From euclidian to hop distance in multi-hop radio networks: a discrete approach

Anthony Busson, Guillaume Chelius, Eric Fleury

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5 60-"+' **-4** +" $*#4 8($#) 2&7[&\0(C 82#] &$*&(&2&$#0" $\mathbb{P}(N_d = n)$ A&$[&&' $*&*03 ' #4$B")&8"' $*&&: 1)2# #8" ' #4$B")&#' K1' #C &'4#0"828"' L1' #C &'4#0"82' #4)(&$&"&$[0(^4_' &3(09# &)204&: \0(C: 2B4 \0($*&C &8" *03 ' #4$B")&A&$[&&' $[0 8(A#$(8(O"0' &4 ' &3&' ' #'Z 0" $*&#(&:)2# #8" ' #4$B")&#' A0$* ' #4)(&$&8"') 0"$#': 0: 4 "&$[0(^4_@0(&Z&'&(822O7[&8240 \0(C 82O3(09&&$*8$(&4: 2$4)0C 3: $&: #' ' #4)(&$&"&$[0(^4 #"' :)&A0: "' 8(#&4 #') 0"$#': 0: 4 0"&4_@0(&09&(784 $*&4$&3 0\ $*&' #4)(&$&"&$[0(^ #4 ' &)(&84&' 7' #4)(&$&:(&4: 2$4)0"9&(Z&$0[8(' 4)0"$#': 0: 4 0"&4_G*#4 3*&' 0C &' 8 V 4$#a &4 $*& C &$*0' 02OZO)0"4#4$#' Z #') 0"4# &(#' Z ' #4)(&$&"&$[0(^4 $0 4$i ' O3(03&($#&4 $*8$8(&*8(' 2O(8)BA2& #') 0"$#': 0: 4 "&$[0(^4_

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5 6’ +/ 6 1 ! "#$%&’()* +&’ &,-. . #,(&(+&*#,0 -1)/ &, 1- ,&1-()#2 $P(N_d = n)$ &2(,&1- +)/(-2’ &&2 2#0 $,&+&/-3(/ &(1- +)/(-2’ &&3’1)+)&22&+-2/ +&(,&(&+35 +)/’ ,&(/ &(’#2()23/ 6 &(78+)0 &2/)#22&1 9 : #3, 1&’-/ 32)8+)0 &2/)#22&1 2#3/ ’#2/)+&,#2/ 1- 1#2<3&3, +"32 . 13/ ’#3,(’=&0)2 -1#,/ >3&+-2/ 1&’-/ $)8+)0 &2/)#22&1 2#3/ 2#3/ ’#2’ &2(,#2/ /3, 1- 1#2<3&3, +&(,#3(&(’#2/(,3)(&(. -, 32 -18 <#,)(=0 &+&,#3(-<&<&#<,-. =)>3&9 ? #3/ &@13#2/ &<-180 &2(1- 1#2<3&3, 0 #A&22&+&(/-3(/ 1&1#2< +&(,#3(&(’,&&&9 : 13/ <&22&-180 &2(; 2#3/ . ,#3@#2/ >3&1&(. ,#. ,)&(&(’-1’ 31&&(+-2/ +&(,&(&+35 +)/’ ,&(/)2+3)/&2(+&($#,2&(>3-2(-35 . ,#. ,)&(&(+&(,&(&+35 ’#2()23/9 B&. 13/; 1#,/>3&1&. -/ +&+)/’ ,&()/-()#2 &(,&+3)(; 1&(@183,/ ’-1’ 31&&(+-2/ 1&(,&(&+35 +)/’ ,&(/ ’#2@&<&2(@&/ ’&11&(+&(,&(&+35 ’#2()23/9 C&. =&22#0 &2&%/()E&1- 0 &(=#+#1#<)&’#2/)/(-2(D’#2/)+&,&(+&(,&(&+35 +)/’ ,&(/ . #3, 1"&(3+&+&. ,#. ,)&(&(>3)/#2(+)*E’)180 &2(’-1’ 31-$1&(+-2/ 1&(,&(&+35 ’#2()23/9

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! " #$%&

!() * #+,-+. #. ' +%/ #01'

" n#$%&'()%(+, -*(.%/ */) ('.0*12 ω %+3(%)'. *4-**1$%+.') $\omega = \{(i_1^j, \dots i_n^j) \in [0,1]^n | j \in [0 \dots C], C \in \mathbb{N}\}$5

6 %'($n \in \mathbb{N}^*$80' -+,, Ω_n .9')'. *4+,, :*))%,' n#$%&'()%(+, -*(.%/ */) ('.0*12)5 6 %'($\lambda \in \mathbb{R}^{+*}$80' -*()%$'1.9' :1*; +; %&%, +-' (Ω_n, Q, P_λ) 09'1' P_λ %.9' :1*; +; %&%< ,+0 =%'(; < -*()%$'1+.%(*4+:*%))*(%($%,1%/.%(*4(*$') 0%&9 %.'()%< λ5

!(! 2 -'30%&+%/ #01'

" n#$%&'()%(+, $%-1'.' ('.0*12 *4)%' S %+4/(-.%($\gamma : [1 \dots S]^n \to \{0,1\}$ $'3('$ *7'1%) 09*,' %:/. $*&+%56 %'(+('.0*12 γ +($+)'. *4%$%') $(i_1, \dots i_n) \in [1 \dots S]^n$80')+< .9+. .9' -**1$%+.' $(i_1, \dots i_n)$ %+*--/:%$ %4 $\gamma(i_1, \dots i_n) = 1$ +($'&:.< %4 $\gamma(i_1, \dots i_n) = 0$5

6 %'($n \in \mathbb{N}^* 8 S \in \mathbb{N}^*$ +($ $p \in [0,1]$80' $'3(' .9' :1*; +; %&%, +-' $(\Gamma_{(n,S)}, Q', P_p)$ 09'1' $\Gamma_{(n,S)}$ %.9')'. *4+,, :*))%,' n#$%&'()%(+, $%-1'.' ('.0*12) *4)%' S5Q' %.9')'. *4+,,)/;)'.) *4$\Gamma_{(n,S)}$?$Q' = 2^{\Gamma_{(n,S)}}$5" (P_p '@+,)?

$$\begin{aligned} P_p : \quad Q' &\to [0,1] \\ A &\mapsto \sum_{\gamma \in A} P_p(\gamma) \end{aligned}$$

09'1' $P_p(\gamma)$ %.9' :1*; +; %&%< *4-9**)%= γ % $\Gamma_{(n,S)}$ =%'(+-**1$%+.' *--/:+(-<:1*; +; %&%< *4 p5 A41(γ) %'@+, .* .9' (/&; '1*4 *--/:%$ -**1$%+.') +($ 0(γ) %'@+, .* .9' (/&; '1*4 '&:.<-**1$%+.') % γ80' .1%%, < 9+7'?

$$P_p(\gamma) = p^{1(\gamma)}.(1-p)^{0(\gamma)}$$

A(-*()'@'(-'80' -+(@+,%+.%', < $'3(' $P_p(A)$ 9+) ;'%='@+, .* .9' :1*; +; %&%< *4.+2%= +(',' &'(.*4 A 09'(-*()%$'1%= +-**1$%+.' *--/:+(-<:1*; +; %&%< *4 p5

!(4 56&$ 76,8 6+$ 8#7 $-',6+3%

6 %'(+$%+(-' || % $\mathbb{R}^n$ +($ =%'(+-*&&/(%+.%(1+(=' T80')+< .9+. .0*('$') +. -**1$%+.') $(i_1^1, \dots i_n^1)$ +($ $(i_1^2, \dots i_n^2)$ -+(-*&&/(%+.' %4 $|(i_1^1, \dots i_n^1), (i_1^2, \dots i_n^2)| \leq T$5

6 %'(+$%-1'.' ('.0*12 γ *1+-*(.%/ */) ('.0*12 ω +($+-*&&/(%+.%(1+(=' T8+7+,%$:+.9 ;'.0 ''(.0* -**1$%+.') x +($ y %+3(%)'@'(-' *4-**1$%+.') $J = \{(j_1^i, \dots j_n^i) | i \in [0 \dots l], l \in \mathbb{N}^*\}$)/-9 .9+.?

- $\forall i \in [1 \dots l-1], \gamma(j_1^i, \dots j_n^i) = 1$ %40' -*()%$'1$\gamma$
- $\forall i \in [1 \dots l-1], (j_1^i, \dots j_n^i) \in \omega$ %40' -*()%$'1$\omega$

- $(j_1^0, \dots j_n^0) = x$
- $(j_1^l, \dots j_n^l) = y$
- $\forall i \in [0 \dots l-1]$"$|(j_1^i, \dots j_n^i), (j_1^{i+1}, \dots j_n^{i+1})| \leq T$

#$%$&' ()*+,-.%$|J|$ &/ , 0,1)(', -$ $J = \{(j_1^i, \dots j_n^i) | i \in [0 \dots l], l \in \mathbb{N}^*\}$)* %'23,1 -&'l4
5 *$&6-#%&+', -$ 7%&8 %&9 -8 &9)0%& -&(%&')*+,-.: 3: 0,1)(', -$ 7%&8 %&9 -$%-8 &9-&(%&'4 ;<
(%&9-)-)&9" -$%6#%&: ,<7%&*0%&6,1*$&6-#%&+', -$* 7%&8 %&9 -8 &9)0%& -&(%&' 73+-$%)6$&' ()*+,-.%&', 6%
%'23,14> %&9-,: %&9-$)* 0,13%&9-$%&$&' ()*+,-.%&9-&-#%($||_H$4
?)0%& , 0,1)(', -$ $J = \{(j_1^i, \dots j_n^i) | i \in [0 \dots l], l \in \mathbb{N}^*\}$"8 %&,1*&(%&9-%&$r_i$,-(e_i ,*/&1188 *@

- $\forall i \in [1 \dots l], r_i = |j_1^{i-1} - j_1^i|$@$%' 6&9 6-#%*)&9 &/ -$%&iA$ $&' &9 -$%&9=6*+,B)*4
- $\forall i \in [1 \dots l], e_i = T - r_i$@$%( )//%6-%&9 %&9 -$%&: ,B): 3: 6,( )&6,-9%&,-(-$%&9=6*+,B)*
' 6&9 6-#%*)&9 &/ -$%&iA$ $&'4

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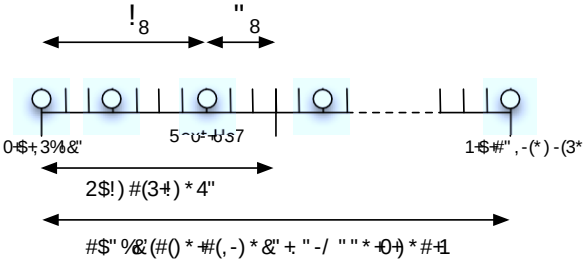


C)936%&D@DA( ): %&*)&9 ,1( )*. 6%&%-%&8 &6E4

#&'7%&9)-8)-$"8 %.&9*)(%&6-$%&*', .%&/ DA( ): %&*)&9 ,1( )*. 6%&%-%&8 &6E* $\Gamma_{(1,S)}$,*(%&'. 6)7%&( )-
%&+)&9 F4F,-(8 %&6<-&%0,13,-%&$%&$&' ()+,-.%$|x,y|_H$ &/ -8 &9-&(%&' x ,-(y 8 $%& $|x,y| = d$4> %
.&9*)(%&6, .&9 : 3-).,+)&9 6,-9%&T ,-(, .&8&6( )-,-%&9.3', -.<'6&7,7)1)+<p4 G -$%&'6&7,7)1)*+).
*', .%&($\Gamma_{(1,S)}, Q', P_p$)"8 %&(%&'-%&-$%&6,-(&0,6),71%&$N^e_{(d,T)}$,*7%&)-9 %'23,1-&-$%&$&' ()*+,-.%7%&A
-8 %&-8 &9-&(%&'x"y8)-$|x,y| = d$,-(x $,0)-9-&-%&)9$7&6z)-yI*( )6%&+)&9 0%&6)/<)-9 $|x,z| \leq e$4
#$%&'%&: &(%&1,-(-&+,+)&9 $,0%&,16%&,(<7%&-3*%&( )-IJK4
/ ,('0 C&6 6%&,( )7)1)+<" $N^0_{(d,T)}$ 8)11,1*&7%&-&%&($N_{(d,T)}$ &6 N_d ,-($N^e_{(d,T)}$:,<7%&-&%&(N^e_d4>)-$
-$%&'%&1,*+-8 &9-&+,)&9 *"8 %&*$&31($&8 %&0%&E%&'-:)-(-$,+N_d (%&%(*&9 T4

