Stability and regulation of nonsmooth dynamical systems

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Abstract: The mathematical analysis of nonsmooth Lagrangian dynamical systems leads to introduce mathematical tools which are unusual in control theory, velocities with locally bounded variations, measure accelerations, measure differential inclusions to name a few. The control theory for such dynamical systems is just beginning to appear, and even the basic Lyapunov stability theory still needs to be stated. We will be able though to propose a Lagrange-Dirichlet theorem for nonsmooth Lagrangian dynamical systems that will be applied through Potential Shaping to the regulation of the position and force of a robotic manipulator.

Key-words: Nonsmooth dynamical systems, functions with locally bounded variations, Lagrange-Dirichlet theorem, Lyapunov stability theorem, LaSalle's invariance theorem, hybrid position and force regulation

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Stabilité et régulation de systèmes dynamiques non-réguliers

Résumé : L'analyse mathématique des systèmes dynamiques Lagrangiens non-réguliers amène à travailler avec des outils mathématiques inhabituels en automatique, des vitesses à variations localement bornées, des mesures abstraites comme accélérations, des inclusions différentielles de mesures pour en citer quelques-uns. L'automatique de tels systèmes dynamiques commence tout juste à apparaître et même la théorie de base de la stabilité selon Lyapunov a encore besoin d'être établie. Nous proposons dans ce rapport un théorème de Lagrange-Dirichlet pour les systèmes dynamiques Lagrangiens non-réguliers qui est ensuite appliqué à la régulation en position et force d'un robot manipulateur.

Mots-clés : Systèmes dynamiques non-réguliers, fonctions à variations localement bornées, théorème de Lagrange-Dirichlet, théorème de Lyapunov, théorème de LaSalle, commande hybride en position et force

I. Introduction

Originating in the analysis of non permanent contact between perfectly rigid bodies, the mathematical analysis of nonsmooth Lagrangian dynamical systems is very recent [1], [2], [3], [4], [5], [6]. It concerns Lagrangian dynamical systems with coordinates constrained to stay inside some closed sets, what leads to introduce mathematical tools which are unusual in control theory, velocities with locally bounded variations, measure accelerations, measure differential inclusions to name a few. We are going therefore to spend some time in section II to present how these mathematical tools combine in the definition and analysis of nonsmooth Lagrangian dynamical systems and how this relates to the usual framework of hybrid systems [7].

The control theory for such dynamical systems is just beginning to appear [8], [9], and even the basic Lyapunov stability theory still needs to be stated. Indeed, it is usually presented for dynamical systems with states that vary continuously with time [10], [11], what is not the case for nonsmooth Lagrangian dynamical systems. We will start therefore by proposing in section III a Lyapunov stability theorem for dynamical systems with state discontinuities. We will see then in section IV that it is possible in some specific cases to propose also a version of LaSalle's invariance theorem.

Building on these theorems, we will be able to propose in section V a Lagrange-Dirichlet theorem for nonsmooth Lagrangian dynamical systems by showing that their energy can be naturally taken as Lyapunov functions. This is done in a very classical way, but in the unusual framework of functions with locally bounded variations and abstract measures.

Note that a Lagrange-Dirichlet theorem for nonsmooth Lagrangian dynamical systems has already been proposed in [9], but with no mention of asymptotic stability and with proofs which are incomplete, mostly by lack of an equivalent to the Lyapunov stability theorem proposed in section III.

The main use of a Lagrange-Dirichlet theorem for the control of a Lagrangian dynamical system is through Potential Shaping. This is applied in section VI to the regulation of the position and force of a robotic manipulator. This regulation problem has already been extensively studied, but either making strong assumptions on the state of the contact between the robot and its environment [12], such as supposing that this contact is permanent [13], or focusing on a contact with finite stiffness [14]. Thanks to the stability framework developped in sections III to V, we can propose in section VI an analysis of this regulation problem in the general case of non-penetrating perfectly rigid bodies.

II. Nonsmooth Lagrangian dynamical systems

Let us introduce first in section II-A what we consider as nonsmooth Lagrangian dynamical systems, Lagrangian systems of finite dimensions the coordinates of which are constrained to stay inside some given closed sets. The velocities of such systems may present discontinuities, so we are led to introduce in section II-B measure differential equations in order to express their dynamics correctly. The interaction between these systems and the constraints is considered then in section II-C to be an inelastic frictionless unilateral interaction, corresponding for example to an interaction between non-penetrating perfectly rigid bodies, a typical example of nonsmooth Lagrangian dynamical system.

A. Constraints on a dynamical system and related cones

Let us consider a dynamical system with n degrees of freedom, $\boldsymbol{q} : \mathbb{R} \to \mathbb{R}^n$ a time-variation of its generalized coordinates and $\dot{\boldsymbol{q}} : \mathbb{R} \to \mathbb{R}^n$ the related velocity:

$$\forall\, t, t_0 \in \mathbb{R},\ \boldsymbol{q}(t) = \boldsymbol{q}(t_0) + \int_{t_0}^{t} \dot{\boldsymbol{q}}(\tau)\, d\tau.$$

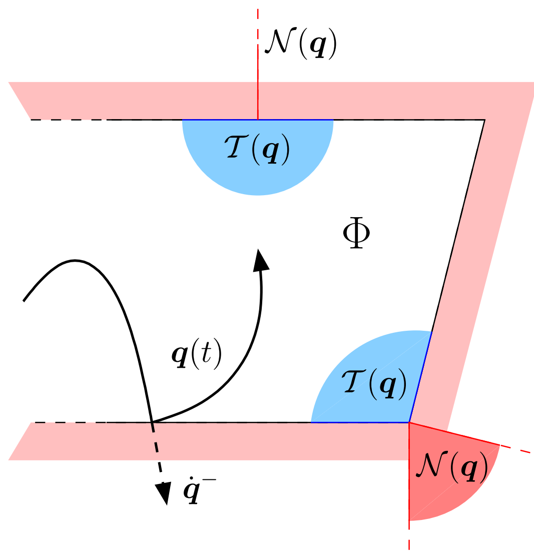

Fig. 1. Examples of tangent cones $\mathcal{T}(\boldsymbol{q})$ and normal cones $\mathcal{N}(\boldsymbol{q})$ to a set Φ, and example of a trajectory $\boldsymbol{q}(t) \in \Phi$ that reaches its boundary with a velocity $\dot{\boldsymbol{q}}^- \notin \mathcal{T}(\boldsymbol{q})$.

