) with static uniform all-to-all traffic (one request for each pair of vertices).

To each request $\{i, j\}$ routed on the path from i to j, we want to assign a wavelength in such a way that at most C requests use the same wavelength on a given edge of the path. Equivalently, each request uses $1/C$ of the bandwidth of the wavelength. C is called the *grooming ratio* (or *grooming factor*). For example, if the request from i to j is one OC-12 and a wavelength can carry an OC-48, the grooming factor is 4. Given the grooming ratio C and the length N of the path, the objective is to minimize the total number of (SONET) ADMs used, denoted $A(P_N, C)$, and so reducing the network cost by eliminating as many ADMs as possible from the "no grooming case".

Figure 1 shows how to groom requests for a grooming factor $C = 2$ and a path P_N with $N = 3, 7, 9$ vertices. For $N = 7$ we have 21 requests. So, a priori, if we give one wavelength to each request we need 42 ADMs. Using the same wavelength for disjoint requests (case $C = 1$), we will see after that 33 ADMs suffice. Indeed two requests may share an ADM if they have a common extremity. For $C = 2$ we will see that the construction given in Figure 1 is optimal and uses 6 wavelengths and 20 ADMs (note that 4 requests share the same ADM in vertex 2).

To the best of our knowledge, the problem for paths has only been studied in [12], where it has been proved NP-complete for a general set of requests (and for given $C \geq 2$) and no other results are known. Other topologies have also been considered and in particular unidirectional rings primarily in the context of variable traffic requirements [8, 14, 19, 27, 29]; but the case of fixed traffic requirements has also been widely studied [2, 3, 5, 6, 15, 17, 18, 21, 22, 24, 28, 30].

In this paper we model the grooming problem on the path as a graph partition problem. We show how a greedy algorithm gives a solution for $C = 1$ and any set of requests. Then, using tools of design theory, we determine exactly the number of ADMs in the case $C = 2$ for the all-to-all set of requests.

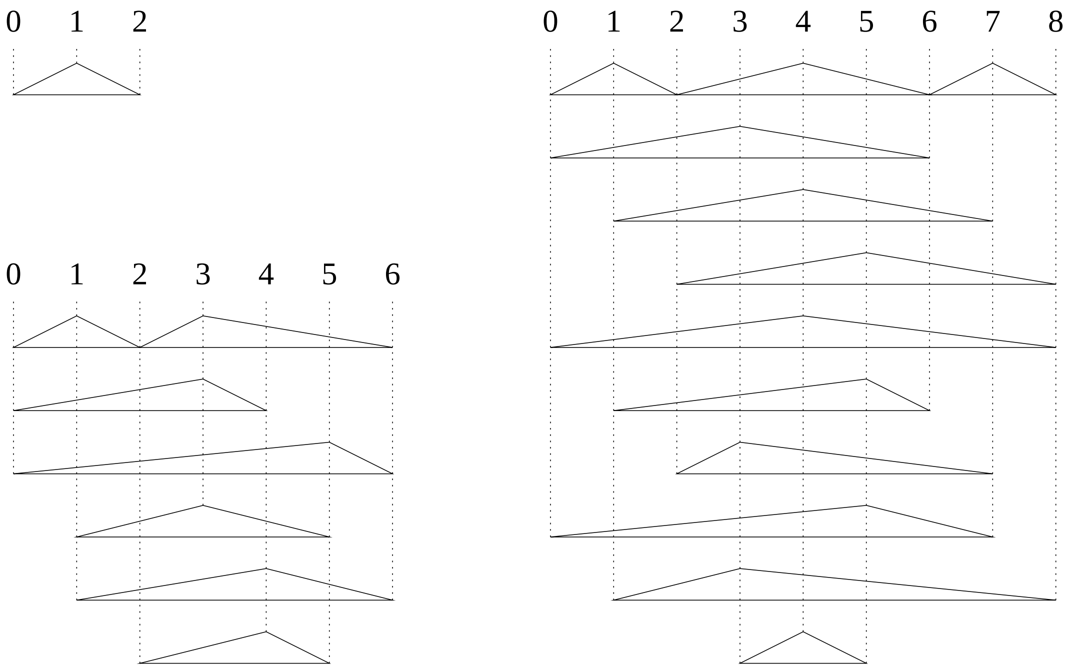

Figure 1: Constructions for $N = 3, 7$ and 9.

2 Modeling

Here we are given a physical graph and a set of requests. The physical graph will be the path P_N with vertex set $V = \{0, 1, 2, \ldots, N-1\}$ and where the edges are the pairs $\{i, i+1\}$, $0 \le i \le N-2$.

The set of requests I is a set of pairs $\{u, v\}$ that we model by a graph $G = (V, E)$ where each edge $e = \{u, v\}$ is associated to the request $\{u, v\}$. Each request is routed along the unique subpath from u to v and we associate to it a wavelength w.

For a subgraph B of requests of G, we define the load of an edge $e = \{i, i+1\}$ of P_N, $L(B, e)$, as the number of requests which are routed through e, that is the number of edges $\{u, v\}$ of B such that $u \le i < v$.

Now let $B_w = (V_w, E_w)$ be the subgraph of G containing all requests carried by wavelength w. The fact that the grooming ratio is C can be expressed as $L(B_w, e) \leq C$ for each edge e of P_N. The number of ADMs used for the wavelength w is nothing else than $|V_w|$.

So the problem corresponds to partition the edges of G (set of requests) into subgraphs B_w (set of requests with wavelength w) such that $L(B_w, e) \leq C$.

It is straightforward to see that minimizing the number W of wavelengths needed to route all requests is equivalent to minimize the number of subgraphs in the partition. Furthermore this is an easy problem since the load $L(G, e)$ is easy to compute. For example if G is the complete graph K_N, $L(K_N, \{i, i+1\}) = (i+1)(N-i-1)$. If $L_{\max}(G)$ is the maximum load over all the edges, $L_{\max}(G) = \max_{e \in P_N} L(G, e)$, then we need at least $\frac{L_{\max}(G)}{C}$ wavelengths and we can assign them in a greedy way. For the complete graph, the number of wavelengths is therefore:

Proposition 2.1 *For the all-to-all set of requests on the path P_N and grooming ratio C, the minimum number of wavelengths needed is $\left\lceil \frac{N^2-\varepsilon}{4C} \right\rceil$, where $\varepsilon = 1$ when N is odd and 0 otherwise.*

Proof: We have $L_{\max}(K_N) = \max_{e \in P_N} L(K_N, e) = \max_{\{i,i+1\}=e \in P_N} (i+1)(N-i-1) = \left\lceil \frac{N^2-\varepsilon}{4} \right\rceil$, where $\varepsilon = 1$ when N is odd and 0 otherwise. □

Here our objective is to minimize the number of ADMs, that is the sum of the number of vertices in the B_w. Thus the problem can be formalized as follows:

Problem 2.2 (Grooming problem on the path)

Inputs :	*a path P_N, a grooming ratio C and a set of requests I modeled by the graph $G = (V, E)$.*
Output :	*a partition of the edges of G into subgraphs $B_w = (V_w, E_w)$, $w = 1, \ldots, W$, such that load$(B_w, e) \leq C$ for each edge e of P_N.*
Objective :	*minimize $\sum_{1 \leq w \leq W} |V_w|$.*

W mainly consider here $G = K_N$ and, following [5], we will denote $A(P_N, C)$ the optimal number of ADMs for a grooming ratio C and the all-to-all set of requests on the path.

We have formalized the problem in its undirected version, but for paths it is the same for directed or symmetric directed versions. Indeed, if we consider a dipath $\overrightarrow{P_N}$ where the arcs are from i to $i+1$, and if the requests are the couples (u, v), with $u < v$, the problem is exactly the same. If we consider a symmetric dipath P_N^* with arcs $(i, i+1)$ and $(i+1, i)$ and the requests are the couples (u, v), we can split the problem into 2 disjoint subproblems, one with the dipath $\overrightarrow{P_N}$ oriented from 0 to $N-1$ with all requests (u, v) with $u < v$, and the second on the dipath $\overleftarrow{P_N}$ oriented from $N-1$ to 0 with requests (u, v) with $v < u$.

To the best of our knowledge, this problem has only been studied in [12] where it has been proved NP-complete, and no other results are known. However, the grooming problem for rings has been extensively studied. For example in [5] we have shown that the grooming problem on the unidirectional ring can be formalized as follows:

Problem 2.3 (Grooming problem on the cycle)

Inputs : *a number of nodes N and a grooming ratio C.*
Output : *a partition of the edges of K_N into subgraphs $B_w = (V_w, E_w)$, $w = 1, \dots, W$, such that $|E_w| \le C$.*
Objective : *minimize* $\sum_{1 \le w \le W} |V_w|$.

Let us denote $A(C_N, C)$ the optimal number of ADMs for a grooming ratio C and all-to-all set of requests on the unidirectional ring.