! '# $ %&'( ) *) +, -. / . +0 1*) -&2/

" # $%& &(#%&)(#* & (+# , $-. # / 0 $\mathbb{P}(N_d = n)$ 1 +&2+ &((+# 3%4$4&&5 (/ +$, # $|x, y|_H = n$ 1 +#' $|x, y| = d$67%&&$–581 # +$, #9

$$\mathbb{P}(N_0^e = 0) = 1, \forall e$$

$$\mathbb{P}(N_d^e = 0) = 0, \forall d > 0, \forall e$$

$$\mathbb{P}(N_d^e = 1) = 0, \forall d > T, \forall e$$

$$\mathbb{P}(N_d^e = 1) = 1, \forall 0 \leq d \leq T, \forall e < d$$

:' $; <*& #')&' $- ' #(1 / %8 (+#)& 3-#)(1 $5 (/ 4. &* $)+/ %(#)(3$(+ &((/ %2. %&,#-5 $)> &(0. %(+#)(' #&2+4/ %&& (+# *#)(&$(&' *&&2(&' (/ 0 ' %. $%& (+# 3$2>#(. ' (& 1 # %&$2+ (+# *#)(&$(&' 6 @&2. %& ; *#) 2%&4#) (+&(%. (&? $-?/ %&+= $))/ 28&(#* (/ (+# ' / ($(&') &(%*. 2#* &)#2(&' AGB $' * .)#* & (+# $' $-5)&67 +# 3%4$4&&5 / 0 x 4#&? $(n +/ 3) 0%= y 1 &+ $|x, y| = d$ &(#C. $- (/ 9

$$\mathbb{P}(N_d^e = n) = \sum_{r=e+1}^{T} \mathbb{P}(N_{d-r}^{T-r} = (n-1)).(1-p)^{T-r} \qquad \text{D;E}$$

@&2. %& A &-.)(%&(#) $\mathbb{P}(N_{d,10} = n)$ 0' %$ ' / *# 3%&)#' 2# 3%4$4&&5 / 0 0.56 :(&(&(#%&)(&? (/ ' / (&2# (+$((+&(3%4$4&&5 &(' / (2/ ' *&&' #* 45 (+# 3%4$4&&5 / 0 $, $-& 3$(+ #F&(#' 2# 4#(1 ##' (+#)/ . %2# $' * (+# *#)(&$(&' 6G) $ 2/ ')#C. #' 2#8 $\sum_{n=1}^{\infty} \mathbb{P}(N_d = n)$ &(' / (' #2#))$%& #C. $- (/ 16

7%&&$–581 &+ $|x, y| = d$8$)+/ %(#)(3$(+ 4#(1 ##' x $' * y 2$' ' / (4# -/ ' ?#%(+$' $2\frac{(d-1)}{T} + 1$ +/ 3)6 :' 2/ ')#C. #' 2#8. ' *#%(+#)+/ %(#)(3$(+ #F&(#' 2# 2/ ' *&&' 8(+# = #$' , $-. # / 0 (+# +/ 3 *&($' 2# & ?&#' 45 (+# 0' –/ 1 &? 0' %. -$9

$$\mathbb{E}[N_d] = \sum_{i=0}^{2\frac{(d-1)}{T}+1} i\mathbb{P}_c(N_d = i)$$

7 +&(0' %. -$ &(' / (2-/)#*6 H/ 1 #, #%81 # 2$' ' . = #%&2$--5 2/ = 3. (# , $-. #) / 0 $\mathbb{E}[N_d]$6 @&2. %& B)+/ 1) $\mathbb{E}[N_{(d,10)}]$ *#3#' *&? / ' (+# #. 2-&&$' *&($' 2# d 0' %$ ' / *# 3%&)#' 2# 3%4$4&&5 / 0 0.56 :0 (+# 0. ' 2(&' -/ / >) -&# $)(#3 0. ' 2(&' 0' %)+/ %, $-. #) / 0 $\frac{d}{T}$8&(C. &2>-5 4#2/ = #) -&#$%&$) d &2%&$)#)6 7 +&(%&). -(&(, #%& &(#%&)(&? $) &(= #$') (+$((+# +/ 3 *&($' 2# 2$' 4# $33%F& $(#* 45 $ -&#$% 0. ' 2(&' / 0 (+# #. 2-&&$' *&($' 2# 1 +&2+ ?%$*&' (&(#C. $- (/ $ /) 0%($1 *) %23(6 7 +&(= #$' +/ 3 -#' ?(+ &(' / (' #2#))$%& #C. $- (/ T ' * &($ 0. ' 2(&' / 0 p6G) $' $33%F& $(&' 81 # 3%3/)#

34+0+)*-*+%#"

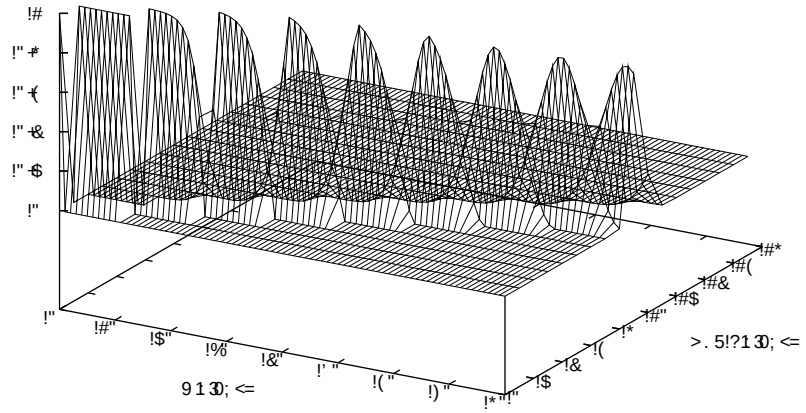
$$\mathbb{E}[N_d] \approx \frac{d}{\mathbb{E}[r]}$$

4

" # ' / 1)(. *5 (+# , $-. # / 0 $\mathbb{E}[r]$6

56" + 786#" ! ———

, -. / 0/ 1234



"#$%&' ( ) *&+, - , ##0 $\mathbb{P}(N_{(d,T)} = n)$1

! '# $ %&' (%) *+ , - &+, .

"# $%&' %($ ') *+, *(' (- ' . ' *# +' #/ (- $0 * - $123 ' 4(, &5 * 6 *'%$) 8- *9# %1%4' #(9#/ (- ' +' #/ (- $0 (- ' 4, 88' 449) ' - $14 9# (- ' 4- $%' 4(1*(- 8%*(' &, 49#/ (- ' *+/ $%'(- . &' 48%'(' &9# (- ' 1%) 9$, 4 4' 8(9$#; <- ' +' #/ (- $0 (- ' n^{th} - $1 94 &' #$(' &: 5 r_n; <- ' %#&$.) *%'*: +' e_n 3 9(- $e_n = T - r_n$ 94 (- ' / *1 : ' (3 ' ' # (- ' n^{th} - $1 *#&(- ' %&$$ %#/ ' T *4 9+, 4(%&9# =/ , % > *#&&' =#' &9# 4' 8(9$# ?;@ A ' *44, . ' (- *((- ' &94(*#8' : ' (3 ' ' # (- ' 4$, %8' *#&(- ' &' 4(9#*(9$# 94 9#=#9(' B$l = +\infty$C <- *(*+$3 4 , 4 ($ 8$#49&' %(- ' *45. 1($(98 : ' - *) 9$%$0 (- ' 6 *'%$) 8- *9#; <$ *) $9&* (%#49' #(6 *'%$) 8- *9#2 3 ' 8$#49&' %(- ' 0*8((- *((- ' % 94 *+3 *54 * 1$9#(9# (- ' %&$$ %#/ ' $0 (- ' 4, 88' 449) ' #$&' 4; D9#8' $e_n = T - r_n$23 ' 8$#49&' %$#+5 (- ' 6 *'%$) 8- *9# $(e_n)_{n\geq 0}$;

/ +%,0%1 2" /%0 123+4) %#+$%0*%), 56 $-789+:) %0 %$3) ; -) #) %<) ; -$=0=+*5 p 0%3 0 -0%9) T8

$$\mathbb{E}[e] = C\left[\frac{T(T-1)}{2}(p-1)^T l + \frac{(-1)^{T-1}}{p^2}\big((1-p) - (1-p)^T(Tp+1-p)\big)\right]$$

6 +5(

$$C = \frac{p}{Tp(p-1)^T + (-1)^{T-1}\left(1-(1-p)^T\right)}$$

0%3

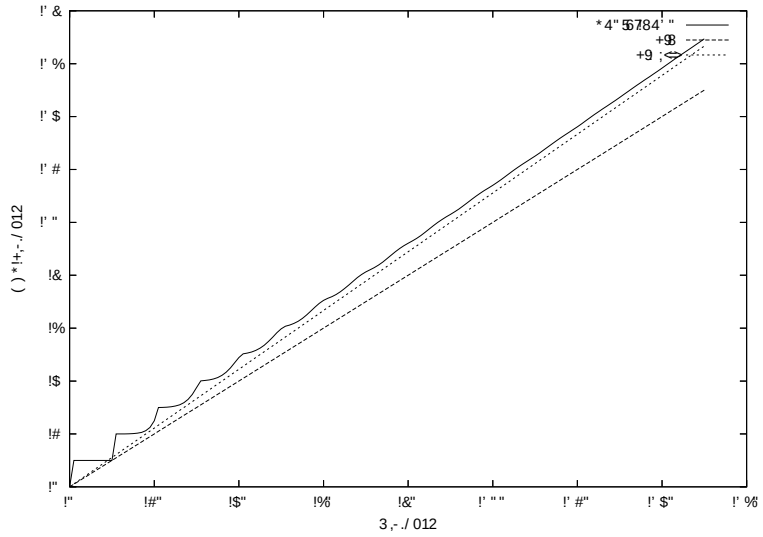
$$\mathbb{E}[r] = T - \mathbb{E}[e]$$

EFGGHI E+' *4' %0' %($ *11' #&9I K; □

3' *%0. 0%&*4, ' " "# H9 , % @23 ' - *) ' &%3 # (- ' . ' *# - $1 &94(*#8' #, . ' %&3 *+5 8$. 1, (' &, 49#/ 0$% , +* > *4 * 0, #8(9$# $0 (- ' ', 8+9&9*# &94(*#8' B$p = 0.52 T = 10$ 9# (- ' =/ , %C; A ' 8$. 1*% (- 94 8, %)' 3 9(- (- ' . 9#9 , . #, . : ' %$0 - $14 L90 #$&' 4 3 ' % 1%4' #(') ' %3 - ' % L' J 1%44' &: 5 $\frac{d}{T}$ *#& (- ' 0$% , +* / 9) ' # : 5 (- ' $%. > *#& 1%$1$49(9$# >2 +) > (- ' +9#' $\frac{d}{\mathbb{E}[r]}$; <- 94 +*4($#' (' #&4 ($ - *) ' * 49 9#*% %&9' #(*4 (- ' 8, %)' 3 - 9&- &' 48%'(' 4 (- ' %*+. ' *# #, . : ' %$0 - $14; <- ') *%'*(9$# : ' (3 ' ' # (- ' 4' 8, %)' 4 94 8$#4(*#(*#&#' / +&B(9 ' ; "(94 &; ' ($ * : 9*4 9#(%&; 8' &: 5 (- ' =%4(- $14; A ' 1$9#($, ((- *($\frac{|x,y|}{\mathbb{E}[r]}$ / 9) ' 4 *# ' J 8' +' #(*11%$J 9 *(9$# ($ (- ' . ' *# #, . : ' %$0 - $14 : ' (3 ' ' # (3 $ #$&' 4 x *#&y; 6 $%$) ' %23 ' / 9) ' $\mathbb{E}[r]$, 49#/ * 8+$4' &0$% , +*;

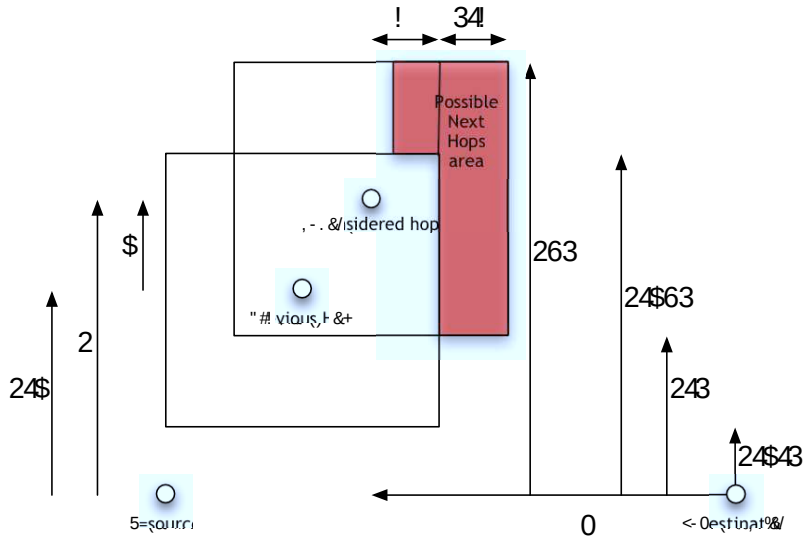
5 6 4780%%#96 1 , : %

A ' #$3 8$#49&' %(- ' 41*8' $0 ?L&9 ' #49$#*+&9#8%(' #' (3 $%4 $\Gamma_{(2,S)}$ *#&4(9+(%5 ($ ') *+, *(' (- ' - $1 &94(*#8' $|x,y|_H$: ' (3 ' ' # (3 $ #$&' 4 x *#&y 3 - ' # (- ' 9%, 8+9&9*# &94(*#8' 94 $|x,y| = d$; M$#(%%5 ($ (- ' >L&9 ' #49$#*+8*4' 2(- ' 4' *%8- 0$%* 4- $%' 4(1*(- : ' (3 ' ' # (3 $ #$&' 4 94 * - *%&(*47 *#&9#) $+) ' 4 * 8$. 1+' J %$, (9#/ 1%$8' 44; "# $%&' %($ 49 1-90$ (- 94 1%: +' . *#&49 9#*%5 ($ (- ' 3 $%' &$#' 9# N+>Q2 3 ' 3 9+-9 9($, %4(, &5 ($ (- ' +' #/ (- $0 %$, ('4 : , 9(, 49#/ * 49 1+' / %' &5 1%$/ %44L; *4' &%$, (9#/ *+/ $%&(- . 49 9#*%($ (- ' 6 HF B' $#5, $-6 0-3 @$" 5*%0 N+PQC$#';