We are interested in dynamical systems with coordinates constrained to stay inside time-invariant closed sets [2]:

$$\forall\, t \in \mathbb{R},\;\; \boldsymbol{q}(t) \in \Phi \subset \mathbb{R}^n.$$

We can define then for all $\boldsymbol{q} \in \Phi$ the cones tangent to these sets [15],

$$\mathcal{T}(\boldsymbol{q}) = \Big\{ \boldsymbol{v} \in \mathbb{R}^n : \;\exists\, \tau_k \to 0,\;\; \tau_k > 0,\;\; \exists\, \boldsymbol{q}_k \to \boldsymbol{q},\;\; \boldsymbol{q}_k \in \Phi \text{ with } \frac{\boldsymbol{q}_k - \boldsymbol{q}}{\tau_k} \to \boldsymbol{v} \Big\},$$

and we can readily observe that if the velocities $\dot{\boldsymbol{q}}(t)$ have a left and right limit at an instant t, then obviously $-\dot{\boldsymbol{q}}^-(t) \in \mathcal{T}(\boldsymbol{q}(t))$ and $\dot{\boldsymbol{q}}^+(t) \in \mathcal{T}(\boldsymbol{q}(t))$. Note that $\mathcal{T}(\boldsymbol{q}) = \mathbb{R}^n$ in the interior of the sets Φ, but they reduce to a half-space or even less on their boundary (Fig. 1). This way, if the systems reach this boundary with velocities $\dot{\boldsymbol{q}}^-(t) \notin \mathcal{T}(\boldsymbol{q}(t))$, then obviously we will have $\dot{\boldsymbol{q}}^+(t) \neq \dot{\boldsymbol{q}}^-(t)$: a discontinuity of the velocities will occur at time t.

The cones polar to the tangent cones [15], the normal cones

$$\mathcal{N}(\boldsymbol{q}) = \big\{ \boldsymbol{u} \in \mathbb{R}^n : \;\forall\, \boldsymbol{v} \in \mathcal{T}(\boldsymbol{q}),\;\; \boldsymbol{u}^T \boldsymbol{v} \leq 0 \big\},$$

will appear in the inclusion (5) as directly related to forces acting on the dynamical systems because of their interaction with the constraints. Note that $\mathcal{N}(\boldsymbol{q}) = \{0\}$ in the interior of the sets Φ, but they contain at least a half-line of $\mathbb{R}^n$ on their boundary (Fig. 1): this will imply that these interaction forces will be experienced only when the dynamical systems are on the boundary of the sets Φ.

B. Nonsmooth Lagrangian dynamics

The dynamics of Lagrangian systems subject to Lebesgues-integrable forces are usually expressed as differential equations

$$\boldsymbol{M}(\boldsymbol{q})\,\frac{d\dot{\boldsymbol{q}}}{dt} + \boldsymbol{N}(\boldsymbol{q},\dot{\boldsymbol{q}})\,\dot{\boldsymbol{q}} = \boldsymbol{f},$$

with $\boldsymbol{M}(\boldsymbol{q})$ the symmetric positive definite inertia matrix that we will suppose to be a C^1 function of $\boldsymbol{q}$, $\boldsymbol{N}(\boldsymbol{q},\dot{\boldsymbol{q}})\,\dot{\boldsymbol{q}}$ the corresponding nonlinear effects and $\boldsymbol{f}$ the Lebesgues-integrable forces, dt being the Lebesgues measure. Classically, solutions to these differential equations lead to smooth motions with locally absolutely continuous velocities $\dot{\boldsymbol{q}}(t)$.

But we have seen that discontinuities of the velocities may occur when the coordinates of such systems are constrained to stay inside closed sets. These classical differential equations must therefore be turned into measure differential equations [2], [4]

$$\boldsymbol{M}(\boldsymbol{q})\,d\dot{\boldsymbol{q}} + \boldsymbol{N}(\boldsymbol{q},\dot{\boldsymbol{q}})\,\dot{\boldsymbol{q}}\,dt = \boldsymbol{f}\,dt + d\boldsymbol{r}, \tag{1}$$

in order to allow for measure accelerations $d\dot{\boldsymbol{q}}$ which may not be Lebesgues-integrable anymore, with velocities $\dot{\boldsymbol{q}}$ not locally absolutely continuous but with locally bounded variations, $\dot{\boldsymbol{q}} \in \mathrm{lbv}(\mathbb{R},\mathbb{R}^n)$ [2], [4] (we will consider moreover that these velocities are right-continuous [3]). Here appears an additionnal force $d\boldsymbol{r}$ acting on the Lagrangian systems as an effect of their interaction with the constraints, and the measure equation (1) shows that it has to be considered also as an abstract measure which may not be Lebesgues-integrable in order to account for the discontinuities of the velocities.

More precisely, functions with locally bounded variations have left and right limits at every instant, and we have for every compact interval $[\sigma,\tau]$

$$\int_{[\sigma,\tau]} d\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}^+(\tau) - \dot{\boldsymbol{q}}^-(\sigma). \tag{2}$$

Considering then the integral of the measure differential equations (1) over a singleton $\{\tau\}$, we have

$$\int_{\{\tau\}} \boldsymbol{M}(\boldsymbol{q})\,d\dot{\boldsymbol{q}} = \boldsymbol{M}(\boldsymbol{q}) \int_{\{\tau\}} d\dot{\boldsymbol{q}} = \boldsymbol{M}(\boldsymbol{q})\big(\dot{\boldsymbol{q}}^+(\tau) - \dot{\boldsymbol{q}}^-(\tau)\big),$$

$$\int_{\{\tau\}} \big(\boldsymbol{N}(\boldsymbol{q},\dot{\boldsymbol{q}})\,\dot{\boldsymbol{q}} - \boldsymbol{f}\big)dt = \big(\boldsymbol{N}(\boldsymbol{q},\dot{\boldsymbol{q}})\,\dot{\boldsymbol{q}} - \boldsymbol{f}\big) \int_{\{\tau\}} dt = 0,$$

leading to a relation between discontinuities of the velocities and atoms of the interaction forces,

$$\boldsymbol{M}(\boldsymbol{q})\,\big(\dot{\boldsymbol{q}}^+(\tau) - \dot{\boldsymbol{q}}^-(\tau)\big) = \int_{\{\tau\}} d\boldsymbol{r}. \tag{3}$$

Note that functions with locally bounded variations can be decomposed into the sum of a continuous function and a countable set of discontinuous step functions. In specific cases [5], the solution to the dynamics (1) can even be shown to be piecewise continuous. In this case, it is possible to focus distinctly on each continuous piece and each discontinuity as in the framework of hybrid systems [16], [7], but this is usually done through a numbering of the discontinuities strictly increasing with time, which is unable to go through accumulations of discontinuities (Zeno behaviours). The framework of functions with locally bounded variations appears therefore to be more powerful (no need to distinguish the continuous pieces) and potentially richer (no need for piecewise continuity).

