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Abstract: We present a new algorithm for computing the topology of a real algebraic surface S, even in singular cases. We use previous algorithms for 2D and 3D algebraic curves and show how properties of the polar variety of S yield a topological complex equivalent to S, or even a simplicial complex isotopic to S. The proof of correctness of the algorithm is detailed. It is based on tools from stratification theory. We construct an explicit Whitney stratification of S, by resultant computation. Using Thom's isotopy lemma, we show how to deduce the topology of S from a finite number of characteristic points on the surface. An analysis of the complexity of the algorithm and effectivity issues conclude the paper.

Key-words: meshing, implicit algebraic surface, isotopy, Thom's lemma, Whitney stratification, singularity.

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Maillage isotopique d'une surface algébrique réelle

Résumé : Nous présentons un nouvel algorithme pour calculer la topologie d'une surface algébrique rélle S, y compris dans le cas où la surface est singulière. Nous utilisons des algorithmes pour calculer la topologie de courbes algébriques 2D et 3D et montrons comment en utilisant la variété polaire de S, on peut déduire un complexe topologique équivalent à S, ou même un complexe simplicial isotope à S. La preuve de l'algorithme est détaillée et fait appel à la théorie des stratifications. Nous construisons une stratification de Whitney explicite de S à partir de calculs de résultants. En utilisant le lemme de Thom, nous montrons comment on arrive a déterminer la topologie de S à partir d'un nombre fini de points sur la surface. Une analyse de la complexité de l'algorithme et des majorations du nombre de classes d'isotopies pour les courbes et surfaces concluent ce rapport.

Mots-clés : maillage, surface algébrique implicite, isotopie, Lemme de Thom, stratification de Whitney, singularité.

1 Introduction

The study of algebraic surfaces is a fascinating area where important developments of mathematics such as singularity theory interact with visualization problems and the rendering of mathematical objects. The classification of singularities [2] provides simple algebraic formula for complicated shapes, which geometry may be difficult to handle. Such models can be visualized through techniques such as ray-tracing[1] in order to produce beautiful pictures of these singularities. Unfortunately, this approach does not allow to exploit the singularity models in applications other than static visualization.

The aim of this paper is to describe an algorithm which produces a mesh of an algebraic surface S, with a guarantee that the topology of the surface is cached correctly. Such a piecewise linear model of a singular surface can be used in interactive visualization, but also coupled with refinement methods, for approximation or simulation purposes. As a by-product, it yields important topological information such as the number of connected components of the surface in a ball or a box, the Euler characteristics, ...

The problem of triangulation of a (semi)-algebraic set has already been studied in the literature [16], [17], mainly from a theoretical point of view. The special case of surfaces in $\mathbb{R}^3$ already received a lot of attention: we refer in particular to [9], [4], [1], but these works deal only with smooth surfaces.

Another trend for tackling this triangulation problem is via Cylindrical Algebraic Decomposition [5], [3]. It consists in decomposing a semi-algebraic set S into cells, defined by sign conditions on polynomial sequences. Such polynomial sequences are obtained by (sub)-resultant computations, corresponding to successive projections from $\mathbb{R}^{k+1}$ to $\mathbb{R}^k$. The degree of the polynomials in these sequences is bounded by $\mathcal{O}(d^{2^{n-1}})$ and their number by $\mathcal{O}((md)^{3^{n-1}})$, where m is the number of polynomials defining the semi-algebraic set S, d is a bound on the degree of these polynomials and n the number of all the variables of these polynomials [3]. This Cylindrical Algebraic Decomposition does not yield directly a triangulation, nor the topology of the set S. An additional work is required to obtain a triangulation of S, (see [6], [3]). For the case of implicit surfaces in $\mathbb{R}^3$ $(m = 1, n = 3)$, this yields a bound of $\mathcal{O}(d^4 \times d^9) = \mathcal{O}(d^{13})$ points to compute in order to get the topology of the surface.

Our aim here is to describe an effective (and efficient) method for the triangulation of real algebraic surfaces of $\mathbb{R}^3$, which requires the computation of $\mathcal{O}(d^7)$ points. We follow a sweeping plane approach, based on stratified Morse theory. It exploits the following idea: choosing a generic sweeping plane direction, the topology of the sections of the surface with this plane will change only for a discrete set of positions C. Computing this set of critical values (or more precisely a sup-set $C' \supset C$) and the topology of the sections at these critical values, will allow us to recover the topology of the surface. For this purpose, we will use the polar variety of S, which is a 3D curve on S.

We are going to consider an algebraic surface S defined by the equation $f(x, y, z) = 0$ (with $f \in \mathbb{R}[x, y, z]$) in a given ball B (Instead of a ball B, we could also consider a box, but

[1] see e.g. `http://www.algebraicsurface.net/`

the description of the method is less simple). We denote by $S_B = S \cap B$ the intersection of S with the closed volume defined by the ball B. Our objective is to compute a mesh of S_B, isotopic to the surface S_B.

Here are some definitions used hereafter. For $f \in \mathbb{R}[x_1, \ldots, x_n]$, we note $\nabla(f) = [\partial_{x_1}(f), \ldots, \partial_{x_n}(f)]$. We denote by π_u ($u \in \{x, y, z\}$), the projection of $\mathbb{R}^3$ along the direction u on the plane (v, w) with $\{u, v, w\} = \{x, y, z\}$. Similarly, we will denote by $\pi_{u,v}$ the projection along the direction (u, v) on the w-axis.

The notion of critical point for a projection is a key notion of our approach, that we define now: for any algebraic variety V in $\mathbb{R}^n$ defined by equations $f_1 = 0, \ldots, f_s = 0$ and any linear map $\pi : \mathbb{R}^n \to \mathbb{R}^n$, a point p of V is said to be critical for the map π if the matrix $[\nabla(f_1)(p), \ldots, \nabla(f_s)(p), \pi]$ is not of maximal rank. A point $p \in \mathbb{R}^3$ of an algebraic variety $V \subset \mathbb{R}^3$ is x-critical (resp. (u, v)-critical) if it is critical for the projection $\pi_{y,z}$ (resp. π_w). If $V \subset \mathbb{R}^3$ is defined once by the polynomial equations $f_1 = 0, \ldots, f_s = 0$, a x-critical point of V is either a singular point or a point where the tangent space of V at this point is in a plane parallel to the (y, z)-plane i.e the multiplicity of intersection of the plane with the ideal $(f_1, \ldots, f_s)$ at p is greater or equal to 2. The corresponding x-coordinate of p is called a x-critical value. If a value is not x-critical, it is called x-regular.

The paper is organized as follows. As our approach is based on algorithms, for computing the topology of implicit curves in 2D and 3D, we first briefly recall these algorithms in section 2. In part 3, we describe the algorithm for surfaces and in particular how to connect two consecutive sections and the triangulation step keeping safe the topology. We will see, in particular, how a discrete description of the polar variety allows us to recover the two dimensional faces of a triangulation of S_B.

