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Random Multi-access Algorithms - A Mean Field analysis

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Abstract: In this paper, using mean field techniques, we present a performance analysis of random back-off algorithms, such as the exponential back-off algorithm, in the case of a finite number of saturated sources. We prove that when the number of sources grows large, the system is decoupled, i.e., source behaviors become mutually independent. These results are applied in the specific case of exponential back-off algorithms, where the transient and stationary distributions of the back-off processes are explicitly characterized. This work provides a theoretical justification of decoupling arguments used by many authors, e.g. Bianchi [4], to analyze the performance of random algorithms in Wireless LANs.

Key-words: Wireless LANs, random multi-access algorithms, mean field analysis, decoupling approach

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Algorithmes d'accès multiple aléatoires - Analyse par champ moyen

Résumé : A l'aide des techniques de champ moyen, nous analysons la performance d'algorithmes d'accès aléatoire à *backoffs* (tels que les algorithmes à backoffs exponentiels) dans le cas où un nombre fini de sources sont en compétition pour l'utilisation d'une ressource partagée. Nous montrons que lorsque le nombre de sources devient important, le système est quasiment découplé, dans le sens où les comportements des sources deviennent statistiquement indépendants. Ces résultats sont appliqués au cas des algorithmes à backoffs exponentiels, et on caractérise alors le comportement transitoire et stationnaire des distributions des processus de backoff. Ce travail permet de justifier théoriquement les arguments heuristiques de découplage utilisés par de nombreux auteurs, tels que Bianchi [4], pour analyser la performance des réseaux WLANs.

Mots-clés : Réseaux WLANs, accés aléatoire multi-utilisateur, analyse en champ moyen, découplage.

1 Introduction

Random multi-access protocols, from the first version of Abramson's ALOHA algorithm [2] to the most recent protocols in IEEE802.xx standards [1], have generated a lot of research interest. Recently, interest has even increased due to the development of decentralized random access protocols for Wireless LANs and Ad-Hoc networks. However, theoretical results characterizing the stability and the performance of such protocols are few, due to the extreme complexity arising from the inherent interaction between sources.

In analyzing random access protocols, we can distinguish between different kinds of models. First consider infinite population models where users are assumed to arrive arbitrarily and randomly in the system according to some point processes. In these models, each user generally has a single packet to transmit and leaves the system after a successful transmission. In these scenarios, stability of protocols received a lot of attention: for example, the exponential back-off algorithm is unstable for all positive user arrival rate, this was first conjectured by Kelly [10] and proved by Aldous [3]. Other algorithms, see for example the proposals by Hajek-van Loon [8], were proved to stabilize the system for non-zero user arrival rates.

Other models consider a finite population: the number of users is fixed. In this class of models, we can further distinguish between two cases: saturated sources where users always have a packet to send and unsaturated sources where packet arrivals are governed by some exogenous arrival processes. The first case corresponds more or less to data traffic, where the congestion control mechanism ensures that buffers are never empty, whereas the second case is more appropriate in modeling streaming traffic. In the case of unsaturated sources, stability is again a major issue. Please refer to the work by Håstad et al. [9] and references therein for a review of existing results and open issues.

In the present paper we consider a finite number of saturated sources. In this context, the stability analysis is simplified, because queues are not considered. The major issue is then to determine the performance of protocols, i.e., we would like to evaluate the throughput of the system and the packet transmission delay of each source. These performance parameters are largely unknown. This is due to the fact that the inherent interactions between sources have proven to be extremely complex to model and analyze. A very popular approach to circumvent this difficulty consists in decoupling the sources, i.e., assuming that the (re)-transmission processes of the different sources are mutually independent. This assumption allows one to derive explicit estimates of the performance. This approach was for example applied by Bianchi [4] to analyze the IEEE 802.11 Decentralized Coordination Function (DCF) algorithm and has been widely used ever since to accurately predict the performance of similar protocols. Using mean field techniques, we prove that, for a wide range of random back-off algorithms, the decoupling assumption is asymptotically exact as the number of sources grows. In the specific case of exponential back-off algorithm (the DCF is based on this algorithm), the mean field analysis provides the transient and stationary distributions of the (re)-transmission processes.