C. Frictionless unilateral interactions

Following [2], we will consider that the interactions with the constraints are perfectly unilateral and frictionless. Expressing the $\mathbb{R}^n$ valued measure $d\boldsymbol{r}$ as the product of a non-negative real measure $d\mu$ and an $\mathbb{R}^n$ valued function $\boldsymbol{r}'_\mu \in L^1_{loc}(\mathbb{R}, d\mu; \mathbb{R}^n)$,

$$d\boldsymbol{r} = \boldsymbol{r}'_\mu\,d\mu, \tag{4}$$

this corresponds to the inclusion

$$\forall\, t \in \mathbb{R},\;\; -\boldsymbol{r}'_\mu(t) \in \mathcal{N}(\boldsymbol{q}(t)) \tag{5}$$

which implies especially a complementarity between the interaction forces $\boldsymbol{r}'_\mu(t)$ and the coordinates $\boldsymbol{q}(t)$ of the system, as has been pointed out in section II-A.

We will consider moreover that the impulsive behaviour of these interactions is ruled by a coefficient of restitution $e \in [0,1]$, perfectly elastic when $e = 1$, perfectly inelastic when $e = 0$. With

$$\dot{\boldsymbol{q}}_e = \dot{\boldsymbol{q}}^+ - \frac{e}{2}(\dot{\boldsymbol{q}}^+ - \dot{\boldsymbol{q}}^-), \tag{6}$$

this corresponds to a complementarity condition between the interaction forces $\boldsymbol{r}'_\mu(t)$ and the velocity $\dot{\boldsymbol{q}}_e(t)$,

$$\forall\, t \in \mathbb{R},\ \ \dot{\boldsymbol{q}}_e(t)^T \boldsymbol{r}'_\mu(t) = 0. \tag{7}$$

For a more in-depth presentation of these concepts and equations which may have subtle implications, the interested reader should definitely refer to [2].

III. Lyapunov's stability theorem for dynamical systems with state discontinuities

The Lyapunov stability theory is usually presented for dynamical systems with states that vary continuously with time [10], [11], but we have seen that it isn't the case for nonsmooth Lagrangian dynamical systems: their state $x(t) = (\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t))$ may have to be discontinuous on the boundary of the set $\Phi \times \mathbb{R}^n$. We must try therefore to propose a Lyapunov stability theorem for dynamical systems with states which may be discontinuous with time. Let us consider then a time-invariant flow on a metric space $\mathcal{X}$, an application $X : \mathbb{R} \times \mathcal{X} \to \mathcal{X}$ such that

$$\forall\, x \in \mathcal{X},\ \ X(0,x) = x,$$
$$\forall\, x \in \mathcal{X},\ \ \forall\, t,s \in \mathbb{R},\ \ X(t, X(s,x)) = X(t+s, x),$$

which may not be differentiable nor even continuous. In this general setting, with $d(x,\mathcal{S})$ the distance between the state x and the set $\mathcal{S}$, we can propose the following Lyapunov stability theorem, strongly inspired by the theorem 12 in [11]:

Theorem 1: A closed set $\mathcal{S} \subset \mathcal{X}$ is Lyapunov stable if and only if there exists a function $V : \mathcal{X} \to \mathbb{R}$ that is

(i) uniformly positive away from $\mathcal{S}$,

$$\forall\, \varepsilon > 0,\ \exists\, \gamma > 0 \text{ such that } \forall\, x \in \mathcal{X},\ \ d(x,\mathcal{S}) \geq \varepsilon \Longrightarrow V(x) \geq \gamma,$$

(ii) uniformly continuous in a neighbourhood of the set $\mathcal{S}$,

$$\forall\, \gamma > 0,\ \exists\, \delta > 0 \text{ such that } \forall\, x \in \mathcal{X},\ \ d(x,\mathcal{S}) \leq \delta \Longrightarrow V(x) \leq \gamma,$$

(iii) and a non-increasing function of time when starting in a neighbourhood of $\mathcal{S}$,

$$\exists\, d_0 > 0 \text{ such that } \forall\, x \in \mathcal{X},\ \ d(x,\mathcal{S}) < d_0 \Longrightarrow \forall\, t \geq 0,\ \ V(X(t,x)) \leq V(x).$$

A function V satisfying these conditions is called a Lyapunov function with respect to the stable set $\mathcal{S}$.

Proof: If there exists a function $V(.)$ satisfying condition (iii) of the theorem, then there exists $d_0 > 0$ such that

$$\forall\, x \in \mathcal{X},\ \ d(x,\mathcal{S}) < d_0 \Longrightarrow \forall\, t \geq 0,\ \ V(X(t,x)) \leq V(x).$$

If this function also satisfies conditions (i) and (ii) then for any $\varepsilon > 0$ there exists $\gamma > 0$ such that

$$\forall\, x \in \mathcal{X},\ d(x, \mathcal{S}) \geq \varepsilon \Longrightarrow V(x) \geq \gamma,$$

what can be written the other way round

$$\forall\, x \in \mathcal{X},\ V(x) < \gamma \Longrightarrow d(x, \mathcal{S}) < \varepsilon,$$

and for this $\gamma > 0$ there exists $0 < \delta < d_0$ such that

$$\forall\, x \in \mathcal{X},\ d(x, \mathcal{S}) \leq \delta \Longrightarrow V(x) \leq \frac{\gamma}{2} < \gamma$$

so that we have

$$\begin{aligned} \forall\, x \in \mathcal{X},\ d(x, \mathcal{S}) \leq \delta &\Longrightarrow \forall\, t \geq 0,\ V(X(t,x)) \leq V(x) < \gamma \\ &\Longrightarrow \forall\, t \geq 0,\ d(X(t,x), \mathcal{S}) \leq \varepsilon, \end{aligned}$$

what indicates that the set $\mathcal{S}$ is Lyapunov stable.

