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Abstract: We obtain error bounds for monotone approximation schemes of a stochastic impulse control problem. This is an extension of the theory for error estimates for the Hamilton-Jacobi-Bellman equation.

For obtaining these bounds we build a sequence of stochastic impulse control problems, and a sequence of monotone approximation schemes. Extending methods of Barles and Jakobsen [2], we give error estimate for each problem of the sequence. Using these bounds we obtain the result. We obtain the same estimate on the rate of convergence as in the equation without impulsions [3], [2].

Key-words: Hamilton-Jacobi-Bellman equation, stochastic impulse control, finite differences, error estimates.

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Estimations d'erreur pour un problème de contrôle impulsionnel stochastique

Résumé : Nous obtenons des estimations d'erreurs pour des schémas d'approximation monotones d'un problème de contrôle stochastique impulsionnel. Ceci constitue une extension de la théorie des estimations d'erreurs pour l'équation de Hamilton-Jacobi-Bellman.

Pour obtenir ces estimations, on construit une suite de problèmes de contrôle stochastique impulsionnel, et une suite associée de schémas d'approximation. En étendant la méthode de Barles et Jakobsen [2], on donne une estimation d'erreur pour chaque problème de la suite. On utilise alors ces estimation pour obtenir le résultat. Nous obtenons les mêmes estimations que pour l'equation sans impulsions.

Mots-clés : Equation de Hamilton-Jacobi-Bellman, contrôle stochastique impulsionnel, différences finies, estimations d'erreur.

1 Introduction

The aim of this paper is to give error bounds for approximation schemes of the impulse control problem. More precisely we consider the following equation

$$\max\{\sup_{\alpha_i\in\mathcal{A}} L^{\alpha_i}(x,\mathcal{D}u); u(x)-\mathcal{M}u(x)\}=0,\ x\in\mathbb{R}^N, \tag{P}$$

where

$$\begin{aligned} L^{\alpha_i}(x,\mathcal{D}u(x)) &= L^{\alpha_i}(x,u(x),Du(x),D^2u(x)),\\ L^{\alpha_i}(x,t,p,X) &= -tr[a^{\alpha_i}(x)X]-b^{\alpha_i}(x)p+c^{\alpha_i}(x)t-f^{\alpha_i}(x). \end{aligned}$$

and

$$\begin{cases} \mathcal{M}u(x) := k+\inf_{\xi\in\mathbb{R}^N_+}\{u(x+\xi)+c(\xi)\},\\ k>0, \quad c:\mathbb{R}^N_+\to\mathbb{R}_+,\\ c(0)=0, \quad c(\xi_1+\xi_2)\le c(\xi_1)+c(\xi_2). \end{cases} \tag{1}$$

Here $\mathcal{A}=\{\alpha_1,\dots,\alpha_M\}$ denotes the set of controls, assumed to be finite. The coefficients $(a^{\alpha_i},b^{\alpha_i},c^{\alpha_i},f^{\alpha_i})$ are, for each $\alpha_i\in\mathcal{A}$, bounded and Lipschitz functions $\mathbb{R}^N\to\mathcal{S}^N\times\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}$, where $\mathcal{S}^N$ denotes the set of $N\times N$ symmetric matrices. Under classical assumptions, (P) has a unique bounded viscosity solution, denoted u. The regularity of u depends on the properties of the coefficients a,b,c,f. We refer to [13], [14] for the existence, the uniqueness and the regularity of u.

Then we consider monotone approximation schemes of (P), of the following form:

$$\max\{S(h,x,u_h(x),u_h); u_h(x)-\mathcal{M}u_h(x)\}=0, \quad x\in\mathbb{R}^N, \tag{S}$$

where $S:\mathbb{R}^N_+\times\mathbb{R}^N\times\mathbb{R}\times C_b(\mathbb{R}^N)\to\mathbb{R}$ is a consistent, monotonic and uniformly continuous approximation of $\sup_{\alpha_i\in\mathcal{A}} L^{\alpha_i}(x,\mathcal{D}u(x))$ (see section 2). We will denote $u_h\in C_b(\mathbb{R}^N)$ the solution of (S), which is the approximation of u, and $h\in\mathbb{R}^N$ the mesh size. This abstract notations was introduced by Barles and Souganidis [4] to display clearly the monotonicity of the scheme: $S(h,x,r,v)$ is non decreasing in r and non increasing in v. Typical approximation schemes that we will consider are Classical Finite Differences [20], Generalized Finite Differences [7] and [6], and Markov Chain Approximations [20].

Until now, results on convergence rates for monotone approximation schemes of the equation without impulsions have been obtained; i.e. for the following equation:

$$\sup_{\alpha_i\in\mathcal{A}} L^{\alpha_i}(x,\mathcal{D}u(x))=0, \qquad x\in\mathbb{R}^N, \tag{2}$$

and the related scheme

$$S(h,x,u_h(x),u_h)=0, \qquad x\in\mathbb{R}^N.$$

Error estimates for this equation have been obtained by Krylov [18],[19] and these results were extended by Barles and Jakobsen [3],[2]. Moreover, results on convergence rate for monotone approximation schemes of a particular Isaac equation have been obtained by the authors [5], and by Jakobsen [16], [15].

Using the method introduced by Ishii [14], to prove the existence of a unique viscosity solution of (P), we approach (P) by a sequence of cascade problems (Pn), $n \geq 1$,

$$\max\{\sup_{\alpha_i} L^{\alpha_i}(x, \mathcal{D}u(x)); u(x) - \mathcal{M}u_{n-1}(x)\} = 0, \quad x \in \mathbb{R}^N, \tag{3}$$

where, for $n = 0$, we have equation (2). Let u_n the viscosity solution of (Pn). In the same way we approach (S) by a sequence of cascade schemes (Sn), $n \geq 1$,

$$\max\{S(h, x, u_h(x), u_h); u_h(x) - \mathcal{M}u_{h(n-1)}(x)\} = 0, \quad x \in \mathbb{R}^N, \tag{4}$$

where, for $n = 0$ we have equation (S0). Let u_{hn} denote the solution of (Sn).

Using the methods introduced by Barles and Jakobsen [2], upper and lower bounds of $u_n - u_{hn}$, for all $n < +\infty$, are obtained. The upper estimate of $u_n - u_{hn}$ is easier to obtain than the lower. The proof involves a "Krylov regularization" of (Pn), i.e. the perturbed equation

$$\max\left\{\sup_{\alpha_i, |e| \leq \epsilon} L^{\alpha_i}(x + e), \mathcal{D}u_n^\epsilon(x)); u_n^\epsilon(x) - \mathcal{M}u_{n-1}(x)\right\} = 0,$$

and its viscosity solution u_n^ϵ. A regularization of u_n^ϵ by convolution gives an approximate smooth sub-solution of (3), denoted $u_{n\epsilon}$, which is also an approximate sub-solution of (4). So, by using the consistency property, we obtain the upper bound of $u_n - u_{hn}$, after choosing an optimal parameter of regularization. Then, we consider $u - u_h$ and we do the following decomposition

$$\sup_x (u(x) - u_h(x)) \leq \sup_x (u(x) - u_n(x)) + \sup_x (u_n(x) - u_{hn}(x)) \\ + \sup_x (u_{hn}(x) - u_h(x)),$$

for all n in $\mathbb{N}$. Finally, choosing the optimal n, we obtain the result.

To obtain the lower estimate, we start by giving lower bound of $u_n - u_{hn}$, for $n \in \mathbb{N}$. We introduce the following switching system which approximates (3)

$$\max\{L^{\alpha_i}(x, \mathcal{D}v_i^n(x)); v_i^n(x) - \min_{j \neq i}\{v_j^n(x) + \ell\}; v_i^n(x) - \mathcal{M}u_{n-1}(x)\} = 0, \tag{5}$$

for $x \in \mathbb{R}^N$, and $i \in \mathcal{I} = \{1, \ldots, M\}$, $\ell \geq 0$. For literature on the switching systems, see [8], [10], [11] and [12]. We consider the viscosity solution $v^n = (v_1^n, \ldots, v_M^n)$ of this system, and give an estimate of the rate of convergence of v^n to u_n. Then we consider a perturbed system

$$\max\{\inf_{|e| \leq \epsilon} L^{\alpha_i}(x + e, \mathcal{D}w_i^{n\epsilon}(x)); w_i^{n\epsilon}(x) - \min_{j \neq i}\{w_j^{n\epsilon}(x) + \ell\}; \\ w_i^{n\epsilon}(x) - \mathcal{M}u_{n-1}(x)\} = 0, \tag{6}$$

for all $i \in \mathcal{I}$ and $x \in \mathbb{R}^N$, and its viscosity solution $w^{n\epsilon} = (w_1^{n\epsilon}, \ldots, w_M^{n\epsilon})$. We regularize $w^{n\epsilon}$ by convolution obtaining $w_{n\epsilon}$, and this function allows to build a local super-solution of (3).

Then, by applying the consistency and the monotonicity of the scheme, we obtain the lower bound of $u_n - u_{hn}$. Finally, since

$$\sup_x (u_h(x) - u(x)) \leq \sup_x (u_h(x) - u_{hn}(x)) + \sup_x (u_{hn}(x) - u_n(x)) + \sup_x (u_n(x) - u(x)),$$

choosing the optimal n, we obtain the result. With our result, we can prove an upper bound of $|h|^{1/2}$ and a lower bound of $|h|^{1/5}$ for classical finite differences scheme and for generalized finite differences scheme. We note that we obtain the same error estimates as in the case without impulsions [3], [2].

The paper is organized as follows: section 2 introduces the notations and gives the main result. Section 3 introduce the cascade approximations of (P) and (S). Section 4 obtains upper bound of $u_n - u_{hn}$, for all $n < +\infty$, and Section 5 gives lower bound of $u_n - u_{hn}$, for all $n < +\infty$. Section 6 is devoted to the proof of the main theorem. Finally the Appendix gives some auxiliary theorems which are used throughout the paper.

2 Notations and main result

We start by introducing some notations we will use in the article. By $|\cdot|$ we mean the standard Euclidean norm in any $\mathbb{R}^M$ type space. In particular, if $X \in \mathcal{S}^N$, then $|X|^2 = tr(XX^\top)$, where $X^\top$ is the transpose of X, i.e. $|X|$ is the Frobenius norm. If g is a bounded function from $\mathbb{R}^N$ into either $\mathbb{R}, \mathbb{R}^M$, or the space of $N \times P$ matrices, we set

$$|g|_0 := \sup_{x \in \mathbb{R}^N} |g(x)|.$$

If g is also Lipschitz continuous, we set

$$[g]_1 := \sup_{\substack{x,y \in \mathbb{R}^N \\ x \neq y}} \frac{|g(x) - g(y)|}{|x - y|}, \qquad |g|_1 := |g|_0 + [g]_1.$$

We denote by $\leq$ the component wise ordering in $\mathbb{R}^N$, and by $\preceq$ the ordering in the sense of positive semidefinite matrices in $\mathcal{S}(N)$. The space $C_b(\mathbb{R}^N)$ (resp. $C_{b,l}(\mathbb{R}^N)$) will denote the space of continuous and bounded functions (resp. bounded and Lipschitz functions) from $\mathbb{R}^N$ to $\mathbb{R}$.

Given $g \in C_{b,l}(\mathbb{R}^N)^M$, $M \geq 1$, we denote by L_g an upper bound of the Lipschitz constant of g, $L_g := \max_{i=1,\ldots,M}[g_i]_1$.

We will use a sequence of mollifiers $(\rho_\epsilon)_\epsilon$ defined as follows:

$$\rho_\epsilon(x) = \epsilon^{-N} \rho(x/\epsilon), \tag{7}$$

where $\rho \in C^\infty(\mathbb{R}^N)$, $\int_{\mathbb{R}^N} \rho = 1$, $\mathrm{supp}\{\rho\} \subseteq \bar{B}(0,1)$ and $\rho \geq 0$. We define the mollification of $g \in C_b(\mathbb{R}^N)$ as follows:

$$g_\epsilon(x) := \int_{\mathbb{R}^N} g(x-e)\rho_\epsilon(e)\mathrm{d}e. \tag{8}$$

Moreover, if g is Lipschitz continuous, then

$$|g(x) - g_\epsilon(x)| \leq L_g \epsilon. \tag{9}$$

If $g \in C_b(\mathbb{R}^N)$ (resp. $C_{b,l}(\mathbb{R}^N)$), then we have

$$|D^i g_\epsilon(x)| \leq C\epsilon^{-i}|g|_0, \qquad (\text{resp. } C\epsilon^{1-i}|g|_0), \quad \forall\, i = 1, \ldots, n. \tag{10}$$

$>$From [14], we have the following properties on $\mathcal{M}$, defined in (1).