Note that in Problem 2.3, for the ring, it is supposed that the two requests (u, v) and (v, u) are assigned to the same wavelength (using thus $1/C$ of the capacity of the wavelength). Clearly, a bound on the number of ADMs for unidirectional ring gives a bound for our problem, but there might be very different (for example $A(C_3, 2) = 5$ but $A(P_3, 2) = 3$) due to capacity constraints.

In fact, the problem for unidirectional rings corresponds to the problem of path "*without erasure*" [12]. In this model a request (u, v) uses $1/C$ of the bandwidth on the whole path and not only on the subpath between u and v. The "load condition" becomes: there are at most C requests in any subgraph B_w which is exactly the constraint of Problem 2.3.

We will show in the next section that the grooming problem on the path with erasure for $C = 1$ and general instances can be solved polynomially, which is not the case on the ring (in the erasure model) [25, 27, 16].

3 Grooming ratio $C = 1$

When the grooming ratio is equal to 1, the grooming problem on the path can be solved optimally for any set of requests in polynomial time. We prove this in Theorem 3.1 and give the exact number of ADMs in the all-to-all case in Corollary 3.2.

Theorem 3.1 $A(P_N, G, 1) = \sum_{i=0}^{N-1} \max\left\{d_G^-(i), d_G^+(i)\right\}$.

Proof: The lower bound is simple since in each node i of the path P_N we can not do better than sharing an ADM between a request ending in this node, that is a request $\{u, i\}$ with $u < i$, and a request starting from it, that is $\{i, v\}$ with $i < v$. Thus $A(P_N, G, 1) \geq \sum_{i=0}^{N-1} \max\left\{d_G^-(i), d_G^+(i)\right\}$.

Now, note that it is always possible to put a request ending in node i and a request starting from i in a same subgraph. Thus we can form the subgraphs using a greedy process: scan the nodes of the path from 0 to $N-2$ and add to each subgraph containing a request ending in i a requests starting from i (if any left), and then create a new subgraph for each remaining request that start from i (if any). So, in each node i, we will use $\max\left\{d_G^-(i), d_G^+(i)\right\}$ ADMs and so the lower bound is attained.

Finally, one may remark that this process will create more subgraphs than necessary, but we can merge two subgraphs if they contains disjoint requests. Doing so we will use the optimal number of subgraphs. □

Corollary 3.2 $A(P_N, 1) = \frac{3N^2-2N-\epsilon}{4}$*, where* $\epsilon = 1$ *when* N *is odd and 0 otherwise .*

The corollary follows from the fact that $d_G^-(i) = i$ and $d_G^+(i) = N-1-i$. Another simple construction is the following. We have $A(P_2, 1) = 2$ and $A(P_3, 1) = 5$. Now let the vertices of P_N be $0, 1, \ldots, N-1$; arrange them in this order, and suppose that $A(P_N, 1) = (3N^2 - 2N - \varepsilon)/4$, where $\varepsilon = 1$ when N is odd and 0 otherwise. Let now the vertices of P_{N+2} be $x, 0, 1, \ldots, N-1, y$ and arrange them in this order. The subgraphs of the partition of K_{N+2} will be: the N subgraphs B_j, $0 \leq j \leq N-1$, each of them containing the edges $\{x, j\}$ and $\{j, y\}$, and so $|V(B_j)| = 3$; the subgraph B_N which contains only the edge $\{x, y\}$, and so $|V(B_N)| = 2$; and the subgraphs of the partition of K_N. So altogether the partition of K_{N+2} contains $2+3N+(3N^2-2N-\varepsilon)/4 = \left(3(N+2)^2 - 2(N+2) - \varepsilon\right)/4$, where $\varepsilon = 1$ when N is odd and 0 otherwise (see Figure 2 for an example).

When the grooming ratio is $C \geq 2$, the problem is NP-complete and difficult to approximate for general instance. In particular, when the grooming ratio is equal to $C = 2$, this problem is similar to partition the edges of G into the maximum number of K_3 (see [11, 20]), although such partition only provides an upper bound of the total number of ADMs (two K_3 may share an ADM). However, for $G = K_N$ we will give in the next sections the exact number of ADMs for $C = 2$.

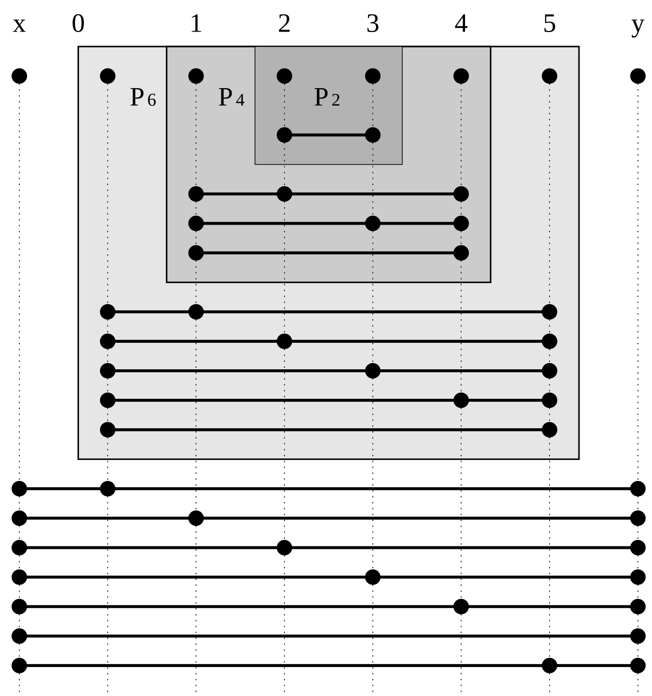

Figure 2: Optimal construction for $A(P_8, 1)$ using the construction for $A(P_6, 1)$.

4 Lower bounds

Consider a valid construction for the Problem 2.2 and let a_p denote the number of subgraphs of the partition with exactly p nodes, A the number of ADMs, and W the number of subgraphs of the partition. We have the following equalities:

$$A = \sum_{p=2}^{N} p a_p \quad (1)$$

$$\sum_{p=2}^{N} a_p = W \quad (2)$$

$$\sum_{w=1}^{W} |E_w| = |E| \quad (3)$$

In the particular case where $G = K_N$, we know by Proposition 2.1 that $W \geq \left\lceil \frac{N^2-\varepsilon}{4C} \right\rceil$, where $\varepsilon = 1$ when N is odd and 0 otherwise, and we have $|E| = \frac{N(N-1)}{2}$.

To obtain accurate lower bounds we need to bound the value of $|E_w|$ for a graph with $|V_w| = p$ vertices, satisfying the load constraint. Let $\gamma(C, p)$ be this maximum number of edges. The determination of $\gamma(C, p)$ is a challenging problem. In a first version of this paper we conjectured that we have to take the edges of smallest length (distance on the path); that corresponds to the intuition that, in order to satisfy the maximum number of requests, one has to choose the smallest ones. This conjecture is true for $C = 1$, as $\gamma(1, p) = p - 1$. We will see that it is true also for $C = 2$, where $\gamma(2, p) = \left\lfloor \frac{3p-3}{2} \right\rfloor$. It is also true for $C = 3$, where $\gamma(3, p) = p - 1 + p - 2 = 2p - 3$ obtained by taking all the edges of length 1 and 2. However, this conjecture is not true in general and has been disproved in [4], were is given a closed formula for $\gamma(C, p)$. For example when $C = \frac{s(s+1)}{2}$ and $p > s(s-1)$ then $\gamma(C, p) = sp - C$.

Equations 2 and 3 become

$$\sum_{p=2}^{N} a_p \geq \left\lceil \frac{N^2 - \varepsilon}{4C} \right\rceil \quad (4)$$

$$\sum_{p=2}^{N} a_p \gamma(C, p) \geq \frac{N(N-1)}{2} \quad (5)$$

For example when $C = 3$ and using the value $\gamma(3,p) = 2p - 3$ we obtain

$$\sum_{p=2}^{N} (2p-3)a_p \geq \frac{N(N-1)}{2} \tag{6}$$

that is

$$2A(P_N, 3) \geq \frac{N(N-1)}{2} + 3\left\lceil \frac{N^2 - \varepsilon}{12} \right\rceil \tag{7}$$

In what follows we will restrict ourselves to the case $C = 2$, which is already non immediate and for which we have been able to obtain exact values. To obtain the right lower bounds when N is even, we need to determine $\gamma(2,p,2h)$ which is the maximum number of edges of a graph B with p vertices with at least $2h$ vertices of odd degree and such that $L(B,e) \leq 2$ for each edge of P_N. Note that $\gamma(2,p) = \gamma(2,p,0)$.

We will denote by $G + H$ the graph obtained by merging the right most node of G with the left most node of H.