"#$%& ( ) * +, - . &# +0 1 ' 2 3 ' ' 0 $\mathbb{E}[N_d]$ 4 $\frac{d}{T}$. 05 $\frac{d}{\mathbb{E}[r]}$ 6

! '# $ %&'( ) *) +, -. / . +0 1 *) -&2/2/ 3 *-. 45/ / 1(5+6-*%4



#$%&'( ) * + (, -. / -0$1$- 2 - 3 14('(%$- 2 5 4('(6 / - 1(2 1$672(814- / , 62 9(3- &2: ;

< (6'($21('(01(: $2 (=78615$2%14(: $0162, ($|x, y|_g$ - 3 14('- &1(9&$71 &0$2%6 %((: > / '- %(00? 960(: '- &1$2%67% '$14. 9(15((2 15 - 2-: (0 x 62: y 5 4(2 $|x, y| = d$; @4$0 67% '$14. $0 0$ $76' 1- A # B C $#0, $-1 2-3 4$"0*%5D5 4$, 4 460 9((2 : (0, '$9(: $2 E) F; G- 20$ ('$2%15 - 2-: (0 61, - - ': $? 261(0 $x = (x_1, x_2)$ 62: $y = (y_1, y_2)$H- &' 67% '$14. 0(7, 10 60 2(814- / 14(2-: ($z = (z_1, z_2)$ 6. - 2% 677 2($%49- '0 - 3 x 5 4$, 4 I '01 . 2 $J(0 14(: $0162, ($|z_1, y_1|$ 62: 0(, - 2: . 2 $J(0 14(: $0162, ($|z_2, y_2|$;

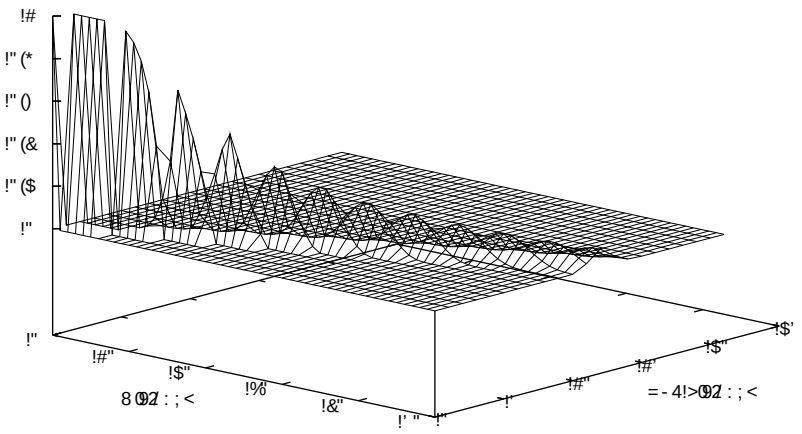
@4$0 67% '$14. $0 6 %(2('67$J61$- 2 - 3 14(67% '$14. &0(: $2 14(!?: $ (20$- 267, 60($2 0(, 1$- 2 K; L 0 - / / - 0(: 1- 14(!?: $ (20$- 267, 60(H14('- &1(0 9&$719> 14$0 67% '$14. 6'(2- 12(, (006'$7> 04- '1(01 / 6140; A - '(- =('H14$0 67% '$14. . 6> 36$71- , '(61(6 '- &1(9(15((2 15 - , - 22(, 1(: 2- : (0;

@4(6'(6 5 4('(5 (, 62 / $ M6 2(814- / $0 0 / 7$1$21- 15 - J- 2(0 60 : (/ $ 1(: - 2 I %&'( ) ; @4(I '01 J- 2($0 : (7$ $1(: 9> $]d - e, d - T] \times [k - T, k + T]$; @4(0(, - 2: J- 2($0 / 6'1$67 62: '(7$(0 - 2 14(, 4- $ (- 3 14(, &'(214- / 61 14(/ '(=$- &0 01(/ $2 14(/ 614 , '(61$- 2; @4$0 , 4- $ ($2: &; (0 14(690(2, (- 3 2- : ($2 62 6'(6 5 4$, 4 $0 , - . . - 2 1- 9- 14 14(7601 4- / 62: 14(, &'(214- / , - =('6% 6'(60; @4(>?9- &2: 6'$(0 - 3 14(/ 6'1$67 J- 2(6'($]k - v + T, k + T]$ $3 v > 0$H$[k - T, k - v - T[$ $3 v < 0$ 62: 14$0 / 6'1$67 J- 2(: - (0 2- 1(8$01$3 $v = 0$; @4(8?9- &2: 6'$(0 - 3 14$0 / 6'1$67 J- 2($0 $[d - 1, d - e]$;

N60(: - 2 14$0 67% '$14. H5 (, 62 : (I 2(14('62: - . =6'$697($N_{(d,k)}^{(e,v)}$ 62: 2& ('$, 677> , - . / &1($\mathbb{P}(N_{(d,k)}^{(e,v)} = n)$ 60 $1 5 60 : - 2($2 0(, 1$- 2 K; #$%&'(O $7801'61(0 $\mathbb{P}(N_d = n)$ 3- ' 6 2- : (/ '(0(2, (/ '- 969$7$1> - 30.05 62: 6 1'620. 00- 2 '62% (- 35; P ('(6%6$2H' - 969$7$1$(0 6'(2- 1, - 2: 1- 2(: 9> 14(

45" (" ' 6 7 5 '

+,- . / . 0 2 3

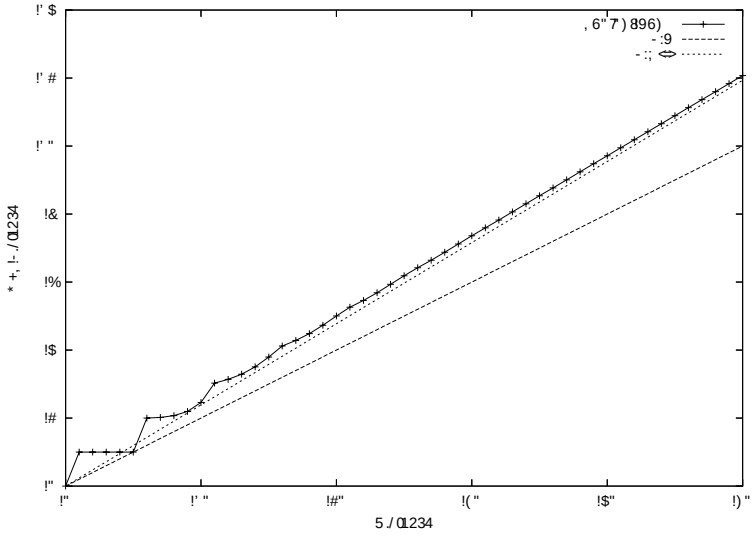


"#$%&' ( ) *&+, -, ##0 $\mathbb{P}(N_{(d,T)} = n)1$

#$%&' &()(*+ %, ' - ')(. #' */ 01(2*0340 &0*5 003 */ 0 2%6$40 ' 3. */ 0 . 02*(3' *(%37 8(96$0 : 2/ %5 2 $\mathbb{E}[N_d]$. 0#03. (39 %3 */ 0 064)(. (' 3 . (2*' 340 d ,%$ ' 3%0 #$0203 40 #$%&' &()(*+ %, 0.05 ' 3. ' *$' 32; (22(%3 $' 390 %, 57 < 0$0 ' 9' (3=*/ 0 ,634*(%3)%&>2)(>0 ' 2*0# ,634*(%3 ,%$ 2/ %$*- ')602 %, $\frac{d}{T}$ ' 3. ?6(4>)+ &04%, 02)(30' $ ' 2 d (34$0' 2027 @@; ()' $)+ *%*/ 0 ! A.(; 032(%3') 4' 20=5 0 3%5 0-')6' *0 */ 0 ; 0' 3 #$%0$022(%3 %, ' / %# %3 */ 0 (x, y) ' 1(2=*/)/ $\mathbb{E}[r]$ ' 2 . 0B30. (3 204*(%3 "7C=(3 %$. 0$*%. 0. 640 */ 0 / %# . (2*' 340 ' 44%$. (39 *%*/ 0 / +#%*/ 02(2 . %30 (3 #$%#%2(*(%3 ! 7

! '# $ %&' ( ) * *+), +%-.) ' /) + 0(%#12 3.-4+%0%5) 3 %6

D3 %$. 0$*%02*(; ' *0 */ 0 #$%&' &()(*+ %, #' */ 01(2*0340 ' 3. */ 0 ; 0' 3 #$%0$022(%3 %, ' / %# (3 */ 0 "Æ . (24$0*0 4' 20=5 0 86(). ' F ' $>%, 4/ ' (3 2(; ()' $*%*/ 0 ! Æ . (24$0*0 4' 207G 0 4%32(. 0$' F ' $>%, H/ ' (3 %, *5 %$' 3. %, - ' $(' &)02 $(e_n, k_n)_{n\geq 0}$7 e_n (2 */ 0 / %&(1%3*') . (,, 0$0340 &0*5 003 */ 0 $' . (%$' 390 J ' 3. */ 0 #$%0$022(%3 %, */ 0 n^{th} / %# ' 2 ())62*$' *0. (3 8(96$0 K7 k_n 4%$02#%3. 2 *%*/ 0 - 0$*(4'))%4' *(%3 %, */ 0 46$$03*3% 0 ', *0$ n / %#27



8(96$0 : LH%, #' $(2%3 &0*5 003 $\mathbb{E}[N_d]/\frac{d}{T}$ ' 3. $\frac{d}{\mathbb{E}[r]}$ (3 2A.(; 032(%3') 30*5 %$>27

D3 %$. 0$*%>00# */ 0 F ' $>%, 4/ ' (3 *$' 4*' &)0=5 0 26##%20 */ ' * */ 0 #$%0$022(%3 %, */ 0 n^{th} / %# r_n ; 62*01400. e_{n-1}7 J / 0 36; &0$ %, #%22(&)0)%4' *(%3 ,%$ */ 0 301*/ %# (2 */ 03 $0. 640. 5 (*/ $09' $. *%*/ 0 %&(9(3') ; %0)7 M5*=63. 0$ */ (2 ' 226; #*(%3=*/ 0 36; &0$ %, #%22(&)0)%4' *(%32 ' * */ 0 n^{th} / %#

#$%&&#' (&)* (&$e_{n-1}$ +&# ,-$. +/0(1 2-+3&$(e_n, k_n)$ 2+&4$ #313#$# 3&;(,5(3&#$%&&#$&; 6 +/0(1 2-+3&'7 8-$ 9/', (&$ $(e_n)_{n\geq 0}$: 5 3)) -$)%;' ,(#$'2/34$,-$ -(/3<(&;+) %(=/$''3(&(>+ -(%+&# $(k_n)_{n\geq 0}$: 2+&-$)%;' ,(',;#* -(5 >+/ ,-$%+,- 3' >(. ,-$)3&$?(3&3&=,-$'(;/2$ +&# ,-$/$2$31$/78-$',;#* (>$(k_n)_{n\geq 0}$ 3' %$'$&;$# 3&'$2,3(&@'7

**! "#$%&& '(** 2, +34*)$ 1, 0)#, +(, 1-5#"678)91, +, #*1/"101, '1/"#: +:)()-; p +, * +"+, 81 T7

$$\begin{aligned}\mathbb{E}[e] = C\Bigg[&\frac{T(T-1)}{2}(-1)^T(1-p)^{(2T+1)T} \\ &+\frac{(-1)^{T-1}}{(1-(1-p)^{2T+1})^2} \\ &\Big((1-p)^{2T+1}-(1-p)^{(2T+1)T} \\ &\big(T(1-(1-p)^{2T+1})+(1-p)^{2T+1}\big)\Big)\Bigg]\end{aligned}$$