The remainder of this paper is organized as follows. In Section 2 we introduce the finite-source model and present the basic principle for random back-off algorithms. In Section 3 we present the mean field results that justify the decoupling approach. Section 4 contains the proofs of the main results.

Notation.

Let $\mathcal{S}$ be a separable, complete metric space, $\mathcal{P}(\mathcal{S})$ denotes the space of probability measures on $\mathcal{S}$. $\mathcal{L}(X)$ is the law of the $\mathcal{S}$-valued random variable X. $D(\mathbb{R}^+, \mathcal{S})$ the space of right-continuous functions with left-handed limits, with the Skorohod topology [5]. $\mathcal{M}_b(\mathcal{S})$ denotes the space of bounded signed measures on $\mathcal{S}$. For $f \in L^\infty(\mathcal{S})$ and $\mu \in \mathcal{M}_b(\mathcal{S})$, we define $\langle f, \mu \rangle = \int f d\mu$.

2 Random back-off algorithms

We consider N users sharing a common access channel in a decentralized manner. Time is slotted and users are assumed to be synchronized. We consider saturated users only, which means that each user has always a packet to transmit. For data traffic, this assumption is more realistic than to assume that each user generates packets according to some predefined point process.

Each user runs a back-off timer. When a user is in stage $s \in \mathbb{N}$, the user's back-off timer is chosen according to a geometric random variable with mean W_s slots. The back-off timer is decremented at each slot when the channel is idle, until it reaches 0 in which case, the user attempts to use the channel. If the transmission is successful, the stage becomes $S(s)$ where $S : \mathbb{N} \to \mathbb{N}$ is a non-increasing function. If a collision occurs the stage becomes $C(s)$, where $C : \mathbb{N} \to \mathbb{N}$ is an increasing function. Then a new back-off timer is chosen. The assumption that the back-off timer is chosen according to a geometric r.v. is not crucial, and we believe the derived results hold for general distributions. The geometric distribution simplifies the analysis, since given that the user's stage is s, a user will attempt to use the channel with probability $p_s = 1/W_s$. Then, at time slot t, the state of the system is fully described by the state random variables $p_i^N(t)$, $i = 1, \ldots, N$ of all users, called the back-off probabilities. The set of possible values for the back-off probabilities is denoted by $\mathcal{B}$. For the sake of simplicity we assume that $\mathcal{B}$ is at most denumerable. We denote by $p_0 = \sup \mathcal{B}$ the maximum back-off probability. We may re-define functions S and $C : \mathcal{B} \to \mathcal{B}$ to express a successful and a collision respectively: Given that the back-off probability of a user is p and that the user attempts to use the channel, the back-off probability becomes $S(p)$ (resp. $C(p)$) in case of successful transmission (resp. collision).

Finally, the evolution of the state of the system is described by the following recursion: for all i and t,

$$\begin{aligned} p_i^N(t+1) &= p_i^N(t) 1_{U_i(t) > p_i^N(t)} + \left(S(p_i^N(t)) \prod_{j \neq i} 1_{U_j(t) > p_j^N(t)} \right. \\ &\left. + C(p_i^N(t))(1 - \prod_{j \neq i} 1_{U_j(t) > p_j^N(t)}) \right) 1_{U_i(t) \leq p_i^N(t)}, \end{aligned} \tag{1}$$

where the $U_i(t)$'s are i.i.d. r.v. uniformly distributed on $[0, 1]$. Having geometric back-offs implies that the back-off probabilities are of course independent of the r.v. $U_i(s)$, $i = 1, \ldots, N$, for all $s < t$.

Example 1 (Exponential back-off) *Most of MAC protocols use a so-called exponential back-off algorithm. This is the case for the Ethernet or for the DCF function in wireless LANs. For these* *protocols the back-off probabilities are values in the set* $\mathcal{B} = \{p_0 \times 2^{-k}, k \in \mathbb{N}\}$ *and the functions* S *and* C *are defined by:*

$$S(p) = p_0 \quad \textit{and} \quad C(p) = \max(p/2, p_0/2^K).$$

K *may be finite or not. Unless otherwise specified,* $K = \infty$*.*

We study the large N asymptotics. With this in mind, we need an adequate renormalization. First note that when the number of users grows, the stationary probability a user attempts to use the channel must decrease and ultimately tend to zero, otherwise the global throughput will become negligible. Then, the mean time between two consecutive changes in the value of the back-off parameter is roughly inversely proportional to the stationary back-off probability. The above observations suggest the following renormalization:

$$q_i^N(t) \equiv N \times p_i^N[N \times t] \text{ where } [x] \text{ denotes the greatest integer in } x. \tag{2}$$