Now, given any set $\mathcal{S}$, there always exists the function

$$V(x) = \sup_{\tau \geq 0}\ d(X(\tau, x), \mathcal{S}).$$

Obviously,

$$\forall\, x \in \mathcal{X},\ V(x) = \sup_{\tau \geq 0}\ d(X(\tau, x), \mathcal{S}) \geq d(X(0, x), \mathcal{S}) = d(x, \mathcal{S}),$$

so that condition (i) is naturally satisfied:

$$\forall\, \varepsilon > 0,\ \forall\, x \in \mathcal{X},\ d(x, \mathcal{S}) \geq \varepsilon \Longrightarrow V(x) \geq \varepsilon.$$

Moreover,

$$\begin{aligned} \forall\, x \in \mathcal{X},\ \forall\, t \geq 0, \quad V(X(t,x)) &= \sup_{\tau \geq 0}\ d(X(\tau, X(t,x)), \mathcal{S}) \\ &= \sup_{\tau \geq 0}\ d(X(\tau + t, x), \mathcal{S}) \\ &= \sup_{\tau \geq t}\ d(X(\tau, x), \mathcal{S}) \\ &\leq \sup_{\tau \geq 0}\ d(X(\tau, x), \mathcal{S}) = V(x), \end{aligned}$$

so condition (iii) is also naturally satisfied for any $d_0 > 0$. Now, if the set $\mathcal{S}$ is Lyapunov stable, for any $\gamma > 0$ there exists $\delta > 0$ such that

$$\begin{aligned} \forall\, x \in \mathcal{X},\ d(x, \mathcal{S}) \leq \delta &\Longrightarrow \forall\, t \geq 0,\ d(X(t,x), \mathcal{S}) \leq \gamma \\ &\Longrightarrow V(x) \leq \gamma, \end{aligned}$$

so condition (ii) is also satisfied. ∎

We can observe that this theorem is very close to the usual one in the continuous case (theorem 12 in [11]): the only difference lies in condition (i) which needs to be satisfied on the whole state space and not only in a neighbourhood of the set $\mathcal{S}$. This is because of the possible discontinuities of the state $x(t)$ which may jump at any moment outside of any neighbourhood: a global condition is therefore required. Note that such a global condition also appears in [7] and [17] for the same reason, even though it is claimed in Remark 4 of [17] that it might not be necessary in the specific case of nonsmooth Lagrangian dynamical systems.

In the framework of class $\mathcal{K}$ functions (strictly increasing continuous definite functions) [10], conditions (i) and (ii) of this theorem are strictly equivalent to the fact that there exist two class $\mathcal{K}$ functions $\alpha(\cdot)$ and $\beta(\cdot)$ such that

(i') $$\forall\, x \in \mathcal{X},\ V(x) \geq \alpha(d(x, \mathcal{S})),$$

(ii') $$\exists\, d_0 > 0 \text{ such that } \forall\, x \in \mathcal{X},\ d(x, \mathcal{S}) < d_0 \Longrightarrow V(x) \leq \beta(d(x, \mathcal{S})).$$

The implications for the Lyapunov function are identical: uniform continuity in a neighbourhood of the set $\mathcal{S}$ and uniform positivity away from it.

IV. LASALLE'S INVARIANCE THEOREM FOR DYNAMICAL SYSTEMS WITH STATE DISCONTINUITIES

A key condition for the statement of LaSalle's invariance theorem is the continuity of the trajectories of the systems with respect to initial conditions. Nonsmooth Lagrangian dynamical systems generally don't present such a continuity [6], but they do in many specific cases [1], [5], [18]. Considering a time-invariant flow as in the previous section, we must try therefore to propose a version of LaSalle's invariance theorem if it is continuous with respect to initial conditions, i.e.

$$\forall\, x,\, x_0 \in \mathcal{X},\ \forall\, t \geq 0,\ x \to x_0 \Longrightarrow X(t, x) \to X(t, x_0).$$

This theorem is built on an analysis of the time variation of a function $V(x)$, what is usually done with the help of its time derivative $\dot{V}(x)$, and such a derivative may not exist for a flow with state discontinuities. We need therefore to state this theorem without the use of time derivatives, in a similar way to what appears in the condition (iii) of theorem 1, what leads to the following variation of the theorem 3.4 in [10], with different notations but with exactly the same conditions, conclusions and proof:

Theorem 2: Let $\Omega \subset \mathcal{X}$ be a compact set and $V : \mathcal{X} \to \mathbb{R}$ a continuous function such that

(i) the set Ω is positively invariant,
$$\forall\, x \in \Omega,\ \forall\, t \geq 0,\ X(t, x) \in \Omega,$$

(ii) the function V is a non-increasing function of time when starting in Ω,
$$\forall\, x \in \Omega,\ \forall\, t \geq 0,\ V(X(t, x)) \leq V(x),$$

(iii) the subset $\mathcal{E} \subset \Omega$ gathers all the states where the function V is stationnary with time,
$$\forall\, x \in \Omega,\ \forall\, t > 0,\ V(X(t, x)) = V(x) \Longrightarrow x \in \mathcal{E}.$$

If the trajectories of the dynamical system are continuous with respect to initial conditions, then every trajectory starting in Ω converges asymptotically as $t \to \infty$ to the largest invariant subset of $\mathcal{E}$.

Proof: The proof is exactly the same as the one of theorem 3.4 in [10]. ■

V. A LAGRANGE-DIRICHLET THEOREM FOR NONSMOOTH LAGRANGIAN DYNAMICAL SYSTEMS

The theorems of the two previous sections can be used now to derive a Lagrange-Dirichlet theorem for nonsmooth Lagrangian dynamical systems. After a preliminary on Stieljes measures in section V-A, this will be achieved mainly by showing in section V-B that when the forces acting on such systems derive from potential functions, the total energies of the systems are non-increasing functions of time. Section V-C concludes about the Lagrange-Dirichlet theorem and section V-D precises the conditions for asymptotic stability. A brief discussion about dissipativity properties is carried out then in section V-E.

A. A preliminary on Stieljes measures

One can observe that both theorems 1 and 2 are based on an analysis of the time variation of a function $V(x)$. In the specific case of nonsmooth Lagrangian dynamical systems, we have seen that the states $x(t) = (\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t))$ have locally bounded variations: $x \in \text{lbv}(\mathbb{R}, \mathbb{R}^n \times \mathbb{R}^n)$. If a function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz continuous, for example if it is convex or $\mathcal{C}^1$, then $V(x(t))$ will also have locally bounded variations: $(V \circ x) \in \text{lbv}(\mathbb{R}, \mathbb{R})$. In this case, following the integration rule (2), the variations of $V \circ x$ will be directly related to the sign of the associated Stieljes measure $d(V \circ x)$. Gathering these results, the following trivial lemma is going to be a cornerstone of this section:

Lemma 1: Let $x \in \text{lbv}(\mathbb{R}, \mathbb{R}^n \times \mathbb{R}^n)$ be a function with locally bounded variations and $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz continuous, then $V \circ x$ is non-increasing if and only if $d(V \circ x) \leq 0$ and constant if and only if $d(V \circ x) = 0$.