Proposition 2.1 *Let $u, v : \mathbb{R}^N \to \mathbb{R}$. Under assumptions (A1)-(A2) we have:*

(1) If $u \leq v$ in $\mathbb{R}$, then $\mathcal{M}u \leq \mathcal{M}v$ in $\mathbb{R}^N$.

(2) $\mathcal{M}(tu + (1-t)v) \geq t\mathcal{M}u + (1-t)\mathcal{M}v$; $t \in [0,1]$.

(3) $\mathcal{M}(u + c) = \mathcal{M}u + c$, for $c \in \mathbb{R}$.

(4) $|\mathcal{M}u - \mathcal{M}v|_0 \leq |u - v|_0$ for all $u, v \in C(\mathbb{R}^N)$. □

The assumption we use on equation (P) are as follows:

(A1) For all $\alpha_i \in \mathcal{A}$, the matrix a^{α_i} can be written $a^{\alpha_i} = \frac{1}{2}\sigma^{\alpha_i}\sigma^{\alpha_i T}$, where σ^{α_i} is a $N \times P$ matrix. There exists a constant K such that, for all $\alpha_i \in \mathcal{A}$,

$$c^{\alpha_i} \geq 1 \quad \text{and} \quad |\sigma^{\alpha_i}|_1 + |b^{\alpha_i}|_1 + |c^{\alpha_i}|_1 + |f^{\alpha_i}|_1 \leq K.$$

(A2) $1 > \sup_{\alpha_i}\{[\sigma^{\alpha_i}]_1^2 + [b^\alpha]_1\}$.

Assumption (A1) ensures that all equations we will use are well-posed; assumption (A2) ensures that all solutions are Lipschitz and bounded functions. Without assumption (A2), we have that all solutions are Hölder and bounded. All our results can be extended to this case.

Result of [14, Theorem 4.2] gives the existence of a viscosity solution of (P). Moreover, generalizing, in the obvious way, proof of [1, Theorem 3.5], which involves only first order impulse control problem, we obtain the following proposition.

Proposition 2.2 *Under the assumptions (A1-A2), (P) has a unique viscosity solution u in $C_{b,l}(\mathbb{R}^N)$. In particular we have*

$$|u|_0 \leq \sup_{\alpha_i} |f^{\alpha_i}|_0. \qquad \square$$

Let $C \geq 0$ a constant, and consider the following equation:

$$\max\{\sup_{\alpha_i} L_C^{\alpha_i}(x, \mathcal{D}u); u(x) - \mathcal{M}u(x)\} = 0,\ x \in \mathbb{R}^N, \tag{PC}$$

where $L_C^{\alpha_i}(x,r,p,X) = L^{\alpha_i}(x,r,p,X) - Cc^{\alpha_i}(x)$. We have then the following lemma, which is given without proof.

Lemma 2.3 *u is a viscosity solution of (P) if and only if $u + C$ is a viscosity solution of (PC).*□

Remark 2.4 *In the sequel we assume that $f^{\alpha_i}(x) \geq 0$, for all x and α_i, since that slightly simplifies the proofs; however, using lemma 2.3, all our results are easily extended to the case when f is not nonnegative.*

We now state assumptions on the discrete scheme (S), which approaches the equation (P):

(S1) Monotonicity: $S(h, x, r + m, u + m) \geq m + S(h, x, r, v)$ for all $h \in \mathbb{R}_+^N$, $r \in \mathbb{R}$, $m \geq 0$, $x \in \mathbb{R}^N$ and u, v in $C_b(\mathbb{R}^N)$ such that $u \leq v$ in $\mathbb{R}^N$.

(S2) Regularity: for all $h \in \mathbb{R}_+^N$ and $\phi \in C_b(\mathbb{R}^N)$, $x \mapsto S(h, x, \phi(x), \phi)$ is bounded and continuous; $r \mapsto S(h, x, r, \phi)$ is uniformly continuous for bounded r, uniformly with respect to $x \in \mathbb{R}^N$.

(S3) There exist $n, k_i > 0$, $i \in J \subseteq \{1, \dots, n\}$ and a constant $K_c > 0$ such that for all $h \in \mathbb{R}_+^N$ and x in $\mathbb{R}^N$, and for every smooth $\phi \in C^n(\mathbb{R}^N)$ such that $|D^i\phi|_0$ is bounded, for every $i \in J$, the following holds:

$$\left| \sup_{\alpha_i} L^{\alpha_i}(x, \mathcal{D}\phi) - S(h, x, \phi(x), \phi) \right| \leq K_c Q(\phi),$$

where $Q(\phi) := \sum_{i \in J} |D^i\phi|_0 |h|^{k_i}$.

(S4) If v is solution of

$$\max\{S(h, x, v(x), v); v(x) - \mathcal{M}v(x)\} = 0, \tag{11}$$

then νv is solution of

$$\max\{S(h, x, \nu v(x), \nu v) + (\nu - 1)f(x); \nu v(x) - \nu\mathcal{M}v(x)\} = 0, \tag{12}$$

where ν is in $(0, 1)$, and f defined in equation (P).

(S5) Let $C \geq 0$ a constant. If v is solution of $\max\{S(h, x, v(x), v); v(x) - \mathcal{M}v(x)\} = 0$,then $v + C$ is solution of $\max\{S_C(h, x, v(x), v); v(x) - \mathcal{M}v(x)\} = 0$, where S_C is defined as S, with the term f^{α_i} is replaced by $f^{\alpha_i} + c^{\alpha_i}C$.

Example 2.5 *An example of a numerical scheme which satisfies these assumptions is the standard Finite Difference Scheme when $N=1$, defined as:*
$S(h,x,r,\phi) :=$

$$\sup_{\alpha_i\in\mathcal{A}}\{-a^{\alpha_i}(x)\Delta\phi(x) - b_+^{\alpha_i}(x)\delta_+\phi(x) + b_-^{\alpha_i}(x)\delta_-\phi(x) + c^{\alpha_i}(x)r - f^{\alpha_i}(x)\}, \tag{13}$$

where

$$\delta_\pm\phi(x) = \frac{\phi(x\pm h)-\phi(x)}{h}, \quad \Delta\phi(x) = \frac{\phi(x+h)-2\phi(x)+\phi(x-h)}{h^2},$$

and $b_+^{\alpha_i}(x) := \max(b^{\alpha_i}(x),0)$, $b_-^{\alpha_i}(x) := \max(-b^{\alpha_i}(x),0)$. Clearly assumptions (S1), (S2), (S4) and (S5) are satisfied. For the consistency hypothesis (S3), we obtain, from a Taylor expansion,

$$Q(\phi) = |D^2\phi|h + |D^4\phi|h^2, \tag{14}$$

i.e. $k_2=1$ and $k_4=2$.

We will say that a function $v\in C_b(\mathbb{R}^N)$ is a sub-solution (resp. super-solution) of the scheme (S) if

$$\max\{S(h,x,v(x),v); v(x)-\mathcal{M}v(x)\} \le 0, \quad (\text{ resp. } \ge 0), \text{ for all } x\in\mathbb{R}^N.$$

The next proposition is a first step for proving uniqueness of solution of (S).

Proposition 2.6 *Let S satisfy (S1-S5), and u, v be the solutions of*

$$\max\{S(h,x,u(x),u); u(x)-\psi_1(x)\} = 0,\ x\in\mathbb{R}^N, \tag{15}$$

$$\max\{S(h,x,v(x),v)+g(x); v(x)-\psi_2(x)\} = 0,\ x\in\mathbb{R}^N, \tag{16}$$

where ψ_1, ψ_2 and g are elements of $C_b(\mathbb{R}^N)$. Then,

$$|u-v|_0 \le \max\{|g|_0; |\psi_1-\psi_2|_0\}. \tag{17}$$

Proof. Since u and v are solutions of (15) and (16) respectively, we have that

$$\max\{S(h,x,u(x),u); u(x)-\psi_1(x)\} \le 0,$$

$$\max\{S(h,x,v(x),v)+g(x); v(x)-\psi_2(x)\} \ge 0,$$

for all x in $\mathbb{R}^N$. Since $\max\{A;B\}-\max\{C;D\} \le \max\{A-C;B-D\}$, (15) and (16) imply

$$0 \le \max\{S(h,x,v(x),v)+g(x)-S(h,x,u(x),u); v(x)-\psi_2(x)-(u(x)-\psi_1(x)\}.$$

Hence we have the two following cases.
a) $u(x)-v(x)\le\psi_1(x)-\psi_2(x)$, which implies $u(x)-v(x)\le|\psi_1-\psi_2|_0$.
b) $S(h,x,u(x),u)\le 0$, and $S(h,x,v(x),v)+g(x)\ge 0$. Then $S(h,x,v(x),v)+|g|_0\ge 0$, and

applying the monotonicity, $S(h, x, v(x) + |g|_0, v + |g|_0) \geq 0$. By [3, Theorem 2.1], obtain $u(x) - v(x) \leq |g|_0$.

Combining the two cases we have

$$\sup_x (u(x) - v(x)) \leq \max\{\sup_x g(x); \sup_x |\psi_1(x) - \psi_2(x)|\}.$$

The converse inequality is obtained in a similar way. □

We can give now the uniqueness result.

Proposition 2.7 *There exists a unique solution $u_h \in C_b(\mathbb{R}^N)$ of (S).*

Proof. Let u_h and v_h be solutions of (S). By (S4), νu_h is a solution of

$$\max\{S(h, x, \nu u_h(x), u_h) + (\nu - 1)f(x); \nu u_h(x) - \nu \mathcal{M} u_h(x)\} = 0, \ x \in \mathbb{R}^N,$$

for $\nu \in (0, 1)$. Apply proposition 2.6 to obtain

$$|\nu u_h - v_h|_0 \leq \max\{|(\nu - 1)f|_0; |\nu \mathcal{M} u_h - \mathcal{M} v_h|_0\}.$$

By [1, Theorem 3.5], we know that $|\nu \mathcal{M} u_h - \mathcal{M} v_h|_0 < |\nu u_h - v_h|_0$, and hence $|\nu u_h - v_h|_0 \leq |(\nu - 1)f|_0$. Letting ν go to 1, we have the result. □

We set, J being defined in (S3):

$$\bar{\gamma} := \min_{i \in J} \left\{ \frac{k_i}{i} \right\}, \qquad \underline{\gamma} := \min_{i \in J} \left\{ \frac{k_i}{3i - 2} \right\}. \tag{18}$$

We explain briefly how we obtain our main result. In the following we build sequences (Pn) and (Sn), $n \geq 0$, of equation of type (3) and (4) respectively, which approximate (P) and (S). Then we have that the sequence of viscosity solutions u_n of (Pn), $n \geq 0$, converges to u, and the sequence of solution u_{hn} of (Sn), $n \geq 0$, converges to u_h. We will give these rates of convergence. Finally, for each n we give an upper and a lower bound of $u_n - u_{hn}$, and we use these bounds to obtain (19).

We state now our main result.

Theorem 2.8 *Assume that (A1-A2) and (S1-S5) hold. Let $u \in C_{b,l}(\mathbb{R}^N)$ and $u_h \in C_b(\mathbb{R}^N)$ be the unique viscosity solution of (P), and the unique solution of (S), respectively. The following two bounds hold:*

$$-C|h|^{\underline{\gamma}} \leq u - u_h \leq C|h|^{\bar{\gamma}}, \tag{19}$$

where C is a bounded constant, which depends on K defined in (A1), and on the rates of convergence of u_n and u_{hn}.

Consider now the finite difference scheme given in example 2.5. We have the following result:

Corollary 2.9 *Let u the solution of (P), for $N = 1$, and let u_h the solution of (S), with S defined as in (13). The following two bounds hold:*

$$-C|h|^{1/5} \leq u - u_h \leq C|h|^{1/2}, \tag{20}$$

where C is a bounded constant, which depends on K defined in (A1), and on the rates of convergence of u_n and u_{hn}.