Lemma 4.1 $\gamma(2,p,2h) = \left\lfloor \frac{3p-3-h}{2} \right\rfloor$

Proof: We prove the lemma by induction. It is true for $p = 2$ as a graph with two vertices has at most one edge. In that case $h = 1$ and we have equality. For $p = 3$ the maximum number of edges is 3, obtained with a K_3, and there is equality for $h = 0$. With $h = 1$, the graph has at most 2 edges and the equality is attained with a P_3. Similarly for $p = 4$, the graph has at most 4 edges. Let the vertices be $\{a,b,c,d\}$ with $a < b < c < d$. For $h = 0$ the equality is attained for example with the graph C_4 consisting of the 4 edges $\{a,b\}$, $\{b,c\}$, $\{c,d\}$ and $\{a,d\}$; for $h = 1$ equality is attained with the graph consisting of an edge joined by a vertex to a K_3 more precisely the 4 edges $\{a,b\}$, $\{b,c\}$, $\{c,d\}$ and $\{b,d\}$; and for $h = 2$ equality is attained with a 3-star $K_{1,3}$ $\{a,b\}$, $\{b,c\}$ and $\{b,d\}$.

Now consider a graph B with p vertices and $2h$ vertices of odd degree. Let $m(B)$ be the number of edges of B, and let u_0 be the first vertex of B (in the order of the path).

1. If u_0 has degree 1, $B - \{u_0\}$ has at least $2h - 2$ vertices of degree 1 and therefore $m(B) \leq \gamma(2, p-1, 2h-2) + 1 = \left\lfloor \frac{3p-3-h}{2} \right\rfloor$
2. If u_0 is of degree 2, let u_1 and u_2 be the 2 neighbors of u_0, with $u_0 < u_1 < u_2$. As $L(B, \{u_1 - 1, u_1\}) \leq 2$ there is no edge $\{u, u_1\}$ with $u < u_1$, and as $L(B, \{u_1, u_1 + 1\}) \leq 2$ there is at most one edge $\{u_1, v\}$ with $v > u_1$.

(a) If there is no edge $\{u_1, v\}$, the graph obtained from B by deleting u_0 and u_1 has at least $2h-2$ vertices of odd degree and so $m(B) \leq \gamma(2, p-2, 2h-2)+2 = \left\lfloor \frac{3p-4-h}{2} \right\rfloor$.

(b) If there is an edge $\{u_1, v_1\}$ 3 subcases can appear.

   i. either $v_1 = u_2$ and the graph obtained from B by deleting u_0 and u_1 (and therefore the K_3 $\{u_0, u_1, v_1\}$) has the same number of vertices of odd degree as B and so $m(B) \leq \gamma(2, p-2, 2h) + 3 = \left\lfloor \frac{3p-3-h}{2} \right\rfloor$.

   ii. or $v_1 < u_2$. Due to the load constraint there is no edge $\{u, v_1\}$ with $u < v_1$ and at most one edge $\{v_1, v\}$ with $v_1 < v$. The graph obtained from B by deleting u_0, u_1, v_1 has at least $2h-2$ vertices of odd degree and 3 or 4 edges less than B. So $m(B) \leq \gamma(2, p-3, 2h-2) + 4 = \left\lfloor \frac{3p-3-h}{2} \right\rfloor$.

   iii. or $v_1 > u_2$ we do the same reasoning by deleting from B the vertices u_0, u_1, u_2 and we obtain $m(B) \leq \left\lfloor \frac{3p-3-h}{2} \right\rfloor$.

So in all cases the bound is proved. Furthermore a careful analysis indicates when the bound is attained. An optimal $(p, 2h)$ graph can be obtained either by adding an edge joined to a vertex of even degree of a $(p-1, 2h-2)$ optimal graph (case 1); or by adding two edges $\{a, b\}$ and $\{a, c\}$ with $a < b < c$, c being a vertex of even degree of an optimal $(p-2, 2h-2)$ graph with $p+h$ even (case 2.a); or by adding a K_3 joined to a vertex of an optimal $(p-2, 2h)$ graph (case 2.b.i); or by adding a C_4 joined to a vertex of an optimal $(p-3, 2h)$ graph (careful analysis of case 2.b.iii).

In particular when p is odd and $h = 0$, the optimal graph is unique and consists of a sequence of $\frac{3p-3}{6}$ K_3's sharing two by two a vertex ($K_3 + K_3 + \cdots + K_3$). □

For any h, equality is attained with the graph consisting of $\frac{3p-3-3h}{6}$ K_3s and h edges merged in the following way $e + K_3 + e + K_3 + \cdots + K_3 + e + K_3 + K_3 + \cdots + K_3$ (with $p \geq h$, and p odd when h even and p even when h odd).

Theorem 4.2

- $A(P_N, 2) \geq \left\lceil \frac{11N^2-8N-3}{24} \right\rceil$ *when N is odd*
- $A(P_N, 2) \geq \left\lceil \frac{N(N-1)}{3} + \left\lceil \frac{N^2}{8} \right\rceil + \frac{N}{6} \right\rceil$ *when N is even.*

Proof: By Lemma 4.1 we know that $|E_w| \leq \gamma(2, p_w, 2h_w) \leq \frac{3p_w-3-h_w}{2}$ for a B_w with p_w vertices and $2h_w$ vertices with odd degree. So

$$\sum_{w=1}^{W} |E_w| \leq \sum_{p=2}^{N} \frac{3p-3}{2} a_p - \sum_{w=1}^{W} \frac{h_w}{2} \tag{8}$$

If N is odd, $\sum_{w=1}^{W} h_w$ can be equal to 0, but when N is even all vertices of K_N being of odd degree, $\sum_{w=1}^{W} 2h_w \geq N$. So Equation 1 and Inequalities 4 and 5 become Equation 9 and Inequalities 10 and 11, where $\varepsilon = 1$ if N is odd and $\varepsilon = 0$ otherwise.

$$A = \sum_{p=2}^{N} p a_p \tag{9}$$

$$\sum_{p=2}^{N} a_p \geq \left\lceil \frac{N^2 - \varepsilon}{8} \right\rceil \tag{10}$$

$$\sum_{p=2}^{N} \frac{3p-3}{2} a_p - (1-\varepsilon)\frac{N}{4} \geq \frac{N(N-1)}{2} \tag{11}$$

Thus Inequality 11 becomes

$$\sum_{p=2}^{N} 3p a_p \geq N(N-1) + 3\sum_{p=2}^{N} a_p + (1-\varepsilon)\frac{N}{2} \tag{12}$$

and so

$$A(P_N, 2) \geq \frac{N(N-1)}{3} + \left\lceil \frac{N^2 - \varepsilon}{8} \right\rceil + (1-\varepsilon)\frac{N}{6} \tag{13}$$

When N is odd, we have $\varepsilon = 1$ and so $A(P_N, 2) \geq \frac{11N^2 - 8N - 3}{24}$, and when N is even, we have $\varepsilon = 0$ and so $A(P_N, 2) \geq \left\lceil \frac{N(N-1)}{3} + \left\lceil \frac{N^2}{8} \right\rceil + \frac{N}{6} \right\rceil$

□

5 Constructions for $C = 2$

5.1 3-GDD

Let $v_1, v_2, \ldots, v_l$ be non negative integers; the *complete multipartite graph with group sizes* $v_1, v_2, \ldots, v_l$ is defined to be the graph with vertex set $V_1 \cup V_2 \cup \cdots \cup V_l$ where $|V_i| = v_i$, and two vertices $u \in V_i$ and $v \in V_j$ are adjacent if $i \neq j$. Using terminology of Design Theory, the graph of type $p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_l^{\alpha_l}$ will be the complete multipartite graph with α_i groups of size p_i. The existence of a partition of this multipartite graph into K_k is equivalent to the existence of a k*-GDD (Group Divisible Design) of type* $p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_l^{\alpha_l}$.

Here we are interested in the existence of 3-GDD's, that is partitions into K_3's.

Theorem 5.1 (Existence of a 3-GDD (see [9])) *There exists a 3-GDD of type* $p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_l^{\alpha_l}$ *if and only if (i) each node of the complete multipartite graph has even degree, and (ii) the number of edges is a multiple of 3.*

Various constructions are explained in [23]. One can find in [9] a collection of multipartite graphs for which there exists a 3-GDD. For example when the total number of nodes is 22, there exists 3-GDDs of type $6^1 4^4$, $6^3 4^1$, $8^1 6^1 4^1 2^2$ and $10^1 2^6$. Some other values are given in Theorem 5.2.

5.2 Constructions for small values of N

We have reported in Table 1 the number $A(P_N, 2)$ of ADMs and the number W of subgraphs of optimal constructions for some small cases. Direct constructions for the value that cannot be obtained in the following constructions are given in Appendix A.

N	2	3	4	5	6	7	8	9	10	11	12	13	16	17	20
$A(P_N, 2)$	2	3	7	10	16	20	28	34	45	52	64	73	115	127	180
W	1	1	2	3	5	6	8	10	13	15	18	21	32	36	50

Table 1: Number of ADMs and number of subgraphs in small cases

5.3 Constructions for odd values

In this section we show that the lower bound is always attained for odd N. To prove that, we use the 3-GDD described in Theorem 5.2 from which we deduce a generic construction in Corollary 5.3. Finally, we show in Theorem 5.4 that the bound is reached for all odd values.