5)-<

$$C = \frac{1-(1-p)^{2T+1}}{A+B}$$

+, *

$$A = T(-1)^T(1-(1-p)^{2T+1})(1-p)^{(2T+1)T}$$

+, *

$$B = (-1)^{T-1}\left(1-(1-p)^{(2T+1)T}\right)$$

+, *

$$\mathbb{E}[r] = T - \mathbb{E}[e]$$

ABCCDEA)$+'$/$>$/,(+%&$&#3F G7 □

**)*-#%%+-+.$*(** H&D3=;/$I:5$'$$,-+,,-$)3&$ $\frac{d}{\mathbb{E}[r]}$ =31$' +1$/* =((#+%%(F3 +,3(&(>,-$. $+& -(%#3',+&2$78-$/$3' ?;', +&$=)32,34)$2(&',+&;'$%+/+,3&=,-$,5(2;/1$' #;$,(,-$9/', -(%7J' >(/,-$!H#3 $&'3(&+)2+'$:%(%('3,3(&! 3' 1$/39$# +&# $\mathbb{E}[r]$ 3' =31$&;'3&=+2)('$#>(/. ;)+7

/(0 1#%&2-33$2-+.$*

H&,-3' '$2,3(& 5$2(. %,$,-$',+,3(&&+/ %(4+43)3* (>+. +/0(1 2-+3&k_n 5-32- #$'2/34$',-$ 1$/,32+)#3',+&2$4$,5 $$&,-$&(#$(>,-$$n^{th}$-(%+&#,-$)3&$?(3&3&=,-$'(;/2$+&#,-$#$',3&+,3(&7 8-$,/+&'3,3(&%(4+43)3,3$' +/$$+'3)* #$#;2$# >(. ,-$=$(=/+%32/(;,3&=+)=(/3,-. 7L$-+1$,-$

#$%&&’( ) *+, (-’*’$(. +$/ , / ’%&*’0-1
#$+$i > T$, (2 $−T ≥ k ≤ T$,

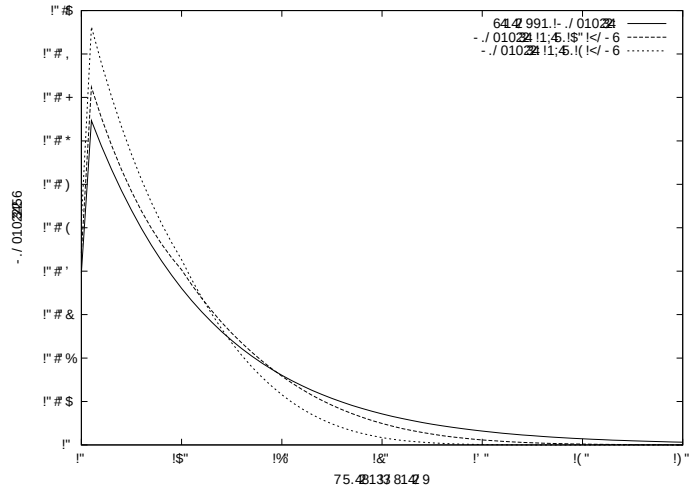
$$P_{i,i-k} = \mathbb{P}\left(k_{n+1} = i - k | k_n = i\right) = \frac{1}{C_i} p(1-p)^{T+k} \quad \text{&’*3 } C_i = \sum_{k=-T}^{T} p(1-p)^{T+k}$$

, (2 #$+$i \leq T$

$$P_{i,j} = \frac{1}{C_i}(1-p)^{2j-1}(1-(1-p)^2) \quad \text{&’*3 } 0 < j \leq T - i$$

$$P_{i,j} = \frac{1}{C_i} p(1-p)^{j+T-i} \quad \text{&’*3 } T - i < j \leq T + i \text{ , (2 } P_{i,0} = \frac{1}{C_i} p$$

$$\text{&’*3 } C_i = \sum_{j=1}^{T-i}(1-p)^{2j-1}(1-(1-p)^2) + \sum_{j=T-i+1}^{T+i} p(1-p)^{j+T-i} + p$$



4’) 5+0 617*, *’$((, +. +$/ , / ’%&8 $#$k_n$92’-*+/ 5*’$($#$v_5$, (2 k_{10} #$+$p = 0.039659 T = 10$:

; 30 -*, *’$((, +. +$/ , / ’%&8 < , 8 / 0 -00(, - *30 2’-*+/ 5*’$($#*30 =0+*’>, %&$>, *’$(, #’0+, (’(? (’*0 (5< / 0+ $# 3$. -: @5*9 ’(+0, %&’+0%&- - < 5%&3$. (0*&$+A *30 (5< / 0+ $# 3$. - ’- %&< ’*02 , (2 $#’0(-<, %&; 30+0#$+09 *30 ’(’*, %&$>, *’$(k_0 B$k_0 = 0$ ’($5+>, -0C3, - , (’< . $+*, (* ’(D50(>09 -’(>0 *30 E , +A$= >3, ’(k_n ’- . 0+*’(0(* $(%& #$+ -<, %&=, %&0 $# n: F(4’) 5+0 6 &0 >$< . , +0 *30 -*, *’$((, + . +$/ , / ’%&8 $# *30 >3, ’(k_n , (2 *30 2’-*+/ 5*’$($#k_5, (2 k_{10} #$+$, . +$/ , / ’%&8 $p = 0.03965$: p 3, -/ 0 >3$-0(-5>3 *3, * *30+0 0G-*, . , *3 $# %&() *3 10 &’*3 . +$/ , / ’%&8 0.99 B3’- =, %&0 $#$p$ ’- $/ *, ’(02

#$ %&' () *+ , -.) (/' 0%1) 2 34-56 4 7 ') #/' *8' %& %8&' 9*) #. #1-1%$ %) #' 0-) /' %) 0 1/ : *' . %* . / %&' 2, + #' *) (&) 9/ 1/ /+ . –; < &' 2 %&' -' 2: %&) (%&' 9. %& 120*' . /' / %&' 9*) #. #1-1%$ %) #' 0-) /' %) %&' . =' >' 0*' . /' / , 2%$– 1%8' . 0&' / %&' . / $+ 9%) %0) >1/ %#, %1) 24 ? &' / %%1) 22. *9*) #. #1-1%$ 1/ %&, / . 9' //1+ 1/ %0) >1/ %#, %1) 2) (k_n@$P(k_\infty \le k) \le P(k_n \le k)\ \forall n > 0$4

A, %1(%&' -' 2: %&) (%&' 9. %& 120*' . /' /; %&' 9*) #. #1-1%$) (%&' ' =1/ %20') (%&' 9. %& >' 0*' . /' /4 7 ' %&12B %& %1%1/ 2) %8' *$ 9' *%12' 2%8) 0) 2/ 1>' * >1/ %#, %1) 2) (%&' 8' *%0. - -) 0. %1) 2 () * %C 0%) *$ < &10&' =1/ % < 1%&. 8' *$ /+ . – 9*) #. #1-1%$ /4 ? &' *' () *' < ' /%>$ %&' >1/ %#, %1) 2) (%&' *. 2>) + 8. *1. #-' k_n ./ %&' -) 2: ' *) (%&' 9. %& n 120*' . /' / . 2> () * . D=' > 9*) #. #1-1%$) (' =1/ %20') (%&' 9. %&4 E) * ' . & 8. -, ') (n; < ' 0) + 9, % %&' 9*) #. #1-1%$) ( ) 00, 9. 20$ p_n /, 0& %& %8&' *' ' =1/ % . 9. %&) (/ 1F' n < 1%& 9*) #. #1-1%$ 0.994 ? &' 2; < ' 0) + 9, % %&' 9*) #. #1-1%$ %& %k_n #' -) 2: / %) $[-T, T]$; #' -) 2: / %) $[-2T, -T[\cup]T, 2T]$. 2> #' -) 2: / %) $[-2T, 2T]$4 ? &' /' 9*) #. #1-1%$ / + . $ #' 12%*9*' %> . / %&' 9*) #. #1-1%$ / %& %k_n 1/ >1/ %2%) (1 &) 9/ ; %) &) 9/ . 2> >1/ %2%) (-' // %& 2 G &) 9/ (*) + 04 H2 E1: , *' I ; < ' 9-) %> %&' / ' 9*) #. #1-1%$ /4 J / < ' 0. 2 /' ' %&' 9*) #. #1-1%$ %) #' -) 2: / %) $[-2T, 2T]$ 8. *$ (*) + 0.93 %) 0.824 K) >' /) (%&' 9. %& & 8' %&') *1: 12 12 %&' 1* *. >1) *. 2: ' L$V_n \in [T, T]$6 < 1%& 9*) #. #1-1%$ 8. *$ 12: (*) + 0.6 %) 0.74 ? &' 9. %& 1/ %&, / . -< . $/ 0-) /' %) %&' %&' . =' C) 1212: %&' /), *0' . 2> %&' >' /%12. %) 24 H%d/ 12%*' /%12: %) + ' 2%) 2 %& %12 . D*' /%%+ ' %&1/ 9*) #. #1-1%$ >' 0*' . /' / < 1%& %&' -' 2: %&) (%&' 9. %& @() * *' . –$ /+ . – 9. %&, %&' % C0%) *$ 1/ %) /&) *0%) %B' . < . $ (*) + %&' . =' 4 H2 . /' 0) 2> %+ '; () * . 2, + #' *) (&) 9/ : *' . %* %& 12 50; %&1/ 9*) #. #1-1%$ 120*' . /' / @() * *' . –$ -) 2: 9. %&, %&' 9*) #. #1-1%$ p_n #' 0) + ' 1+ 9) *%2%12 /, 0&. < . $ %&' *' ' =1/ %2) >' / 12 %&' 2' 1: &#) *&)) > < &10&: ' % %&' % C0%) *$ 0-) /' * %) %&') *1: 124 ? &' *() *' ; < &' 2 1% =1/ %. 9. %& %&' *' 1/ . &1: &9*) #. #1-1%$ %& %%&' % C0%) *$ /%$ *' .–$ 0-) /' %) %&' . =' C) 1212: %&' /), *0' . 2> %&' >' /%12. %) 24

! " #$%&' ( ) ' (* +& $ - +.) / #) 0 %) (#) . +. $ +) ' $

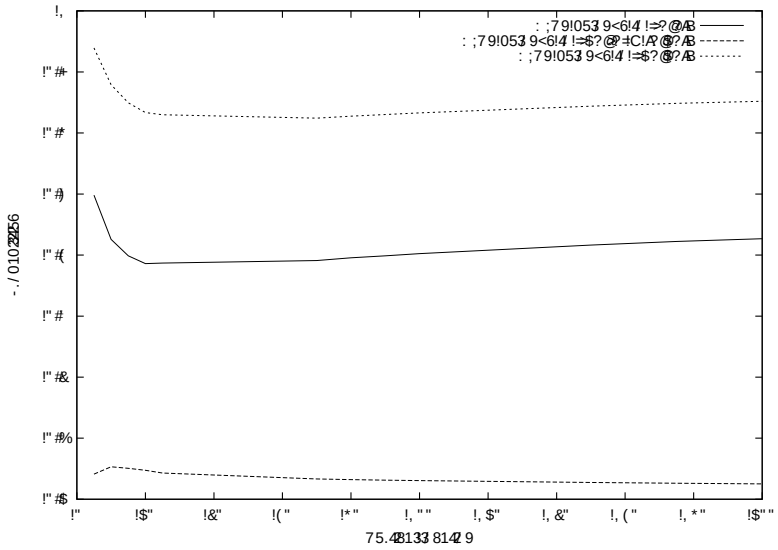
? &' *' . *' . %' . / %%&*' ' 12%*' /% 12 /%>$ 12: >1/ 0*' % 2' %) *B/ . / . 99*) =1+ . %) 2/) (0) 2%12,) , /) 2' /4 ? &' D*/ %) 2' 1/ %& %1%1/) (%2 ' . / 1' * %) / %>$ 9*) 9' *%1 / 12 . D21%>1/ 0*' % ' 281*) 2+ ' 2%%& 2 12 . 2 12D21%0) 2%12,) , /) 2' 4 ? &' / ' 0) 2>) 2' 1/ %& %/, *9*1/ 12: -$; >1/ 0*' % 2' %) *B/ & 8' /) + ' 9*) 9' *%1 / %& %+ . $ #' +) *' *' . -1/ %0) %& 2 0) 2%12,) , /) 2' /4 E) * ' =. + 9-' ; MN& / /&) < 2 %& %8&' 2) >' >' : *' ' 12 . OPQ J *. >1) 2' %) *B 1/ #) , 2>' >4 H%d/ %&' 0. / ' 12 >1/ 0*' % 2' %) *B/ < &' *' . / 12 0) 2%12,) , /) 2' /; 2) >' / & 8' . 2 , 2#) , 2>' > >' : *' ' 4 ? &' %&1*>) 2' 1/ %& %. / < ' /&) < 12 %&1/ / ' 0%1) 2; *' /, -% 12 %&' >1/ 0*' % +) >' - + . $ 12>, 0' #) , 2>. *1' / 12 %&' 0) 2%12,) , / <) *->4