The set of possible values for $q_i^N(t)$ is $\mathcal{B}$. In case of Example 1, it means that p_0 is replaced by p_0/N. In general, it means that the set of possible values for $p_i^N(t)$ becomes $\{p/N, p \in \mathcal{B}\}$. The process $q_i^N(.)$ may be seen as a $D(\mathbb{R}^+, \mathcal{B})$-valued process.

3 Convergence theorems

3.1 Transient behavior

The following theorem constitutes the main result of the paper and states that when the number of sources become large, the back-off timers of all sources evolve independently of each other. In short, users behave independently when the number of users grows.

Theorem 1 *We assume that the initial values* $q_i^N(0)$*,* $i = 1, \ldots, N$*, are exchangeable and chaotic. This means the empirical measure* μ_0^N *of the* $q_i^N(0)$ *converges weakly to a deterministic limit* Q_0*. In practice the assumption that the* $q_i^N(0)$ *are i.i.d. suffices. Then there exists a probability measure* Q *on* $D(\mathbb{R}^+, \mathcal{B})$ *such that for all subset* $I \subset \mathbb{N}$ *of finite cardinal* $|I|$*,*

$$\lim_{N\to\infty} \mathcal{L}\left((q_i^N(.))_{i\in I}\right) = Q^{\otimes |I|} \quad \textit{weakly in } \mathcal{P}(D(\mathbb{R}^+, \mathcal{B})^{|I|}). \tag{3}$$

To establish this result, we use the method developed by Sznitman [11] and further investigated by Graham [7]. The empirical measure on path space, with samples in $\mathcal{P}(D(\mathbb{R}^+, \mathcal{B}))$, is defined as

$$\mu^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{q_i^N}, \tag{4}$$

where q_i^N stands for the process $q_i^N(\cdot)$. From Sznitman [11] Proposition 2.2, we know that (3) is equivalent to:

$$\lim_{N\to\infty} \mathcal{L}(\mu^N) = \delta_Q \quad \text{weakly in } \mathcal{P}(\mathcal{P}(D(\mathbb{R}^+, \mathcal{B}))). \tag{5}$$

Moreover, (5) allows us to consider functionals on $D(\mathbb{R}^+, \mathcal{B})$ such as the indicator of the event that a sample path backs off more than b in a finite it time interval $[0,T]$. Let X denote a canonical trajectory in $D(\mathbb{R}^+, \mathcal{B})$. Let ϕ denote this functional on X. Then $\langle \mu^N, \phi \rangle$ is the proportion of the users that back off more than b over the time interval $[0,T]$. If we studied the weaker convergence of $\mu_t^N \to Q(t)$ then we could not obtain sample path information like this.

For a large class of protocols such as the exponential back-off protocol, we are also able to characterize the evolution of the distribution of the back-off of a source when the number of sources grows large, i.e., to evaluate the marginals of Q.

Theorem 2 *In case of exponential back-off, the marginals of Q satisfy the following set of differential equations: define by $Q_k(t) \equiv Q(t)(\{2^{-k}p_0\})$,*

$$\frac{dQ_k}{dt}(t) = 2^{1-k}p_0 \left(Q_{k-1}(t)\big(1 - \exp(-\sum_{i=0}^{\infty} 2^{-i}p_0 Q_i(t))\big) - \frac{Q_k(t)}{2} \right), \quad \textit{for all } k \geq 1, \tag{6}$$

$$\frac{dQ_0}{dt}(t) = \sum_{k=0}^{\infty} 2^{-k}p_0 Q_k(t) \exp(-\sum_{i=0}^{\infty} 2^{-i}p_0 Q_i(t)) - p_0 Q_0(t). \tag{7}$$

3.2 Stationary regimes

We now investigate the system behavior in equilibrium. We consider here exponential back-off protocols only. We first prove that when the number of sources N is fixed, the back-off process $(q_i^N(t))_i$ is ergodic. The theorem below is proved assuming $p_0 < \ln 2$, which is consistent with usual MAC protocols (in case of the DCF, $p_0 = 1/32$). The results actually hold for any p_0, but we lack of space for a full proof. In the following we use the subscript st to represent the stationary behavior of the system.