Theorem 5.2 (1.26 page 190 of [9]) *Let u and v be positive integer with $v \leq u$. Then a 3-GDD of type $u^1 v^1 1^u$ exists if and only if $(u, v) \equiv (1,1), (3,1), (3,3), (3,5), (5,1) \pmod{(6,6)}$.*

Corollary 5.3 *Given u and v satisfying the condition of Theorem 5.2 and an optimal construction for both u and v, we can build an optimal construction for $N = 2u + v$.*

Proof: Let the nodes of K_N be numbered from left to right $0, 1, \ldots, u-1, u, \ldots, u+v-1, \ldots, 2u+v-1 = N$ and let $A = \{0, 1, \ldots, u-1\}$, $B = \{u, u+1, \ldots, u+v-1\}$ and $C = \{u+v, u+v+1, \ldots, 2u+v-1\}$.

The examples of Figure 1 for $N = 7$ (resp. $N = 9$) are obtained with this construction using $u = 3$ and $v = 1$ (resp. $v = 3$).

The 3-GDD of type $u^1 v^1 1^u$ has $\frac{3u^2-u+4uv}{6}$ K_3, and we say that the K_3s are of type ABC or ACC or CCC depending of their number of nodes in A, B and C. There are uv K_3 of type ABC, $\frac{u(u-v)}{2}$ K_3 of type ACC and $\frac{u(v-1)}{6}$ K_3 of type CCC.

Each node of A is the left most node of $v + \frac{u-v}{2} = \frac{u+v}{2}$ K_3 of type ABC or ACC. Since each node of A is the right most node of at most $\frac{u-1}{2}$ subgraphs of the decomposition of K_u, we can merge each of the $\frac{u^2-1}{8}$ subgraphs with one K_3 and so we save $\frac{u^2-1}{8}$ ADMs.

Each node of C is the right most node of v K_3 of type ABC. It is also involved in $u-v$ K_3 of type ACC and in $\frac{u-1-(u-v)}{2} = \frac{v-1}{2}$ K_3 of type CCC. Thus we can merge each K_3 of type CCC with a K_3 of type ABC and so we save $\frac{u(v-1)}{6}$ more ADMs.

Note that since each node of B is the middle node of a K_3 of type ABC, we can not merge the subgraphs of the partition of K_v.

Thus, the number of ADMs used in this construction is

$$\frac{3u^2 - u + 4uv}{2} + A(P_u, 2) - \frac{u^2 - 1}{8} - \frac{u(v-1)}{6} + A(P_v, 2) \tag{14}$$

Since for $w = u$ or v, we have $A(P_w, 2) = \frac{11w^2-8w-3}{24} + \varepsilon_w$, where $\varepsilon_w = \frac{1}{3}$ for $w \equiv 5 \pmod{6}$ and 0 otherwise, Equation 14 become

$$\begin{aligned} & \frac{3u^2 - u + 4uv}{2} + \frac{11u^2 - 8u - 3}{24} + \varepsilon_u \\ & -\frac{u^2 - 1}{8} - \frac{u(v-1)}{6} + \frac{11v^2 - 8v - 3}{24} + \varepsilon_v \\ = \quad & \frac{11(2u+v)^2 - 8(2u+v) - 3}{24} + (\varepsilon_u + \varepsilon_v) \end{aligned} \tag{15}$$

Finally, if $(u, v) \equiv (1,1),(3,1),(3,3) \pmod{(6,6)}$, then we have $\varepsilon_u = \varepsilon_v = 0$ and we obtain the lower bound, and if $(u, v) \equiv (3,5)$ or $(5,1) \pmod{(6,6)}$, then $2u + v \equiv 5 \pmod 6$ but $\varepsilon_u + \varepsilon_v = \frac{1}{3}$ and we get again the lowerbound.

Note that, as expected, the number of subgraphs in the partition is

$$\frac{3u^2 - u + 4uv}{6} - \frac{u(v-1)}{6} + \frac{v^2 - 1}{8} \quad = \quad \frac{(2u+v)^2 - 1}{8} \tag{16}$$

□

We can now prove that the bound is attained for all odd values.

Theorem 5.4 *When N is odd, $A(P_N, 2) = \left\lceil \frac{11N^2 - 8N - 3}{24} \right\rceil$. Furthermore, the construction contains $\frac{N^2-1}{8}$ subgraphs.*

Proof: For $N = 3, 5, 13, 17$ we give direct constructions in Lemmas A.1, A.3, A.11 and A.13. For other values we will use Corollary 5.3 using induction on u.

- When $N = 12t + 1$, $t \geq 2$, let $u = 6t - 3$ and $v = 7$. Since $(6t - 3, 7) \equiv (3,1) \pmod{(6,6)}$, we can use Corollary 5.3.
- When $N = 12t + 3$, $t \geq 0$, we can use Corollary 5.3 with $u = 6t + 1$ and $v = 1$
- When $N = 12t + 5$, $t \geq 3$, we can use Corollary 5.3 with $u = 6t - 3$ and $v = 11$, and for $t = 2$, that is $N = 29$ we can use Corollary 5.3 with $u = 11$ and $v = 7$
- When $N = 12t + 7$, $t \geq 0$, we can use Corollary 5.3 with $u = 6t + 3$ and $v = 1$
- When $N = 12t + 9$, $t \geq 0$, we can use Corollary 5.3 with $u = 6t + 3$ and $v = 3$.
- When $N = 12t + 11$, $t \geq 1$, we can use Corollary 5.3 with $u = 6t + 3$ and $v = 5$. Finally, we can also use Corollary 5.3 for $N = 11$ with $u = 5$ and $v = 1$

□

5.4 Construction for even values

In view of the lower bound, an optimal partition will have exactly $\left\lceil \frac{N^2}{8} \right\rceil$ subgraphs and each vertex will appear once with odd degree and otherwise the value $\frac{3p-3}{2}$ is attained. So we will have mainly K_3's, plus $\frac{N}{2}$ graphs $K_3 + e$ (except for some congruence classes where one edge is isolated) some of these K_3's or $K_3 + e$ being merged together.

Lemma 5.5 *There exists a 3-GDD of type* $(2u)^1(2v)^1 2^u$ *when* $u \geq v \geq 1$ *and* $u(v-1) \equiv 0 \pmod 3$.

Proof: To deduce the lemma from Theorem 5.1, one has to check that all nodes have even degree (which is true) and that the total number of edges $4u^2 + 4uv + 4uv + 4\frac{u(u-1)}{2} = 6u^2 + 6uv + 2u(v-1)$ is a multiple of 3 which follows from $u(v-1) \equiv 0 \pmod 3$. □

Theorem 5.6 *When* N *is even,* $A(P_N, 2) = \left\lceil \frac{N(N-1)}{3} + \left\lceil \frac{N^2}{8} \right\rceil + \frac{N}{6} \right\rceil = \frac{11N^2-4N}{24} + \varepsilon_N$, *where* $\varepsilon_N = \frac{1}{2}$ *when* $N \equiv 2$ *or* $6 \pmod{12}$, $\varepsilon_N = \frac{1}{3}$ *when* $N \equiv 4 \pmod{12}$, $\varepsilon_N = \frac{5}{6}$ *when* $N \equiv 10 \pmod{12}$, *and 0 when* $N \equiv 0$ *or* $8 \pmod{12}$. *Furthermore, the construction contains* $\left\lceil \frac{N^2}{8} \right\rceil$ *subgraphs.*

Proof: First of all, the theorem is true for $N = 2, 4, 8, 12, 16, 20$ by Lemmas A.1, A.2, A.6, A.10, A.12 and A.14 (see Appendix A).

Now suppose that the result is true for $2u$ and $2v$, that is for $w = u$ or v,

$$A(P_{2w}, 2) = \left\lceil \frac{2w(2w-1)}{3} + \left\lceil \frac{4w^2}{8} \right\rceil + \frac{2w}{6} \right\rceil = \frac{44w^2 - 8w}{24} + \varepsilon_w \tag{17}$$

where $\varepsilon_w = \frac{1}{2}$ when $2w \equiv 2$ or $6 \pmod{12}$, $\varepsilon_w = \frac{1}{3}$ when $2w \equiv 4 \pmod{12}$, $\varepsilon_w = \frac{5}{6}$ when $2w \equiv 10 \pmod{12}$, and 0 otherwise. Furthermore, the number of subgraph is $\left\lceil \frac{4w^2}{8} \right\rceil$.

Let now $N = 4u + 2v$, where u and v are such that there exists a 3-GDD of type $(2u)^1(2v)^1 2^u$. Let also the nodes be $A \cup B \cup C_1 \cup C_2 \cup \cdots \cup C_u$ with $|A| = 2u$, $|B| = 2v$ and $|C_i| = 2$, $1 \leq i \leq u$, and let $C = \cup_{i=1}^{u} C_i$.

To simplify the notation, we say that an edge is of type CC if it has one node in C_i and another in C_j with $i \neq j$.

The 3-GDD of type $(2u)^1(2v)^1 2^u$ has $\frac{6u^2-2u+8uv}{3}$ K_3: $4uv$ of type ABC, $\frac{2u(2u-2v)}{2} = 2u(u-v)$ of type ACC and $\frac{2u(v-1)}{3}$ of type CCC.