Proof. Applying (14), we obtain $\underline{\gamma} = 1/5$ and $\bar{\gamma} = 1/2$. Then we can use the precedent theorem to obtain the result. □

Remark 2.10 *Corollary 2.9 can be extended to the Finite Differences scheme in dimension $N > 1$ [20], and to the Generalized Finite Differences scheme in dimension $N \geq 1$ [6], [7]. The bounds that we obtain are the same as (20), where now h is the vector of space steps along each component of x.*

3 The cascade approximations

In this section we will present the approximations of (P) and (S), and we will study their main properties.

3.1 Cascade for the HJB equation

We will approach equation (P) by a sequence of obstacle problems. We will use the same methods as in [14, Proof of theorem 4.2], to prove that the solutions of the sequence of equations converge to the solution of (P).

By remark 2.4, we have that $u \equiv 0$ is a viscosity sub-solution of (P).

Consider the following problem:

$$\sup_{\alpha_i} L^{\alpha_i}(x, \mathcal{D}u(x)) = 0, \quad x \in \mathbb{R}^N. \tag{P0}$$

Under assumptions (A1-A2), this equation has a unique viscosity solution u_0 in $C_{b,l}(\mathbb{R}^N)$. Since $u \equiv 0$ is a viscosity sub-solution of (P0), the comparison principle (see [14, Theorem 3.3]) implies $0 \leq u_0$. Consider the following problem:

$$\max\{\sup_{\alpha_i} L^{\alpha_i}(x, \mathcal{D}u(x)); u(x) - \mathcal{M}u_0(x)\} = 0, \quad x \in \mathbb{R}^N. \tag{P1}$$

Since $\mathcal{M}u_0$ is continuous, under assumptions (A1-A2), there exists a unique viscosity solution u_1 of (P1) in $C_{b,l}(\mathbb{R}^N)$. Similarly, for $n = 2, 3, \cdots$, let $u_n \in C_{b,l}(\mathbb{R}^N)$ be the unique viscosity solution of

$$\max\{\sup_{\alpha_i} L^{\alpha_i}(x, \mathcal{D}u(x)); u(x) - \mathcal{M}u_{n-1}(x)\} = 0, \quad x \in \mathbb{R}^N. \tag{Pn}$$

It is easy to check that u_1 is a viscosity sub-solution of $(P0)$. By the comparison principle, $u_1 \leq u_0$. Moreover, we know that $u \equiv 0$ is a sub-solution of $(P1)$ in $\mathbb{R}^N$, and then $0 \leq$

$u_1 \le u_0$ in $\mathbb{R}^N$. Proposition 2.1,(1), implies that $\mathcal{M}u_1 \le \mathcal{M}u_0$, then we can say that u_2 is a viscosity sub-solution of (P1), and also $u_2 \le u_1$ in $\mathbb{R}^N$. By induction over n, we obtain:

$$0 \le \cdots \le u_n \le \cdots \le u_2 \le u_1 \le u_0. \tag{21}$$

We can see that, if $|u_0|_0 \le k$, then u_0 is a viscosity solution of (P) and then we refer to [3], [2] for error estimates. Suppose now that $|u_0|_0 > k$, and let $\mu \in (0,1)$ such that set $\mu |u_0|_0 < k$.

Theorem 3.1 *We have that, for all n,*

$$u_n - u_{n+1} \le (1-\mu)^n |u_0|_0. \tag{22}$$

Proof: Let $n \in \mathbb{N}$, and $\theta_n \in (0,1]$ be such that

$$u_n - u_{n+1} \le \theta_n u_n, \quad \text{in } \mathbb{R}^N. \tag{23}$$

(By (21), this holds at least for $\theta_n = 1$.) Rewriting (23) as $(1-\theta_n)u_n \le u_{n+1}$, and using proposition 2.1, get

$$(1-\theta_n)\mathcal{M}u_n + \theta_n k \le (1-\theta_n)\mathcal{M}u_n + \theta_n \mathcal{M}0 \le \mathcal{M}[(1-\theta_n)u_n] \le \mathcal{M}u_{n+1}. \tag{24}$$

We now prove that

$$(1-\theta_n+\mu\theta_n)u_{n+1} \le u_{n+2}, \tag{24a}$$

where u_{n+2} is the viscosity solution of (Pn+2). Since u_{n+1} is the viscosity solution of (Pn+1), and $f^{\alpha_i}(x) \ge 0$, for all x and for all α_i, we have that $(1-\theta_n+\mu\theta_n)u_{n+1}$ is a viscosity sub-solution of $\sup_{\alpha_i} L^{\alpha_i}(x, \mathcal{D}v(x)) = 0$. Moreover, by the construction of the sequence (21), and by (24), we have

$$(1-\theta_n+\mu\theta_n)u_{n+1} \le (1-\theta_n)u_{n+1} + \mu\theta_n |u_0|_0, \tag{25a}$$

$$\mathcal{M}u_{n+1} \ge (1-\theta_n)\mathcal{M}u_n + \theta_n k. \tag{25b}$$

Taking the difference between (25a) and (25b), and knowing that u_{n+1} is the viscosity solution of (Pn), we have

$$\begin{aligned} &(1-\theta_n+\mu\theta_n)u_{n+1}(x_0) - \mathcal{M}u_{n+1}(x_0) \\ &\le (1-\theta_n)u_{n+1}(x_0) + \mu\theta_n|u_0|_0 - (1-\theta_n)\mathcal{M}u_n(x_0) - \theta_n k \\ &\le (1-\theta_n)u_{n+1}(x_0) + \theta_n k - (1-\theta_n)\mathcal{M}u_n(x_0) - \theta_n k \le 0. \end{aligned}$$

So we can say that $(1-\theta_n+\mu\theta_n)u_{n+1}$ is a viscosity sub-solution of (Pn+2). The comparison principle implies (24a), or equivalently

$$u_{n+1} - u_{n+2} \le \theta_n(1-\mu)u_{n+1}. \tag{26}$$

As in [14, Proof of theorem 4.2], by the inequalities $u_0 - u_1 \leq u_0$ in $\mathbb{R}^N$, we obtain $u_1 - u_2 \leq (1-\mu)u_1$ in $\mathbb{R}^N$. Then we can take $\theta_1 = 1-\mu$ and we obtain $u_2 - u_3 \leq (1-\mu)^2 u_2$, and by induction we have

$$u_{n+1} - u_{n+2} \leq (1-\mu)^{n+1} u_{n+1} \leq (1-\mu)^{n+1} |u_0|_0. \qquad \square \tag{27}$$

By (21) and (22), we can find a function $u \in C(\mathbb{R}^N)$, such that $|u_n - u|_0 \to 0$, when $n \to +\infty$. Proposition 2.1 and the stability of solutions imply that u is a viscosity solution of (P). Then we can say that u_n converges to u, the unique viscosity solution of (P), when $n \to +\infty$. We want to estimate an upper bound of $u_n - u$ for an arbitrary n. By (22) and since $(1-\mu) < 1$, we obtain that, for all $n \geq 0$,

$$u_n - u \leq \sum_{i=n}^{+\infty} (1-\mu)^i |u_0|_0 = \frac{(1-\mu)^n}{1-(1-\mu)} |u_0|_0 = \frac{(1-\mu)^n}{\mu} |u_0|_0. \tag{28}$$

3.2 Cascade for the numerical scheme

As we have done for the equation (P), we will approach (S) by a sequence of equations.

Let $u_{h0} \in C_b(\mathbb{R}^N)$ be the unique solution of

$$S(h, x, u_h(x), u_h) = 0, \quad x \in \mathbb{R}^N. \tag{S0}$$

Since $\mathcal{M}u_{h0}$ is continuous, there exists a unique solution $u_{h1} \in C_b(\mathbb{R}^N)$ of

$$\max\{S(h, x, u_h(x), u_h); u_h(x) - \mathcal{M}u_{h0}(x)\} = 0, \quad x \in \mathbb{R}^N. \tag{29}$$

For $n = 2, 3, \cdots$, we note u_{hn} the unique continuous and bounded solution of

$$\max\{S(h, x, u_h(x), u_h); u_h(x) - \mathcal{M}u_{h(n-1)}(x)\} = 0, \quad x \in \mathbb{R}^N. \tag{Sn}$$

The function u_{h1} is a sub-solution of (S0), and then $u_{h1} \leq u_{h0}$ in $\mathbb{R}^N$. Using remark 2.4 and assumption (S5), we verify that $u_h \equiv 0$ is a sub-solution of (29) in $\mathbb{R}^N$, and then we have $0 \leq u_{h1} \leq u_{h0}$ in $\mathbb{R}^N$. Proposition 2.1 implies that $0 \leq \mathcal{M}u_{h1} \leq \mathcal{M}u_{h0}$, then u_{h2} is a sub-solution of (29), and hence $u_{h2} \leq u_{h1}$ in $\mathbb{R}^N$. By induction on n, we obtain

$$0 \leq \cdots \leq u_{hn} \leq \cdots \leq u_{h2} \leq u_{h1} \leq u_{h0}. \tag{30}$$

As in subsection 3.1, we suppose that $|u_0|_0 > k$. Then, since $u_{h0} \to u_0$, we have also $|u_{h0}|_0 > k$ and we can choose $\mu \in (0,1)$ such that $\mu |u_0|_0 < k$, and $\mu |u_{h0}|_0 < k$.

Theorem 3.2 *For all n we have*

$$u_{hn} - u_{h(n+1)} \leq (1-\mu)^n |u_{h0}|_0. \tag{31}$$

Proof: We use the same methods as in theorem 3.1, taking some θ_n. The unique difference is that we have to show that $(1-\theta_n-\mu\theta_n)u_{h(n+1)}$ is a sub-solution of (Sn+2), which can be written

$$\max\{S(h,x,u_h(x),u_h);u_h(x)-\mathcal{M}u_{h(n+1)}(x)\}=0,\quad x\in\mathbb{R}^N.$$

With the monotonicity of S, we obtain the result. □

We have proved that u_{hn} converges to the solution u_h of (S), for $n\to+\infty$. Moreover we have

$$u_{hn}-u_h\le\sum_{i=n}^{+\infty}(1-\mu)^i|u_{h0}|_0=\frac{(1-\mu)^n}{\mu}|u_{h0}|_0. \tag{32}$$

4 The upper bound for the cascade problems

In this section we will use the methods of [3], [2], to obtain an upper bound of u_n-u_{hn}, for all n. We start with the case $n=0$, and then we will study the general case $n\ge 1$. Finally, we will use these estimates to obtain the upper bound of $u-u_h$.

4.1 Problem without impulsions

Consider the problem (P0) and its viscosity solution $u_0\in C_{b,l}(\mathbb{R}^N)$. Let

$$L_{u_0}:=\sup_{\alpha_i}\frac{[c^{\alpha_i}]_1|u_0|_0+[f^{\alpha_i}]_1}{1-[\sigma^{\alpha_i}]_1^2-[b^{\alpha_i}]_1}.$$

We recall here the result of [17, Lemma A.1].

Lemma 4.1 L_{u_0} *is an upper bound of the Lipschitz constant of* u_0. □

Consider the scheme (S0) and its solution $u_{h0}\in C_b(\mathbb{R}^N)$. We recall that L^{α_i} and S satisfy assumptions (A1-A2) and (S1-S5). An upper bound of u_0-u_{h0} has been obtained in [3]. Here we need to rewrite some ideas of this paper, in order to detail constants which appear in various proofs. The auxiliary equation (see [18])

$$\sup_{\alpha_i\in\mathcal{A},|e|\le\epsilon}L^{\alpha_i}(x+e,\mathcal{D}u_0^\epsilon(x))=0,\qquad x\in\mathbb{R}^N, \tag{P0P}$$

has a unique viscosity solution $u_0^\epsilon\in C_{b,l}(\mathbb{R}^N)$. Let $u_{0\epsilon}$ the mollification of u_0^ϵ, defined as in (8). We give now a lemma useful in the sequel.