In part 4, we prove the correctness of the algorithm, showing as a new result, how resultant computation yields a Whitney stratification of S. The proof of correctness of the algorithm of connection is given and the isotopy between the surface S and its triangulation is made explicit. Finally, we detail the effectivity and complexity issues.

2 Topology of curves

In this section, we recall the algorithm for computing the topology of 2D and 3D curves, defined by polynomial equations.

2.1 Computing the topology of curves in 2D and 3D

Algorithm in 2D. We first recall a classical algorithm in 2D, which computes a graph of points, isotopic to the curve [15], [13]. We consider, as input, an algebraic curve $\mathcal{C}$ in $\mathbb{R}^2$, defined by a polynomial equation $f(x, y) = 0$. It outputs a 2D graph of points having the topology of $\mathcal{C}$. We assume that there is no two x-critical points having the same x-coordinate. This is possible by a generic linear change of coordinates.

Algorithm 1 — **Topology of 2D curve $\mathcal{C}$ defined by** $f(x, y) = 0$

INPUT: *A polynomial* $f(x, y)$.

- *Compute the* x*-critical points of* $\mathcal{C}$ *and their* x*-coordinates* $\Sigma := \{\sigma_1, \ldots, \sigma_k\}$ *with* $\sigma_1 < \cdots < \sigma_k$.
- *Check the generic position; If the curve is not in a generic position, apply a random change of variables and restart from the first step.*
- *Compute values* μ_i *such that* $\mu_0 < \sigma_1 < \mu_1 < \sigma_2 < \cdots < \sigma_k < \mu_k$ *and insert them in* Σ.
- *For each* $\sigma \in \Sigma$*, compute the number of branches of* $\mathcal{C}$ *above* $x = \sigma$ *(resp. above the critical point if it exists).*
- *If there is no critical points above* σ*, compute also the solutions of* $f(\sigma, y) = 0$*, which yields one point per branch of* $\mathcal{C}$*, above* $x = \sigma$.
- *Connect the branches above two consecutive values* σ.

OUTPUT: *The graph of 2D points of the curve connected by segments, with the same topology as the curve* $\mathcal{C}$.

The connection of branches between two successive sections, one corresponding to a x-critical value, the other to a regular value, is performed as follows: we consider the section corresponding to the x-critical value and connect sequentially the branches above (resp. under) the critical points with the points of the regular section starting from the upper (resp. smaller) one. The critical point is connected to the remaining points (see figure 1). The algorithm can be easily adapted to compute the topology of $\mathcal{C}$ in a disc or a box $B \subset \mathbb{R}^2$.

Algorithm in 3D. We recall here the main results of [10] which are needed for our algorithm of topology of algebraic surfaces. The algorithm of topology of curves in 3D takes as input two implicit surfaces S_1, S_2 and outputs a 3D-graph of points of the curve connected by segments, having the same topology as the curve intersection of S_1 and S_2.

The algorithm starts by computing the x-critical points of the 3D curve, the singular points of the projections of the curve on the xy-plane and xz-plane. We dispose of a list Σ of these x-critical values. We add to this list, regular values in between two critical values. We compute the points on the curve corresponding to all the sections $x = \sigma$ for $\sigma \in \Sigma$. An algorithm of connection creates the segments between the points in two successive sections.

We assume here also that we are in a generic position, i.e there is at most one critical point per section computed. In this case, there is only one way to connect two successive sections (see [10]).

Algorithm 2 — **Topology of a 3D curve $\mathcal{C}$ defined by** $f_1(x, y, z) = f_2(x, y, z) = 0$

INPUT: *Polynomials* $f_1(x, y, z), f_2(x, y, z)$.

- *Compute the* x*-critical points of* $\mathcal{C}$ *and their* x*-coordinates* $\Sigma := \{\sigma_1, \ldots, \sigma_k\}$ *with* $\sigma_1 < \cdots < \sigma_k$.

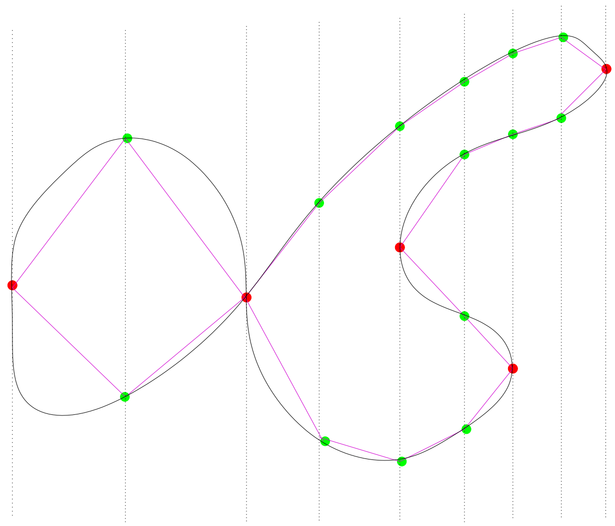

Fig. 1. Topology of a planar curve.

- *Check the generic position; If the curve is not in a generic position, apply a random change of variables and restart from the first step.*
- *Compute the square-free part* $g(x, y)$ *of* $\mathrm{Res}_z(f_1, f_2)$.
- *Compute the square-free part* $h(x, z)$ *of* $\mathrm{Res}_y(f_1, f_2)$.
- *Compute the singular points of the curves* $g(x, y) = 0$ *and* $h(x, z) = 0$ *and insert their* x*-coordinate in* Σ.
- *Compute an ordered sequence* $\alpha_1 < \cdots < \alpha_l$ *containing* $\Sigma = \{\sigma_1 < \cdots < \sigma_r\}$*, two values* $\delta_0 < \sigma_1$ *and* $\delta_1 > \sigma_r$ *and regular values* μ_i *for* $i = 1, \ldots, r-1$ *with* $\sigma_i < \mu_i < \sigma_{i+1}$. *Above each* α_i *for* $i = 1, \ldots, l$*, compute the set of points* L_i *on the curve* $\mathcal{C}$.
- *For each* $i = 0, \ldots, l-1$*, connect the points* L_i *to those of* L_{i+1}.

OUTPUT: *The graph of 3D points of the curve connected by segments, with the same topology as the curve* $\mathcal{C}$.

An interesting property of this graph is that by construction, in between two successive values of Σ, the graph does not cross according to the projection on the xy-plane and xz-plane.

2.2 The polar variety

Hereafter, we will use the properties of the polar variety of $S_B = S \cap B$. The polar variety for the projection π_z in the z-direction is the loci of the critical points of S for the projection along the direction z.

If S is defined by $f(x,y,z)=0$, this polar variety is defined by the equation $f(x,y,z)=\partial_z f(x,y,z)=0$.