We observe that each node of C is the right most node of $2v$ K_3 of type ABC and is involved in $2u - 2v$ K_3 of type ACC and $v - 1$ K_3 of type CCC. Thus, we can merge each K_3 of type CCC with a K_3 of type ABC and so save $\frac{2u(v-1)}{3}$ ADMs. Furthermore, we can merge each edge $\{c_i^1, c_i^2\}$ such that $c_i^1, c_i^2 \in C_i$, $1 \leq i \leq u$, with a K_3 of type ABC or ACC and so save u more ADMs.

Each node of A is the left most node of $2v + u - v = u + v$ K_3 of type ABC or ACC and is the right most node of at most $\frac{2u-2}{2} + 1 = u$ subgraphs of the optimal construction for $2u$. Thus we can merge each subgraph and save $\left\lceil \frac{4u^2}{8} \right\rceil$ more ADMs.

By hypothesis we have

$$A(P_{2u},2) - \left\lceil \frac{4u^2}{8} \right\rceil = \left\lceil \frac{2u(2u-1)}{3} + \frac{2u}{6} \right\rceil = \left\lceil \frac{u(4u-1)}{3} \right\rceil = \frac{u(4u-1)}{3} + \alpha_u \quad (18)$$

where $\alpha_u = \frac{1}{3}$ when $u \equiv 2 \pmod 3$ and 0 otherwise.

Altogether the construction has the following number of ADMs.

$$\begin{aligned} A(P_N,2) &\leq A(P_{2u},2) - \left\lceil \frac{4u^2}{8} \right\rceil + A(P_{2v},2) + (6u^2 - 2u + 8uv) - \frac{2u(v-1)}{3} \\ &\quad +2u - u \\ &\leq \frac{u(4u-1)}{3} + \alpha_u + \frac{44v^2 - 8v}{24} + \varepsilon_v + \frac{18u^2 - u + 22uv}{3} \quad (19) \\ &\leq \frac{11(4u+2v)^2 - 4(4u+2v)}{24} + \alpha_u + \varepsilon_v \quad (20) \end{aligned}$$

Now we have to check that $\alpha_u + \varepsilon_v = \varepsilon_N$ in all cases. For that, observe that the conditions of Lemma 5.5 are satisfied when $v = 1$ and when $v = 4$, assuming that $u \geq v \geq 1$. So we have reported in the following table all cases that satisfies the above construction.

N	condition	u	v	α_u	ε_v	ε_N
$12t+2$	$t \geq 1$	$3t$	1	0	$\frac{1}{2}$	$\frac{1}{2}$
$12t+4$	$t \geq 2$	$3t-1$	4	$\frac{1}{3}$	0	$\frac{1}{3}$
$12t+6$	$t \geq 0$	$3t+1$	1	0	$\frac{1}{2}$	$\frac{1}{2}$
$12t+8$	$t \geq 2$	$3t$	4	0	0	0
$12t+10$	$t \geq 0$	$3t+2$	1	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{5}{6}$
$12t+12$	$t \geq 1$	$3t+1$	4	0	0	0

Furthermore, the number of subgraphs in our construction for $N = 4u + 2v$ is equal to the number of K_3 of type ABC, plus the number of K_3 of type ACC, plus the number of subgraphs in the construction for $2v$, that is $4uv + 2u(u-v) + \left\lceil \frac{4v^2}{8} \right\rceil = \left\lceil \frac{(4u+2v)^2}{8} \right\rceil$.

In conclusion, Theorem 5.6 is true for all even N. □

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References

[1] B. Beauquier, J-C. Bermond, L. Gargano, P. Hell, S. Pérennes, and U. Vaccaro. Graph problems arising from wavelength-routing in all-optical networks. In *IEEE Workshop on Optics and Computer Science*, Geneva, Switzerland, April 1997.

[2] J-C. Bermond and S. Ceroi. Minimizing SONET ADMs in unidirectional WDM ring with grooming ratio 3. *Networks*, 41(2):73–82, February 2003.

[3] J.-C. Bermond, C.J. Colbourn, A. Ling, and M.-L. Yu. Grooming in unidirectional rings : $K_4 - e$ designs. *Discrete Mathematics, Lindner's Volume*, 284(1-3):57–62, 2004.

[4] J-C. Bermond, M. Cosnard, D. Coudert, and S. Perennes. Optimal solution of the maximum all request path grooming problem. Technical report, INRIA Research Report 5627 and I3S Research Report I3S/RR-2005-18-FR, 2005.

[5] J-C. Bermond and D. Coudert. Traffic grooming in unidirectional WDM ring networks using design theory. In *IEEE ICC*, Anchorage, USA, May 2003.

[6] J-C. Bermond, D. Coudert, and X. Muñoz. Traffic grooming in unidirectional WDM ring networks: the all-to-all unitary case. In *IFIP ONDM*, pages 1135–1153, February 2003.

[7] L. Braud. Groupage de trafic sur le chemin. Rapport de stage de MIM 1, encadrant D. Coudert, ENS Lyon, 2004.

[8] A. L. Chiu and E. H. Modiano. Traffic grooming algorithms for reducing electronic multiplexing costs in WDM ring networks. *IEEE/OSA Journal of Lightwave Technology*, 18(1):2–12, 2000.

[9] C.J. Colbourn and J.H. Dinitz, editors. *The CRC handbook of Combinatorial designs*. CRC Press, 1996.

[10] D. Coudert and H. Rivano. Lightpath assignment for multifibers WDM optical networks with wavelength translators. In *IEEE Globecom*, Taipei, Taiwan, November 2002.

[11] D Dor and M. Tarse. Graph decomposition is NP-complete: a complete proof of Holyer's conjecture. *SIAM Journal on Computing*, 26(4):1166–1187, 1997.

[12] R. Dutta, S. Huang, and N. Rouskas. On optimal traffic grooming in elemental network topologies. In *Opticomm*, pages 13–24, Dallas, USA, October 2003.

[13] R. Dutta and N. Rouskas. A survey of virtual topology design algorithms for wavelength routed optical networks. *Optical Networks Magazine*, 1(1):73–89, 2000.

[14] R. Dutta and N. Rouskas. On optimal traffic grooming in WDM rings. *IEEE Journal of Selected Areas in Communications*, 20(1):1–12, 2002.

[15] R. Dutta and N. Rouskas. Traffic grooming in WDM networks: Past and future. *IEEE Network*, 16(6):46–56, 2002.

[16] L. Epstein and A. Levin. Better bounds for minimizing SONET ADMs. In *WAOA*, Budapest, Hungary, September 2004.

[17] O. Gerstel, P. Lin, and G. Sasaki. Wavelength assignment in a WDM ring to minimize cost of embedded SONET rings. In *IEEE Infocom*, pages 94–101, San Francisco, USA, 1998.

[18] O. Gerstel, R. Ramaswani, and G. Sasaki. Cost-effective traffic grooming in WDM rings. *IEEE/ACM Transactions on Networking*, 8(5):618–630, 2000.

[19] O. Goldschmidt, D. Hochbaum, A. Levin, and E. Olinick. The SONET edge-partition problem. *Networks*, 41(1):13–23, 2003.

[20] I. Holyer. The NP-completeness of some edge-partition problems. *SIAM Journal on Computing*, 10(4):713–717, 1981.

[21] J.Q. Hu. Optimal traffic grooming for wavelength-division-multiplexing rings with all-to-all uniform traffic. *OSA Journal of Optical Networks*, 1(1):32–42, 2002.

[22] J.Q. Hu. Traffic grooming in wdm ring networks: A linear programming solution. *OSA Journal of Optical Networks*, 1(11):397–408, 2002.

[23] C.C. Lindner and C.A. Rodger. *Design Theory*. CRC Press, 1997.

[24] E. Modiano and P. Lin. Traffic grooming in WDM networks. *IEEE Communications Magazine*, 39(7):124–129, 2001.

[25] M. Shalom and S. Zaks. A $10/7 + \epsilon$ approximation for minimizing the number of ADMs in SONET rings. In *IEEE BroadNets*, pages 254–262, San José, USA, October 2004.

[26] A. Somani. Survivable traffic grooming in WDM networks. In D.K. Gautam, editor, *Broad band optical fiber communications technology – BBOFCT*, pages 17–45, Jalgaon, India, December 2001.

[27] P-J. Wan, G. Calinescu, L. Liu, and O. Frieder. Grooming of arbitrary traffic in SONET/WDM BLSRs. *IEEE Journal of Selected Areas in Communications*, 18(10):1995–2003, 2000.

[28] J. Wang, W. Cho, V. Vemuri, and B. Mukherjee. Improved approches for cost-effective traffic grooming in WDM ring networks: ILP formulations and single-hop and multihop connections. *IEEE/OSA Journal of Lightwave Technology*, 19(11):1645–1653, 2001.