Lemma 4.2 *Let* $g\in C_{b,l}(\mathbb{R}^N)$*, and its mollification* g_ϵ*. Set* $\epsilon=|h|^{\bar\gamma}$*. Then,* J *being defined in (S3),*

$$Q(g_\epsilon)\le|J|K_c|g|_0|h|^{\bar\gamma}. \tag{33}$$

Proof: Using (10), get

$$Q(g_\epsilon) = K_c|g|_0 \sum_{i\in J} \epsilon^{1-i}|h|^{k_i} = K_c|g|_0 \sum_{i\in J} |h|^{\bar\gamma(1-i)+k_i}.$$

Since $\bar\gamma(1-i)+k_i \geq \bar\gamma$, for all $i \in J$, we obtain the result. □

We recall here the result of [2, Proposition 3.2], where we detail some constants.

Proposition 4.3 *Let $u_0 \in C_{b,l}(\mathbb{R}^N)$ the viscosity solution of (P0), and $u_{h0} \in C_{b,l}(\mathbb{R}^N)$ the solution of (S0). Then we have*

$$u_0(x) - u_{h0}(x) \leq \bar C_0 |h|^{\bar\gamma}, \quad \forall\, x \in \mathbb{R}^N, \tag{$\bar E0$}$$

$$\bar C_0 := |J| K_c |u_0^\epsilon|_0 + R, \tag{34}$$

where R depends only on the constant K of assumption (A1).

Proof: In [3] the authors verify that $u_{0\epsilon}$ is a classical sub-solution of (P0). By the consistency hypothesis (S3), (10) and lemma 4.2,

$$S(h, x, u_{0\epsilon}(x), u_{0\epsilon}) \leq Q(u_{0\epsilon}) \leq |J| K_c |u_0^\epsilon|_0 |h|^{\bar\gamma}, \quad x \in \mathbb{R}^N.$$

Monotonicity implies that $u_{0\epsilon} - |J|K_c|u_0^\epsilon|_0|h|^{\bar\gamma} \leq u_{h0}$. By [3, Lemma A.1], we have that $|u_0 - u_{0\epsilon}| \leq R\epsilon$, where R depends only on K defined in (A1). So we have the result. □

4.2 Problem with n impulsions, $n \geq 1$

Consider now the problem with n impulsions (Pn), for $n \geq 1$, and its viscosity solution $u_n \in C_{b,l}(\mathbb{R}^N)$. We generalize here the method of [3], by introducing the perturbed equation

$$\max\left\{ \sup_{\alpha_i, |e|\leq\epsilon} L^{\alpha_i}(x+e), \mathcal{D}u_n^\epsilon(x)); u_n^\epsilon(x) - \mathcal{M}u_{n-1}(x) \right\} = 0, \tag{PnP}$$

that has a unique viscosity solution $u_n^\epsilon \in C_{b,l}(\mathbb{R}^N)$. The next result, proved in the appendix, gives upper bounds of Lipschitz constants of u_n and u_n^ϵ.

Lemma 4.4 *Let u_n and u_n^ϵ denote the viscosity solutions of (Pn) and (PnP) respectively, for $n \geq 1$. Then we have*

$$L_{u_n} = L_{u_0}, \tag{35}$$

$$L_{u_n^\epsilon} = \max\left(L_{u_0}; \sup_{\alpha_i} \frac{[c^{\alpha_i}]_1 |u_n^\epsilon|_0 + [f^{\alpha_i}]_1}{1 - [\sigma^{\alpha_i}]_1^2 - [b^{\alpha_i}]_1}\right). \qquad \square \tag{36}$$

Using the same methods as for sequence (21), we can show that

$$0 \leq \cdots \leq u_n^\epsilon \leq \cdots \leq u_2^\epsilon \leq u_1^\epsilon. \tag{37}$$

Combining with (36), get

$$0 \leq \cdots \leq L_{u_n^\epsilon} \leq \cdots \leq L_{u_2^\epsilon} \leq L_{u_1^\epsilon}. \tag{38}$$

The following result is proved in the appendix.

Proposition 4.5 *Let u_n and u_n^ϵ be the viscosity solutions of (Pn) and (PnP) respectively, and A_{u_n,u_n^ϵ} be defined in (61). Then*

$$|u_n - u_n^\epsilon|_0 \le A_{u_n,u_n^\epsilon}\epsilon.$$

Relations (38), (61), (21) and (36) imply the following result.

Lemma 4.6 $0 \le \cdots \le A_{u_n,u_n^\epsilon} \le \cdots \le A_{u_2,u_2^\epsilon} \le A_{u_1,u_1^\epsilon}$. □

Proof: This follows from the expression of coefficients A_{u_i,u_i^ϵ}, $i = 1, \ldots, n$, given in (61), combined with lemma 4.4 and relation (38). □

We can give now the error estimate.

Proposition 4.7 *Let $u_n \in C_{b,l}(\mathbb{R}^N)$ be the unique viscosity solution of (Pn), and $u_{hn} \in C_b(\mathbb{R}^N)$ the unique solution of (Sn), $n \ge 1$. Then we have*

$$u_n(x) - u_{hn}(x) \le \bar{C}_n |h|^{\bar{\gamma}}, \qquad (\bar{E}n)$$

$$\bar{C}_n = \bar{C}_{n-1} + A_{u_n,u_n^\epsilon} + L_{u_n^\epsilon} + L_{u_0}. \qquad (39)$$

Proof: For all $n \in \mathbb{N}$ and $\epsilon > 0$, we denote by $u_{n\epsilon}$ the mollification of u_n^ϵ. We prove the proposition by induction over n. Take $n = 1$. We show that $u_{1\epsilon} - \bar{C}_0|h|^{\bar{\gamma}} - L_{u_0}\epsilon$ is a sub-solution of (29). Applying the classical methods (see [3], [2], [5]), since $L_{u_1} = L_{u_0}$, we have that $u_{1\epsilon} - L_{u_0}\epsilon$ is a classical sub-solution of (P1). Using the consistency hypothesis (S3), proposition 4.3, the equality $Q(u_{1\epsilon} - L_{u_0}\epsilon) = Q(u_{1\epsilon})$, and the monotonicity of S, obtain

$$\begin{cases} S(h, x, u_{1\epsilon}(x) - L_{u_0}\epsilon - Q(u_{1\epsilon}), u_{1\epsilon} - L_{u_0}\epsilon - Q(u_{1\epsilon}) \le 0 \\ u_{1\epsilon}(x) - L_{u_0}\epsilon - \bar{C}_0|h|^{\bar{\gamma}} \le \mathcal{M}u_{h0}(x). \end{cases}$$

We deduce that $u_{1\epsilon}(x) - L_{u_0}\epsilon - \max\{\bar{C}_0|h|^{\bar{\gamma}}, Q(u_{1\epsilon})\}$ is sub-solution of (S1). By lemma 4.2, and by (37) and (34), we obtain

$$Q(u_1^\epsilon) \le |J|K_c|u_1^\epsilon|_0|h|^{\bar{\gamma}} \le |J|K_c|u_0^\epsilon|_0|h|^{\bar{\gamma}} \le \bar{C}_0|h|^{\bar{\gamma}}.$$

Then $\max\{\bar{C}_0|h|^{\bar{\gamma}}, Q(u_{1\epsilon})\} = \bar{C}_0|h|^{\bar{\gamma}}$, which implies $u_{1\epsilon}(x) - \bar{C}_0|h|^{\bar{\gamma}} - L_{u_0}\epsilon \le u_{h1}(x)$, for all x. Hence, with (9) and proposition 4.5,

$$\begin{aligned} u_1(x) - u_{h1}(x) &= u_1(x) - u_1^\epsilon(x) + u_1^\epsilon(x) - u_{1\epsilon}(x) + u_{1\epsilon}(x) - u_{h1}(x) \\ &\le A_{u_1,u_1^\epsilon}\epsilon + L_{u_1^\epsilon}\epsilon + L_{u_0}\epsilon + \bar{C}_0|h|^{\bar{\gamma}}. \end{aligned}$$

Setting $\epsilon = |h|^{\bar{\gamma}}$, we obtain that (39) holds for $n = 1$.

Now we suppose the proposition true for $n - 1$. The same methods as before, the assumption of induction and lemma 4.4 give us the result. □

So we have obtained that, for all $n \ge 1$, $u_n - u_{hn} \le \bar{C}_n|h|^{\bar{\gamma}}$. We set

$$\bar{D}_{n-1} := \bar{C}_n - \bar{C}_{n-1} = A_{u_n,u_n^\epsilon} + L_{u_n^\epsilon} + L_{u_0}.$$

Lemma 4.6 and relation (38) imply that $\bar{D}_n \le \bar{D}_0$, and hence, by (39):

$$\bar{C}_n \le \bar{C}_0 + n\bar{D}_0. \qquad (40)$$

5 The lower bound for the cascade problems

In this section we will use the methods of [3], [2], to obtain a lower bound of $u_n - u_{hn}$, for all n. We start with the case $n = 0$, and then we will study the general case $n \geq 1$. Finally, we will use these estimates to obtain the lower bound of $u - u_h$.

5.1 Problem without impulsions

Consider problem (P0) of solution $u_0 \in C_{b,l}(\mathbb{R}^N)$, and the scheme (S0) of solution $u_{h0} \in C_b(\mathbb{R}^N)$. We recall that L^{α_i} and S satisfy assumptions (A1-A2) and (S1-S5). A lower bound of $u_0 - u_{h0}$ has been obtained in [3]. Here we need to rewrite some parts of this paper, in order to give explicit bounds of constants which appear in the different proofs. Consider the following switching system, which approaches (P0),

$$\max\{L^{\alpha_i}(x, \mathcal{D}v_i^0(x)); v_i^0(x) - \min_{j\neq i}\{v_j^0(x) + \ell\}\} = 0, \tag{SS0}$$

for $x \in \mathbb{R}^N$, $i \in \mathcal{I} = \{1, \dots, M\}$, and $\ell \geq 0$. Let $v^0 = (v_1^0, \dots, v_M^0)$ be the unique viscosity solution of (SS0), $v^0 \in C_{b,l}(\mathbb{R}^N)^M$. By remark 2.4, we have that $(0, \dots, 0)$ is a viscosity sub-solution of (SS0), hence $0 \leq v_i^0(x)$, for all $i \in \mathcal{I}$ and $x \in \mathbb{R}^N$.

For every i, v_i^0 converges to u_0, when $\ell \to 0$. We rewrite here the result of [2, Theorem 2.3], which give this rate of convergence.

Lemma 5.1 *Let u_0 and v^0 be the viscosity solutions of (P0) and (SS0) respectively. Then, for all i, we have*

$$0 \leq v_i^0 - u_0 \leq C\ell^{1/3}, \tag{H0}$$

where C depends only on K, defined in (A1). □

5.1.1 Error Estimate

Consider the following perturbed switching system (SS0)

$$\max\left\{\inf_{|e|\leq\epsilon} L^{\alpha_i}(x+e, \mathcal{D}w_i^{\epsilon 0}(x)); w_i^{\epsilon 0}(x) - \min_{j\neq i}\{w_j^{\epsilon 0}(x) + \ell\}\right\} = 0. \tag{SS0P}$$

We denote by $w^{0\epsilon} = (w_1^{0\epsilon}, \dots, w_M^{0\epsilon})$ the unique viscosity solution of (SS0P) in $C_{b,l}(\mathbb{R}^N)^M$. We have $0 \leq w_i^{0\epsilon}(x)$, for all i and for all x.

The following result is proved in the appendix.

Lemma 5.2 *Let v^0 and $w^{0\epsilon}$ be the viscosity solutions of (SS0) and (SS0P) respectively. Then,* $\max_i |v_i^0 - w_i^{0\epsilon}|_0 \leq \epsilon A_{v^0,w^{0\epsilon}}$, *where $A_{v^0,w^{0\epsilon}}$ is defined in (61).* □

Consider $\underline{\gamma}$ defined in (18). We have the following result.

Lemma 5.3 *Given $g \in C_{b,l}(\mathbb{R}^N)$, its mollification g_ϵ, and $\epsilon = |h|^{3\underline{\gamma}}$, we have that, for J defined in (S3),*

$$Q(g_\epsilon) \le |J| K_c |g|_0 |h|^{\underline{\gamma}}. \tag{41}$$

Proof: By (10), we know that

$$Q(g_\epsilon) = K_c |g|_0 \sum_{i \in J} \epsilon^{1-i} |h|^{k_i} = K_c |g|_0 \sum_{i \in J} |h|^{3(1-i)\underline{\gamma} + k_i}.$$

Since $3(1-i)\underline{\gamma} + k_i \ge \underline{\gamma}$, for all $i \in J$, we obtain the result. □

We recall here the result of [2, Theorem 3.5], where we detail some constants.