7 ' 0) 2/ 1>' * $S \in \mathbb{N}^*$. 2> 0. – $D_S : \Omega_n \to \Gamma_{(n,S)}$ %&' /. + 9-12:) (Ω_n 12 $\Gamma_{(n,S)}$ >' D2' > . / () –) < /@

$$D_S(\omega)(i_1, \dots i_n) = \begin{cases} 1 & 1(\omega \cap \Pi_{k=1}^n [\frac{i_k - 1}{S}, \frac{i_k}{S}[\neq \emptyset \\ 0 &) \%&' *< 1/' \end{cases}$$

1 2' +& 3 45 2 1 ' #, 0) *1" $\lambda \in \mathbb{R}^{+*}$ 3 $n \in \mathbb{N}^*$ 3 $S \in \mathbb{N}^*$ 42 1 +(0# ' #, 0) *1" -5" 11 6&; ' -) #, 0 $V_c : \Omega_n \to \mathbb{R}$ +, * $V_d^1, V_d^2 : \Gamma_{(n,S)} \to \mathbb{R}$ 0& 2 5 -5 +- 6#" +(($\omega \in \Omega_n$ 37 1 5 +81 $V_d^1 \circ D_S(\omega) \le V_c(\omega) \le V_d^2 \circ D_S(\omega)$4 V_c)0 + "+, *#$ 8+") +9(1), -51 / "#9+9)(0-)' 0/ +' 1 (Ω_n, Q, P_λ) +, * V_d^1, V_d^2 +"1 "+, *#$ 8+") +9(10), $(\Gamma_{(n,S)}, Q', P_{p_\lambda})$ 7 51" 1 $p_\lambda = 1 - \exp(-\frac{\lambda}{S^n})$4: 51, 37 1 5 +81;

$$\mathbb{E}[V_d^1] \le \mathbb{E}[V_c] \le \mathbb{E}[V_d^2]$$



#$%&( ) *+’, - . - $$01 02. 0 $k_n \in [-T, T] 3 k_n \in [-2T, -T[U]T, 2T]$. 45 $k_n \in [-2T, 2T]$ 6 7 7 0 7 n . 45 6 $01 2$. 8’, - . - $$01 , 9, : : &8. 4: 1 ; & 2 02. 00 2(’((<$ 0. 8. 02 6 $02 8’, - . - $$01 $0.997 T = 107$

#$%%&' #()*+) ,)-), ./ *00)1234 5 6 □

!"#$%&'$#' #() "* 783+.8)/,)9 :3,)+<+*=,+. <+)-<(.//(-/, .8) +.<2>/-?/1.31</<+:,*08+0,/0@),.3)+' ,)+<(.+31 ?/1.31</<+:,*08+*12 31 0*,.3?<(*, .8) 9)*1;*(<) /-*,*12/9 ;*,3*A() ?*1 A) A/<12)2 A>,)+<(.+?/9 0<.)2 31 23+?,).) :,*08+*12 31 0*,.3?<(*, .8) 9)*1;*(<)+/-)B<3,*()1.,*1@ 2/9 ;*,3*A()+6 C+*1 *00(3?*.3/1/-.83+.8)/,)9 D31 +)?.3/1 "DE) E3((+.<2> .8) (*E /-.8) 8/0 23+.*1?) 31 ?/1.31</<+1).E/,F+6

+ , %)- . (/0%$#$ "$#1)%2/ #) 0) "#("3) 3/) "$/

C1 +)?.3/1 H*12 +)?.3/1 IDE) 8*;) ?/9 0<.)2 $\mathbb{P}(N_d = n)$D3+9)*1;*(<) *12 .8) *;),*:) 0,/:)+3/1 /-*8/0 31 23+?,).) 1).E/,F+6C +39 3(*, +.<2>3+.,*?.*A() 31 ?/1.31</<+/1)+A<.9 /,) ?/9 0()4 JKD'I6 M/,)/;), .8) ?/1.31</<+?*+) A)?/9)+;),>?/9 0(3?*.)2 31 n@39)1+3/1+E38 $n \geq 2$ JKL*12 .8*. 3+ E8> E) 0,)-), ./ <+) ,)+<(.+?/9 0<.)2 31 *23+?,).))1;3/19)1. ./:).8), E38 .8)/,)9 H./ 2)2<?) A/<12*,3)+/1 $\mathbb{P}(N_d = n)$ 31 ?/1.31</<+1).E/,F+6

C1 +)?.3/1+H*12 IDE) 8*;) 2)=1)2 .8) ,*12/9 ;*,3*A()+$N_{(d,T)}$ -/, 239)1+3/1+1 *12 26 N) :)1),*(3O) .83+2)=133/1 ./ .8) n@39)1+3/1 0,/A*A3(3+.3? +0*?) $(\Gamma_{(n,S)}, Q', P_p)$ 6 N) ?*1 *(+/).)12 .83+-<1?.3/1+-*9 3(> ./ .8) ?/1.31</<+?*+) E38 .8) 2)=133/1 /-$\overline{N}_{(d,T)}$ 31 (Ω_n, Q, P_λ)6 $\overline{N}_{(d,T)}$ 3+.8) ,*12/9 ;*,3*A() A)31:)B<*(./ .8) 8/0 23+.*1?) /-.E/ 1/2)+xDy +<?8 .8*. $|x, y| = d$ 31 Ω_n E38 *.,*1+93+3/1,*1:))B<*(./ T6

45$)%$- 6* !#" +,2 n3*)$1,0)#,+(0/+'1454)61, $T, d \in]0, 1]4\lambda \in \mathbb{R}^{+*}$ +,* $S \in \mathbb{N}^*$47101- $T_S = \lfloor T.S \rfloor 4 d_S = \lfloor d.S \rfloor$89 1#:-+),), $(\Gamma_{(n,S)}, Q', P_{p_\lambda})$ +,* (Ω_n, Q, P_λ);

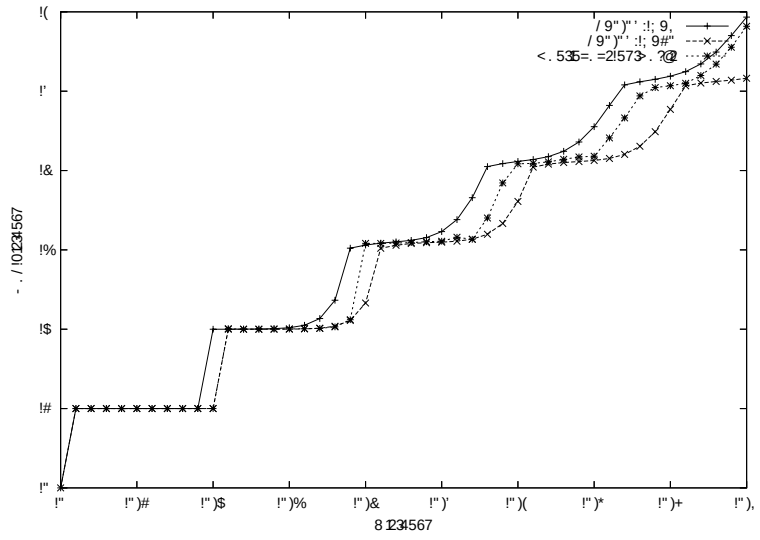
$$\mathbb{E}[N_{(d_S, T_S+1)}] \leq \mathbb{E}[\overline{N}_{(d,T)}] \leq \mathbb{E}[N_{(d_S, T_S)}]$$

#$%%&' #()*+) ,)-), ./ *00)1234 P 6 □

!"#$%&'$#' #() "* 783+.8)/,)9 3123?*.)+.8*.:3,)1.8) *00,/0,3*.) .,*1+-/,9 *.3/1D.8) *;),*:) 8/0 23+.*1?) 31 *-*9 3(>/-?/1.31</<+1).E/,F+3+A/<12)2 A>.8) *;),*:) 8/0 23+.*1?) 31 .E/ ?(/+) 23+?,).) 1).E/,F-*9 3(3)+6 C+*1)40),39)1.*(;),3=?*.3/1/-.83+.8)/,)9 31 .8) Q@39)1+3/1 ?*+)D =:<,) R3((<+.,*.)+.8) *1*(>.3?*(;*(<)+/-$\mathbb{E}[N_{(d_S, T_S)}]D\mathbb{E}[N_{(d_S, T_S+1)}]$?/9 0<.)2 <+31:.8) -/,9 <(*) :3,)1 31 +)?.3/1 I6 *12)40),39)1.*(;*(<)+/-$\mathbb{E}[\overline{N}_{(d,T)}]$ -/, $T = 0.2$D$\lambda = 129$ *12 $S = 506$ 78) *1*(>.3?*(,)+<(.+?/9 0<.)2 31 23+?,).) 1).E/,F+?()*,(> A312.8) +39 <(*.)2/1)+31 ?/1.31</<+1).E/,F+6

45$)%$- 7* !#" +,2 n3*)$1,0)#,+(0/+'1454)61, $T, d \in]0, 1]4\lambda \in \mathbb{R}^*$ +,* $S \in \mathbb{N}^*$47101- $T_S = \lfloor T.S \rfloor 4 d_S = \lfloor d.S \rfloor$89 1<+61), $(\Gamma_{(n,S)}, Q', P_{p_\lambda})$ +,* (Ω_n, Q, P_λ);

$$\lim_{S \mapsto \infty} N_{(d_S, T_S)} = \overline{N}_{(d,T)}$$



#$%&’( ) * +, -&(. / 0 $\mathbb{E}[N_{(d,9)}]$ 1 $\mathbb{E}[N_{(d,10)}]$, 23 $\overline{N}_{(d,0.2)}$ 0’ , 2/ 3(4’/ 5(.. $26(2.$7 $\lambda = 1291$, 6, 2.8 $. $’ 2 ’, %($T = 0.2$, 23 , 3$ 5’(6(.6(4 / 0 $\frac{1}{50}$ 9

#$%%&' #()*+) ,)-), ./ *00)1234 5 6 □

! "#$%&'$#' #() "* 783+.8)/,)9 :3,)+<+* (*+. <+)-<(.//(./ +.<2=>/1.31</<+1).?/,@6 A.?) *,) *B() ./ 0,/;32) * >(/+)2 -/,9 <(* -/, .8) 9)*1 8/0 23+.*1>) 31 23+>,).) 1).?/,@C* ?/,@.8*. ?) 8*;),)*(3D)2 31 +)>.3/1 E*12 +)>.3/1 FC?) >*1)*+3(=2)2<>) .8) 9)*1 8/0 23+.*1>) 31 >/1.31</<+ 1).?/,@-*+* (39 3./-.8) 23+>,).) >*+)6783+?/,@3+2/1) 31 .8) -/((/?31: +)>.3/16

A3+31.),)+.31: ./ 1/.3>) .8*. .8)/,)9 F*12 .8)/,)9 G9 *=B) *00(3)2 ./ /.8), ,*12/9 ;*,3*B()+ .8*1 .8) 8/0 23+.*1>) B).?))1 .?/ 1/2)+6

+ , &&&(.'#() "#) #/ $/)&0(1#" ". $

A1 .83+ +)>.3/1C?) *00(=.8) .8)/,).3>*(,)+<(.+/-+)>.3/1 G*12 +)>.3/1 H./ .8) +.<23)+/-+)>.3/1 E *12 +)>.3/1 F31 23+>,).) :,*08+31/,2), ./ 2)2<>) >(/+)2 -/,9 <(*+31 .8) >*+)/-+>/1.31</<+:,*08+6

+*2 341/("5 #/ $-(6 (#(" 278 (6 $"1() "'-.)"#("4)41 "$#9)%1

I)>.3/1 E6I 0,)+)1.+* >(/+)2 -/,9 <(* -/, $\mathbb{E}[r]$ 31 .8) !K23)9)1+3/1*(23+>,).) >*+)6L)*,) 31.),)+.)2 31 .83+-/,9 <(* ?8)1 $S \to \infty$6 A12))2C.8)/,)9 G+*=+.8*. 3 +8/<(2()*2 <+./ *-/,9 <(* 31 .8) >/1.31</<+?/,(26M+ $S \to \infty C\frac{1}{S} \to 0$ *12?) >*1 <+) +/9)(39 3)2 2);)(/09)1.+'

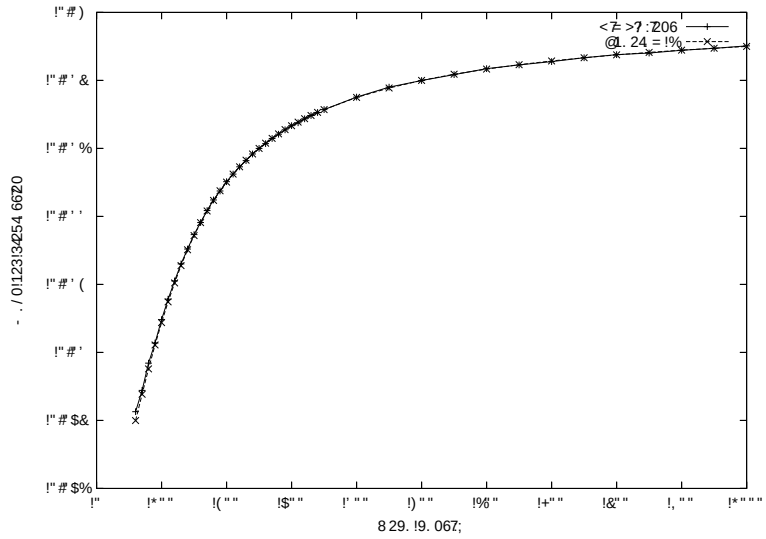
$$p = 1 - e^{-\frac{\lambda}{S}} = 1 - (1 - \frac{\lambda}{S} + o(\frac{1}{S})) = \frac{\lambda}{S} + o(\frac{1}{S}))$$