In order to take into account the restriction of S to B, we use the following definition:

Definition 1. *We denote by* $\mathrm{VP}_z(S_B)$ *the union of*

- *the set of points* $p \in B$ *on the polar variety of* S *in the* z*-direction,*
- *the intersection of* S *with the border of the ball* B*.*

The equations of the intersection of S with the border of B are obtained by the algorithm 2 for the 3D curve defined by $f(x,y,z)=0$ and $Q(x,y,z)=0$, where Q is the quadratic polynomial of the sphere associated to B.

3 The algorithm for singular algebraic surfaces

We will need to assume that the surface is in a generic position:

Definition 2. *We say that the surface is in generic position if the polar variety of the surface has at most one* x*-critical point in a section* $x=$ *constant.*

This condition can be checked during the algorithm. If it is not fulfilled, we perform a random change of coordinates and a restart of the algorithm. This process will eventually stop with a high probability, and yields a surface in a generic position.

For simplicity of the presentation, we assume hereafter that the polar variety $\mathrm{VP}_z(S_B)$ for the z-direction has at most one x-critical point in a (y,z)-section.

Let us first outline briefly the algorithm for algebraic surfaces, before going into the details.

The first step consists in computing the polar variety for the projection in the z-direction. We apply algorithm 2 for 3D-curves with $f_1=f$, $f_2=\partial_z f$, which computes a polygonal approximation of the polar variety which is isotopic to it. Doing this, the algorithm computes x-critical values corresponding to x-critical points of the 3D curve and singular points of the projection of the polar variety on the xy-plane and the xz-plane.

For each σ of this set Σ of x-critical values, we compute the topology of the corresponding sections of the surface, by applying algorithm 1 for the topology of 2D curves.

Next, we compute regular values between two critical values and the topology of the corresponding sections. Here again, we use the 2D algorithm for implicit curves (see algorithm 1).

The following step consists in connecting two consecutive sections, using the topology of the polar variety (see details in section 3.1).

Finally, we mesh the resulting patches of the surface, by computing a set of points, open segments and open triangles, which are not self-intersecting, defining a simplicial complex, isotopic to the surface (see details in section 3.2).

We summarize the algorithm as follows:

Algorithm 3 — **Topology of an algebraic surface S in a ball B.**

INPUT: *A polynomial $f(x, y, z)$ defining S and a ball B.*

- *Compute the topology of the polar variety for the projection in the z-direction, using algorithm 2.*
- *Compute the topology of the sections, using algorithm 1.*
- *Connect two consecutive sections, by exploiting the topology of the polar variety.*
- *Triangulate the resulting surface patches, avoiding self-intersection of segments and triangles.*

OUTPUT: *A simplicial complex isotopic to S_B.*

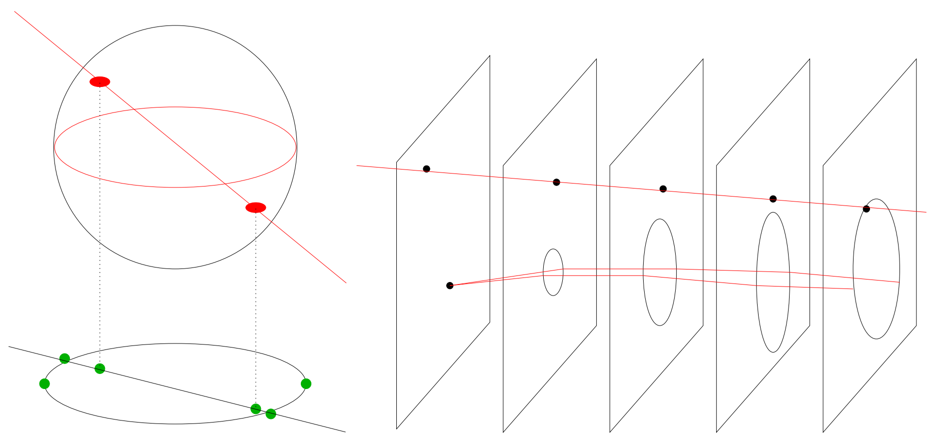

Fig. 2. Polar variety and first connections for the union of a sphere and a line defined by one equation.

Let us now details the two last steps of this algorithm.

3.1 Algorithm of connection

We denote by $\mathcal{V}$ the topological description of $\mathrm{VP}_z(S)$ and $\mathcal{K} := \mathcal{V}$ the initial value of the topological complex describing S. We are going to update these complexes as follows, explaining how we define the connections between the points of two successive sections of S, a regular one which is regular S_r and a critical one S_c, containing a x-critical point of $\mathrm{VP}_z(S_B)$.

Let us denote by $p_1, \ldots, p_r$ (resp. $q_1, \ldots, q_s$) the points of $\pi_z(\mathcal{V} \cap S_r)$ (resp. $\pi_z(\mathcal{V} \cap S_c)$) ordered by increasing y-coordinates, which are also on the projection of an arc of $\mathcal{V}$ connecting S_r and S_c.

Hereafter, we use the convention that $p_0, p_{l+1}, q_0, q_{m+1}$ are points on the border of the ball B. We denote by $\mathcal{A}_i$ $(i = 0, \ldots, l)$ the set of arcs of S_r which projects onto $[p_i, p_{i+1}]$. We denote by $\mathcal{B}_j$ $(j = 0, \ldots, m)$ the set of arcs of S_c, which connects a point projecting at q_j to a point projecting at q_{j+1}. If, moreover there is a critical point R in between, we require that if this arc is to the k^{th} branch arriving at R on the left, then it is also the k^{th} branch starting from R on the right, if this branch exists.

The arcs in $\mathcal{A}_i$ (resp. B_j) are naturally ordered according to their z-position: an arc is bigger than another if it is above the other (see figure 3). We treat incrementally the

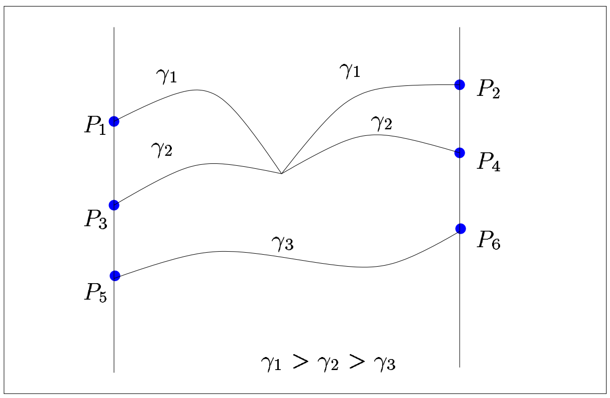

Fig. 3. Order on the arcs.

points p_i, starting from p_0. Let us denote by $q_{\nu(i)}$ the point connected to p_i by an arc of $\pi_z(\mathcal{K}) \supset \pi_z(\mathcal{V})$.