Proposition 5.4 *Let $u_0 \in C_{b,l}(\mathbb{R}^N)$ be the viscosity solution of (P0) and $u_{h0} \in C_{b,l}(\mathbb{R}^N)$ the solution of (S0). Then, we have*

$$u_{h0}(x) - u_0(x) \le \underline{C}_0 |h|^{\underline{\gamma}}, \quad \forall\, x \in \mathbb{R}^N, \tag{$\underline{\mathrm{E0}}$}$$

$$\underline{C}_0 = |J| K_c |w^{0\epsilon}|_0 + R, \tag{42}$$

where R depends only on K defined in (A1), and J is defined in (S3).

Proof: We recall the ideas of [2, Theorem 3.5]. We set

$$m := \sup_{y \in \mathbb{R}^N} \{u_{h0}(y) - g_0(y)\},$$

where $g_0 = \min_{i \in \mathcal{I}} w^0_{\epsilon i}$. Computations of [2, Theorem 3.5], combined with lemma 5.3, gives

$$m \le |J| K_c |w_i^{\epsilon 0}|_0 |h|^{\underline{\gamma}}, \tag{43}$$

where J is defined in (S3). Applying lemma 5.1, lemma 5.2, and (43), we have that, for all $i \in \mathcal{I}$,

$$\begin{aligned}
\sup_x (u_{h0}(x) - u_0(x)) &\le m + \sup_x (w^0_{\epsilon i}(x) - w_i^{0\epsilon}(x)) + \sup_x (w_i^{0\epsilon}(x) - v_i^0(x)) \\
&\quad + \sup_x (v_i^0(x) - u_0(x)) \\
&\le |J| K_c |w_i^{\epsilon 0}|_0 \sum_{i \in J} \epsilon^{1-i} |h|^{k_i} + C\epsilon + A_{v^0, w^{0\epsilon}} \epsilon + C l^{1/3},
\end{aligned}$$

where C depends only on K defined in (A1). Choose $\epsilon = |h|^{3\underline{\gamma}}$ and $l = 4\epsilon L_{w^{0\epsilon}}$, where $L_{w^{0\epsilon}}$ is an upper bound of the Lipschitz constant of $w^{0\epsilon}$. By lemma 5.3, we have

$$\sup_x (u_{h0}(x) - u_0(x)) \le R_0 (2|h|^{3\underline{\gamma}} + |h|^{\underline{\gamma}}) + |J| K_c |w_0^{0\epsilon}|_0 |h|^{\underline{\gamma}},$$

where R_0 depends only on K defined in (A1). Setting $R = 3R_0$, we obtain the result. □

5.2 Problem with n impulsions, $n \geq 1$

We generalize here the methods of [3]. Consider problem (Pn) and its solution $u_n \in C_{b,l}(\mathbb{R}^N)$, defined in section 3.1. We know that L_{u^0} is an upper bound of the Lipschitz constant of u_n, for all n.

Then consider the scheme (Sn) of solution $u_{hn} \in C_b(\mathbb{R}^N)$, defined in section 3.2. We recall that L^{α_i} and S satisfy assumptions (A1-A2), (S1-S5). Consider the following switching system which approach (Pn):

$$\max\{L^{\alpha_i}(x, \mathcal{D}v_i^n(x)); v_i^n(x) - \min_{j \neq i}\{v_j^n(x) + \ell\}; v_i^n(x) - \mathcal{M}u_{n-1}(x)\} = 0, \qquad \text{(SSn)}$$

for $x \in \mathbb{R}^N$ and $i \in \mathcal{I} = \{1, \dots, M\}$. Under assumptions (A1-A2), (SSn) has a unique viscosity solution $v^n = (v_1^n, \dots, v_M^n) \in C_{b,l}(\mathbb{R}^N)^M$. By remark 2.4, it is easy to see that $(0, \dots, 0)$ is a viscosity sub-solution of (SSn), and that v^n is a viscosity sub-solution of (SS(n-1)), for all n. We can build, then the following sequence

$$0 \leq \cdots \leq v_i^n(x) \leq \cdots \leq v_i^1(x) \leq v_i^0(x),$$

for all i, and for all x.

5.2.1 Convergence of the switching system

Using the same methods as in [2, Theorem 2.3], we introduce an auxiliary switching system

$$\max\{\sup_{|e| \leq \epsilon} L^{\alpha_i}(x+e, \mathcal{D}v_i^{n\epsilon}(x)); v_i^{n\epsilon}(x) - \min_{j \neq i}\{v_j^{n\epsilon}(x) + \ell\}; v_i^{n\epsilon}(x) - \mathcal{M}u_{n-1}(x)\}, \qquad (44)$$

and denote by $v^{n\epsilon} = (v_1^{n\epsilon}, \dots, v_M^{n\epsilon})$ its viscosity solution in $C_{b,l}(\mathbb{R}^N)^M$. As before, we have that $n \mapsto v_i^{n\epsilon}(x)$ is non-increasing, for all i and for all x.

We can give now the following result about the convergence.

Proposition 5.5 *Let u_n and v^n be the solutions of (Pn) and (SSn) respectively. Then, for all i, we have*

$$0 \leq v_i^n - u_n \leq H_{v^n, v^{n\epsilon}} \ell^{1/3}, \qquad \text{(Hn)}$$

where $H_{v^n, v^{n\epsilon}}$ is defined in (61).

Proof: We start by giving the proof for $n = 1$. Consider $w = (u_1, \dots, u_1)$ (a vector with M components equal to u_1). Then, for every i, we have:

$$\begin{cases} L^{\alpha_i}(x, \mathcal{D}u_1(x)) \leq 0 \\ u_1(x) \leq u_1(x) + \ell \\ u_1(x) \leq \mathcal{M}u_0(x) \end{cases} \Rightarrow u_1 \text{ is a sub-solution of (SS1) } \Rightarrow u_1(x) \leq v_i^1(x),$$

for all $x \in \mathbb{R}^N$, $i \in \mathcal{I}$. We show that, for all i, $v_{i\epsilon}^1 - C\epsilon^{-2} - L_{u_0}\epsilon$ is a sub-solution of (P1), where

$$C = C_\rho \ell \sup_{\alpha_i}(|\sigma^{\alpha_i}|_0 + |b^{\alpha_i}|_0 + |c^{\alpha_i}|_0). \qquad (45)$$

With classical methods (see [3], [2], [5]), we have that $v^1_{i\epsilon}$ is, for all i, a sub-solution, in the classical sense, of

$$L^{\alpha_i}(x, \mathcal{D}v(x)) = 0,\ \forall\, x \in \mathbb{R}^N. \tag{46}$$

The definition of switching system implies that $|v_i^{1\epsilon} - v_j^{1\epsilon}| \leq \ell$, for all i, j. Combining with (10), we obtain

$$|L^{\alpha_i}(x, \mathcal{D}v^1_{\epsilon j}(x)) - L^{\alpha_i}(x, \mathcal{D}v^1_{\epsilon i}(x))| \leq \frac{C}{\epsilon^2}, \forall\, i, j \in \mathcal{I},\ \text{et}\ \forall\, x \in \mathbb{R}^N.$$

Since $v^1_{\epsilon i}$ is a sub-solution of (46), this implies

$$L^{\alpha_i}(x, \mathcal{D}v^1_{\epsilon j}(x)) \leq \frac{C}{\epsilon^2}, \quad \forall i, j,\ \text{and}\ \forall\, x \in \mathbb{R}^N. \tag{46a}$$

Consequently $v^1_{\epsilon i} - C\epsilon^{-2}$ is a classical sub-solution of $\sup_{\alpha_i} L^{\alpha_i}(x, \mathcal{D}w(x)) = 0$. Moreover, by the definition of the auxiliary system, we have that $v_i^{1\epsilon}(x) - \mathcal{M}u_0(x) \leq 0$, for all $i \in \mathcal{I}$, and for all $x \in \mathbb{R}^N$. Let $u_{\epsilon 0}$ be the mollification of u_0, defined as in (8). Then, we have $v^1_{\epsilon i}(x) - \mathcal{M}u_{0\epsilon}(x) \leq 0$, which implies $v^1_{\epsilon i}(x) - \mathcal{M}u_0(x) \leq L_{u_0}\epsilon$, and also

$$v^1_{\epsilon i}(x) - L_{u_0}\epsilon - C\epsilon^{-2} - \mathcal{M}u_0(x) \leq 0,\ \forall\, x \in \mathbb{R}^N.$$

Hence, for all $x \in \mathbb{R}^N$, we have

$$\begin{cases} \sup_{\alpha_i} L^{\alpha_i}(x, \mathcal{D}(v^1_{\epsilon i} - C)(x)) \leq L_{u_0}\epsilon, \\ v^1_{\epsilon i}(x) - \mathcal{M}u_0(x) \leq L_{u_0}\epsilon + C. \end{cases}$$

So $v^1_{\epsilon i} - L_{u_0}\epsilon - C$ is a viscosity sub-solution of (P1), and we have $v^1_{\epsilon i}(x) - L_{u_0}\epsilon - C \leq u_1(x)$, for all $x \in \mathbb{R}^N$. Finally we obtain

$$v^1_i(x) - u_1(x) \leq \frac{C_\rho \ell}{\epsilon^2} \sup_{\alpha_i}(|\sigma^{\alpha_i}|_0 + |b^{\alpha_i}|_0 + |c^{\alpha_i}|_0) + (L_{u_0} + L_{v^{1\epsilon}} + A_{v^1, v^{1\epsilon}})\epsilon,$$

for all x in $\mathbb{R}^N$. Minimizing with respect to ϵ, obtain

$$v^1_i(x) - u_1(x) \leq H_{v^1, v^{1\epsilon}} \ell^{1/3}.$$

The result for $n > 1$ can be proved similarly, using $L_{u_{n-1}} = L_{u_0}$ as an upper bound of the Lipschitz constant of u_{n-1}. □

5.2.2 Error Estimates

Consider the following perturbed switching system which approaches (Pn),

$$\max\{\inf_{|e|\leq\epsilon} L^{\alpha_i}(x + e, \mathcal{D}w_i^{n\epsilon}(x)); w_i^{n\epsilon}(x) - \min_{j\neq i}\{w_j^{n\epsilon}(x) + \ell\};$$

$$w_i^{n\epsilon}(x) - \mathcal{M}u_{n-1}(x)\} = 0, \tag{SSnP}$$

and its unique viscosity solution $w^{n\epsilon} \in C_{b,l}(\mathbb{R}^N)^M$. As before, we can prove that $0 \leq \cdots \leq w_i^{n\epsilon}(x) \leq \cdots \leq w_i^{1\epsilon}(x) \leq w_i^{0\epsilon}(x)$, for all i and x. Let $g^n := v^n,\ v^{n\epsilon},\ w^{n\epsilon}$. Then, we set

$$L_{g^n} := \max\left(\sup_{\alpha_i, i} \frac{[c^{\alpha_i}]_1 |g_i^n|_0 + [f^{\alpha_i}]_1}{1 - [\sigma^{\alpha_i}]_1^2 - [b^{\alpha_i}]_1} ; L_{u_0} \right). \tag{47}$$

We have the following results, which are showed in the appendix.

Lemma 5.6 *Let $g^n := v^n,\ v^{n\epsilon},\ w^{n\epsilon}$. Then $\max_i [g_i^n]_1 \leq L_{g^n}$.* □

Lemma 5.7 *Let v^n, $v^{n\epsilon}$ and $w^{n\epsilon}$ be the viscosity solutions of (SSn), (44) and (SSnP) respectively. Then, we have*

$$\max_i |v_i^n - v_i^{n\epsilon}|_0 \leq A_{v^n, v^{n\epsilon}} \epsilon, \quad \max_i |v_i^n - w_i^{n\epsilon}|_0 \leq A_{v^n, w^{n\epsilon}} \epsilon,$$

where $A_{v^n,v^{n\epsilon}}$ and $A_{v^n,w^{n\epsilon}}$ are defined in (61). □

The following result in proved in theorems A.1 and B.3.