**;/$) %$6 <*** 2, +34*)$ 1,0)#, +(,1-5#"6778)91, +-"+,0$)00)#, "+,81 $T \in [0,1]$ +,*+,#*1*1,0)-: λ75 1;+91

$$\mathbb{E}[r] = \frac{T - \frac{1}{\lambda} - e^{-\lambda T}(\frac{\lambda T^2}{2} - \frac{1}{\lambda})}{1 - e^{-\lambda T}(\lambda T + 1)}$$

#$%%&' L)31.,/2<>).8) (39 3)2 2);)(/09)1./-p_λ 31 .8) -/,9 <(*/-+)>.3/1 F6I6 □

! "#$%&'$#' #() "* 78)/,)9 H0,)+)1.+* >(/+)2 -/,9 <(* -/, .8) *;),*:) 8/0 0,/:,)+3/1/-* +8/,.)+. 0*.8 31 * !K23)9)1+3/1*(>/1.31</<+1).?/,@6 7/:).8), ?38 0,/0/+33/1 !C3 ()*2+./ *;),=::/2 *00,/43)9 *.3/1/-.8) 9)*1 8/0 23+.*1>) 31 +<>8 1).?/,@6 &3<,) !N>/9 0*,)+.8) ;*(<) /-$\mathbb{E}[r]$:3,)1 B=.83+-/,9 <(*?38.8) /1) .*@)1-,/9 +39 <(*.3/1+C$T = 0.05$C100000 3),*.3/1+B6M+?)>*1 +))C+39 <(*.3/1+*12 *1*(=.3>*(,)+<(.+*,) +39 3(*,6



#$%&( ) "* + (, - . / 0 0’/ %(11$’ - 2/ 3 0&4(5 6’/ 3 1$3 &7, 4$’ - 1, - 5 6’/ 3 4 (6’ ’3 &7, / 64 (/ ’(3 8 $-
4 ( ) 9 5$3 (- 1$’ - , 72, 1(:$T = 0.05$;<

! "# $ %&' () * +' , -(. (+() #/0 (. ,) &(1) 2- 31) +() %1 %&) , +4 156 &

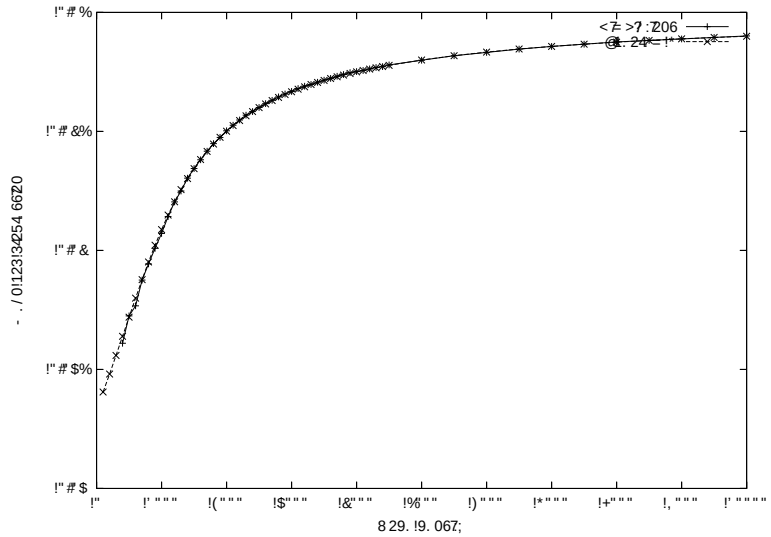
#$%$$&(&*+ , - . , /0, 1&2. , 3 42(%- & 24 . , 1/$25 67 42($\mathbb{E}[r]$ /28, /0, (+ $0 ' &$%$', 3 3, 9, &2: %, 5/ $5 /0, ! ;3$%, 5. $25' &1' . , <

$$p = 1 - e^{-\frac{\lambda}{S^2}} = 1 - (1 - \frac{\lambda}{S^2} + o(\frac{1}{S^2})) = \frac{\lambda}{S^2} + o(\frac{1}{S^2}))$$

7 ' , 15, . ! " 2, + 34*)$ 1, 0)#, +(, 1-5 #"678)91, + -"+, 0$)00)#, "+, 81 $T \in [0, 1]$ +, * + , #*1 *1, 0)-: λ75 1 ; +91

$$\mathbb{E}[r] = \frac{T - \frac{1}{2\lambda T} - e^{-2\lambda T^2}(\lambda T^3 - \frac{1}{2\lambda T})}{1 - e^{-2\lambda T^2}(2\lambda T^2 + 1)}$$

=>? ? @<A , $5/(23- 1, /0, &$%$', 3 3, 9, &2: %, 5/ 24 p_λ $5 /0, 42(%- & 24 . , 1/$25 67 7 □



@$8- (, ""<B , ' 5 02: : (28(, . . $25 12%: - /, 3 4(2% . $%- &/$25. ' 53 4(2% /0, 42(%- & 24/0, 2(, % C $5 /0, ! ;3$%, 5. $25' &1' . , $T = 0.05$7

8) +, 595, +2+(1) " F0, 2(, % C: (, . , 5/. ' 1&2. , 3 42(%- & 42(/0, ' 9, (' 8, 02: : (28(, . . $25 24' : ' /0 G- $& - . $58 /0, : (, . , 5/, 3 8(, , 3) 8, 28(' : 0$1 ' &2($0% $5 ! ;3$%, 5. $25' &125/$5- 2- . 5, /+ 2(H ' 53 + $0 '

"#$%& &%( )* +*, '&- . "$&%+' 0 1 " /*&23' 45( )6 ' - ")*- %89 ' +*6 : %& 23' %+%7, 2)+%7, %5$' " 9)23 23' *- ' " *<2%)- ' (/&*6 ")6 $7%2)*- " 0 =2)" : &"' - 2' ( )- >. $& 44 ?$T = 0.058100000$)2' &2)*- "@0 A' & %+%)- 8")6 $7%2)*- " , %7)(%2' *$&23' *&6 0

! " #$ %&()#$

= 23)" %&2)+7 89 ' 3%,' "3*9 - 23%223' "2$(; */ ( )"+&2' - ' 29 *&B")" %. **(%: &*%+3 /*&23' "2$(; */ +*- 2)- $*$" *- ' "0 C)&2 2* .), ' <*$- (%&)" *- 23' : &*: ' &2)" */ +*- 2)- $*$" - ' 29 *&B" %+("' +*- (2* (' &, ' +7*"' (/*&6 $7%)- +*- 2)- $*$" - ' 29 *&B" 9 3' - : $"3)- . ( )"+&2' &"$72' 2* 23' 7)6)20 D' 3%,' /*&6 %7', : &*, ' (23)" %: &*%+30 1 " %( )&+2 %: 7)+%2)*- */ *$&%+%7, ")"89 ' 3%,' : &"' - 2' (% "2$(; */ 23' 3*: ( )"2%+' <' 29 ' ' - 29 * - *(' ")- 45( )6 ' - ")*- %7 &%+(*6 . ' *6 ' 2&)+ . &%3"0 H3' "' . &%3" %& 7%&' 7, $"' (2* 6 *(' 7)I ' 6 $72) 3*: " 9)&7 "" - ' 29 *&B" "$+3 % %(3*+ *&"' - "*&- ' 29 *&B"0 H3)" %: &*%+3)" */ +*$&' - *2 <*$- (2* 23' "2$(; */ 23' 3*: ( )"2%+' %+(+*$7(%7"* <' %: 7)' (2* "2$(; *23' &: &*: ' &2)" " */ +*- 2)- $*$" . &%3"0

C*&6 %7)I %2)*- */ $\mathbb{E}[N_d]$ 6 %, <' &%7,)- 2' &"2)- . /*&%7%&' - $6 <' &*/ *: 2)6)I %2)*- : &*<7 6 " % 9 ' 77 % /*&23' "2$(; */ . ' *. &%3)+ &*$2)- . %7 *&2)36 : ' &*"&6 %+' "0 1 7%&' - $6 <' &*/ %&2)+7 " ?)/0/ J4K84LM+*- ")(' &23%2 $\mathbb{E}[N_d] = \frac{d}{T}$ 9 3)+3 3% <' ' - "3*9 -)- 23)" %&2)+7 2* <')- %+$&2' 0 N$& &"$72' 9)77%7*9 2* &>- ' %+' &2%)- $6 <' &*/ "2$( )' " "$+3 %8/*&' G%6 : 7' 823' *: 2)6 %7&%. ' /*& . ' *. &%3)+ &*$2)- . 9 3' - +*- ")(' &)- . 23' ' - ' &; +*- "$6 : 2)*- J4KM1

* +, +- +$ %+(

J4MC0 E %+' 7)8 F 0 O7 ')- 8 F 0 P ' <*$&' "8 %(Q0 R$; ', 0 Q2*+3%2)+ . ' *6 ' 2&+ %(%&+3)2' +2$& */ +*6 6 $-)+%2)*- - ' 29 *&B"0 1/ 2) *) 3$4 4 " %+356$%7. #6)4 #8 ST! UVW ! S84 VVS0

J! MC0 E %+' 7)8 O0 H+3*$6 %2+3' - B*8 %(Q0 R$; ', 0 F %&B*, : %23" *- 23' X*)"" *- 5Y ' 7%$- %, . &%3 9)23 %: 7)+%2)*- " 2* &$2)- .)- 6 *<)7 - ' 29 *&B"0 8 9: / 8 ; ; */ <-$=5=/8K! ?4@4 VW4 ZZ8 ! UUU0

JKMC0 E %+' 7) %(Q0 R$; ', 0 Q2*+3%2)+ . ' *6 ' 2&+ 6 *(' 7' */ 6 *<)7 +*6 6 $-)+%2)*- - ' 29 *&B"0 = , -$%6+)-# +%>") ") +%)8: %' " ! ! SW[K0 \ ] \ X&""8 E *+%] %2*- 8 CP 84 VVS0

J[M\ 0 E ' 22' 2 2' &0 N- 23' 6)-)6 $6 - *(' (' . &' %(+*- - ' +2),)2, */ %9)&7 "" 6 $72)3*: - ' 29 *&B0 = 8 ' ? ? $=+@$3 ABC3 P %$I %- ' 8 Q9)2 ' &%(8^$- ' ! UU! 0

JLME 0 E !%I +I ; "I ; - %( ] 0 Q+3*20 1 : : &*G)6 %2' (' +*6 : *")2)*- */ "*6 ' 6 *($7%2' (5X*)""*- _ *&- *) 2' "" ' 7%)*- "0 8 9: / +%8 ; ; */ <-$=5=/8KLTZ[SW2' ! 8 ! UUK0

J' MP 0 E 7%+',)+0 735*5=*) D$" 6+%0 <-$6$3$*# E +6(8 ; ; *+356$%# 6$? $=+*6 0 X3 Y 23' ")"8 a +*7 X*7, 2' +3-)#$' Cb(b&2' 7 (' P %$"%- ' 8 ! UU! 0 H3' ")" c * ! L4S0

#$%&'( ) (*+, - . , / 0) 1 23+45 667) (4758. 9 82, , +857: 7; <*+, 2= +, . 7, = >957*2< 4, / 72 = 2/ +9?) 2%%%3"+, 0+' -)#, 0 #, 4 #$ $ &,)' +-)#, 0@' AB$$DE$$$@F>9, GHI H)

#I %J) K2>??+@L) M 88+997@, / N) 0 *74, ,) O= <. 85 2P 7, 5+4P+4+, 8+? 2, 5*+ 82, , +857: 7; 2P. / *28 , +5Q24R?) O, 5 "#' 6#7 2%%%28 ! 9 4 9 : @S. , L4, 87?82@ DD') 52 . <<+. 4 7, CTTT 0 4, ?. 852, ? 2, U+5Q24R7, -)

#HP%J) K2>??+@N) 0 *74, , @, / V) W. ?9+4) (2, , +857: 7; 7, . / '*28 . , / *; 347 , +5Q24R?) O, 5 "#' 6#7 2%%%28 ! 9 4 9 : @U+Q &24R@XSY@F>, + ! DD!)