- If p_{i+1} is connected to $q_{\nu(i)}$ by an arc of $\pi_z(\mathcal{K})$, for any arc $\gamma = (P, P')$ of $\mathcal{A}_i$, such that P is connected by $\mathcal{K}$ to Q, we add the arc (P', Q) and the face (P, P', Q) to $\mathcal{K}$.
- If p_{i+1} is not connected to $q_{\nu(i)}$, it is connected to $q_{\nu(i)+1}$. We consider the smallest arc $\gamma = (P, P')$ of $\mathcal{A}_i$, the smallest arc $\eta = (Q, Q')$ of $\mathcal{B}_{\nu(i)}$. The arc (P, Q) is in $\mathcal{K}$. We add the arc (P', Q') and the face (P, P', Q', Q) to $\mathcal{K}$. Then we remove these smallest arcs γ and δ, respectively from $\mathcal{A}_i$ and $\mathcal{B}_{\nu(i)}$ and go on until $\mathcal{A}_i$ is empty.

This procedure is applied iteratively, until we reach the point p_r, so that we move to the next section S'_r, S'_c.

3.2 Algorithm of triangulation

The final step is the triangulation of the different faces computed previously.

Assume that in the algorithm of connection (section 3.1), we have connected an arc $\gamma = (P_1, P_2)$ of S_r to an arc (or point) $\eta = (T_1, T_2)$ in S_c, by a face of $\mathcal{K}$.

The algorithm of triangulation works as follows (see figure 4): we start simultaneously from the starting point P_1 of γ and T_1 of η (i.e the point with the smaller y-coordinate). Those two points are connected by an arc of $\mathcal{K}$. We consider the two next points of $\mathcal{K}$ which are respectively on γ and η. We connect these two points by a segment and triangulate the new corresponding face. This procedure is applied iteratively on γ and η until there is no more point on one of the arc. If there's less points on one arc than on the other, we connect the remaining points on one side by adjacent triangles sharing the same vertex (see figure 5). After this step we obtain triangles or quadrangles, which we subdivide in order to obtain the required triangulation.

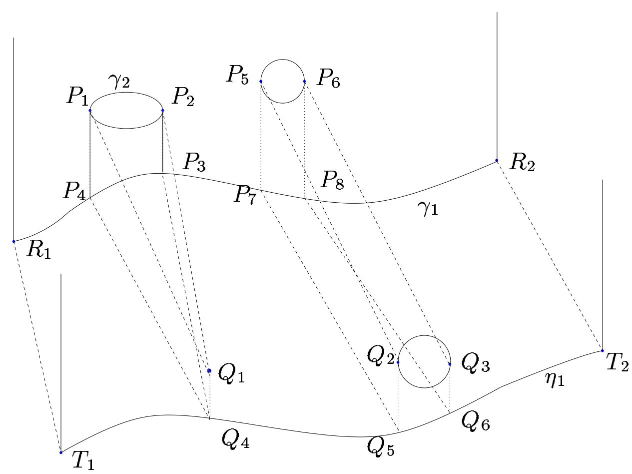

Fig. 4. Division of the space with vertical walls.

4 Why we get the topology

As mentioned previously, the general idea of this sweeping algorithm is to detect where *some topological changes* in the intersection of S with the sweeping plane happen so that in-between the topology is fixed. We are going to prove that in-between the events that we have computed in the previous section, the topology is locally trivial and use this result to describe explicitly the isotopy between the mesh and the surface.

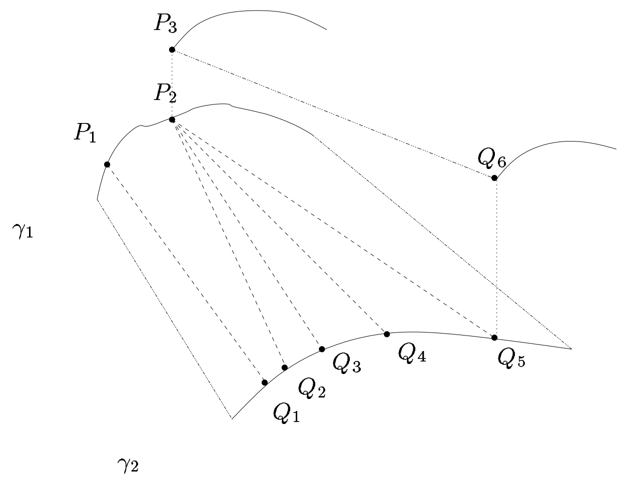

Fig. 5. Meshing.

To prove the correctness of the algorithm we will use results from stratified Morse theory. We refer to [14], [7] for more details.

4.1 Local triviality

The fundamental notion is the Whitney stratification. It is a decomposition of the variety into smooth parts that fit together "regularly". Here are some definitions:

Definition 3. *[7] A stratification of a (semi-algebraic) variety $A \subset \mathbb{R}^n$ is a locally finite partition of A into smooth submanifolds.*

Definition 4. *[7], [14] Let (X, Y) be two strata and $p \in \overline{X} \cap Y \subset \mathbb{R}^n$. X is Whitney-regular at p along Y if for any sequences $x_n \in X$, $y_n \in Y$ converging to p, $l = \lim_{n\to+\infty} \overline{x_n y_n} \subset \mathcal{T} = \lim_{n\to+\infty} T_{x_n}X$, where T_xX is the tangent space of X at the point x.*

A Whitney stratification of a variety S is a stratification of S so that all the couples of strata are Whitney-regular.

It can be noticed that we don't consider the frontier condition for the strata as we only need to use Thom's lemma and it plays no role in its proof [11].

Proposition 1. *Any semi-algebraic stratum S is Whitney regular over a zero-dimensional stratum.*

Proof. The idea is to use the Curve Selection Lemma. See [7].

Definition 5. *A differential map $f : \mathbb{R}^m \to \mathbb{R}^n$ is a submersion at a point p of $\mathbb{R}^m$ if the differential map of f at p is surjective.*

Definition 6. *A continuous map $f : \mathbb{R}^m \to \mathbb{R}^n$ is proper if the inverse image of any compact of $\mathbb{R}^m$ is a compact of $\mathbb{R}^n$.*

The main theorem, that we will use is Thom's lemma [14].

Theorem 1 (Thom's first isotopy). *Let Z be a Whitney stratified subset of $\mathbb{R}^m$ and $\pi : Z \to \mathbb{R}^n$ be a proper stratified submersion. Then there is a stratum preserving homeomorphism*

$$h : Z \to (\pi^{-1}(0) \cap Z) \times \mathbb{R}^n$$

which is smooth on each stratum and commutes with the projection to $\mathbb{R}^n$.

$$\begin{array}{ccc} Z & \xrightarrow{\pi} & \mathbb{R}^n \\ {\scriptstyle h}\downarrow & \nearrow_{\nu} & \\ (\pi^{-1}(0) \cap Z) \times \mathbb{R}^n & & \end{array}$$

This implies that Z is homeomorphic to the cylinder of basis $\pi^{-1}(0) \cap Z$.

In our case, we will apply the theorem with $Z = S_B$, $m = 3$, $n = 1$ and π the projection on the x-axis which is automatically proper as we work in a ball B which is compact.