Theorem 3 *In case of exponential back-off, the Markov process $(q_i^N(t))_i$ is positive recurrent. Furthermore the set of laws of $q_i^N(0)$ in equilibrium is tight, i.e., $\mathcal{L}_{\mathrm{st}}(q_i^N(0))$ is tight.*

Proof. It can be easily proved that the Markov process considered is irreducible. Consider an arbitrary source, say source 1. Its back-off is less than that in a fictive system, where all other sources attempt to use the channel with highest probability p_0/N. In this fictive system, $\log(Nq_1^N(t)/p_0)$ evolves as a Markov process with generator $P = (p_{ij})$ defined by: $p_{01} = p_0 \times a/N$, $p_{i0} = p_0 2^{-i}(1-a)/N$ and $p_{ii+1} = p_0 2^{-i}a/N$. $a = 1 - (1 - p_0/N)^{N-1} < 1/2$ (since $p_0 < \ln 2$). The analysis of this Markov process is straightforward, it is positive recurrent and we can also prove that in this fictive system, the mean value of $q_1^N(0)$ in the stationary regime is $p_0(1-2a)/(1-a)$. We deduce that $(q_i^N(t))$ in the actual system is also ergodic and that $E_{\mathrm{st}}[q_1^N(0)] \leq p_0(1-2a)/(1-a)$, which provides the desired tightness. □

It can be easily proved that the dynamic system described by differential equations (6)-(7) admits a unique equilibrium point $q_{\mathrm{st}} = (q_k)_k$ defined by:

$$\forall k, \quad q_k = (2(1-e^{-S}))^k q_0, \quad q_0 = Se^{-S} \quad \text{and} \quad S = \sum_k 2^{-k} q_k.$$

The stability of this equilibrium point is obtained remarking that the gradient matrix of the dynamical system has eigenvalues with strictly negative real part at the equilibrium point. Due to lack of space, we leave the study of the global stability of (6)-(7) for future work. In the following we will assume that it is globally stable. We are now able to characterize the system behavior in equilibrium when the number of sources grows large.

Theorem 4 *In equilibrium, for all subset $I \subset \mathbb{N}$ of finite cardinal $|I|$,*

$$\lim_{N\to\infty} \mathcal{L}_{\mathrm{st}}\left((q_i^N(.))_{i\in I}\right) = q_{\mathrm{st}}^{\otimes|I|} \quad \textit{weakly in } \mathcal{P}(D(\mathbb{R}^+,\mathcal{B})^{|I|}), \tag{8}$$

This theorem is proved in Section 4. It states that in equilibrium, the behavior of sources are independent. It then implies that the decoupling approach described in the introduction is asymptotically correct when the number of sources grows large.

4 Proof of Theorems 1 and 2

4.1 Proof of Theorems 1 and 2

Step 1. The sequence $\mathcal{L}(\mu^N)$ is tight in $\mathcal{P}(\mathcal{P}(D(\mathbb{R}^+,\mathcal{B})))$. Thanks again to Sznitman [11] Proposition 2.2, we only have to prove that $\mathcal{L}(q_1^N(.))$ is tight in $\mathcal{P}(D(\mathbb{R}^+,\mathcal{B}))$. By assumption, $\mathcal{L}(q_1^N(0))$ is tight. The jumps of $q_1^N(.)$ are included in those of a Poisson process of intensity p_0. the jump sizes are bounded (by 1). We conclude by the tightness criterion in Ethier-Kurtz [6] p 128.