#GD%F) Z 74RQ22/ . , / ( ) [. ; , +) N+4829 572, 7, 82, 57, >2>? ?; ?5+= ?) N4+<47, 5 "\ @0, ?5) V . 5*) Y<<9)@ X, 7:) 2PV 7, , +?25. @QHI ")

#GG%0) N*79?<?@S) N. , Q. 4@, / Y) 0. , 5 Q7) (2, , +857: 7; <42<+457+? 2P. <. 8R+54, / 72 , +5Q24R = 2/ +9) 2%%%3"+, 0+' -)#, 0 #, 2, 7#"$ +-)#, 3; 1#"<@] A]BCGD\ \ EGD\ $@S+<5+= 3+4 GHI H)

#G! %N) N74+5) J , 5*+ 82, , +857: 7; 2P 4, / 72 , +5Q24R?) 2%%%3"+, 0+' -)#, 0 #, 2, 7#"$ +-)#, 3; 1#"<@ "$A]BCG\ HDEG\ H! @S+<5+= 3+4 GHHG)

#G"%O) S52^= +, 2: 78 . , / _) ' 7,) N2Q+4 . Q. 4+ 928. 976+/ 42>57, - 7, Q74+9+??, +5Q24R?) 2%%%3"+, 0= +' -)#, 0 #, 5+"+((1(+, * >)0-")?&*1* @0-1$ 0@! AGGBCGG! ! EGG'"@ DDG)

#G\ %W) 0. R. - 7. , / ') Z 9+7, 428R) J <57= . 954, ?= 7??72, 4, , -+? P244, , / 2= 9, / 7?547 3>5+/ <. 8R+54, / 72 5+4= 7, . 9?) 2%%%3"+, 0+' -)#, 0 #, 4 #$ $ &,)' +-)#, 0@' ! A'BC\ aE!] $@QHI \ )

#G]%[ ) b *. , - . , / c >2*2, - (. 2) J <57= 767, - 04++ 1 +82, d- >4. 572, P24 V 2379+ 0. 4- +504. 8R7, - 7, S+, ?24 U+5Q24R?) O, 2, 7#' #$ ABBC@W2, - Z 2, - @' *7, . @V . 48* ! DD\ ) CTTT)

! "#$$%&%'()*$+ , -.

/ 0'$#'1 2- 2, + D=*)$ 1, 0)#, +(, 1-E#"FGH)I 1, + , #*1 / "101, '1 / "#?+?)()-< p +, * + "+, H1 TG

$$\mathbb{E}[e] = C\left[\frac{T(T-1)}{2}(p-1)^T l + \frac{(-1)^{T-1}}{p^2}\big((1-p) - (1-p)^T(Tp+1-p)\big)\right]$$

+, *

$$\mathbb{E}[r] = T - \mathbb{E}[e]$$

N1 J J LCL24 $n \geq 0$. , / $i \leq T-1$ @8*+ 54, , ?752, <423. 37957+? . 4+ C

$$\mathbb{P}\left(e_{n+1} = k \middle| e_n = i\right) = 0 \text{ P24 } k > T - i - 1$$

$$\mathbb{P}\left(e_{n+1} = k \middle| e_n = i\right) = \frac{p(1-p)^k}{1-(1-p)^{T-i}}$$

25*+4Q7?+)

#$ %&' ( )*+' ', -.'//0*$/ $1 - (1 - p)^{T-i}$ 0' %&' -.*) ( ) 0 1 0%2 %&(%%&'.' (%1'(/%*$' -*0$%0$ $T - i$ /1*%3/0$4' $e_{n-1} = i$ 5 '($/ %&(%%&'.' (.' i /1*%6 0%&*7%$*8'9($8 $p(1-p)^k$ 0' %&' -.*) ( ) 0 1 0%2 %&(%%&' :(.'/%$*8' 3$n^{th}$ &*-9:.*5 %&' .(80*.($;' T 0' (%80'%$4' k<

=&' > (.?*+ 4&(0$ 0' &*5 *;'$'*7/@0.'8740)1' ($8 &(/ (A$0%$75)'. *: /%%/@%&'.':*.' %&' (/25 -*%%4 80'%0) 7%0*$ $(\pi_i)_{i=0,..,T-1}$ *: %&' 4&(0$ 0' %&' /*17%0*$ *: $\pi P = \pi$<

B'%)' e (.($8*5 +(.0( )1' 6 0%&80'%0) 7%0*$ π@6' &(+' :*. $i \in \{0, .., T-1\}$@

$$\mathbb{P}(e = i) = \pi_i = C\left[(p-1)^T + (-1)^{(T-1)}(1-p)^i\right]$$

6 &.'

$$C = \frac{p}{Tp(p-1)^T + (-1)^{T-1}\left(1 - (1-p)^T\right)}$$

C 0%&%&' ?$*6 1'8;' *: %&' 80'%0) 7%0*$ π@6' (.' ( )1' %8'874' %&' (+'.(;' *: e ($8 r 36 0%& $r = T - e$9<

$$\begin{aligned}\mathbb{E}[e] &= C\Bigg[\frac{T(T-1)}{2}(p-1)^T \\ &+ \frac{(-1)^{T-1}}{p^2}\left((1-p) - (1-p)^T(Tp+1-p)\right)\Bigg]\end{aligned}$$

$$\mathbb{E}[r] = T - \mathbb{E}[e]$$

□

! "#$$%$%&()*$+ , -.

/ 0'$#'1 .- /%0 123+4)%#+$%0*%)56$-789+:)%0 %$3) ; -)#)%-) ; -$=0=-*5 p 0%3 0 -0%9) T8

$$\begin{aligned}\mathbb{E}[e] = C\Bigg[&\frac{T(T-1)}{2}(-1)^T(1-p)^{(2T+1)T} \\ &+\frac{(-1)^{T-1}}{(1-(1-p)^{2T+1})^2} \\ &\Big((1-p)^{2T+1} - (1-p)^{(2T+1)T} \\ &\left(T(1-(1-p)^{2T+1}) + (1-p)^{2T+1}\right)\Big)\Bigg]\end{aligned}$$

0%3

$$\mathbb{E}[r] = T - \mathbb{E}[e]$$

#$%%&' ( ) * + *, - .*-/0) 12,) 13 45 6*7-*6 4- 0)45 5*8041- , 5 $T - e$ 9) *:* e 45, :, -61+ ;, :4; <.* 9 40) 0) *, 5=+ 3010 8 6450 4<>041- 12 0) * ? , :@1; 8), 4- $(e_n)_{n\geq 0}$ A B-6*: 0) *, 55>+ 3041- 12, 3, 0) *C450*-8*D0) * 0, -5401 1- 3:1<, <4404*5 12 $(e_n)_{n\geq 0}$, :*

$$\mathbb{P}\left(e_{n+1}=k \middle| e_n=i\right)=0 \text{ 21: } k>T-i-1 \text{ , -6 } k\neq T$$

$$\mathbb{P}\left(e_{n+1}=k \middle| e_n=i\right)=\frac{\left(1-(1-p)^{2T+1}\right)(1-p)^{(2T+1)k}}{1-(1-p)^{(T-i)(2T+1)}}$$

21: $k\leq T-i-1$ AE 40) $u=(1-p)^{2T+1}$ D9 * 1<0, 4-

$$\mathbb{P}\left(e_{n+1}=k \middle| e_n=i\right)=0 \text{ 21: } k>T-i-1 \text{ , -6 } k\neq T$$

$$\mathbb{P}\left(e_{n+1}=k \middle| e_n=i\right)=\frac{(1-u)\,u^k}{1-u^{T-i}} \text{ 21: } k\leq T-i-1$$

( ) * 0, -5401 1- 3:1<, <4404*5, :* 0) * 5, + *, 50) * 1-*54- 5*8041- FA A( ) *:*21:* 0) * 21:+ >., *, <1>0 0) * 50, 041- -, : 3:1<, <440=, -6 $\mathbb{E}[e]$ 120) * 5*8041- FA 8, - <* >5*6 9 40) $u=(1-p)^{2T+1}$ AE * 1<0, 4- '

$$\begin{aligned}\mathbb{P}\left(e=i\right) &= \pi_i = C\Big[(-1)^T(1-p)^{(2T+1)T} \\ &+ (-1)^{(T-1)}(1-p)^{(2T+1)i}\Big]\end{aligned}$$

$$C=\frac{1-(1-p)^{2T+1}}{A+B}$$

9 40)

$$A=T(-1)^T(1-(1-p)^{2T+1})(1-p)^{(2T+1)T}$$

, -6

$$B=(-1)^{T-1}\left(1-(1-p)^{(2T+1)T}\right)$$

$$\begin{aligned}\mathbb{E}\left[e\right] = C\Bigg[&\frac{T(T-1)}{2}(-1)^T(1-p)^{(2T+1)T} \\ &+\frac{(-1)^{T-1}}{(1-(1-p)^{2T+1})^2} \\ &\Big((1-p)^{2T+1}-(1-p)^{(2T+1)T} \\ &\left(T(1-(1-p)^{2T+1})+(1-p)^{2T+1}\right)\Big)\Bigg]\end{aligned}$$

$$\mathbb{E}[r] = T - \mathbb{E}[e]$$

□

! "#$$%&$%&()*$+ ,

$ %&' ()*$+ $S \in \mathbb{N}^*$, ' * %-- $D_S : \Omega_n \to \Gamma_{(n,S)}$ ./ $ (, 0 1-)' 2 &3 Ω_n)' $\Gamma_{(n,S)}$ *$4' $* , (3&+-&5 (6

$$D_S(\omega)(i_1, \dots i_n) = \begin{cases} 1 &)3\ \omega \cap \Pi_{k=1}^n [\frac{i_k - 1}{S}, \frac{i_k}{S}[\neq \emptyset \\ 0 & \&/\ \$+5\ )(\$ \end{cases}$$

- '. . / **01** D_S -)/ 0( ) # +# 2 ($*) $" 13" 14$5 / +%6)7$76

$$\forall \gamma \in \Gamma_{(n,S)}, \exists \omega \in \Omega_n, D_S(\omega) = \gamma$$

8 9 : : ; 6 <&1+&=$ ./)(($' .$' %$>5 $ %&' ()*$+ $\gamma \in \Gamma_{(n,S)}$, ' * (/ &5 ./ , . $\omega = \{(\frac{i_1}{S}, \dots, \frac{i_n}{S}) | \gamma((i_1, \dots, i_n) = 1\}$ =$+)4$($D_S(\omega) = \gamma$7 □

$ *$4' $ $D_S^{-1}(\gamma) = \{\omega | D_S(\omega) = \gamma\}$7

- '. . / **21** $\{D_S^{-1}(\gamma) | \gamma \in \Gamma_{(n,S)}\}$ +# / 7%+1) 3/ -1+1$%$8 Ω_n9

8 9 : : ; 6 D_S)(*$4' $* &=$+).(5 / &$)' 1?. *&0 ,)' , ' * +$, % $( ).(5 / &$ &?.1?. *&0 ,)' @$$ -$0 0 , AB5 /)%)(4').$7 □

$ %&' ()*$+ $\lambda \in \mathbb{R}^{+*} > n \in \mathbb{N}^* > S \in \mathbb{N}^*$7 # $, -(&%&' ()*$+ ./ +$$ 3?' %)&' ($V_c : \Omega_n \to \mathbb{R}$, ' * $V_d^1, V_d^2 : \Gamma_{(n,S)} \to \mathbb{R}$ (?% ./ , . 3&+, -- $\omega \in \Omega_n > 5$ $ / , =$

$$V_d^1(D_S(\omega)) \leq V_c(\omega) \leq V_d^2(D_S(\omega))$$

V_c)(, +, ' *&0 =, +), C-$)' ./ $ 1+&C, C)-)(.)%(1, %$ (Ω_n, Q, P_λ) , ' * V_d^1, V_d^2 , +$ +, ' *&0 =, +), C-$( )' $(\Gamma_{(n,S)}, Q', P_{p_\lambda})$ 5 / $+$ $p_\lambda = 1 - \exp(-\frac{\lambda}{S^n})$7

$ *$4' $ .5 &3?' %)&' ($V_{D_S^{-1}}^1, V_{D_S^{-1}}^2$ (?% ./ , .6

$$\begin{aligned} V_{D_S^{-1}}^1 : \quad \Omega_n &\to \mathbb{R} \\ \omega &\mapsto V_d^1 \circ D_S(\omega) \end{aligned}$$

, ' *

$$\begin{aligned} V_{D_S^{-1}}^2 : \quad \Omega_n &\to \mathbb{R} \\ \omega &\mapsto V_d^2 \circ D_S(\omega) \end{aligned}$$

$V_{D_S^{-1}}^1$, ' * $V_{D_S^{-1}}^2$, +$ +, ' *&0 =, +), C-$( )' ./ $ 1+&C, C)-)(.)%(1, %$ (Ω_n, Q, P_λ)7

! "# # $ %&2)31, -41 / "13)#&0 *15,)-)#, 067 1 4+31

$$\mathbb{E}[V^1_{D_S^{-1}}] \leq \mathbb{E}[V_c] \leq \mathbb{E}[V^2_{D_S^{-1}}]$$