Lemma 5.8 *Let $g^i := v^i,\ v^{i\epsilon},\ w^{i\epsilon}$, and let L_{g^i} be defined as in (47). Then*

$$L_{g^n} \leq \cdots \leq L_{g^2} \leq L_{g^1},$$

$$A_{v^n,w^{n\epsilon}} \leq \cdots \leq A_{v^1,w^{1\epsilon}} \leq A_{v^0,w^{0\epsilon}},$$
$$A_{v^n,v^{n\epsilon}} \leq \cdots \leq A_{v^1,v^{1\epsilon}} \leq A_{v^0,v^{0\epsilon}}. \qquad \square.$$

We can give now the lower bound.

Proposition 5.9 *Let $u_n \in C_{b,l}(\mathbb{R}^N)$ the viscosity solution of (Pn) and let $u_{hn} \in C_{b,l}(\mathbb{R}^N)$ the solution of (Sn), $n \geq 1$. Then we have*

$$u_{hn}(x) - u_n(x) \leq \underline{C}_n |h|^{\underline{\gamma}}, \quad \forall\, x \in \mathbb{R}^N, \tag{\underline{En}}$$

$$\underline{C}_n = \underline{C}_{n-1} + 12 L_{w^{n\epsilon}} + 4 L_{u_0} + A_{v^n,w^{n\epsilon}} + H_{v^n,v^{n\epsilon}} (6 L_{w^{n\epsilon}})^{1/3}. \tag{48}$$

Proof: The proof is by induction over n. Let $n = 1$, and let

$$m := \sup_{y \in \mathbb{R}^N} \{u_{h1}(y) - g(y)\}, \tag{49}$$

where $g = \min_{i \in \mathcal{I}} w_{\epsilon i}^1$. For $l \geq 0$, let

$$m_l := \sup_{y \in \mathbb{R}^N} \{u_{h1}(y) - g(y) - l\phi(y)\},$$

where $\phi(x) = (1+|x|^2)^{1/2}$. Let x_0 be such that $m_l = u_{h1}(x_0) - g(x_0) - l\phi(x_0)$. Then we have also $m_l = u_{h1}(x_0) - w^1_{\epsilon i_0}(x_0) - l\phi(x_0)$, where $w^1_{\epsilon i_0}(x_0) = \min_{j\in\mathcal{I}} w^1_{\epsilon j}(x_0)$. After some computations (see [2, Theorem 3.4]), we can say that, if $\epsilon \le (6L_{w^{1\epsilon}})^{-1}l$, then

$$w^{1\epsilon}_{i_0}(y) - \min_{j\neq i_0}\{w^{1\epsilon}_j(y) + l\} < 0, \quad \forall\, y \in B(x_0, 2\epsilon). \tag{50}$$

Then, equation i_0 in the system (SS1P) becomes

$$\max\{\inf_{|e|\le\epsilon} L^{\alpha_{i_0}}(y+e, \mathcal{D}w^{1\epsilon}_{i_0}(y)); w^{1\epsilon}_{i_0}(y) - \mathcal{M}u_0(y)\} = 0, \quad y \in B(x_0, 2\epsilon). \tag{51}$$

We have to study two cases.
<u>CASE 1:</u> There exists $\bar{x} \in B(x_0, 2\epsilon)$ such that

$$w^{1\epsilon}_{i_0}(\bar{x}) = \mathcal{M}u_0(\bar{x}), \quad \text{i.e.} \quad w^{1\epsilon}_{i_0}(\bar{x}) = k + \inf_{\xi}\{u_0(\bar{x}+\xi) + c(\xi)\}.$$

Then, for all $y \in B(x_0, 2\epsilon)$,

$$w^{\epsilon 1}_{i_0}(y) + 4(L_{w^{1\epsilon}} + L_{u_0})\epsilon \ge k + \inf_{\xi}\{u_0(y+\xi) + c(\xi)\}.$$

Consider now $\mathcal{M}u_{h0}(y) - \mathcal{M}u_0(y)$. By proposition 5.4, we have that $\mathcal{M}u_0(y) \ge \mathcal{M}u_{h0}(y) - \underline{C}_0|h|^{\underline{\gamma}}$. Then, we obtain

$$w^{\epsilon 1}_{i_0}(y) + 4(L_{w^{1\epsilon}} + L_{u_0})\epsilon + \underline{C}_0|h|^{\underline{\gamma}} \ge k + \inf_{\xi}\{u_{h0}(y+\xi) + c(\xi)\}, \quad \forall\, y \in B(x_0, 2\epsilon).$$

Since $u_{h1}(y) \le k + \inf_\xi\{u_{h0}(y+\xi) + c(\xi)\}$, for all $y \in B(x_0, 2\epsilon)$, hence

$$u_{h1}(x_0) - w^1_{\epsilon i_0}(x_0) \le 4(L_{w^{1\epsilon}} + L_{u_0})\epsilon + \underline{C}_0|h|^{\underline{\gamma}} + L_{w^{1\epsilon}}\epsilon = (5L_{w^{1\epsilon}} + 4L_{u_0})\epsilon + \underline{C}_0|h|^{\underline{\gamma}},$$

which implies

$$m_l \le (5L_{w^{1\epsilon}} + 4L_{u_0})\epsilon + \underline{C}_0|h|^{\underline{\gamma}} - l\phi(x). \tag{52}$$

<u>CASE 2:</u> For all $y \in B(x_0, 2\epsilon)$, we have

$$w^{1\epsilon}_{i_0}(y) < \mathcal{M}u_0(y).$$

The classical methods (see [2], [5]) imply that

$$\sup_{\alpha_i} L^{\alpha_i}(x_0, \mathcal{D}w^1_{\epsilon i_0}(x_0)) \ge 0.$$

We can apply the consistency hypothesis, to obtain

$$-Cl \le S(h, x_0, (w^1_{\epsilon i_0} + l\phi)(x_0)), w^1_{\epsilon i_0} + l\phi) + Q(w^1_{\epsilon i_0} + l\phi)$$

$$\Rightarrow S(h, x_0, (w^1_{\epsilon i_0} + l\phi)(x_0), w^1_{\epsilon i_0} + l\phi) \ge -Q(w^1_{\epsilon i_0}) + O(l).$$

Monotonicity implies that

$$\begin{aligned} S(h, x_0, (w^1_{\epsilon i_0} + l\phi)(x_0), w^1_{\epsilon i_0} + l\phi) &\leq S(h, x_0, u_{h1}(x_0) - m_l, u_{h1} - m_l) \\ &\leq -m_l + S(h, x_0, u_{h1}(x_0), u_{h1}) \\ &\leq -m_l. \end{aligned}$$

The last inequality follows from the definition of (S1). Then, we have

$$m_l \leq Q(w^1_{\epsilon i_0}) + O(l). \tag{53}$$

CONCLUSION:
By (52) and (53), we obtain that

$$m_l \leq \max\left\{ 5L_{w^{1\epsilon}} + 4L_{u_0})\epsilon + \underline{C}_0|h|^{\underline{\gamma}} - l\phi(x); Q(w^1_{\epsilon i_0}) + O(l)\right\}.$$

Then, if l goes to 0, we can conclude that

$$m \leq \max\left\{ (5L_{w^{1\epsilon}} + 4L_{u_0})\epsilon + \underline{C}_0|h|^{\underline{\gamma}}; K_c|w^{1\epsilon}|_0 \sum_{i\in J} \epsilon^{1-i}|h|^{k_i}\right\}.$$

Hence

$$\begin{aligned} u_{h1} - u_1 &= u_{h1} - w^1_{\epsilon i} + w^1_{\epsilon i} - u_1 \\ &\leq m + w^1_{\epsilon i} - w^{1\epsilon}_i + w^{1\epsilon}_i - v^1_i + v^1_i - u_1 \\ &\leq \max\left\{ (5L_{w^{1\epsilon}} + 4L_{u_0})\epsilon + \underline{C}_0|h|^{\underline{\gamma}}; K_c|w^{1\epsilon}|_0 \sum_{i\in J} \epsilon^{1-i}|h|^{k_i}\right\} \\ &\quad + \ell + L_{w^{1\epsilon}}\epsilon + A_{v^1, w^{1\epsilon}}\epsilon + H_{v^1, v^{1\epsilon}}\ell^{1/3}. \end{aligned}$$

If we set $\epsilon = |h|^{\underline{\gamma}}$, and $\ell = (6Lw^{1\epsilon})$, as in the case without impulsions, we obtain

$$u_{h1} - u_1 \leq \max\{(12L_{w^{1\epsilon}} + 4L_{u_0})|h|^{3\underline{\gamma}} + (\underline{C}_0 + H_{v^1, v^{1\epsilon}}(6L_{w^{1\epsilon}})^{1/3})|h|^{\underline{\gamma}};$$

$$(7L_{w^{1\epsilon}})|h|^{3\underline{\gamma}} + (|J|K_c|w^{1\epsilon}|_0 + H_{v^1, v^{1\epsilon}}(6L_{w^{1\epsilon}})^{1/3})|h|^{\underline{\gamma}}\}.$$

Since $|J|K_c|w^{1\epsilon}|_0 \leq |J|K_c|w^{0\epsilon}|_0 \leq \underline{C}_0$, the "max" is realized by the first term. Then we have the result.

Suppose now that (En) and (48) hold for $n-1$. The same methods as before, the induction and the fact that $L_{u_{n-1}} = L_{u_0}$ give the result. □

We set

$$\underline{D}_{n-1} := \underline{C}_n - \underline{C}_{n-1} = 12L_{w^{n\epsilon}} + 4L_{u_0} + A_{w^n} + H_n(6L_{w^{n\epsilon}})^{1/3}. \tag{Dn}$$

Lemma 5.8 implies that

$$\underline{C}_n \leq \underline{C}_0 + n\underline{D}_0. \tag{54}$$

6 Proof of theorem 2.8

Before giving the proof of theorem 2.8, consider the following result. Let $\phi : \mathbb{R} \to \mathbb{R}$, $\phi(x) = a^x + bx$, where $0 < a < 1$, $b \in \mathbb{R}^+$. Let $m := \min_{n\in\mathbb{N}} \phi(n)$. Then we have the following elementary lemma that we state without proof.

Lemma 6.1 *(i) ϕ attains its minimum over $\mathbb{R}$ at $r := \log_a\left(-\frac{b}{\ln a}\right)$, where $-b/\ln a > 0$, since $a < 1$.*

(ii) If $-\frac{b}{\ln a} \geq 1$, then $r \leq 0$, and hence $m = \phi(0) = 1 \leq -\frac{b}{\ln a}$.

(iii) If $-\frac{b}{\ln a} < 1$, then
$m \leq \phi(\lceil r\rceil) = a^{\lceil r\rceil} + b\lceil r\rceil \leq a^r + b(r+1) = -\frac{b}{\ln a} + b\Big(\log_a\left(\frac{b}{\ln a}\right) + 1\Big)$. □

Proof of theorem 2.8, page 7 We start by proving the upper bound. Consider the following decomposition:

$$\sup_x (u(x) - u_h(x)) \leq \sup_x (u(x) - u_n(x)) + \sup_x (u_n(x) - u_{hn}(x)) \qquad (55)$$
$$+ \sup_x (u_{hn}(x) - u_h(x)),$$

for all $n < +\infty$. Using $u - u_n \leq 0$, $u_n - u_{hn} \leq \bar{C}_n|h|^{\bar{\gamma}}$, $u_{hn} - u_{h\infty} \leq \frac{(1-\mu)^n}{\mu}|u_{h0}|_0$, and (40), obtain

$$\sup_x (u(x) - u_h(x)) \leq (\bar{C}_0 + n\bar{D}_0)|h|^{\bar{\gamma}} + \frac{(1-\mu)^n}{\mu}|u_{h0}|_0. \qquad (55b)$$

Let $\phi(n) = (\bar{C}_0 + n\bar{D}_0)|h|^{\bar{\gamma}} + \frac{(1-\mu)^n}{\mu}|u_{h0}|_0$, and let $m := \min_{n\in\mathbb{N}} \phi(n)$. Applying lemma 6.1 and the fact that $r \leq \lceil r\rceil \leq r+1$, we obtain that

- $u - u_h \leq \left(\bar{C}_0 + \frac{\bar{D}_0}{(-\ln(1-\mu))}\right)|h|^{\bar{\gamma}}$, if $-\frac{\bar{D}_0\mu|h|^{\bar{\gamma}}}{|u_{h0}|_0 \ln(1-\mu)} \geq 1$;
- $u - u_h \leq \left[-\frac{\bar{D}_0}{\ln(1-\mu)} + \bar{C}_0 + \bar{D}_0\Big(\log_{(1-\mu)}\Big(-\frac{\mu\bar{D}_0|h|^{\bar{\gamma}}}{|u_{h0}|_0\ln(1-\mu)}\Big) + 1\Big)\right]|h|^{\bar{\gamma}}$, otherwise.