Step 2. We can mimic the Step 2 in [7]. We show that any accumulation point of $\mathcal{L}(\mu^N)$ satisfies a certain martingale problem. For $f \in L^\infty(\mathcal{B})$, the bounded and forcibly measurable functions of $\mathcal{B} \to \mathbb{R}$, define $f^s(q) = f(S(q)) - f(q)$ and $f^c(q) = f(C(q)) - f(q)$ (s and c stand for *success* and *collision*, respectively). Now, for $f \in L^\infty(\mathcal{B})$,

$$\begin{aligned} f(q_i^N(t)) - f(q_i^N(0)) &= \sum_{k=0}^{[Nt]-1} (f(q_i^N(\frac{(k+1)}{N} - f(q_i^N(\frac{k}{N})) \\ &= \sum_{k=0}^{[Nt]-1} \mathcal{G}^{f,i,N}(k) + M^{f,i,N}(t), \end{aligned}$$

where

$$\begin{aligned} M^{f,i,N}(t) &= \sum_{k=0}^{[Nt]-1} f^s(q_i^N(\frac{k}{N})) \Bigg(1_{\{NU_i(k)\leq q_i^N(\frac{k}{N})\}} \prod_{j\neq i} 1_{\{NU_j(k)>q_j^N(\frac{k}{N})\}} \\ &- \frac{q_i^N(\frac{k}{N})}{N} \prod_{j\neq i}(1 - \frac{q_j^N(\frac{k}{N})}{N}) \Bigg) \end{aligned} \tag{9}$$

$$+f^c(q_i^N(\frac{k}{N}))\left(1_{\{NU_i(k)\le q_i^N(\frac{k}{N})\}}\big(1-\prod_{j\neq i}1_{\{NU_j(k)>q_j^N(\frac{k}{N})\}}\big)-\frac{q_i^N(\frac{k}{N})}{N}\big(1-\prod_{j\neq i}(1-\frac{q_j^N(\frac{k}{N})}{N})\big)\right).$$

and where

$$\mathcal{G}^{i,N}f(k)=\left[f^s(q_i^N(\frac{k}{N}))\frac{q_i^N(\frac{k}{N})}{N}\prod_{j\neq i}(1-\frac{q_j^N(\frac{k}{N})}{N})+f^c(q_i^N(\frac{k}{N}))\frac{q_i^N(\frac{k}{N})}{N}\left(1-\prod_{j\neq i}(1-\frac{q_j^N(\frac{k}{N})}{N})\right)\right],$$

The proofs of the two following lemmas are given at the end of this section.

Lemma 1 *For the martingale $M^{f,i,N}(t)$ defined at (9), the Doob-Meyer Brackets $\langle M^{f,i,N},M^{f,j,N}\rangle=t||f||_\infty\mathcal{O}(1/N)$ as $N\to\infty$ uniformly in $i\neq j$.*

Lemma 2 *The martingale $M^{f,i,N}(t)$ defined at (9) satisfies*

$$M^{f,i,N}(t)=f(q_i^N(t))-f(q_i^N(0))-\int_0^t\mathcal{G}f(\mu_v^N,q_i^N(v))dv+\epsilon^{f,i,N}(t) \tag{10}$$

where

$$\mathcal{G}f(\mu,q)=q\cdot(f^s(q)-f^c(q))\exp(-\langle Id,\mu\rangle)+q\cdot f^c(q) \tag{11}$$

and where $\epsilon_t^{f,i,N}=t||f||_\infty\mathcal{O}(1/N)$ uniformly in i.

If indeed μ^N does converge to Q then (10) will converge to a solution of a martingale problem. Recall that X denotes a canonical trajectory in $D(\mathbb{R}^+,\mathcal{B})$. A probability Q in $\mathcal{P}(D(\mathbb{R}^+,\mathcal{B}))$ solves the non-linear martingale problem if

$$M_t^f=f(X_t)-f(X_0)-\int_0^t\mathcal{G}f(Q(s),X_s)ds \tag{12}$$

is a Q-martingale. It solves the martingale problem starting at Q_0 if $Q(0)=Q_0$.

Let Π^∞ be an accumulation point of $\mathcal{L}(\mu^N)$. Let $R\in\mathcal{P}(D(\mathbb{R}^+,\mathcal{B}))$ belong to the support of Π^∞. Recall as in [7] that the projection map $X\to X_t$ is R-a.s. continuous for all t except perhaps in at most a countable subset D_R of R_+. Further it is shown easy to show that $D=\{t\in R_+:\Pi^\infty(\{R:t\in D_R)>0\}$ is at most denumerable.