4.2 Computation of a Whitney stratification

For a projection π_z in the direction z on the (x, y) plane, we define the discriminant $\Delta(\pi_z, S)$ of π_z as the set of zeroes of the squarefree part of the resultant $\mathrm{Res}_z(f, \partial_z f)$.

Proposition 2. *For a generic projection π_z, let*

- *S^0 be the set of points of $\mathrm{VP}_z(S)$ that projects by π_z onto singular points of $\Delta(\pi_z, S)$, each point is considered as a stratum,*
- *S^1 the set of the connected components of $\mathrm{VP}_z(S) - S^0$, (each connected component is a stratum),*
- *S^2 the set of the connected components of $S - \mathrm{VP}_z(S)$ (each connected component is a stratum).*

Then (S^0, S^1, S^2) is a Whitney stratification of S.

From proposition 1, we deduce that showing that (S^0, S^1, S^2) is a Whitney stratification of S boils down to showing that (S^1, S^2) is Whitney-regular.

Whether we consider the polynomial f defining S over $\mathbb{R}$ or $\mathbb{C}$, we obtain a real variety $S = S_{\mathbb{R}}$ or a complex variety $S_{\mathbb{C}}$, as the set of zeroes of f. We will use the results of equisingularity over $\mathbb{C}$ and the notion of "permissible" projection to prove the proposition.

Speder gave in [20] a definition of permissible projection, stronger than the original one of Zariski [21]. We will consider only the case of codimension 1, for which both definitions coincide, so hereafter we will just consider the definition of permissible projection of Zariski:

Definition 7. *A permissible direction of projection for the couple (X,Y) with $Y \subset X$ at $Q \in Y$ is an element of $\mathbb{PC}^3$ so that the line passing through Q defined by this direction is neither included in a neighborhood of Q nor in the tangent space at Y in Q.*

Proposition 3. *For a given algebraic surface S, a generic direction of projection is permissible for (S^1, S^2) at every point of S^1.*

Proof. For an algebraic variety, the local inclusion of a line into the surface is equivalent to a global inclusion. We deduce that the directions of projection to avoid are included into the union of:

- directions of lines included into the surface
- directions of the tangents to the smooth part of the singular locus of the variety.

We consider the first set of directions of lines included into the surface S, defined by one equation $f(x,y,z) = 0$. We consider the surface embedded into the projective space. The directions of lines included into S considered as points of the projective space are included into the intersection of S with the hyperplan at infinity which is a projective curve. Thus the directions corresponding to the first set are included into a set of dimension 1 and are generically avoided.

Now let us consider the second set. We consider an arc of the smooth part of the polar variety (there exists a finite number of such arcs for an algebraic surface). We consider a semi-algebraic parameterization of this arc $(x(s), y(s), z(s))$. Thus we obtain a semi-algebraic parameterization $(x'(s), y'(s), z'(s))$ of a set of unit vectors corresponding to the directions of the tangents to the curve. We deduce that the set to avoid (for tangency condition) corresponds to a semi-algebraic curve on the unit sphere of $\mathbb{R}^3$ and is generically avoided.

Proposition 4. *If the variety S is in generic position (see definition 2) then the projection π_z along the z-axis is a permissible projection.*

Proof. First, there is no line parallel to the z-axis in S because if it were the case, all the vertical line would be included in the polar variety and we would not be in generic position. The second point to check is that the z-direction is not a direction of a tangent of S^1 which is the case as by construction the points of the polar variety with vertical tangents project onto singular points of $\Delta(\pi_z, f)$ and are thus in S^0.

We also recall the notion of equisingularity:

Definition 8. *[20], [21] Let $X \in \mathbb{C}^n$ a hypersurface, Y a smooth submanifold of X of codimension c, P be a point of Y. We say that X is equisingular at P along Y if either $c = 0$ and X is smooth at P or $c > 0$, $Y \subset X_{\text{sing}}$ and there exists a permissible projection π_z such that $\Delta(\pi_z, X))$ is equisingular at $\pi_z(P)$ along $\pi_z(Y)$.*

The main result that we use is the following:

Proposition 5. *[21] If the hypersurface X is equisingular along Y (in codimension 1) then the couple $(X_{smooth} - Y, Y)$ fulfills the Whitney conditions along Y.*

This also us to check the Whitney condition on $\mathbb{C}$. We need to check it on $\mathbb{R}$:

Proposition 6. *X and Y two strata of a Whitney stratification of $S_{\mathbb{C}}$ with $\dim X = 2$ and $\dim Y = 1$, then $X_{\mathbb{R}} = X \cap \mathbb{R}^3$ and $Y_{\mathbb{R}} = Y \cap \mathbb{R}^3$ are Whitney regular.*

Proof. Let P be a point in $Y_{\mathbb{R}} \cap X_{\mathbb{R}}$. Consider a sequence x_n of points of $X_{\mathbb{R}}$ and y_n of points of $Y_{\mathbb{R}}$, both sequences converging to P. We assume that the sequence of secants $\overline{x_n y_n}$ converges to a limit $l \in \mathbb{R}^3$ and the sequence of tangent planes $T_{x_n} X_{\mathbb{R}}$ converges to a limit T. If we consider x_n and y_n in $\mathbb{C}^3$, the sequence of secants converges also to a complex line $l_{\mathbb{C}}$ because $x_n \wedge y_n$ converges to a limit L in $\mathbb{P}(\Lambda^2 \mathbb{R}^4)$ which is embedded in $\mathbb{P}(\Lambda^2 \mathbb{C}^4)$. The convergence of the sequence $T_{x_n} X_{\mathbb{R}}$ is equivalent to the convergence of $T_{x_n} X$: the sequence of normals defined by the orthogonal vectors ∇f converges equivalently in $\mathbb{R}$ or $\mathbb{C}$. Thus $\lim_{n\to\infty} \overline{x_n y_n}_{\mathbb{R}} \subset \lim_{n\to\infty} \overline{x_n y_n}_{\mathbb{C}} \subset \lim_{n\to\infty} T_{x_n} X$ (since (X, Y) is Whitney regular). We deduce that $\lim_{n\to\infty} \overline{x_n y_n}_{\mathbb{R}} \subset \lim_{n\to\infty} T_{x_n} X \cap \mathbb{R}^3$. We know that $\lim_{n\to\infty} T_{x_n} X_{\mathbb{R}} \subset \lim_{n\to\infty} T_{x_n} X \cap \mathbb{R}^3$. As x_n is a sequence of real points, $\lim_{n\to\infty} T_{x_n} X$ is defined as the orthogonal in $\mathbb{C}$ of a real vector. We deduce that $\lim_{n\to\infty} T_{x_n} X \cap \mathbb{R}^3$ is a real space of dimension less or equal to 2 containing $\lim_{n\to\infty} T_{x_n} X_{\mathbb{R}}$ which is of dimension 2, thus the two linear spaces are equal. So we deduce that $\lim_{n\to\infty} \overline{x_n y_n}_{\mathbb{R}} \subset \lim_{n\to\infty} T_{x_n} X_{\mathbb{R}}$ and that $(X_{\mathbb{R}}, Y_{\mathbb{R}})$ is Whitney regular.