Hence we have the result. We prove now the lower bound. Consider the following decomposition:

$$\sup_x (u_h(x) - u(x)) \leq \sup_x (u_h(x) - u_{hn}(x)) + \sup_x (u_{hn}(x) - u_n(x)) \qquad (56)$$
$$+ \sup_x (u_n(x) - u(x)),$$

for all $n < +\infty$. Since $u_h - u_{hn} \leq 0$, $u_{hn} - u_n \leq \underline{C}_n|h|^{\underline{\gamma}}$, $u_n - u \leq \frac{(1-\mu)^n}{\mu}|u_0|_0$, and (54), we obtain

$$u_h - u \leq \frac{(1-\mu)^n}{\mu}|u_0|_0 + \underline{C}_0|h|^{\underline{\gamma}} + n\underline{D}_0|h|^{\underline{\gamma}}. \tag{56b}$$

Applying lemma 6.1, we obtain that

- $u_h - u \leq \left(\underline{C}_0 + \frac{\underline{D}_0}{(-\ln(1-\mu))}\right)|h|^{\underline{\gamma}}$, if $-\frac{\underline{D}_0\mu|h|^{\underline{\gamma}}}{|u_0|_0\ln(1-\mu)} \geq 1$;
- $u_h - u \leq \left[-\frac{\underline{D}_0}{\ln(1-\mu)} + \underline{C}_0 + \underline{D}_0\left(\log_{(1-\mu)}\left(-\frac{\mu\underline{D}_0|h|^{\underline{\gamma}}}{|u_0|_0\ln(1-\mu)}\right) + 1\right)\right]|h|^{\underline{\gamma}}$, otherwise.

Hence we have the result. □

A The upper bounds of Lipschitz constants

Proof of lemma 4.4 page 12. We prove this lemma by induction. Let $n = 1$, and set

$$m_{\epsilon_1} := \sup_{x,y} \phi(x,y) := \sup_{x,y\in\mathbb{R}^N}\{u_1(x) - u_1(y) - \delta|x-y|^2 - \epsilon_1(|x|^2 + |y|^2)\}.$$

Let $m_{\epsilon_1} = \phi(x_0, y_0)$. By Ishii's lemma (see [9]), there exist $X,\ Y \in \mathcal{S}^N$ such that

$$0 \leq \max\{\sup_{\alpha_i} L^{\alpha_i}(y_0, u_1(y_0), p_y, Y); u_1(y_0) - \mathcal{M}u_0(y_0)\}$$
$$- \max\{\sup_{\alpha_i} L^{\alpha_i}(x_0, u_1(x_0), p_x, X); u_1(x_0) - \mathcal{M}u_0(x_0)\}, \tag{57}$$

where

$$p_x = 2\delta(x_0 - y_0) + 2\epsilon_1 x_0,\ p_y = 2\delta(x_0 - y_0) - 2\epsilon_1 y_0, \tag{58}$$

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \delta \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + 2\epsilon_1 \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{59}$$

Then, (57) implies

$$0 \leq \max\{\sup_{\alpha_i}[L^{\alpha_i}(y_0, u_1(y_0), p_y, Y) - L^{\alpha_i}(x_0, u_1(x_0), p_x, X)];$$
$$u_1(y_0) - \mathcal{M}u_0(y_0) - u_1(x_0) - \mathcal{M}u_0(x_0)\}.$$

We can reduce us to study two different cases.
<u>CASE 1</u>: $u_1(y_0) - \mathcal{M}u_0(y_0) - (u_1(x_0) - \mathcal{M}u_0(x_0)) \geq 0$.
This last inequality implies that $u_1(x_0) - u_1(y_0) \leq L_{u_0}|x_0 - y_0|$. Then we deduce that

$$m_{\epsilon_1} \leq L_{u_0}|x_0 - y_0| - \delta|x_0 - y_0|^2. \tag{60}$$

Setting $r := |x_0 - y_0|$, and noting that $\max_r(L_{u_0} r - \delta r^2) = L_{u_0}^2/4\delta$, we obtain

$$m_{\epsilon_1} \le \frac{L_{u_0}^2}{4\delta}.$$

Applying [17, Lemma 2.3], for fixed δ, we have that

$$\lim_{\epsilon_1 \to 0} m_{\epsilon_1} = \sup_{x,y \in \mathbb{R}^N} \{u_1(x) - u_1(y) - \delta|x-y|^2\} := m,$$

and hence

$$m \le \frac{L_{u_0}^2}{4\delta}.$$

Then we have, by definition of m,

$$u_1(x) - u_1(y) \le \frac{L_{u_0}^2}{4\delta} + \delta|x-y|^2, \; \forall\, x, y \in \mathbb{R}^N.$$

Use $\min_\delta \left(\frac{L_{u_0}^2}{4\delta} + \delta|x-y|^2 \right) = L_{u_0}|x-y|$, to obtain

$$u_1(x) - u_1(y) \le L_{u_0}|x-y|, \; \forall\, x, y \in \mathbb{R}^N.$$

CASE 2: $\sup_{\alpha_i} L^{\alpha_i}(y_0, u_1(y_0), p_y, Y) - \sup_{\alpha_i} L^{\alpha_i}(x_0, u_1(x_0), p_x, X) \ge 0.$
This is the standard case (see [17, Lemma A.1]), and we have that

$$u_1(x) - u_1(y) \le \sup_{\alpha_i} \frac{[c^{\alpha_i}]_1 |u_1|_0 + [f^{\alpha_i}]_1}{1 - [\sigma^{\alpha_i}]_1^2 - [b^{\alpha_i}]_1} |x-y|, \; \forall\, x, y, \in \mathbb{R}^N.$$

In conclusion, we obtain

$$L_{u_1} = \max\left\{ L_{u_0}; \sup_{\alpha_i} \frac{[c^{\alpha_i}]_1 |u_1|_0 + [f^{\alpha_i}]_1}{1 - [\sigma^{\alpha_i}]_1^2 - [b^{\alpha_i}]_1} \right\}.$$

Since by (21) $|u_1|_0 \le |u_0|_0$, using the definition of L_{u_0}, we have $L_{u_1} = L_{u_0}$.

We compute now $L_{u_1^\epsilon}$. With the same methods as before, we obtain

$$L_{u_1^\epsilon} = \max\big(L_{u_0}; \sup_{\alpha_i} \frac{[c^{\alpha_i}]_1 |u_1^\epsilon|_0 + [f^{\alpha_i}]_1}{1 - [\sigma^{\alpha_i}]_1^2 - [b^{\alpha_i}]_1} \big).$$

In this case we have not estimate between $|u_0|_0$ and $|u_1^\epsilon|_0$, hence we must give the result in this form.

Suppose now that lemma is true for $n-1$, i.e.

$$L_{u_{n-1}} = L_{u_0}, \; L_{u_{n-1}^\epsilon} = \max\big(L_{u_0}; \sup_{\alpha_i} \frac{[c^{\alpha_i}]_1 |u_{n-1}^\epsilon|_0 + [f^{\alpha_i}]_1}{1 - [\sigma^{\alpha_i}]_1^2 - [b^{\alpha_i}]_1} \big).$$

Applying the same method as before, we can show that

$$L_{u_n} = \max\big(L_{u_{n-1}}; \sup_{\alpha_i} \frac{[c^{\alpha_i}]_1 |u_{n-1}|_0 + [f^{\alpha_i}]_1}{1 - [\sigma^{\alpha_i}]_1^2 - [b^{\alpha_i}]_1} \big).$$

Induction and definition of (21) give the result. The same for $L_{u_n^\epsilon}$. □

Proof of lemma 5.6: We start by computing L_{v^1}. We set

$$m_{\epsilon_1} := \sup_{i,x,y} \phi_i(x,y) := \sup_{x,y\in\mathbb{R}^N, i\in\mathcal{I}} \{v_i^1(x) - v_i^1(y) - \delta|x-y|^2 + \epsilon_1(|x|^2+|y|^2)\}.$$

Let $m = \phi_j(x_0,y_0)$, i.e. (j,x_0,y_0) attains the supremum. Let $A := \{i \in \mathcal{I},\ (i,x_0,y_0)$ attains the supremum$\}$. Then, by [2, Lemma A.2], there exists $i_0 \in A$, such that $v_{i_0}^1(y_0) < \min_{j\neq i_0}\{v_j^1(y_0)+l\}$. The definition of viscosity solution, and Ishii's lemma imply the existence of $X,\ Y \in \mathcal{S}^N$ such that

$$\max\{L^{\alpha_{i_0}}(x_0, v_{i_0}^1(x_0), p_x, X); v_{i_0}^1(x_0) - \min_{j\neq i}\{v_j^1(x_0)+l\};$$
$$v_{i_0}^1(x_0) - \mathcal{M}u_0(x_0)\} \le 0,$$
$$\max\{L^{\alpha_{i_0}}(y_0, v_{i_0}^1(y_0), p_y, X); v_{i_0}^1(y_0) - \mathcal{M}u_0(y_0)\} \ge 0,$$

where $p_x,\ p_y,\ X\ Y$ satisfy (58) and(59). The we can reduce us to study two cases.
CASE 1: $v_{i_0}^1(y_0) - \mathcal{M}u_0(y_0) - (v_{i_0}^1(x_0) - \mathcal{M}u_0(x_0)) \ge 0.$
This last inequality implies that $v_{i_0}^1(x_0) - v_{i_0}^1(y_0) \le L_{u_0}|x_0-y_0|$. From now on, we continue as the case 1 of the precedent proof, and we have

$$v_i^1(x) - v_i^1(y) \le L_{u_0}|x-y|,\ \forall\, x,y\in\mathbb{R}^N,\ \forall\, i\in\mathcal{I}.$$

CASE 2: $L^{\alpha_{i_0}}(y_0, v_{i_0}^1(y_0), p_y, Y) - L^{\alpha_{i_0}}(x_0, v_{i_0}^1(x_0), p_x, X) \ge 0.$
This is the standard case (see [2, Lemma A.2]), and we have

$$v_i^1(x) - v_i^1(y) \le \sup_{\alpha_i,i} \frac{[c^{\alpha_i}]_1|v_i^1|_0 + [f^{\alpha_i}]_1}{1-[\sigma^{\alpha_i}]_1^2 - [b^{\alpha_i}]_1}|x-y|,\ \forall\, x,y,\in\mathbb{R}^N,\ \forall\, i\in\mathcal{I}.$$

Then we obtain

$$L_{v^1} = \max\big(L_{u_0}; \sup_{\alpha_i,i} \frac{[c^{\alpha_i}]_1|v_i^1|_0 + [f^{\alpha_i}]_1}{1-[\sigma^{\alpha_i}]_1^2 - [b^{\alpha_i}]_1}\big).$$

The same computations lead us to obtain $L_{v^{1\epsilon}}$, and $L_{w^{1\epsilon}}$. For $n > 1$, we apply exactly the same method. We only need to recall that $L_{u_{n-1}} = L_{u_0}$. □

Theorem A.1 *The sequences* $(L_{v^n})_n$, $(L_{v^{n\epsilon}})_n$, $(L_{w^{n\epsilon}})_n$ *are non increasing.*

Proof: We prove this theorem for $(L_{v^n})_n$, the other cases are similar. Using lemma 5.6, and since $(v_i^n)_n$ is a decreasing sequence, we obtain that $(L_{v^n})_n$ is decreasing, and then we have the result. □

B Constants A_i

We begin this section by introducing the following notation. Let $\psi,\varphi \in C_{b,l}(\mathbb{R}^N)^M$, $M \ge 1$. We define constants $A_{\psi,\varphi}$ and $H_{\psi,\varphi}$ as follows

$$A_{\psi,\varphi} := 2k_1\sqrt{k_2^{\psi,\varphi}} + k_3^{\psi,\varphi}, \quad H_{\psi,\varphi} := \frac{3}{2^{2/3}}h_1 h_2^{\psi,\varphi}, \tag{61}$$

where

$k_1 = \sup_{\alpha_i}\{[\sigma^{\alpha_i}]_1 + [b^{\alpha_i}]_1\}$,

$k_2^{\psi,\varphi} = \sup_{\alpha_i}\{\frac{1}{4}(L_\psi + L_\varphi)^2(2[\sigma^{\alpha_i}]_1^2 + 4 + 2[b^{\alpha_i}]_1) + \frac{1}{2}(L_\psi + L_\varphi)(|\psi|_0 \wedge |\varphi|_0[c^{\alpha_i}]_1 + [f^{\alpha_i}]_1 + L_{u_0})\}$,

$k_3^{\psi,\varphi} = \sup_{\alpha_i}\{|\psi|_0 \wedge |\varphi|_0[c^{\alpha_i}]_1 + [f^{\alpha_i}]_1\}$.