B. Energy as a non-increasing function of time

Let us consider now that the Lebesgues-integrable forces $\boldsymbol{f}$ acting on the dynamics (1) derive from a C^1 potential function $P(\boldsymbol{q})$, with a dissipative term $\boldsymbol{h}$:

$$\boldsymbol{f} = -\frac{dP}{d\boldsymbol{q}}(\boldsymbol{q}) + \boldsymbol{h}, \quad \text{with} \quad \dot{\boldsymbol{q}}^T \boldsymbol{h} \leq 0. \tag{8}$$

With

$$K(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \frac{1}{2}\, \dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\, \dot{\boldsymbol{q}}$$

the kinetic energy of the Lagrangian dynamics (1), the total energy of the system $K(\boldsymbol{q}, \dot{\boldsymbol{q}}) + P(\boldsymbol{q})$ can be shown to be a non-increasing function of time in the following way. First of all, let us note that all conditions are fulfilled to apply lemma 1 and focus on the Stieljes measures of both the kinetic and the potential energies. Applying then calculus rules specific to the differentiation of functions with bounded variations [3], [2] and the fact that $\frac{1}{2}(\dot{\boldsymbol{q}}^+ + \dot{\boldsymbol{q}}^-)\, dt = \dot{\boldsymbol{q}}\, dt$, we have

$$dK = \frac{1}{2} \dot{\boldsymbol{q}}^T \dot{\boldsymbol{M}}(\boldsymbol{q}, \dot{\boldsymbol{q}})\, \dot{\boldsymbol{q}}\, dt + \frac{1}{2} (\dot{\boldsymbol{q}}^+ + \dot{\boldsymbol{q}}^-)^T \boldsymbol{M}(\boldsymbol{q})\, d\dot{\boldsymbol{q}}.$$

With the dynamics (1), this leads to

$$dK = \frac{1}{2} \dot{\boldsymbol{q}}^T \big(\dot{\boldsymbol{M}}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - 2\, \boldsymbol{N}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \big)\, \dot{\boldsymbol{q}}\, dt + \frac{1}{2} (\dot{\boldsymbol{q}}^+ + \dot{\boldsymbol{q}}^-)^T d\boldsymbol{r} + \dot{\boldsymbol{q}}^T \boldsymbol{f}\, dt,$$

but since $\dot{\boldsymbol{M}}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - 2\, \boldsymbol{N}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ is an antisymmetric matrix, the first term here is void:

$$dK = \frac{1}{2} (\dot{\boldsymbol{q}}^+ + \dot{\boldsymbol{q}}^-)^T d\boldsymbol{r} + \dot{\boldsymbol{q}}^T \boldsymbol{f}\, dt, \tag{9}$$

what relates the variation of the kinetic energy dK to the power exerted by the forces $d\boldsymbol{r}$ and $\boldsymbol{f}\, dt$. Now, with the help of relations (6), (4) and (3), we have at every time τ

$$\begin{aligned}
\frac{1}{2} (\dot{\boldsymbol{q}}^+ + \dot{\boldsymbol{q}}^-)^T d\boldsymbol{r} &= \dot{\boldsymbol{q}}_e^T d\boldsymbol{r} - \frac{1-e}{2} (\dot{\boldsymbol{q}}^+ - \dot{\boldsymbol{q}}^-)^T d\boldsymbol{r} \\
&= \dot{\boldsymbol{q}}_e^T d\boldsymbol{r} - \frac{1-e}{2} \Big[\int_{\{\tau\}} d\boldsymbol{r} \Big]^T \boldsymbol{M}(\boldsymbol{q})^{-1} d\boldsymbol{r} \\
&= \dot{\boldsymbol{q}}_e^T \boldsymbol{r}'_\mu\, d\mu - \frac{1-e}{2} {\boldsymbol{r}'_\mu}^T \boldsymbol{M}(\boldsymbol{q})^{-1} \boldsymbol{r}'_\mu \Big[\int_{\{\tau\}} d\mu \Big] d\mu,
\end{aligned}$$

where the first term is void because of (7) and the second is non-positive since $d\mu$ is a non-negative real measure and $\boldsymbol{M}(\boldsymbol{q})$ is a positive definite matrix. We see here that the interaction forces $d\boldsymbol{r}$ are passive, so that the equation (9) can be reduced to

$$dK \leq \dot{\boldsymbol{q}}^T \boldsymbol{f}\, dt.$$

Now, with forces $\boldsymbol{f}$ defined as in (8), this results in

$$dK \leq -\dot{\boldsymbol{q}}^T \frac{dP}{d\boldsymbol{q}}(\boldsymbol{q})\, dt + \dot{\boldsymbol{q}}^T \boldsymbol{h}\, dt,$$

and the derivative of the potential energy being

$$dP = \dot{\boldsymbol{q}}^T \frac{dP}{d\boldsymbol{q}}(\boldsymbol{q})\, dt,$$

we end up with

$$dK + dP \leq \dot{\boldsymbol{q}}^T \boldsymbol{h}\, dt \leq 0. \tag{10}$$

Through lemma 1, this implies that the energy of the system $K(\boldsymbol{q}, \dot{\boldsymbol{q}}) + P(\boldsymbol{q})$ is a non-increasing function of time.

C. Energy as a Lyapunov function

Being a non-increasing function of time, this energy satisfies condition (iii) of theorem 1 for any $d_0 > 0$ and appears as a good candidate for a Lyapunov function, but in order to conclude, conditions (i) and (ii) of this theorem must also be satisfied. These conditions can be dispatched as conditions on both the kinetic and potential energies, so that we can state the following Lagrange-Dirichlet theorem:

Theorem 3: If the kinetic energy $K(\boldsymbol{q}, \dot{\boldsymbol{q}})$ of a nonsmooth Lagrangian dynamical system is uniformly continuous in a neighbourhood of the set $\{\boldsymbol{q} \in \Phi,\ \dot{\boldsymbol{q}} = 0\}$,