Lemma 3 *R satisfies the non-linear martingale problem (12)*

The proof of this lemma is postponed to the end of the section. The continuity of $X\to X_0$ implies $R_0=Q_0$, Π^∞-a.s.

Step 3. We now show the solution to (12) is unique so $R=Q$. It is done as in Theorem 3.3 in [7]. We remark that:

$$\mathcal{G}f(\mu,q)=\int_{\mathcal{B}}(f(y)-f(x))J(\mu,x,dy)$$

where

$$J(\mu, x, dy) = x \exp(-\langle Id, \mu\rangle)(\delta_{S(x)}(dy) - \delta_{C(x)}(dy)) + x\delta_{C(x)}(dy).$$

Next, $||J(\mu, x, \cdot)|| \leq x \leq p_0$ and

$$\begin{aligned} ||J(\alpha, x, \cdot) - J(\beta, x, \cdot)|| &\leq x|\exp(-\langle Id, \alpha\rangle) - \exp(-\langle Id, \beta\rangle)| \\ &\leq x|1 - \exp(\langle Id, \alpha\rangle - \langle Id, \beta\rangle)| \\ &\leq xC|\langle Id, \alpha\rangle - \langle Id, \beta\rangle \\ &\leq p_0 C||\alpha - \beta|| \end{aligned}$$

where $||\cdot||$ denotes the total variation norm. As in Theorem 3.3 in [7] we use Proposition 2.3 in [7] to establish the solution to the martingale problem (12) is unique.

Step 4. We have now proved convergence because any subsequence converges to the same limit Q. We can now identify this limit for the protocol considered in Example 1. If Q satisfies the martingale problem then $(Q(t))_{t\geq 0}$ solves the non-linear Kolmogorov equation derived by taking the expected value:

$$\langle f, Q(t)\rangle - \langle f, Q(0)\rangle = \int_0^t \langle \mathcal{G}f(Q(s), q); Q(s)\rangle ds. \tag{13}$$

Applying (13) to $f = 1_{p_0 2^{-k}}$ for all k, we get the set differential equations (6)-(7) in case of exponential back-off.

Proof of Lemma 1.

By the Dynkin formula, $M^{f,i,N}(t)$ is a martingale (more precisely $M^{f,i,N}(k/N)$ is a discrete martingale, but since we consider only time epochs of the form k/N we allow this *abuse of language*). Next, we define the following variables :

$$S^N_{i,k} = 1_{\{NU_i(k) \leq q^N_i(\frac{k}{N})\}} \prod_{j\neq i} 1_{\{NU_j(k) > q^N_j(\frac{k}{N})\}}$$

and

$$C^N_{i,k} = 1_{\{NU_i(k) \leq q^N_i(\frac{k}{N})\}}\big(1 - \prod_{j\neq i} 1_{\{NU_j(k) > q^N_j(\frac{k}{N})\}}\big).$$

$S^N_{i,k} = 1$ if the user i is the only user accessing the channel at time k and $C^N_{i,k} = 1$ if a collision occurs for user i at time k/N. Both these events have a probability less than p_0/N. Note in particular for $i \neq j$ that :

$$S^N_{i,k}C^N_{i,k} = 0, \quad S^N_{i,k}S^N_{j,k} = 0, \quad S^N_{i,k}C^N_{j,k} = 0 \text{ and } P(C^N_{i,k}C^N_{j,k} = 1) = \mathcal{O}(1/N^2), \tag{14}$$

where the last equation stands uniformly in i, j.

In the sequel, $E_{U(k)}(.)$ will denote $E(.|(p^N_i(k))_i)$. $E_{U(k)}(S^N_{i,k})$ and $E_{U(k)}(S^N_{i,k})$ are bounded by p_0/N. We now rewrite (9) as :

$$M^{f,i,N}(t) = \sum_{k=0}^{[Nt]-1} f^s(q_i^N(\frac{k}{N}))\left(S_{i,k}^N - E_{U(k)}S_{i,k}^N\right) + f^c(q_i^N(\frac{k}{N}))(C_{i,k}^N - E_{U(k)}C_{i,k}^N) \quad (15)$$