[29] X. Yuan and A. Fulay. Wavelength assignment to minimize the number of SONET ADMs in WDM rings. In *IEEE ICC*, New York, USA, April 2002.

[30] X. Zhang and C. Qiao. An effective and comprehensive approach for traffic grooming and wavelength assignment in SONET/WDM rings. *IEEE/ACM Transactions on Networking*, 8(5):608–617, 2000.

A Small cases

Remark that all the subgraphs that we consider in the constructions satisfy $L(B_w, e) \leq 2$. It is clear for a K_3 $\{u, v, w\}$, where we suppose $u < v < w$. For a graph $e + K_3$, where the edge $\{t, u\}$ is glued with the K_3 $\{u, v, w\}$, we suppose that $t < u < v < w$. For a graph $K_3 + e$, where the K_3 $\{u, v, w\}$ is glued with the edge $\{w, x\}$, we suppose that $u < v < w < x$.

Lemma A.1 $A(P_2, 2) = 2$ *and* $A(P_3, 2) = 3$.

Lemma A.2 $A(P_4, 2) = 7$.

Proof: The first subgraph is the $e + K_3$ $\{0, 1\}$+$\{1, 2, 3\}$, and the second subgraph contains the two edges $\{0, 2\}$ and $\{0, 3\}$. □

Lemma A.3 $A(P_5, 2) = 10$.

Proof: The subgraphs of the decomposition are the 2 K_3 $\{0, 2, 4\}$ and $\{0, 1, 3\}$, plus the subgraph B_3 containing the 4 edges $\{1, 2\}$, $\{2, 3\}$, $\{3, 4\}$ and $\{1, 4\}$. This construction uses 10 ADMs, which fits the lower bound. □

Lemma A.4 $A(P_6, 2) = 16$.

Proof: Let the vertices be $a_0, a_1, a_2, a_3, a_4, a_5$. Using a 3-GDD of type 2^3, our construction consists in the 2 K_3 $\{a_0, a_2, a_5\}$ and $\{a_1, a_3, a_5\}$, plus the 2 K_3+e $\{a_0, a_3, a_4\}$+$\{a_4, a_5\}$ and $\{a_0, a_1\}$+$\{a_1, a_2, a_4\}$, plus the edge $\{a_2, a_3\}$. This construction use 16 ADMs. □

Lemma A.5 $A(P_7, 2) = 20$

Proof: Let the vertices of P_7 be $\mathbb{Z}_7$. The construction is obtained using the partition of K_7 into the 7 K_3 $\{i, i + 1, i + 3\}$, indices being taken modulo 7, and the remark that the 2 K_3 $\{0, 1, 3\}$ and $\{3, 4, 6\}$ fit in a same subgraph. This construction uses 20 ADMs which is equal to the lower bound. □

Lemma A.6 $A(P_8, 2) = 28$

Proof: Let the nodes be $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2$. We have 4 groups of 2 consecutive nodes and we use a 3-GDD of type 2^4. Our construction consist on the 4 K_3 $\{a_2, b_2, c_2\}$, $\{b_1, c_2, d_1\}$, $\{a_1, c_2, d_2\}$ and $\{a_1, b_2, d_1\}$ plus the 2 $e + K_3$ $\{a_1, a_2\} + \{a_2, b_1, d_2\}$ and $\{b_1, b_2\}+\{b_2, c_1, d_2\}$, and the two K_3+e $\{a_1, b_1, c_1\}+\{c_1, c_2\}$ and $\{a_2, c_1, d_1\}+\{d_1, d_2\}$. This construction has 28 ADMs. □

Lemma A.7 $A(P_9, 2) = 34$

Proof: Let the vertices of P_9 be $\mathbb{Z}_9$. The construction is obtained using the partition of K_9 into the 9 K_3 $\{i, 3+j, 6+k\}$, $i, j \in \mathbb{Z}_3$ and $k = i + j \pmod 3$, and the 3 K_3 $\{l, l+1, l+2\}$, $l = 0, 3, 6$, and the remark that the 3 K_3 $\{0, 1, 2\}$, $\{2, 3, 6\}$ and $\{6, 7, 8\}$ fit in a same subgraph. This construction use 34 ADMs which is equal to the lower bound. □

Lemma A.8 $A(P_{10}, 2) = 45$

Proof: Let the vertices of P_{10} be $\{a_1, a_2\} \cup \{b_1, b_2\} \cup \{c_1, c_2\} \cup \{0, 1, 2, 3\}$. Using a 3-GDD of type $2^3 4^1$ (see [9] page 189), we obtain a partition into the 13 following subgraphs (K_3, edges and union of K_3 and edges) $\{a_1, b_2, 1\}$, $\{a_1, c_1, 2\}$, $\{a_1, c_2, 3\}$, $\{a_1, a_2\}$+$\{a_2, b_2, 3\}$, $\{a_2, b_1, 2\}$, $\{a_2, c_1, 1\}$, $\{b_1, c_1, 3\}$, $\{b_1, c_2, 1\}$, $\{b_2, c_2, 2\}$, $\{a_2, c_2, 0\}$+$\{0, 1\}$+$\{1, 2, 3\}$, $\{a_1, b_1, 0\}$+$\{0, 2\}$, $\{b_1, b_2\}$+$\{b_2, c_1, 0\}$+$\{0, 3\}$ and $\{c_1, c_2\}$. Altogether this partition use 45 ADMs. □

Lemma A.9 $A(P_{11}, 2) = 52$

Proof: Let the vertices of P_{11} be $\mathbb{Z}_{11}$. We can partitioned the edges of $K_{11} - K_5$ into 15 K_3 (existence of a 3-GDD of type $5^1 1^6$, see [9] page 189), and from Lemma A.3 we can partition K_5 into 2 K_3 and 1 C_4. If the nodes of the K_5 are $0, 1, 2, 3, 4$, each node is the left most node of 3 K_3's of the partition of $K_{11} - K_5$. So me can merge each subgraph of the partition of K_5 with one K_3, and we saved 3 ADMs. Altogether, we use $15 \times 3 + 10 - 3 = 52$ ADMs, which is equal to the lower bound. □

Lemma A.10 $A(P_{12}, 2) = 64$

Proof: Let the nodes of P_{12} be $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, e_1, e_2, f_1, f_2$ and arrange them in this order.

The decomposition contains the 2 subgraphs $K_3 + K_3$ $\{a_1, b_1, c_2\} + \{c_2, e_2, f_1\}$ and $\{a_2, c_2, d_2\} + \{d_2, e_1, f_2\}$, plus the 3 $e + K_3$ $\{a_1, a_2\} + \{a_2, b_2, f_1\}$, $\{b_1, b_2\} + \{b_2, c_1, d_2\}$ and $\{c_1, c_2\} + \{c_2, d_1, e_1\}$, and the 3 $K_3 + e$ $\{a_2, c_1, d_1\} + \{d_1, d_2\}$, $\{a_2, b_1, e_1\} + \{e_1, e_2\}$ and $\{a_1, d_2, f_1\} + \{f_1, f_2\}$, and plus the 10 K_3 $\{b_1, d_1, f_1\}$, $\{b_2, d_1, e_2\}$, $\{a_1, c_1, e_2\}$, $\{b_1, c_1, f_2\}$, $\{a_1, d_1, f_2\}$, $\{b_2, c_2, f_2\}$, $\{a_1, b_2, e_1\}$, $\{b_1, d_2, e_2\}$, $\{c_1, e_1, f_1\}$ and $\{a_2, e_2, f_2\}$. Altogether, it has $2 \times 5 + 6 \times 4 + 10 \times 3 = 64$ ADMs. □

Lemma A.11 $A(P_{13}, 2) = 73$

Proof: Let the vertices of P_{13} be $\mathbb{Z}_{13}$ and remark that K_{13} can be partitioned into the 26 K_3 $\{i, i+1, i+4\}$ and $\{i, i+5, i+7\}$, $i \in \mathbb{Z}_{13}$. Our decomposition contains the subgraph $K_3 + K_3 + K_3$ $\{0, 1, 4\} + \{4, 5, 8\} + \{8, 9, 12\}$, plus the 3 subgraphs $K_3 + K_3$ $\{i, i+1, i+4\} + \{i+4, i+5, i+8\}$, $i = 1, 2, 3$, plus the 4 K_3 $\{j, j+1, j+4\}$, $j = 9, 10, 11, 12$, and plus the 13 K_3 $\{k, k+5, k+7\}$, $k \in \mathbb{Z}_{13}$. Altogether the construction has $7 + 3 \times 5 + 17 \times 3 = 73$ ADMs. □

Lemma A.12 $A(P_{16}, 2) = 115$

Proof: Let the vertices of P_{16} be $A \cup B \cup C$, where $A = \{a_0, a_1, a_2, a_3, a_4, a_5\}$, $B = \{b_0, b_1, b_2, b_3\}$ and $C = \{c_0, c_1, c_2, c_3, c_4, c_5\}$. Our construction is based on the existence of a 3-GDD of type $6^1 4^1 2^3$, which consist on 24 K_3 of type ABC, 6 K_3 of type ACC and 2 K_3 of type CCC, and by merging the 5 subgraphs of the decomposition of K_6 with K_3s of type ABC, the 2 K_3 of type CCC and the 3 edges $\{c_i, c_{i+1}\}$, $i = 0, 1, 2$, with K_3s of type ABC. Altogether this construction uses 115 ADMs and the subgraphs of the decomposition are:

- The 4 subgraphs $K_3 + K_3$ $\{a_0, b_0, c_0\}$ + $\{c_0, c_2, c_4\}$, $\{a_1, b_1, c_1\}$ + $\{c_1, c_3, c_5\}$, $\{a_0, a_2, a_5\}$ + $\{a_5, b_1, c_0\}$ and $\{a_1, a_3, a_5\}$ + $\{a_5, b_3, c_3\}$,so 20 ADMs.
- The 3 $K_3{+}e$ $\{a_2, b_2, c_0\}$ + $\{c_0, c_1\}$, $\{a_3, b_3, c_2\}$ + $\{c_2, c_3\}$ and $\{a_4, b_2, c_4\}$ + $\{c_4, c_5\}$, and the $e + K_3$ $\{a_2, a_3\}$ + $\{a_3, b_1, c_3\}$, so 16 ADMs.
- The 2 subgraphs on 6 vertices, the $K_3 + e + K_3$ $\{a_0, a_3, a_4\}$ + $\{a_4, a_5\}$ + $\{a_5, b_0, c_2\}$ and the $e + K_3 + K_3$ $\{a_0, a_1\}$ + $\{a_1, a_2, a_4\}$ + $\{a_4, b_0, c_1\}$, so 12 ADMs.
- The 21 K_3 $\{a_0, b_1, c_5\}$, $\{a_0, b_2, c_3\}$, $\{a_0, b_3, c_4\}$, $\{a_0, c_1, c_2\}$, $\{a_1, b_0, c_5\}$, $\{a_1, b_2, c_2\}$, $\{a_1, b_3, c_0\}$, $\{a_1, c_3, c_4\}$, $\{a_2, b_0, c_3\}$, $\{a_2, b_1, c_4\}$, $\{a_2, b_3, c_1\}$, $\{a_2, c_2, c_5\}$, $\{a_3, b_0, c_4\}$, $\{a_3, b_2, c_1\}$, $\{a_3, c_0, c_5\}$, $\{a_4, b_1, c_2\}$, $\{a_4, b_3, c_5\}$, $\{a_4, c_0, c_3\}$, $\{a_5, b_2, c_5\}$, $\{a_5, c_1, c_4\}$ and $\{b_0, b_2, b_3\}$, so 63 ADMs.
- The star $\{b_0, b_1\}$ + $\{b_1, b_2\}$ + $\{b_1, b_3\}$, 4 ADMs.

□

Lemma A.13 $A(P_{17}, 2) = 127$

Proof: The decomposition is based on the existence of a 3-GDD of type $3^2 5^1 3^2$ (which was kindly given to us by C.J. Colbourn) and the subgraphs are:

- The 9 subgraphs $K_3 + K_3$ $\{0,1,2\}+\{2,3,11\}$, $\{3,4,5\}+\{5,13,15\}$, $\{1,4,11\}+\{11,12,13\}$, $\{2,4,14\}+\{14,15,16\}$, $\{0,5,6\}+\{6,11,14\}$, $\{2,5,7\}+\{7,11,16\}$, $\{0,4,8\}+\{8,11,15\}$, $\{1,5,9\}+\{9,13,14\}$ and $\{0,3,10\}+\{10,12,14\}$, so altogether 45 ADMs.
- The 24 K_3s $\{4,6,12\}$, $\{1,6,13\}$, $\{2,6,15\}$, $\{3,6,16\}$ $\{1,7,12\}$, $\{4,7,13\}$, $\{3,7,15\}$, $\{0,7,14\}$ $\{2,8,12\}$, $\{3,8,13\}$, $\{1,8,16\}$, $\{5,8,14\}$ $\{3,9,12\}$, $\{4,9,15\}$, $\{2,9,16\}$, $\{0,9,11\}$ $\{2,10,13\}$, $\{1,10,15\}$, $\{4,10,16\}$, $\{5,10,11\}$ $\{1,3,14\}$, $\{0,12,15\}$, $\{0,13,16\}$ and $\{5,12,16\}$, so 72 ADMs.
- The 3 graphs of the decomposition of the K_5 on $6,7,8,9,10$: the 2 K_3 $\{6,8,10\}$ and $\{6,7,9\}$ and the C_4 $\{7,8,9,10\}$, so 10 more ADMs.

In summary our construction has 127 ADMs. □

Lemma A.14 $A(P_{20},2)=180$

Proof: The decomposition is based on a 3-GDD of type $2^3 8^1 2^3$ in which the vertices are labeled $a_0,a_1,b_0,\ b_1,c_0,c_1,0,1,\dots,7,d_0,d_1,e_0,e_1,f_0,f_1$ and ranked in this order. The subgraphs are:

- The 2 subgraphs K_3+K_3 $\{a_1,c_0,0\}+\{0,3,6\}$ and $\{0,5,7\}+\{7,d_0,f_1\}$, and the 3 subgraphs $e+K_3+e$ $\{a_0,a_1\}+\{a_1,4,d_0\}+\{d_0,d_1\}$, $\{b_0,b_1\}+\{b_1,4,e_0\}+\{e_0,e_1\}$ and $\{c_0,c_1\}+\{c_1,4,f_0\}+\{f_0,f_1\}$, so 25 ADMs.
- The 4 subgraphs on 6 vertices: the two K_3+e+K_3 $\{a_0,b_1,0\}+\{0,1\}+\{1,2,7\}$ and $\{2,5,6\}+\{6,7\}+\{7,e_1,f_0\}$, the K_3+K_3+e $\{b_0,c_1,0\}+\{0,2,4\}+\{4,5\}$ and the $e+K_3+K_3$ $\{2,3\}+\{3,4,7\}+\{7,d_1,e_0\}$ so 24 ADMs.
- The 2 subgraphs $K_3+K_3+K_3$ $\{a_0,b_0,c_0\}+\{c_0,2,d_0\}+\{d_0,e_0,f_0\}$ and $\{a_1,b_1,c_1\}+\{c_1,2,d_1\}+\{d_1,e_1,f_1\}$, so 14 ADMs.
- The 39 K_3 $\{1,4,6\}$, $\{1,3,5\}$, $\{0,d_0,e_1\}$, $\{0,e_0,f_1\}$, $\{0,d_1,f_0\}$, $\{a_0,c_1,7\}$, $\{a_1,b_0,7\}$, $\{b_1,c_0,7\}$, $\{a_0,1,d_0\}$, $\{b_0,1,e_0\}$, $\{c_0,1,f_0\}$, $\{a_1,1,d_1\}$, $\{b_1,1,e_1\}$, $\{c_1,1,f_1\}$, $\{a_0,2,e_0\}$, $\{b_0,2,f_0\}$, $\{a_1,2,e_1\}$, $\{b_1,2,f_1\}$, $\{a_0,3,f_0\}$, $\{b_0,3,d_0\}$, $\{c_0,3,e_0\}$, $\{a_1,3,f_1\}$, $\{b_1,3,d_1\}$, $\{c_1,3,e_1\}$, $\{a_0,4,d_1\}$, $\{b_0,4,e_1\}$, $\{c_0,4,f_1\}$, $\{a_0,5,e_1\}$, $\{b_0,5,f_1\}$, $\{c_0,5,d_1\}$, $\{a_1,5,e_0\}$, $\{b_1,5,f_0\}$, $\{c_1,5,d_0\}$, $\{a_0,6,f_1\}$, $\{b_0,6,d_1\}$, $\{c_0,6,e_1\}$, $\{a_1,6,f_0\}$, $\{b_1,6,d_0\}$ and $\{c_1,6,e_0\}$, so 117 more ADMs

Altogether this construction has 180 ADMs. □

Lemma A.15 $A(P_{23}, 2) = 235$

Proof: The proof is similar to the proof of Lemma A.9.

Let the vertices of P_{23} be $\mathbb{Z}_{23}$. We can partitioned the edges of $K_{23} - K_{11}$ into 66 K_3 (existence of a 3-GDD of type $11^1 1^{12}$, see [9] page 189), and from Lemma A.9 we can partition K_{11} into 15 subgraphs (K_3s and union of K_3 and K_4). If the nodes of the K_{11} are $0, 1, \ldots, 10$, each node is the left most node of 6 K_3's of the partition of $K_{23} - K_{11}$. So me can merged each subgraph of the partition of K_{11} with one K_3, and we saved 15 ADMs. Altogether, we use $66 \times 3 + 52 - 15 = 235$ ADMs, which is equal to the lower bound. □

Lemma A.16 $A(P_{47}, 2) = 997$

Proof: The proof is similar to the proof of Lemma A.15.