(a) $\quad \forall\, \gamma > 0,\ \exists\, \delta > 0$ such that $\forall\, (\boldsymbol{q}, \dot{\boldsymbol{q}}) \in \Phi \times \mathbb{R}^n,\ ||\dot{\boldsymbol{q}}|| \leq \delta \Longrightarrow K(\boldsymbol{q}, \dot{\boldsymbol{q}}) \leq \gamma,$

and uniformly positive away from it,

(b) $\quad \forall\, \varepsilon > 0,\ \exists\, \gamma > 0$ such that $\forall\, (\boldsymbol{q}, \dot{\boldsymbol{q}}) \in \Phi \times \mathbb{R}^n,\ ||\dot{\boldsymbol{q}}|| \geq \varepsilon \Longrightarrow K(\boldsymbol{q}, \dot{\boldsymbol{q}}) \geq \gamma,$

and if the forces $\boldsymbol{f}$ acting on this system derive as in (8) from a potential function $P(\boldsymbol{q})$ that has a minimum $\min\limits_{\Phi}\ P(\boldsymbol{q})$ such that the function $P(\boldsymbol{q}) - \min\limits_{\Phi}\ P(\boldsymbol{q})$ is uniformly continuous in a neighbourhood of the set $\mathcal{S}' = \operatorname{Arg}\min\limits_{\Phi}\ P(\boldsymbol{q})$,

(c) $\quad \forall\, \gamma > 0,\ \exists\, \delta > 0$ such that $\forall\, \boldsymbol{q} \in \Phi,\ d(\boldsymbol{q}, \mathcal{S}') \leq \delta \Longrightarrow P(\boldsymbol{q}) - \min_{\Phi}\ P(\boldsymbol{q}) \leq \gamma,$

and uniformly positive away from it,

(d) $\quad \forall\, \varepsilon > 0,\ \exists\, \gamma > 0$ such that $\forall\, \boldsymbol{q} \in \Phi,\ d(\boldsymbol{q}, \mathcal{S}') \geq \varepsilon \Longrightarrow P(\boldsymbol{q}) - \min_{\Phi}\ P(\boldsymbol{q}) \leq \gamma,$

then the set $\mathcal{S} = \mathcal{S}' \times \{\boldsymbol{0}\}$ is Lyapunov stable.

Proof: We need now to show that the function

$$V(\boldsymbol{q}, \dot{\boldsymbol{q}}) = K(\boldsymbol{q}, \dot{\boldsymbol{q}}) + P(\boldsymbol{q}) - \min_{\Phi}\ P(\boldsymbol{q})$$

satisfies conditions (i) and (ii) of theorem 1 with respect to the set $\mathcal{S} = \mathcal{S}' \times \{\boldsymbol{0}\}$. Note first that since the state space $\mathbb{R}^n \times \mathbb{R}^n$ is a real vector space of finite dimension, it makes no difference which norm is used on it. We will make use therefore of the norm $||(\boldsymbol{q}, \dot{\boldsymbol{q}})|| = ||\boldsymbol{q}|| + ||\dot{\boldsymbol{q}}||$ so that we have

$$d\big((\boldsymbol{q}, \dot{\boldsymbol{q}}), \mathcal{S}\big) = d(\boldsymbol{q}, \mathcal{S}') + ||\dot{\boldsymbol{q}}||,$$

what will help simplify the proof.

Given now any $\gamma > 0$, let us choose $\delta_1 > 0$ such that

$$\forall\, (\boldsymbol{q}, \dot{\boldsymbol{q}}) \in \Phi \times \mathbb{R}^n,\ ||\dot{\boldsymbol{q}}|| \leq \delta_1 \Longrightarrow K(\boldsymbol{q}, \dot{\boldsymbol{q}}) \leq \frac{\gamma}{2}$$

according to (a) and $\delta_2 > 0$ such that

$$\forall\, \boldsymbol{q} \in \Phi,\ d(\boldsymbol{q}, \mathcal{S}') \leq \delta_2 \Longrightarrow P(\boldsymbol{q}) - \min_{\Phi}\, P(\boldsymbol{q}) \leq \frac{\gamma}{2}$$

according to (c). With $\delta = \min\{\delta_1, \delta_2\} > 0$, we obviously have

$$\forall\, (\boldsymbol{q}, \dot{\boldsymbol{q}}) \in \Phi \times \mathbb{R}^n,\ d\big((\boldsymbol{q}, \dot{\boldsymbol{q}}), \mathcal{S}\big) \leq \delta \Longrightarrow \begin{cases} ||\dot{\boldsymbol{q}}|| \leq \delta_1 \\ d(\boldsymbol{q}, \mathcal{S}') \leq \delta_2 \end{cases}$$
$$\Longrightarrow V(\boldsymbol{q}, \dot{\boldsymbol{q}}) = K(\boldsymbol{q}, \dot{\boldsymbol{q}}) + P(\boldsymbol{q}) - \min_{\Phi}\, P(\boldsymbol{q}) \leq \gamma,$$

so condition (ii) of theorem 1 is satisfied.

Given any $\varepsilon > 0$, let us choose $\gamma_1 > 0$ such that

$$\forall\, (\boldsymbol{q}, \dot{\boldsymbol{q}}) \in \Phi \times \mathbb{R}^n,\ ||\dot{\boldsymbol{q}}|| \geq \frac{\varepsilon}{2} \Longrightarrow K(\boldsymbol{q}, \dot{\boldsymbol{q}}) \geq \gamma_1$$

according to (b) and $\gamma_2 > 0$ such that

$$\forall\, \boldsymbol{q} \in \Phi,\ d(\boldsymbol{q}, \mathcal{S}') \geq \frac{\varepsilon}{2} \Longrightarrow P(\boldsymbol{q}) - \min_{\Phi}\, P(\boldsymbol{q}) \geq \gamma_2$$

according to (d). With $\gamma = \min\{\gamma_1, \gamma_2\} > 0$, we obviously have

$$\forall\, (\boldsymbol{q}, \dot{\boldsymbol{q}}) \in \Phi \times \mathbb{R}^n,\ d\big((\boldsymbol{q}, \dot{\boldsymbol{q}}), \mathcal{S}\big) \geq \varepsilon \Longrightarrow \begin{cases} \text{either } ||\dot{\boldsymbol{q}}|| \geq \dfrac{\varepsilon}{2} \\ \text{or } d(\boldsymbol{q}, \mathcal{S}') \geq \dfrac{\varepsilon}{2} \end{cases} \text{but in both cases}$$
$$\Longrightarrow V(\boldsymbol{q}, \dot{\boldsymbol{q}}) = K(\boldsymbol{q}, \dot{\boldsymbol{q}}) + P(\boldsymbol{q}) - \min_{\Phi}\, P(\boldsymbol{q}) \geq \gamma,$$

so condition (i) of theorem 1 is also satisfied and we can conclude that the set $\mathcal{S}$ is Lyapunov stable. ■

The idea in dispatching the conditions (i) and (ii) of theorem 1 on both the kinetic and potential energies is that except for pathological cases, the conditions on the kinetic energy are always satisfied. It is therefore only necessary to check them for the potential energy, and we can observe that they are satisfied for very broad classes of functions such as for example for coercive (radially unbounded) continuous functions: a coercive continuous function always has a minimum, with a compact $\mathrm{Arg\,min}$ so that uniform continuity in an neighbourhood of it is immediate, and coercivity is a much stronger property than uniform positivity which is therefore automatically satisfied.