$h_1 := \left(C_\rho \sup_{\alpha_i}(|\sigma^{\alpha_i}|_0 + |b^{\alpha_i}|_0 + |c^{\alpha_i}|_0)\right)^{1/3}$, C_ρ depends only on ρ.

$h_2^{\psi,\varphi} := (L_\varphi + A_{\psi,\varphi} + L_{u_0})^{2/3}$.

We give here an extension of the comparison principle of [3, Lemma A.1].

Proposition B.1 *Let u_n and v_n the viscosity solutions of two equations like (Pn), for $n \geq 1$, with coefficients σ, b, c, f and $\bar{\sigma}, \bar{b}$, $\bar{c}$, $\bar{f}$, respectively. Then, we have*

$$\sup_x\{u_n(x) - v_n(x)\} \leq 2k_1(k_2^{u_n,v_n})^{1/2} + k_3^{u_n,v_n},$$

where

- $k_1 = \sup_{\alpha_i}\{|\bar{\sigma}^{\alpha_i} - \sigma^{\alpha_i}|_0^2 + |\bar{b}^{\alpha_i} - b^{\alpha_i}|_0^2\}$,
- $k_2^{u_n,v_n} = \sup_{\alpha_i}\{\frac{(L_{u_n}+L_{v_n})^2}{4}(2[\sigma^{\alpha_i}]_1^2 + 4 + 2[b^{\alpha_i}]_1) + \frac{(L_{u_n}+L_{v_n})}{2}(|u_n|_0 \wedge |v_n|_0[c^{\alpha_i}]_1 + [f^{\alpha_i}]_1 + L_{u_0})\}$,
- $k_3^{u_n,v_n} = \sup_{\alpha_i}\{|u_n|_0 \wedge |v_n|_0|\bar{c}^{\alpha_i} - c^{\alpha_i}|_0 + |\bar{f}^{\alpha_i} - f^{\alpha_i}|_0\}$.

Proof. We prove the proposition for $n = 1$. We apply the same methods as in [3, Theorem A.1]; we set

$$m := \sup_{x,y}\phi(x,y) := \sup_{x,y}\{u_1(x) - v_1(y) - \delta|x-y|^2 - \epsilon_1(|x|^2 + |y|^2)\}.$$

Let $m = \phi(x_0, y_0)$. Applying the notion of viscosity solution and Ishii's lemma, there exist $X, Y \in \mathcal{S}^N$ such that

$$\begin{aligned} 0 \quad \leq \quad & \max\{\sup_{\alpha_i} \bar{L}^{\alpha_i}(y_0, v_1(y_0), p_y, Y); v_1(y_0) - \mathcal{M}u_0(y_0)\} \\ & - \max\{\sup_{\alpha_i} L^{\alpha_i}(x_0, u_1(x_0), p_x, X); u_1(x_0) - \mathcal{M}u_0(x_0)\}, \end{aligned} \tag{62}$$

where $(p_x,\, p_y,\, X,\, Y)$ satisfy (58)-(59). Using $2\phi(x_0, y_0) \geq \phi(x_0, x_0) + \phi(y_0, y_0)$, obtain

$$|x_0 - y_0| \leq \frac{L_{u_1} + L_{v_1}}{2}\delta^{-1}. \tag{63}$$

Now we have to study two different cases.
<u>CASE 1</u>: $v_1(y_0) - \mathcal{M}u_0(y_0) - (u_1(x_0) - \mathcal{M}u_0(x_0)) \geq 0$.

This last inequality implies that $u_1(x_0) - v_1(y_0) \le L_{u_0}|x_0 - y_0|$, and, using (63), we have $u_1(x_0) - v_1(y_0) \le L_{u_0}(L_{u_1} + L_{v_1})(2\delta)^{-1}$, which implies

$$m \le \frac{1}{2}(L_{u_1} + L_{v_1})L_{u_0}\delta^{-1}. \tag{64}$$

CASE 2: $\sup_{\alpha_i} L^{\alpha_i}(y_0, v_1(y_0), p_y, Y) - \sup_{\alpha_i} L^{\alpha_i}(x_0, u_1(x_0), p_x, X) \ge 0$.
This is the standard case, and we use the same computations as in [3, Theorem A.1], detailing all constants. For the bounds of $-tr[\bar{a}^{\alpha_i}(y_0)Y - a^{\alpha_i}(x_0)X]$, $(b^{\alpha_i}(x_0)p_x - \bar{b}^{\alpha_i}(y_0)p_y)$, $(\bar{c}^{\alpha_i}(y_0)v_1(y_0) - c^{\alpha_i}(x_0)u_1(x_0))$, $(f^{\alpha_i}(x_0) - \bar{f}^{\alpha_i}(y_0))$, we use the estimates given in [3, Theorem A.1]. Finally we obtain

$$m \le 2\delta \sup_{\alpha_i}\{|\bar{\sigma}^{\alpha_i} - \sigma^{\alpha_i}|_0^2 + |\bar{b}^{\alpha_i} - b^{\alpha_i}|_0^2\} + \frac{1}{\delta}\sup_{\alpha_i}\{(2[\sigma^{\alpha_i}]_1^2 + 4 + 2[b^{\alpha_i}]_1)\big(\frac{L_{v_1} + L_{u_1}}{2}\big)^2$$

$+(|u_1|_0[c^{\alpha_i}]_1 + [f^{\alpha_i}]_1)\big(\frac{L_{v_1}+L_{u_1}}{2}\big)\} + \sup_{\alpha_i}\{|v_1|_0|\bar{c}^{\alpha_i} - c^{\alpha_i}|_0 + |\bar{f}^{\alpha_i} - f^{\alpha_i}|_0\} + \epsilon_1(1+|x_0|^2+|y_0|^2)$.
If we add the two cases, we have

$$m \le 2k_1\delta + \frac{k_2}{\delta} + k_3 + \epsilon_1 k_4,$$

where

- $k_1 = \sup_{\alpha_i}\{|\bar{\sigma}^{\alpha_i} - \sigma^{\alpha_i}|_0^2 + |\bar{b}^{\alpha_i} - b^{\alpha_i}|_0^2\}$,
- $k_2 = \sup_{\alpha_i}\{\frac{(L_{u_1}+L_{v_1})^2}{4}(2[\sigma^{\alpha_i}]_1^2 + 4 + 2[b^{\alpha_i}]_1) + \frac{(L_{u_1}+L_{v_1})}{2}(|u_1|_0[c^{\alpha_i}]_1 + [f^{\alpha_i}]_1 + L_{u_0})\}$,
- $k_3 = \sup_{\alpha_i}\{|v_1|_0|\bar{c}^{\alpha_i} - c|_0 + |\bar{f}^{\alpha_i} - f^{\alpha_i}|_0\}$,
- $k_4 = (1 + |x_0|^2 + |y_0|^2)$.

Since $\min_\delta\{2k_1\delta + \frac{k_2}{\delta}\} = \sqrt{2k_1k_2}$, letting ϵ_1 go to 0, we obtain

$$m \le \sqrt{2k_1k_2} + k_3.$$

Inverting $|u_1|_0$ and $|v_1|_0$, we have also the symmetric inequality, hence we have the result, with $k_i^{u_1,v_1}$ defined as before. For the general case, we have only to recall that $L_{u_{n-1}} = L_{u_0}$, for all n.□

Proof of proposition 4.5. We apply the precedent proposition, using that $|\bar{g} - g| \le [g]_1\epsilon$, for $g = \sigma, b, c, f$. Then we have the result.□

Consider now the switching systems. We give here an extension of [3, Lemma A.1].

Proposition B.2 *Let v^n and w^n be solutions of two equations (SSn), for $n \ge 1$, with coefficients σ, b, c, f and $\bar{\sigma}, \bar{b}$, $\bar{c}$, $\bar{f}$, respectively. Then, we have*

$$\sup_{x,i}\{v_i^n(x) - w_i^n(x)\} \le (2k_1k_2^{v^n,w^n})^{1/2} + k_3^{v^n,w^n},$$

where

- $k_1 = \sup_{\alpha_i}\{|\bar\sigma^{\alpha_i} - \sigma^{\alpha_i}|_0^2 + |\bar b^{\alpha_i} - b^{\alpha_i}|_0^2\}$,
- $k_2^{v^n,v^n} = \sup_{\alpha_i}\{\frac{(L_{v^n}+L_{w^n})^2}{4}(2[\sigma^{\alpha_i}]_1^2 + 4 + 2[b^{\alpha_i}]_1)$ $+\frac{(L_{u_n}+L_{v_n})}{2}(|v^n|_0 \wedge |w^n|_0[c^{\alpha_i}]_1 + [f^{\alpha_i}]_1 + L_{u_0})\}$,
- $k_3^{v^n,w^n} = \sup_{\alpha_i}\{|v^n|_0 \wedge |w^n|_0|\bar c^{\alpha_i} - c^{\alpha_i}|_0 + |\bar f_i^{\alpha} - f^{\alpha_i}|_0\}$.

Proof. We prove the proposition for $n = 1$. We apply the same methods as in [3, Theorem A.1]; we set

$$m := \sup_{x,y,i} \phi_i(x,y) := \sup_{x,y,i}\{v_i^1(x) - w_i^1(y) - \delta|x-y|^2 - \epsilon_1(|x|^2+|y|^2)\}.$$

Let $m = \phi_j(x_0,y_0)$, i.e. (j,x_0,y_0) attains the supremum. Let $A := \{i \in \mathcal{I}, (i,x_0,y_0)$ attains the supremum$\}$. Then, by [2, Lemma A.2], there exists $i_0 \in A$, such that $w_{i_0}^1(y_0) < \min_{j\neq i_0}\{w_j^1(y_0)+l\}$. Applying the notion of viscosity solution, and Ishii's lemma, there exist $X, Y \in \mathcal{S}^N$ such that

$$\begin{aligned} 0 \le \max\{&\bar L^{\alpha_{i_0}}(y_0, w_{i_0}^1(y_0), p_y, Y); w_{i_0}^1(y_0) - \mathcal{M}u_0(y_0)\} \\ &- \max\{L^{\alpha_{i_0}}(x_0, v_{i_0}^1(x_0), p_x, X); v_{i_0}^1(x_0) - \min_{j\neq i_0}\{v_j^1(x_0)+\ell\}; v_{i_0}^1(x_0) - \mathcal{M}u_0(x_0)\}, \end{aligned} \tag{65}$$

where p_x, p_y, X and Y satisfy (58) and (59). Continuing as in proposition B.1, we obtain the result. □

Proof of lemma 5.7. We apply the precedent theorem, using that $|\bar g - g| \le [g]_1\epsilon$, for $g = \sigma,\ b,\ c,\ f$. □

Theorem B.3 *We have that*

$$A_{v^n} \le \cdots \le A_{v^2} \le A_{v^1},$$

$$A_{w^n} \le \cdots \le A_{w^2} \le A_{w^1}.$$

Proof: The form of A_g and L_g, $g = v^i, w^i$, $i \ge 0$, defined in (61) and (47) respectively, imply the result. □

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