To prove Lemma 1, we first show that $E[M^{f,1,N}(t)M^{f,2,N}(t)]$ tends to 0. Since $(M^{f,i,N}(t))$ is a martingale this product is equal to :

$$\begin{aligned} EM^{f,1,N}(t)M^{f,2,N}(t) &= \sum_{k=0}^{[Nt]-1} Ef^s(q_1^N(\frac{k}{N}))\left(S_{1,k}^N - E_{U(k)}S_{1,k}^N\right) f^s(q_2^N(\frac{k}{N}))\left(S_{2,k}^N - E_{U(k)}S_{2,k}^N\right) \\ &\quad + Ef^c(q_1^N(\frac{k}{N}))\left(C_{1,k}^N - E_{U(k)}C_{1,k}^N\right) f^c(q_2^N(\frac{k}{N}))\left(C_{2,k}^N - E_{U(k)}C_{2,k}^N\right) \\ &\quad + 2Ef^c(q_1^N(\frac{k}{N}))\left(C_{1,k}^N - E_{U(k)}C_{1,k}^N\right) f^s(q_2^N(\frac{k}{N}))\left(S_{2,k}^N - E_{U(k)}S_{2,k}^N\right). \end{aligned}$$

Using (14), it appears easily that each term of the sum is bounded by $||f||_\infty \mathcal{O}(1/N^2)$ and the lemma follows.

Proof of Lemma 2.

$$\begin{aligned} \int_0^t \mathcal{G}f(\mu_s^N, q_i^N(s))ds &= \sum_{k=0}^{[Nt]-1} \int_{[k/N]}^{(k+1)/N} \mathcal{G}f(\mu_s^N, q_i^N(s))ds + \int_{[Nt]/N}^t \mathcal{G}f(\mu_s^N, q_i^N(v))ds \\ &= \sum_{k=0}^{[Nt]-1} \int_k^{k+1} \mathcal{G}f(\mu_{w/N}^N, q_i^N(\frac{w}{N}))dw + \frac{1}{N}\int_{[Nt]}^{Nt} \mathcal{G}f(\mu_{w/N}^N, q_i^N(\frac{w}{N}))dw \end{aligned}$$

Hence

$$\begin{aligned} |\epsilon_t^{f,i,N}| &\leq \sum_{k=0}^{[Nt]-1} |\mathcal{G}^{i,N}f(k) - \frac{1}{N}\int_k^{k+1} \mathcal{G}f(\mu_{w/N}^N, q_i^N(\frac{w}{N}))dw| + \frac{1}{N}|\int_{[Nt]}^{Nt} \mathcal{G}f(\mu_{w/N}^N, q_i^N(\frac{w}{N}))dw| \\ &\leq \sum_{k=0}^{[Nt]-1} |\mathcal{G}^{i,N}f(k) - \frac{1}{N}\mathcal{G}f(\mu_{k/N}^N, q_i^N(\frac{k}{N}))| + \frac{1}{N}|\mathcal{G}f(\mu_{k/N}^N, q_i^N(\frac{k}{N}))|. \end{aligned}$$

The last term of this inequality is obviously less than $3p_0||f||_\infty/N$ (since $q_i^N(t) \leq p_0/N$). To finish the proof, it remains to bound the terms appearing in the sum and, taking a closer look at $\mathcal{G}$, it suffices to bound $\exp(-1/N\sum_{i=1}^N q_i^N(k/N)) - \prod_{i\neq j}(1 - q_i^N(k/N)/N)$ by $\mathcal{O}(1/N)$ uniformly in i, j. We have :

$$|e^{-\frac{1}{N}\sum_{i=1}^N q_i^N(k/N)} - \prod_{i\neq j}(1 - \frac{q_i^N(k/N)}{N})| \leq e^{-p_0}|1 - e^{\frac{q_i^N(k/N)}{N} + \sum_{j\neq i}|\ln(1-\frac{q_j^N(k/N)}{N}) + \frac{q_j^N(k/N)}{N}|}|.$$

Using the inequality $|\ln(1-x)+x| \leq \frac{x^2}{2(1-x)^2}$ for $x \in (0,1)$, we easily deduce the required bound.

Proof of Lemma 3.