Corollary 1: Excluding pathological behaviours of the kinetic energy (if conditions (a) and (b) of theorem 3 are satisfied), if the forces $\boldsymbol{f}$ acting on a nonsmooth Lagrangian dynamical system derive as in (8) from a coercive C^1 potential function $P(\boldsymbol{q})$, then the set $\mathcal{S} = \{\mathrm{Arg}\min_{\Phi}\, P(\boldsymbol{q})\} \times \{\boldsymbol{0}\}$ is Lyapunov stable.

D. Attractivity of the equilibrium points and asymptotic stability

Being a non-increasing function of time, the energy of the system naturally satisfies condition (ii) of theorem 2 whatever the set Ω. We can observe also through lemma 1 and condition (10) that if it is constant over a time interval then on this interval,

$$dK + dP = 0 \implies \dot{\boldsymbol{q}}^T \boldsymbol{h}\, dt = 0.$$

Now, if the dissipative term $\boldsymbol{h}$ is strictly dissipative,

$$\dot{\boldsymbol{q}}^T \boldsymbol{h} = 0 \implies \dot{\boldsymbol{q}} = \boldsymbol{0}.$$

In this case, condition (iii) of theorem 2 is satisfied by the set of states with zero velocity,

$$\mathcal{E} = \Omega \cap (\Phi \times \{\boldsymbol{0}\}),$$

of which the largest invariant subset is by construction the set of equilibrium points that lie inside Ω, what leads to the following application of theorem 2:

Theorem 4: If the forces $\boldsymbol{f}$ acting on a nonsmooth Lagrangian dynamical system derive as in (8) from a C^1 potential function $P(\boldsymbol{q})$ with a *strictly* dissipative term and if the trajectories of this system are continuous with respect to initial conditions, then if there is a compact set $\Omega \subset \Phi \times \mathbb{R}^n$ that is positively invariant, every trajectory starting in this set converges asymptotically as $t \to \infty$ to the equilibrium points of the system that lie inside this set.

The set $\mathcal{S}$ that is proved to be Lyapunov stable in theorem 3 is a set of equilibrium points, so if there are no other equilibrium points in a compact positively invariant neighbourhood of this set, the theorem 4 can show additionnaly that it is asymptotically stable. Now, we can observe that under the conditions of corollary 1, there exist compact sublevel sets of the energy

$$\Omega_\lambda = \{(\boldsymbol{q}, \dot{\boldsymbol{q}}) \in \Phi \times \mathbb{R}^n \mid K(\boldsymbol{q}, \dot{\boldsymbol{q}}) + P(\boldsymbol{q}) \leq \lambda\}$$

which are naturally positively invariant neighbourhoods of $\mathcal{S}$. We can propose therefore the following corollary:

Corollary 2: Under the assumptions of corollary 1, if the dissipative term in (8) is strictly dissipative, if the trajectories of the dynamical system are continuous with respect to initial conditions and if there are no other equilibrium points in a neighbourhood of the set $\mathcal{S} = \{\operatorname{Arg}\min_{\Phi} P(\boldsymbol{q})\} \times \{\boldsymbol{0}\}$, then it is asymptotically Lyapunov stable.

E. A dissipativity point of view

One can see in section V-B that both

$$dP \leq \dot{\boldsymbol{q}}^T(-\boldsymbol{f})\, dt$$

and

$$dK \leq \boldsymbol{f}^T \dot{\boldsymbol{q}}\, dt,$$

what allows to consider the system studied there as a negative feedback interconnection of a passive dynamical system with an input $\dot{\boldsymbol{q}}$, an output $-\boldsymbol{f}$ and a dynamics (8), with $P(\boldsymbol{q})$ as a storage function, to a passive dynamical system with an input $\boldsymbol{f}$, an output $\dot{\boldsymbol{q}}$ and a dynamics (1)-(7), with $K(\boldsymbol{q}, \dot{\boldsymbol{q}})$ as a storage function (Figure 2). The stability results obtained in the previous sections can be seen therefore as a trivial conclusion of passivity properties.

Yet, the dissipativity theory hasn't been developped so far for nonsmooth dynamical systems, but since it is intimately related to Lagrangian dynamical systems [19], it wouldn't be surprising that its extension to the case of nonsmooth Lagrangian dynamical systems be straightforward, with the help of tools such as those presented here.

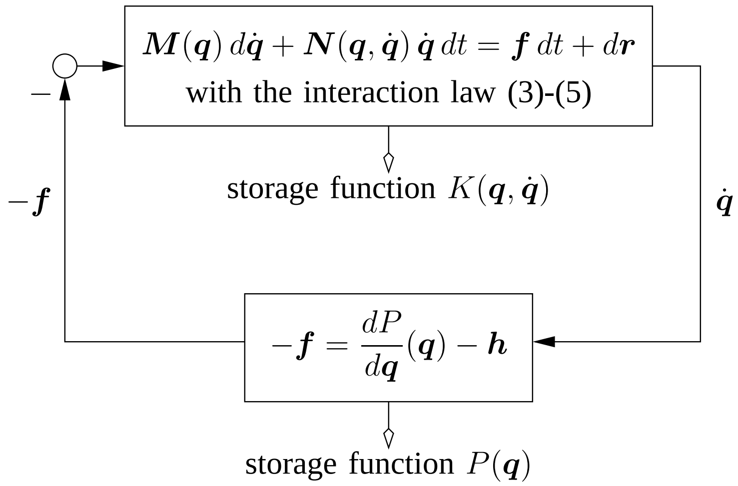

Fig. 2. A negative feedback interconnection of two passive dynamical systems.

VI. Regulation of the position and force of a robotic manipulator through Potential Shaping

The main use of a Lagrange-Dirichlet theorem such as theorem 3 for the control of a Lagrangian dynamical system is through Potential Shaping, what is presented here in section VI-A. This is how the stability of a control law for the regulation of the position and force of a robotic manipulator is proved in section VI-B in the framework of non-penetrating perfectly rigid bodies.