Proof of proposition 2. The construction (2) over $\mathbb{C}$, yields a stratification of $S_{\mathbb{C}}$. We consider its restriction to $\mathbb{R}^3$. By proposition 1, we only need to check the Whitney condition for a 1-dimensional strata $S^1_{\mathbb{R}}$ and a 2-dimensional strata $S^2_{\mathbb{R}}$ of $S_{\mathbb{R}}$. Let $p \in S^1_{\mathbb{R}} \cap \overline{S^2_{\mathbb{R}}}$. If p is a smooth point of S, the Whitney condition is trivialy satisfied. If p is singular, by proposition 5, we have the Whitney condition for $(S^2_{\mathbb{C}}, S^1_{\mathbb{C}})$ at p. And by proposition 6, we deduce the Whitney condition for $(S^2_{\mathbb{R}}, S^1_{\mathbb{R}})$ at p. This proves that $(S^0_{\mathbb{R}}, S^1_{\mathbb{R}}, S^2_{\mathbb{R}})$ is a Whitney stratification of $S_{\mathbb{R}}$.

4.3 Connection of the sections

We have described in section 3 an algorithm to connect two successive sections. Now we are going to justify what this algorithm does.

By proposition 2 and using Thom's lemma, we deduce that in between two consecutive critical sections, the topology of the sections is constant. We have computed the topology of regular sections, in between two successive critical ones. So now, in order to prove the isotopy of the surface and the mesh, we have three things to verify:

a) From a topological point of view, we define the good connections between the sections.
b) The triangulation that we propose is valid i.e there is no self-intersection that do not exist in reality.

c) The mesh is isotopic to the surface.

The point c) will be made explicit in subsection 4.4. We prove now the first two points:

a) We are going to justify the algorithm of connection described in section 3.1. Let us recall the notations of section 3.1.

We denote by $p_1, \ldots, p_l$ (resp. $q_1, \ldots, q_m$) the points of $\pi_z(\mathcal{V} \cap S_r)$ (resp. $\pi_z(\mathcal{V} \cap S_c)$) ordered by increasing y-coordinates, which are on the projection of an arc of $\mathcal{V}$ connecting S_r and S_c. Notice that we have $s \le r$.

We denote by $\mathcal{A}_i$ $(i = 0, \ldots, l)$ the set of arcs of S_r which projects onto $[p_i, p_{i+1}]$ and by $\mathcal{B}_j$ $(j = 0, \ldots, m)$ the set of arcs of S_c which projects onto $[q_j, q_{j+1}]$, with the convention that $p_0, p_{l+1}, q_0, q_{m+1}$ are on the border of the ball B.

The point p_i is connected to $q_{\nu(i)}$ by an arc δ_i of the projection of $\mathcal{K}$. We note Θ_i the open planar domain between δ_i and δ_{i+1}. See figure 6.

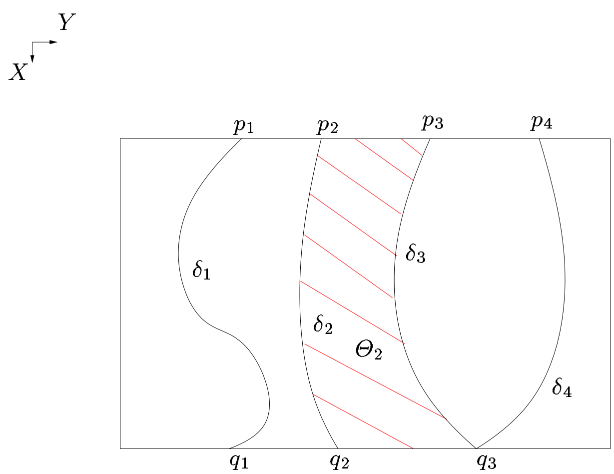

Fig. 6. Projection of the polar variety

Proposition 7. *If the topology of $\pi_z^{-1}(\delta_i) \cap S$, S_r, S_c is determined, then algorithm 3.1 computes the topology of $\pi_z^{-1}(\delta_{i+1}) \cap S$ and of $\pi_z^{-1}(\Theta_i) \cap S$.*

Proof. Let us consider an arc γ in $\mathcal{A}_i$ connecting a point P to a point P'. If we apply Thom's lemma to $S \cap \pi_z^{-1}(\Theta)$, we deduce that $S \cap \pi_z^{-1}(\Theta)$ is topologically trivial i.e made of a family of sheets one above the other and that the border of each sheet contains an arc θ_i in $\pi_z^{-1}(\delta_i)$ and an arc θ_{i+1} in $\pi_z^{-1}(\delta_{i+1})$. We denote hereafter by F the sheet associated to γ.

There are two cases to consider:

1. δ_i and δ_{i+1} intersect in $q_{\nu(i)}$.

2. δ_i and δ_{i+1} do not intersect.

In the first case, we denote by $Q = \theta_i \cap \theta_{i+1}$ the point of $\overline{F}$ which projects onto $q_{\nu(i)}$. By induction hypothesis, as the topology of $\overline{F_i} \cap \pi_z^{-1}(\delta_i)$ is determined by algorithm 3.1, the arc θ_i is represented in $\mathcal{K}$ as the connection of P to Q. The arc θ_{i+1} corresponds to the connection (P', Q), produced by the algorithm, as well as the face (P, P', Q) corresponding to F.

We have

$$\pi_z^{-1}(\delta_{i+1}) \cap S = \left(\mathrm{VP}_z(S) \cap \pi_z^{-1}(\delta_{i+1})\right) \cup \left(\overline{\pi_z^{-1}(\Theta_i) \cap S} \cap \pi_z^{-1}(\delta_{i+1})\right)$$

According to the previous paragraph, the arcs of $\overline{\pi_z^{-1}(\Theta_i) \cap S} \cap \pi_z^{-1}(\delta_{i+1})$ are thus obtained by algorithm 3.1. The arcs of $\mathrm{VP}_z(S) \cap \pi_z^{-1}(\delta_{i+1})$ are obtained by algorithm 2. Thus the algorithm 3.1 compute the topology of $\pi_z^{-1}(\delta_i) \cap S$.

In the second case, we denote again by $Q = \theta_i \cap S_c$ the point of $\overline{F}$ which projects onto $q_{\nu(i)}$ and by $Q' = \theta_{i+1} \cap S_c$ the point of $\overline{F}$ which projects onto $q_{\nu(i)+1}$. The intersection $\overline{F} \cap S_c$ is an arc connecting Q to Q', which exists in $\mathcal{K}$, by hypothesis.