#$%%&' $V_c(\ V^1_{D_S^{-1}}\ )$*+ $V^2_{D_S^{-1}}$),- +-. *-+ /* 01- 2)3 - 4,56)6/7208 24)8-)*+ /*)++/05*(95,)77 $\omega \in \Omega_n(\ V^1_{D_S^{-1}}(\omega) \leq N_c(\omega) \leq V^2_{D_S^{-1}}(\omega)$: ;* 85*2-<=-*8-(95,)77 7*-), 54-,)05,2)*+ 95, 01- ->4-80-+?)7=- /* 4),08=7),(01- /*-<=)70@157+2: □

! "# # $ '&2)31, $\lambda \in \mathbb{R}^{+*}6n \in \mathbb{N}^*6S \in \mathbb{N}^*$ +, * $\gamma \in \Gamma_{(n,S)}$67 1 4+318

$$P_\lambda(D_S^{-1}(\gamma)) = P_{p_\lambda}(\gamma)$$

#$%%&' &5,) A?-* γ(B- 853 4=0- $P_\lambda(D_S^{-1}(\gamma))$: C - . ,20 +/?/+- $[0,1]^n$ /* 23)77-, ?57=3 -2 59 855,+/*)0-2 $[\frac{i_1-1}{S}, \frac{i_1}{S}[\times \cdots \times [\frac{i_n}{S}, \frac{i_n+1}{S}[$ B/01 01- +/999-,-*0i_j ?),@*A/* $[1 \ldots S]$: $D_S^{-1}(\gamma)$ /201- 2-0 59)77 85*0*5=2*-0B5,D2 2=81 01)001-,- /2*5 *5+- /*)77?57=3 -2 $[\frac{i_1}{S}, \frac{i_1+1}{S}] \times \cdots \times [\frac{i_n}{S}, \frac{i_n+1}{S}]$ 95, B1/81 $\gamma(i_1 \ldots i_n) = 0$)*+ 5*- 5, 3 5,- *5+-2/* 01- 501-,2: E/?-* 01- 45/225*/)* 4,58-22/*0-*2/0@ λ(01- 4,56)6/70@591)?/*A*5 *5+- /* 5*- 23)77?57=3 - /2 $exp(-\frac{\lambda}{S^n})$)*+ 01- 4,56)6/70@591)?/*A 5*- 5, 3 5,- *5+-2/2 $1 - exp(-\frac{\lambda}{S^n})$: ;* 85*2-<=-*8-(B- 1)?- 01- 9575B/*A-<=)05*'

$$\begin{aligned} P_\lambda(D_S^{-1}(\gamma)) &= exp(-\frac{\lambda}{S^n})^{0(\gamma)}.(1 - exp(-\frac{\lambda}{S^n}))^{1(\gamma)} \\ &= p_\lambda^{1(\gamma)}.(1-p_\lambda)^{0(\gamma)} \\ &= P_{p_\lambda}(\gamma) \end{aligned}$$

□

( ) "*+"# %&9 1 '#, 0)*1" $\lambda \in \mathbb{R}^{+*}6n \in \mathbb{N}^*6S \in \mathbb{N}^*$: 9 1 +(0# '#, 0)*1" -4"11 ;&; '-)#, 0 $V_c : \Omega_n \to \mathbb{R}$ +, * $V_d^1, V_d^2 : \Gamma_{(n,S)} \to \mathbb{R}$ 0&'4 -4+- ;#" +(($\omega \in \Omega_n$67 1 4+31 $V_d^1 \circ D_S(\omega) \leq V_c(\omega) \leq V_d^2 \circ D_S(\omega)$: V_c)0 + "+, *#$ 3+")+<(1), -41 / "#<+<)(0-)' 0/ +' 1 (Ω_n, Q, P_λ) +, * V_d^1, V_d^2 +"1 "+, *#$ 3+")+<(10), $(\Gamma_{(n,S)}, Q', P_{p_\lambda})$ 7 41"1 $p_\lambda = 1 - \exp(-\frac{\lambda}{S^n})$: =41, 67 1 4+318

$$\mathbb{E}[V_d^1] \leq \mathbb{E}[V_c] \leq \mathbb{E}[V_d^2]$$

#$%%&' &,53 7-3 3) !(B- D*5B 01)0$\{D_S^{-1}(\gamma) | \gamma \in \Gamma_{(n,S)}\}$ /2) . */0- 4),0'05* 59 Ω_n: ;* 85*2-F <=-*8-(

$$\mathbb{E}[V^1_{D_S^{-1}}] = \sum_{\gamma \in \Gamma_{(n,S)}} P_\lambda(D_S^{-1}(\gamma)).V_d^1(\gamma)$$

&,53 7-3 3) C(B- 8)* +-+=8- 01)0

$$\begin{aligned}\mathbb{E}[V^1_{D_S^{-1}}] &= \sum_{\gamma\in\Gamma_{(n,S)}} P_{p_\lambda}(\gamma).V^1_d(\gamma)\\ &= \mathbb{E}[V^1_d]\end{aligned}$$

$ %&' ( ) %*+), -& , . / 01. 0

$$\mathbb{E}[V^1_{D_S^{-1}}] \leq \mathbb{E}[V_c] \leq \mathbb{E}[V^2_{D_S^{-1}}]$$

2

3 &4(. –5) 60 7(

$$\mathbb{E}[V^1_d] \leq \mathbb{E}[V_c] \leq \mathbb{E}[V^2_d]$$

□

! "#$$%&&$%&()*$+ ,

- . '$#' / 01 , $- / % n01+2)%#+$%*#3/ 4)56+7)%$T, d \in]0,1]$5 $\lambda \in \mathbb{R}^{+*}$ / %1 $S \in \mathbb{N}^*$58) #)9 $T_S = \lfloor T.S \rfloor$5 $d_S = \lfloor d.S \rfloor$: ;) $<9 +%+%(\Gamma_{(n,S)}, Q', P_{p_\lambda})$ / %1 (Ω_n, Q, P_λ)=

$$\mathbb{E}[N_{(d_S,T_S+1)}] \leq \mathbb{E}[\overline{N}_{(d,T)}] \leq \mathbb{E}[N_{(d_S,T_S)}]$$

8 9 : ; <= > ? @A>>?B <) ($7C&+ $n \in \mathbb{N}^*$ D$T, d \in]0,1]$ D$\lambda \in \mathbb{R}^{+*}$.(C $S \in \mathbb{N}^*$2 D;) E+) F&01 7$ 01 8) +& D%&%7– G$&01&+8$G-0 HF&(65 01 8) +& / %701 V_c 6&7(H&1G - 0) $\overline{N}_{(d,T)}$.(C V_1 .(C V_2 6&7(H+8$E&1 0F&-5 &1G - 0) $N_{(d_S,T_S+1)}$.(C $N_{(d_S,T_S)}$2# $ J) ($&1G&(J&D70 7$ &( ) GH1 0) $1) % 01. 0*) + . HF&($\omega \in \Omega_n$ D

$$N_{(d_S,T_S+1)} \circ D_S(\omega) \leq \overline{N}_{(d,T)}(\omega) \leq N_{(d_S,T_S)} \circ D_S(\omega)$$

<) ($7C&+ ω 7(Ω_n2 K&0 $J_c = \{(j^i_1, \ldots j^i_n) | i \in [0 \ldots l], l \in \mathbb{N}^*\}$ 6&. $1) +0$0 E. 01 6&0%&&(0%) () C&$ x .(C y 7(Ω_n2 3 &J) ($7C&+ $\omega' \in \Omega_n$ %17J1 7$ 01&(&0%) +' J), E) $&C 65 01&J)) +C7(. 0$ J) (0 7(&C 7(J2 3 &J) ($7C&+ $J_d = D_S(\omega')$2 3 &J. ($1) % 01. 0 J_d 7$. F. -7C E. 01 7($N_{(d_S,T_S+1)} \circ D_S(\omega)$2I(J) ($&1G&(J&D%&1. F&$N_{(d_S,T_S+1)} \circ D_S(\omega) \leq \overline{N}_{(d,T)}(\omega)$2

;) $1) % 01. 0 $\overline{N}_{(d,T)} \leq N_{(d_S,T_S)} \circ D_S$D%&$GEE) $&01. 0 01&+&&M$0$ ω $GJ1 01. 0 $\overline{N}_{(d,T)}(\omega) > N_{(d_S,T_S)} \circ D_S(\omega)$2 3 &J) ($7C&+ . $1) +0$0 E. 01 J_d 7($D_S(\omega)$.(C 6G-0*+), J_d . E. 01 J_c 7(ω $GJ1 01. 0 $|J_c| = |J_d|$2 L0-& C$ 0) . J) (0+. C7 07) (%17J1 E+) F&$ 01&01&)+& 2 □

2 '/ / 3 41 , $- / % n01+2)%#+$%*1+#4-)9) #3/ 4)56+7)%$T, d \in]0,1]$5 $\lambda \in \mathbb{R}^{+*}$ / %1 $S \in \mathbb{N}^*$58) #)9 $T_S = \lfloor T.S \rfloor$5 $d_S = \lfloor d.S \rfloor$ / %1 8) (/ 7) +%($\Gamma_{(n,S)}, Q', P_{p_\lambda}$)=

$$\lim_{S\mapsto\infty} \mathbb{P}(N_{(d_S,T_S+1)} = N_{(d_S,T_S)}) = 1$$

#$%%&' ( )*+,-./ $T \in]0,1]$ 0*- $\lambda \in \mathbb{R}^*$12. 3*)4 5605 $N_{(d_S,T_S+1)} \leq N_{(d_S,T_S)}$176. 8/)909,:,5;)<60=,*> $N_{(d_S,T_S+1)} \neq N_{(d_S,T_S)}$,+,*<,/,)/5) 56. 8/)909,:,5; 560556. /)?5;*> 0:>)/,56@9?,:-+54) -,≪,/.*5805 6+4,56 T_S 0*- $T_S + 1$176,+86.*)@.*0)AA?/+46.* 56./.,+)*. *)-. ,* 56. /0*>. $T_S + 1$ 5605,+*)5,* 56. /0*>. T_S 0*- 5605,+A)+./5) 56. -.+5,*05,)* 560* 56. *.B56)8 A6)+.*,* 56. /0*>. T_S12. A0* 9,*- 56,+8/)909,:,5; 9; 56.)*.)<60=,*>)*. *)-. 05/0*>. $T_S + 1$' $(1-p_\lambda)^{(2.T_S+1)^{n-1}}$1C)/.)=./D4. 3*)4 <)@0+,@,:0/ /.0+)*,*> 560* 56.)*.)<+.A5,)* E1F 5605 <)/0:: d 0*- TDN$_{(d,T)} = O((\frac{d}{T})^n)$1G* 80/5,A?:0/D<)/ d_S 0*- T_SD

$$N_{(d_S,T_S)} = O((\frac{d_S}{T_S})^n) = O((\frac{\lfloor d.S \rfloor}{\lfloor T.S \rfloor})^n) = O(1)$$

G* A)*+.H?.*A D

$$\begin{aligned} \mathbb{P}(N_{(d_S,T_S+1)} &= N_{(d_S,T_S)}) \geq ((1-p_\lambda)^{(2.T_S+1)^{n-1}})^{O(1)} \\ &= N_{(d_S,T_S)}) \geq (1-p_\lambda)^{O(S^{n-1})} \end{aligned}$$

I $+ p_\lambda = 1 - exp(-\frac{\lambda}{S^n})$D4. 60=.

$$\begin{aligned} \mathbb{P}(N_{(d_S,T_S+1)} = N_{(d_S,T_S)}) &\geq exp(-\frac{\lambda}{S^n})^{O(S^{n-1})} \\ &\geq exp(-\frac{\lambda}{S^n}.O(S^{n-1})) \\ &\geq exp(-O(\frac{1}{S})) \end{aligned}$$

&,*0::; D

$$1 \geq \mathbb{P}(N_{(d_S,T_S+1)} = N_{(d_S,T_S)}) \geq exp(-O(\frac{1}{S}))$$
$$1 \geq \lim_{S \to \infty} \mathbb{P}(N_{(d_S,T_S+1)} = N_{(d_S,T_S)}) \geq 1$$

46,A6 8/)=.+56. :. @@01 □

! "#$%&& '(!#" +,2 n3*)$1,0)#,+(0/+'145)61, $T, d \in]0,1]4 \lambda \in \mathbb{R}^*$ +,* $S \in \mathbb{N}^*$471 01- $T_S = \lfloor T.S \rfloor 4 d_S = \lfloor d.S \rfloor$ +,*718+61), $(\Gamma_{(n,S)}, Q', P_{p_\lambda})$ +,* (Ω_n, Q, P_λ)9

$$\lim_{S \mapsto \infty} N_{(d_S,T_S)} = \overline{N}_{(d,T)}$$

#$%%&' 7/,=,0:>,=.* 56.)/.@J 0*- :. @@0K1 □

! "#$%& &(&)*&()*&+, - +. - *0"&1. 23&4
566789&': &' &2<: (03&1 =>==? @0"$A0""0$B8#'$1+4C #&(DE(8") &F