Let the vertices of P_{47} be $\mathbb{Z}_{47}$. We can partitioned the edges of $K_{47} - K_{23}$ into 276 K_3 (existence of a 3-GDD of type $23^1 1^{24}$, see [9] page 189), and from Lemma A.15 we can partition K_{23} into 66 subgraphs. If the nodes of the K_{23} are $0, 1, \ldots, 23$, each node is the left most node of 12 K_3's of the partition of $K_{47} - K_{23}$. So me can merged each subgraph of the partition of K_{23} with one K_3, and we saved 66 ADMs. Altogether, we use $276 \times 3 + 235 - 66 = 997$ ADMs, which is equal to the lower bound. □

B Another constructions for $N \equiv 1, 3 \pmod 6$

When $u, z \equiv 1, 3 \pmod 6$, and given an optimal decomposition for both K_u and K_z, we can obtain an optimal decomposition of K_{uz}. For that, we will use the following construction:

- We replace each node of K_u by a group of z nodes and each edge of K_u by the corresponding complete bipartite graph $K_{z,z}$
- From the optimal decomposition of K_u we deduce an optimal decomposition of the graph of type z^u, that is the complete multipartite graph with u groups of size z, $K_{z\times u}$. So the decomposition will have z^2 times more subgraphs and so z^2 times more ADMs
- Since each node in each group of size z has degree $z-1$, it is involved in at most $\frac{z-1}{2}$ subgraphs of the optimal decomposition of K_z. Furthermore, it is also involved in at least z subgraphs of the optimal decomposition of $K_{z\times u}$ (external subgraphs). Moreover, we will see in Lemma B.1 that in exactly $u-1$ groups of nodes, each node is the left or right most node of z external subgraphs and so we can merge each internal subgraph with an external one.
- It remains to decompose one K_u.

Altogether, this construction will use $z^2 A(P_u, 2) + (u-1)\left(A(P_z, 2) - \left\lceil \frac{z^2-1}{8} \right\rceil\right) + A(P_z, 2)$ ADMs which is equal to $A(P_{zu}, 2)$.

Lemma B.1 *When $N \equiv 1, 3 \pmod 6$ and $C = 2$, each node $i \neq \frac{N-1}{2}$ of P_N is the left or right most node of at least one subgraph of the optimal decomposition of K_N*

Proof: Let the nodes be numbered from 0 to $N-1$ from left to right, and let $d_l(i)$ (resp. $d_r(i)$) denotes the left (resp. right) degree of node i, that is the number of nodes on the left or on the right of i. We have $d_l(i) = i$ and $d_r(i) = N - i - 1$.

According to the optimal construction obtain in Theorem 5.4, when a node is in a subgraph, it contributes for 2 or 4 edges, that is one on each side (middle node) or 2 on the same side (left or right most node) or 2 on each side (union node).

When $y = \frac{N-1}{2}$ we have $d_l(y) = d_r(y)$ and so node y is always a middle or union node for a subgraph. To show that, suppose that y is the right most node of one subgraph. Since it is also the middle node of α subgraphs and a union node for β subgraphs, and since $d_l(y) = d_r(y) = \alpha + 2\beta + 2$, y is also the left most node of one subgraph which is in contradiction with the optimality of the construction.

For all other node $i \neq y$, we have $d_l(i) \neq d_r(i)$ and so node i is the left or right most node of at least one subgraph of the construction.

So in any optimal construction for $N \equiv 1, 3 \pmod 6$ and $C = 2$, each node $i \neq \frac{N-1}{2}$ of P_N is the left or right most node of at least one subgraph of the optimal decomposition of K_N. □

We have circled in Figure 3 some left and right most nodes in the optimal decomposition for $N = 3, 7$ and 9.

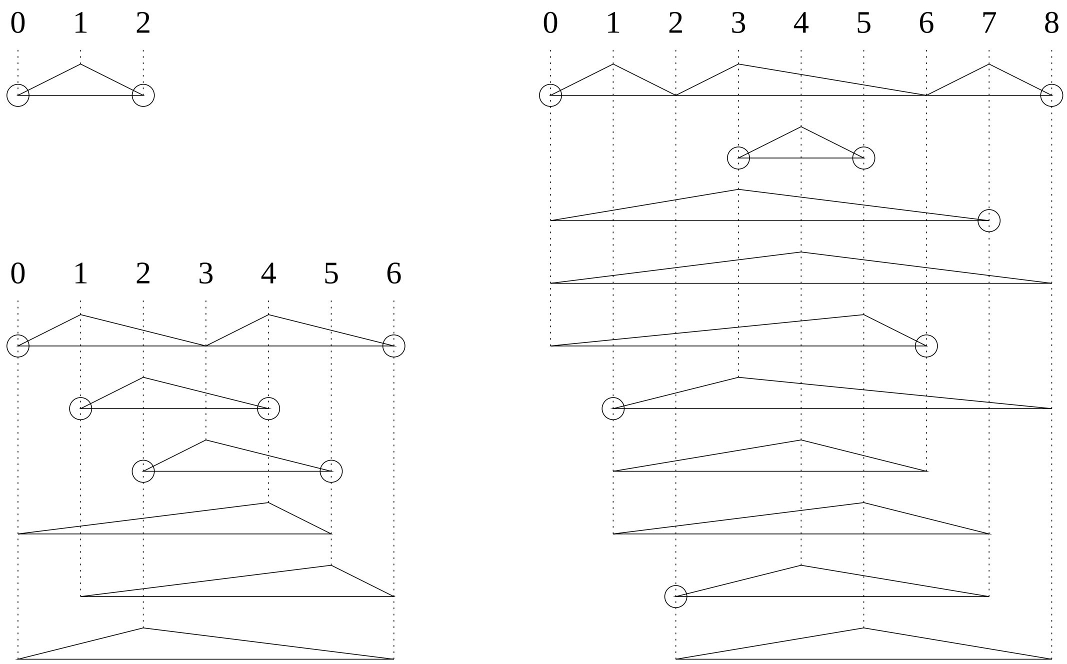

Figure 3: Construction for $N = 3, 7$ and 9. Left and right most nodes have been circled.

Theorem B.2 *Given $u, z \equiv 1, 3 \pmod 6$ and an optimal decomposition for both K_u and K_z, we can obtain an optimal decomposition of K_{uz}. Furthermore, each node of P_{uz} except node $\frac{uz-1}{2}$ is the left or right most node of at least one subgraph of the decomposition.*

Proof: According to Lemma B.1, $u - 1$ nodes of the optimal construction for u are left or right most nodes of some subgraphs.

Now we replace each node of K_u by a group of z nodes and each edge of K_u by the corresponding complete bipartite graph $K_{z,z}$.

From the optimal decomposition of K_u we can deduce an optimal decomposition of the resulting complete multipartite graph with u groups of size z, $K_{z\times u}$. To see that, remark that the complete tripartite graph $K_{z,z,z}$ can be decompose into z^2 K_3s. Thus for a pair of K_3s of the decomposition of K_u that shared we will obtain z^2 pairs of K_3s sharing a node. So the decomposition of $K_{z\times u}$ will have $z^2\left\lceil\frac{u^2-1}{8}\right\rceil$ subgraphs and use $z^2A(P_u,2)$ ADMs.

In each group of z nodes except group $\frac{u-1}{2}$, each node is the left or right most node of at least z subgraphs of the decomposition of $K_{z\times u}$. Since it is also the left or right most node of at most $\frac{z-1}{2}$ subgraphs of the decomposition of K_z, we can merge each subgraph of the decomposition of K_z with a subgraph of the decomposition of $K_{z\times u}$. So, we will save $(u-1)\left\lceil\frac{z^2-1}{8}\right\rceil$ ADMs.

Altogether, this construction use the following number of ADMs

$$z^2A(P_u,2)+(u-1)\left(A(P_z,2)-\left\lceil\frac{z^2-1}{8}\right\rceil\right)+A(P_z,2) \tag{21}$$

$$= z^2\frac{11u^2-8u-3}{24}+u\frac{11z^2-8z-3}{24}-(u-1)\frac{z^2-1}{8} \tag{22}$$

$$= \frac{11(uz)^2-8uz-3}{24} \tag{23}$$

$$= A(P_{uz},2) \tag{24}$$

and has the following number of subgraphs

$$z^2\frac{u^2-1}{8}+\frac{z^2-1}{8} = \frac{(uz)^2-1}{8} \tag{25}$$

Finally, since $\frac{z-1}{2}<z$, each node of the $u-1$ groups different from group $\frac{u-1}{2}$ will be the left or right most node of some subgraph of the decomposition, and since $z-1$ nodes of the optimal construction for z are left or right most nodes of some subgraphs, $uz-1$ nodes will be left or right most node of some subgraphs of the decomposition of K_{uz}. □

One may remark that the decomposition of K_9 drawn in Figure 3 has been obtain using above construction with $u=z=3$.

Corollary B.3 *The lower bound is attained for all N such that $N=p_1^{\alpha_1}p_2^{\alpha_2}\dots p_k^{\alpha_k}$, where $p_i\equiv 1,3 \pmod 6$, $1\le i\le k$ and $\alpha_i\ge 0$.*

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