Conversely, as the surface is in generic position (see definition 2), an arc of $S_c \cap \pi_z^{-1}([q_{\nu(i)}, q_{\nu(i)+1}]) = \mathcal{B}_{\nu(i)}$ is in the closure of only one sheet defined by an arc in $S_r \cap \pi_z^{-1}([p_i, p_{i+1}]) = \mathcal{A}_i$. So there is a one to one correspondence between the arcs in $\mathcal{A}_i$ and the arcs in $\mathcal{B}_{\nu(i)}$. Moreover, this correspondence respects the z-order on the arcs, since there is not point of $\mathrm{VP}_z(S_B)$ above Θ_i.

In particular, the smallest arc $\gamma = (P, P')$ in $\mathcal{A}_i$ is connected to the smallest arc $\eta = (Q, Q')$ in $\mathcal{B}_{\nu(i)}$ by a face (P, P', Q', Q) corresponding to F, as computed by algorithm 3.1.

The arc θ_{i+1} connects the point P' to Q', as computed by the algorithm 3.1, so that the topology of $\pi_z^{-1}(\delta_{i+1}) \cap S$ is determined by the algorithm.

This proves that if the topology of $\pi_z^{-1}(\delta_i) \cap S$, S_r, S_c are determined, then algorithm 3.1 compute the topology of S above $\overline{\Theta_i}$.

b) We have to ensure that our triangulation is valid. It is clear that the triangulation we compute does not create holes, because the triangulation refines the topological complex $\mathcal{K}$. Let us check now that we do not create intersection of the open segments and open triangles.

As the algorithm proceeds iteratively on the cylinders $\overline{\pi_z^{-1}(\Theta_i)}$, we have only to check this property above $\overline{\Theta_i}$. By construction, the projection by π_z of open segments and open triangles are either disjoint or included one in the other.

If these projections are disjoint, they cannot self-intersect.

Otherwise, since these are linear objects, their intersection would imply an inversion of the z-position of the corresponding arcs (resp. points) in the section S_r and S_c, which is not possible by Thom's isotopy lemma.

This shows that the triangulation of S is valid.

4.4 The isotopy

We detail here an explicit isotopy between the original surface and the polygonal approximation.

Definition 9. *We say that two surfaces S and S' are isotopic if there exists an application $F : S \times [0,1] \longrightarrow \mathbb{R}^3$ such that:*

1. *F is continuous*
2. *$F(.,0)$ = Identity*
3. *$F(S,1) = S'$*
4. *$\forall t \in [0,1]$, $F(.,t)$ is an homeomorphism onto its image.*

By construction of the mesh, it is sufficient to explain the isotopy between these two successive sections S_r, S_c. We are going to make explicit the isotopy between the surface and the triangulation between two successive sections.

The surface between two consecutive sections can be interpreted as a collection of smooth faces bordered by arcs of the polar variety, such that their projection in the z-direction is either disjoint or identical. We are going to describe the isotopy above the closure of the projection of a given patch, that is in a cylindrical region above the closure of this projection.

The projection of a patch is a region defined by the projection of two arcs of the polar variety. We call δ_1 and δ_2 the two projected arcs. These are semi-algebraic functions of x on $[a,b]$, since $VP_z(S_B)$ is regular for $x \in [a,b[$.

Consider the application :

$$F : S \times [0,1] \longrightarrow \mathbb{R}^3$$

defined by

$$(x, \lambda\delta_1(x) + (1-\lambda)\delta_2(x), z, t) \longrightarrow$$
$$(x, \lambda((1-t)\delta_1(x) + t\frac{y_B - y_A}{x_B - x_A}(x - x_A) + (1-\lambda)((1-t)\delta_2(x) + t\frac{y_D - y_C}{x_D - x_C}(x - x_D)), z)$$

This continuous transformation F is an isotopy from the patch S of the original surface onto $S' = F(S,1)$. This transformation is made such that in projection onto the xy-plane, S' and the triangulation coincide.

After this step, S' and the triangulation T "correspond" in projection i.e the algorithm of connection had associated to each patch of S a patch of T and the images by the isotopy of the faces of S coincide with those of T in projection. As S' is made of n faces with semi-algebraic graphs $z = \phi_1^1(x,y) \ldots z = \phi_1^n(x,y)$ and similarly for T to $z = \phi_2^1(x,y) \ldots z = \phi_2^n(x,y)$. Then, the continuous transformation

$$(x, y, \lambda) \longrightarrow \lambda(\phi_1^1(x,y) + \cdots + \phi_1^n(x,y)) + (1-\lambda)(\phi_2^1(x,y) + \cdots + \phi_2^n(x,y))$$

is an isotopy between S' and T. Combining the two isotopy, we obtain a global isotopy between our variety and the triangulation we have computed.

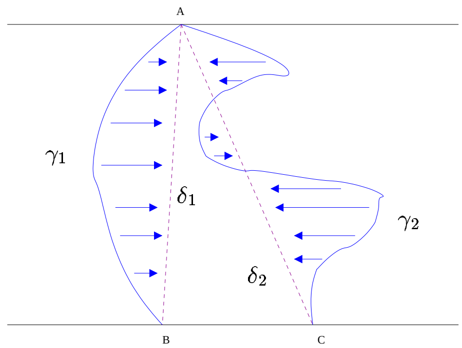

Fig. 7. First step of the isotopy

5 Complexity and effectivity

The algorithm of Cylindrical Algebraic Decomposition (CAD) computes in the case of a polynomial $f(x,y,z)=0$, at most $\mathcal{O}(d^{3^2})$ polynomials of degree at most $\mathcal{O}(d^{2^2})$ [3], which yields at most $\mathcal{O}(d^{13})$ points to compute. In this case, we have the following result.

Proposition 8. *The number of points needed to compute the topology of an algebraic surface S of degree d is at most $\mathcal{O}(d^6)$.*

Proof. As described in the previous sections, we are able to deduce the topology of the surface from its sections at the x-coordinates of a singular point of the projection of the polar variety. A set of points containing these characteristic points is defined by the polynomials systems:

$$\begin{cases} g(x,y)=0 \\ \partial_y g(x,y)=0 \\ f(x,y',z')=0 \\ \partial_z f(x,y',z')=0 \end{cases} \quad \begin{cases} h(x,z)=0 \\ \partial_z h(x,z)=0 \\ f(x,y',z')=0 \\ \partial_z f(x,y',z')=0 \end{cases} \tag{1}$$

where $g(x,y)=\mathrm{Res}_z(f(x,y,z),\partial_z f(x,y,z))$ $h(x,z)=\mathrm{Res}_y(f(x,y,z),\partial_z f(x,y,z))$ and y',z' are new variables.

Indeed, from this set of points, using the 2D algorithm we compute the topology of the x-critical sections. We obtain one sample point per arc connecting two x-critical points, and the topology of the sections as graphs of points, containing at most $\mathcal{O}(d^2)$ points and arcs (since the curves S_r are S_c are of degree d). Using the algorithm of curves in $3D$, we also deduce from this set of points, the topology of the polar variety. The topology of the surface is then deduced from this set of points and arcs.