We prove this lemma as in Step 2 of Theorem 3.4 of [7]. Take $0 \le s_1 < s_2 < \cdots s_k \le s < t$ outside D and $g \in L^\infty(\mathcal{B}^k)$. Take $f \in L^\infty(\mathcal{B})$. The map

$$G : R \in \mathcal{P}(D(\mathbb{R}^+, \mathcal{B})) \to \langle \left(f(X_t) - f(X_s) - \int_s^t \mathcal{G}f(R_u, X_u)du \right) g(X_{s_1}, \ldots, X_{s_k}); R\rangle$$

is Π^∞-a.s. continuous.

Let Π^N be the law of μ^N, following [7], we write :

$$\begin{aligned}\langle G^2, \Pi^N\rangle &= E(G(\mu^N)^2) \\ &= E(G(\frac{1}{N}\sum_{i=1}^N \delta_{q_i^N})^2)\end{aligned}$$

$$= E\left(\frac{1}{N}\sum_{i=1}^N \left(f(q_i^N(t)) - f(q_i^N(s)) - \int_s^t \mathcal{G}f(\mu_u^N, q_i^N(u))du \right) g(q_i^N(s_1), \ldots, q_i^N(s_k))\right)^2$$

$$= E\left(\frac{1}{N}\sum_{i=1}^N \left(M^{f,i,N}(t) - M^{f,i,N}(s) - (\epsilon^{f,i,N}(t) - \epsilon^{f,i,N}(s))\right) g(q_i^N(s_1), \ldots, q_i^N(s_k))\right)^2. \tag{16}$$

Let $g^{i,N} = g(q_i^N(s_1), \ldots, q_i^N(s_k))$ and for a process Y let $Y_{s,t} = Y_t - Y_s$. Using exchangeability, from (16) we obtain :

$$\langle G^2, \Pi^N\rangle = E\big(\frac{1}{N}\sum_{i=1}^N \left(M_{s,t}^{f,i,N} - \epsilon_{s,t}^{f,i,N}\right) g(q_i^N(s_1), \ldots, q_i^N(s_k))\big)^2$$

$$= \frac{1}{N}E\big((M_{s,t}^{f,1,N} - \epsilon_{s,t}^{f,1,N})g^{1,N}\big)^2 + \frac{N-1}{N}E(M_{s,t}^{f,1,N} - \epsilon_{s,t}^{f,1,N})g^{1,N}(M_{s,t}^{f,2,N} - \epsilon_{s,t}^{f,2,N})g^{2,N}.$$

Lemmas 1 and 2 imply that $\langle G^2, \Pi^N\rangle$ tends to 0. From Fatou's lemma, $\langle G^2, \Pi^\infty\rangle \le \lim_N \langle G^2, \Pi^N\rangle = 0$ and thus Π^∞-a.s, $G(R) = 0$. Since this holds for arbitrary $0 \le s_1 < s_2 < \cdots s_k \le s < t$ outside D and $g \in C_b(\mathcal{B}^k)$ it follows that R satisfies the non-linear martingale problem (12).

4.2 Proof of Theorem 4

Let $(q_i^N(0))_i$ have the invariant law of the system with N users. By symmetry, $(q_i^N(0))_i$ is exchangeable. We define $\Pi^N = 1/N\sum_{i=1}^N q_i^N$. By Theorem 3, Π_0^N is tight. Consider an accumulation point Π_0^∞. We cannot apply directly Theorem 1 since we do not know whether the subsequence of Π_0^N converges weakly toward a deterministic limit.

We now circumvent this difficulty. As in step 1 in the proof of Theorem 1, we deduce from Sznitman [11] Proposition 2.2, that Π^N is tight in $\mathcal{P}(\mathcal{P}(D(\mathbb{R}^+, \mathcal{B})))$. Let R in $\mathcal{P}(D(\mathbb{R}^+, \mathcal{B}))$ in the support of an accumulation point of Π^N. We can prove similarly that Lemma 3 still holds for R.

By Step 3 of Theorem 1, the solution of the martingale problem is unique and R solves it with initial condition R_0. The global stability of (6)-(7) implies that $\lim_{t\to+\infty} R_t = q_{st}$. However, by stationarity, Π_t^N and Π_0^N are equal, we deduce immediately that the support Π_0^N is reduced to q_{st} and $R_0 = q_{st}$.

Theorem 4 is then a consequence of Theorem 1.

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