A. Potential Shaping

When the Lebesgues-integrable forces $\boldsymbol{f}$ acting on the dynamics (1) include control forces $\boldsymbol{f}_c$ that can be set to any desired value and uncontrolled forces $\boldsymbol{f}_u$ that we suppose to know perfectly well in advance,

$$\boldsymbol{f} = \boldsymbol{f}_c + \boldsymbol{f}_u,$$

the control forces can be set in the following way,

$$\boldsymbol{f}_c = -\frac{dP}{d\boldsymbol{q}}(\boldsymbol{q}) - \boldsymbol{C}\,\dot{\boldsymbol{q}} - \boldsymbol{f}_u, \tag{11}$$

with a term deriving from a C^1 potential function $P(\boldsymbol{q})$, a dissipative term where $\boldsymbol{C}$ is a positive definite matrix, and a compensation of the uncontrolled forces $\boldsymbol{f}_u$. This way, the total forces $\boldsymbol{f}$ acting on the dynamics match exactly the forces considered in (8) and the theorem 3 can be used to conclude if the set $\mathcal{S} = \left\{ \operatorname*{Arg\,min}_{\Phi}\ P(\boldsymbol{q}) \right\} \times \{\boldsymbol{0}\}$ is Lyapunov stable. The point in doing so is that the potential function that appears in the control law (11) can be shaped at will if the stability of a specific set $\mathcal{S}$ is particularly looked for.

B. Regulation of the position and force of a robotic manipulator

As an example of potential shaping for a nonsmooth Lagrangian dynamical system, let us consider the problem of regulating both the position $\boldsymbol{q}$ and the force $d\boldsymbol{r}$ of a robotic manipulator to some desired values $\boldsymbol{q}_d$ and $\boldsymbol{r}_d\,dt$, inspiring on the regulation law proposed in [13]. First of all, note that these desired position and force have to be consistent with the interaction law described in section II-C, and especially with the unilaterality and absence of friction expressed by the inclusion (5):

$$-\boldsymbol{r}_d \in \mathcal{N}(\boldsymbol{q}_d). \tag{12}$$

Let us consider then a coercive C^1 potential function $P(\boldsymbol{q})$ such that

$$\frac{dP}{d\boldsymbol{q}}(\boldsymbol{q}_d) = \boldsymbol{r}_d. \tag{13}$$

This corresponds together with (12) to the necessary condition for $\boldsymbol{q}_d$ to be a minimum of $P(\boldsymbol{q})$ over the set Φ (theorem 6.12 of [20]), a condition which is even sufficient in specific cases such as the convex case. Since the function $P(\boldsymbol{q})$ is coercive, it has a minimum over the closed set Φ. It is reasonnable therefore to suppose under mild conditions on the shape of both $P(\boldsymbol{q})$ and Φ that this minimum is reached at the unique position $\boldsymbol{q}_d$:

$$\operatorname*{Arg\,min}_{\Phi}\ P(\boldsymbol{q}) = \{\boldsymbol{q}_d\}.$$

The corollary 1 can be applied then to conclude that the set $\{(\boldsymbol{q}_d, \boldsymbol{0})\}$ is Lyapunov stable. This implies more particularly that this set is invariant, what means here that the state $(\boldsymbol{q}_d, \boldsymbol{0})$ is an equilibrium state: in this state, $\dot{\boldsymbol{q}} = \boldsymbol{0}$, $d\dot{\boldsymbol{q}} = \boldsymbol{0}$, so the dynamic equation (1) with the control forces (11) turn into a static equation

$$\boldsymbol{0} = -\frac{dP}{d\boldsymbol{q}}(\boldsymbol{q}_d)\, dt + d\boldsymbol{r},$$

that is, through (13),

$$d\boldsymbol{r} = \boldsymbol{r}_d\, dt.$$

We have proved then without making any assumptions on the state of the contacts between the robot and its environment that the state $(\boldsymbol{q}_d, \boldsymbol{0})$ is a stable equilibrium state where the contact force is $\boldsymbol{r}_d\, dt$, as desired. Moreover, if the trajectories of the dynamical system are continuous with respect to initial conditions and if there are no other equilibrium points in a neighbourhood of $(\boldsymbol{q}_d, \boldsymbol{0})$, the corollary 2 can be applied to conclude that this state is asymptotically Lyapunov stable.

Note though that on the contrary to what appears in [13], the contact force $\boldsymbol{r}_d\, dt$ can't be stable in our case if it is strictly greater than zero because of the unilaterality condition (5): the contact force has to be zero in every neighbourhood of every state of the system, what wouldn't be compatible with Lyapunov stability.

Now, following [13], we can observe that for a potential function such as the convex quadratic function

$$P(\boldsymbol{q}) = \frac{1}{2}(\boldsymbol{q} - \boldsymbol{q}_d)^T \boldsymbol{W}\, (\boldsymbol{q} - \boldsymbol{q}_d) + \boldsymbol{r}_d^T(\boldsymbol{q} - \boldsymbol{q}_d),$$

with $\boldsymbol{W}$ a symmetric positive definite matrix, the control forces (11) realise a simple proportionnal-derivative feedback law

$$\boldsymbol{f}_c = -\boldsymbol{W}\, (\boldsymbol{q} - \boldsymbol{q}_d) - \boldsymbol{r}_d - \boldsymbol{C}\,\dot{\boldsymbol{q}} - \boldsymbol{f}_u,$$

which is Lyapunov stable by direct application of the previous results.

VII. Conclusion

The mathematical analysis of nonsmooth Lagrangian dynamical systems requires to work with tools which are unusual in control theory, functions with locally bounded variations and abstract measures to name a few. The calculus rules in this framework are a bit more intricate than in the usual case of locally absolutely continuous functions but we have seen that their use is the same and they lead easily to results that would be harder to obtain in the usual framework of hybrid systems (especially when having to go through accumulations of impacts).

We have been able to propose in this way a Lagrange-Dirichlet theorem, and it wouldn't be surprising that the extension of other similar results from control theory to the case of nonsmooth

Lagrangian dynamical systems be straightforward with the help of tools such as those presented here. As an example, the case of dissipativity has been discussed in section V-E.

Some care must be taken though about the particularities of nonsmooth dynamical systems: the Lyapunov stability theorem that we have proposed in section III isn't identical to the usual one in the smooth case since one condition on the Lyapunov function candidate had to be strengthened, to be satisfied on the whole state space and not only in a neighbourhood of the putative stable set, and the version of LaSalle's invariance theorem that we have proposed in section IV can't be applied in the general case since nonsmooth Lagrangian dynamical systems generally don't present a continuity of the trajectories with respect to initial conditions.

Still, we have been able to use our Lagrange-Dirichlet theorem in order to prove that the position and force control law proposed in [13] is asymptotically stable in the framework of nonsmooth dynamics with no need for any assumptions concerning the state of the contact between the robot and its environment.

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