As $\deg(f)=d$, the degree of $g(x,y)$ is bounded by d^2. By Bezout theorem, the number of (real) solutions of these systems is bounded by

$$2\,d^2(d^2-1)d(d-1)=2\,d^3(d-1)^2(d+1)=\mathcal{O}(d^6).$$

Proposition 9. *The number of points needed to compute a simplicial complex, isotopic to an algebraic surface S of degree d is at most $\mathcal{O}(d^7)$.*

Proof. In order to obtain a triangulation isotopic to the surface, we have in addition, to compute in each section, sample points for the arc passing above the y-critical and regular values. Since there are $\mathcal{O}(d^2)$ critical points on each section, the number of sample points per section is bounded by $\mathcal{O}(d^2) \times d = \mathcal{O}(d^3)$. As there are at most d^4 critical sections (corresponding to the solutions of $g(x, y) = 0, \partial_y g(x, y) = 0$), the number of sample points is at most $\mathcal{O}(d^3) \times d^4 = \mathcal{O}(d^7)$.

From an effective point of view, we have to compute an approximate or exact representation of the real roots of the systems (1) and then to compare their coordinates in order to deduce the connections. This can be performed effectively by using a rational univariate representation of the roots and Sturm (Habicht) sequences [12], [3], [8].

In [19], [18] an analysis of the number of isotopy types of a smooth plane algebraic curve of degree d is given. It is shown that this number is exponentially weakly equivalent to $exp(d^2)$ when $d \to \infty$. A function f is said to be exponentially weakly equivalent to (resp. bounded by) f if $\log(f) = \Theta(\log(g))$ (resp. $\log(f) = \mathcal{O}(\log(g))$).

Using algorithm 1, we can prove that the number of isotopy classes for general planar curves of degree d is exponentialy weakly bounded by $exp(d^3)$. The proof is similar to the one that we detail now for surfaces:

Proposition 10. *The number of isotopy types of an algebraic surface of degree d is exponentially weakly bounded by $exp(d^7)$.*

Proof. We consider $2\,d^4$ sections of d^2 vertical lines, each containing d points. This yields a total of $2\,d^7$ points. To each of these points, we attribute the value

- 0 if it is not in the section of S,
- r if it is a regular point of the section of S,
- c if it is on the polar variety.

From this information, the algorithm determines a unique topological complex equivalent to the surface. This shows that the number of isotopy classes of algebraic surfaces of degree d is bounded by $3^{2\,d^7}$, which proves the proposition.

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References

1. L. Alberti, G. Comte, and B. Mourrain. Meshing implicit algebraic surfaces: the smooth case. Technical Report 5405, INRIA Sophia-Antipolis, 2004.

2. V. I. Arnol'd, S. M. Guseĭn-Zade, and A. N. Varchenko. *Singularities of differentiable maps. Vol. I*, volume 82 of *Monographs in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1985. The classification of critical points, caustics and wave fronts, Translated from the Russian by Ian Porteous and Mark Reynolds.
3. S. Basu, R. Pollack, and M.-F. Roy. *Algorithms in Real Algebraic Geometry*. Springer-Verlag, Berlin, 2003. ISBN 3-540-00973-6.
4. J.D Boissonnat, D. Cohen-Steiner, and G. Vegter. Isotopic implicit surface meshing. In *Proceedings of STOC 2004*, pages 301–309, 2004.
5. G. E. Collins. Quantifier elimination for real closed fields by cylindrical algebraic decomposition. In *Proc. 2nd GI Conference on Automata Theory and Formal Languages*, volume 33 of *Lecture Notes Comput. Sci.*, pages 134–183. Springer-Verlag, 1975.
6. M. Coste. An introduction to semi-algebraic geometry. RAAG network school, 2002.
7. Alexandru Dimca. *Singularities and topology of hypersurfaces*. Universitext. Springer-Verlag, New York, 1992.
8. M. Elkadi and B. Mourrain. Symbolic-numeric tools for solving polynomial equations and applications. In A. Dickenstein and I. Emiris, editors, *Solving Polynomial Equations: Foundations, Algorithms, and Applications.*, volume 14 of *Algorithms and Computation in Mathematics*, pages 125–168. Springer, 2005.
9. E. Fortuna, P. Gianni, P. Parenti, and C. Traverso. Computing the topology of real algebraic surfaces. In *Proceedings of the 2002 international symposium on Symbolic and algebraic computation*, 2003.
10. G. Gatellier, A. Labrouzy, B. Mourrain, and J.-P. Técourt. Computing the topology of 3-dimensional algebraic curves. In *Computational Methods for Algebraic Spline Surfaces*, pages 27–44. Springer-Verlag, 2004.
11. Christopher G. Gibson, Klaus Wirthmüller, Andrew A. du Plessis, and Eduard J. N. Looijenga. *Topological stability of smooth mappings*. Springer-Verlag, Berlin, 1976. Lecture Notes in Mathematics, Vol. 552.
12. L. Gonzalez-Vega, F. Rouillier, and M.F. Roy. *Symbolic Recipes for Polynomial System Solving*. Some Tapas of Computer Algebra. Springer, 1997.
13. Laureano González-Vega and Ioana Necula. Efficient topology determination of implicitly defined algebraic plane curves. *Comput. Aided Geom. Design*, 19(9):719–743, 2002.
14. M. Goresky and R. MacPherson. *Stratified Morse Theory*. Springer-Verlag, 1988.
15. T. A. Grandine and F. W. Klein. A new approach to the surface intersection problem. *Computer Aided Geometric Design*, 14:111–134, 1997.
16. Robert M. Hardt. Triangulation of subanalytic sets and proper light subanalytic maps. *Invent. Math.*, 38(3):207–217, 1976/77.
17. Heisuke Hironaka. Triangulations of algebraic sets. In *Algebraic geometry (Proc. Sympos. Pure Math., Vol. 29, Humboldt State Univ., Arcata, Calif., 1974)*, pages 165–185. Amer. Math. Soc., Providence, R.I., 1975.
18. V. M. Kharlamov and S. Yu. Orevkov. The number of trees half of whose vertices are leaves and asymptotic enumeration of plane real algebraic curves. *J. Combin. Theory Ser. A*, 105(1):127–142, 2004.
19. S. Yu. Orevkov and V. M. Kharlamov. Growth order of the number of classes of real plane algebraic curves as the degree grows. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 266(Teor. Predst. Din. Sist. Komb. i Algoritm. Metody. 5):218–233, 339, 2000.
20. J. P. Speder. Équisingularité et conditions de Whitney. *Amer. J. Math.*, 97(3):571–588, 1975.

21. Oscar Zariski. Studies in equisingularity. II. Equisingularity in codimension 1 (and characteristic zero). *Amer. J. Math.*, 87:972–1006, 1965.

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