Branching cells as local states for event structures and nets: probabilistic applications

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N°5347

Octobre 2004

Systèmes communicants

Rapport de recherche

ISSN 0249-6399 ISRN INRIA/RR--5347--FR+ENG

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Systèmes communicants
Projets DistribCom

Rapport de recherche n ° 5347 — Octobre 2004 — 24 pages

Abstract: We study the concept of choice for true concurrency models such as prime event structures and safe Petri nets. We propose a dynamic variation of the notion of cluster previously introduced for nets. This new object is called a *branching cell*, defined within the class of *locally finite* event structures. Our aim is to show that branching cells are the right notion of "local state", for a concurrent system.

We illustrate the above claim through applications to probabilistic concurrent models. In this respect, our results extends in part previous work by Varacca-Völzer-Winskel on probabilistic confusion free event structures. We propose a construction for probabilities over locally finite event structures that makes concurrent processes probabilistically independent—simply attach a dice to each branching cell; dices attached to *concurrent* branching cells are thrown independently. Furthermore, we provide a true concurrency generalization of Markov chains, called *Markov nets*. Unlike in existing variants of stochastic Petri nets, our approach randomizes Mazurkiewicz traces, not firing sequences. We show in this context the Law of Large Numbers (LLN), which confirms that branching cells deserve the status of local state.

Our study was motivated by the stochastic modeling of fault propagation and alarm correlation in telecommunications networks and services. It provides the foundations for probabilistic diagnosis, as well as the statistical distributed learning of such models.

Key-words: probabilistic event structure

(Résumé : tsvp)

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Les cellules de branchement comme états locaux pour les structures d'événements et les réseaux de Petri saufs : applications probabilistes

Résumé : On étudie le concept de choix pour des modèles *true-concurrent* tels que les structures d'événements premières et les réseaux de Petri 1-saufs. On propose une variante dynamique de la notion de cluster, bien connue pour les réseaux. Ce nouvel object est appelé "cellule de branchements", et est défini pour la classe des structures d'événements "localement finies". Notre but est de montrer que les cellules de branchements constituent la bonne notion d'"état local" pour les sytèmes concurrents.

On illustre cette affirmation par des applications aux modèles probabilistes concurrents. En ce sens, nos résultats étendent une partie d'un travail précédent par Varacca-Völzer-Winskel sur les structures d'événements sans confusion. On propose une construction de probabilité pour les structures d'événements localement finies qui rend indépendents en probabilités certains processus locaux concurrents. On propose une généralisation *true-concurrent* des chaînes de Markov en temps discret (réseaux markoviens). On montre la Loi forte des grands nombres pour les réseaux markoviens, ce qui confirme le statut d'état local que tiennent les cellules de branchement.

Notre étude est motivée par la modélisation stochastique de propagation de fautes dans les réseaux de télécommunication. Elle fournit des bases pour le diagnostic probabiliste et pour l'apprentissage distribué dans des modèles probabilistes distribués.

Mots-clé : structure d'événements probabiliste

1 Introduction

The study we present in this paper was motivated by algorithmic problems of distributed nature encountered in the area of telecommunications network and service management [3], in particular distributed alarm correlation and fault diagnosis. This problem consists in reconstructing the hidden history of the distributed system from partial observations (the alarms). The supervision architecture is distributed and comprises several supervisors acting as peers and communicating asynchronously.

True concurrency is essential in these algorithms: interleaving semantics is not adequate for such large distributed systems. States need to be local. Time is totally ordered at each network node, but only partially ordered by causality between nodes. Due to unavoidable ambiguity in diagnosis, nondeterminism is solved by seeking for the "most likely" solutions of the diagnosis problem. This requires having a probabilistic setting at hand.

While searching for existing models in the litterature, we found very few approaches meeting our requirements. *Stochastic Petri nets* [5] and their variants are useful for performance evaluation. This model typically randomizes the holding time in places or the firing time at transitions. Making reference to a global time causes some probabilistic coupling to occur between subsystems that otherwise do not interact. *Probabilistic process algebras* [6] or *probabilistic automata* [10] are related to so-called Markov Decision Processes from applied probability theory, they rely on interleaving semantics and do not meet our needs either. In those models, interactions occur via synchronized actions and are subject to nondeterminism. In contrast, probabilistic choices are purely private, occur between interactions and do not conflict with these. Whereas this is perfectly adequate, e.g, for testing or security protocols [7, 8], this is not convenient for modeling the uncertain occurrence and propagation of faults and alarms in telecommunications networks.

Concurrent probabilistic models is a recent area of research meeting our requirements. Runs of concurrent systems are randomized without reference to a global clock, and with a true-concurrent semantics. Fundamental difficulties have lead to restrict to models with limited concurrency, e.g., *confusion free* event structures [14, 13]. *Distributed probabilistic event structures* and *Markov nets* are studied in [1], based on technical earlier results of [2]; these approaches address event structures with confusion.

From the results of [1], it occurred to us that the very key for the analysis of probabilistic choices in true-concurrent models was the informal concept of "concurrent local state". In this paper, we investigate this concept for safe Petri nets and prime event structures. We show that so-called *branching cells* introduced in [1] for event structures provide the answer. Informally, for an event structure, branching cells are minimal subsets of events closed under immediate conflict. Processes are *dynamically* decomposed by branching cells: in different executions, the same event can belong to different branching cells. Branching cells differ from clusters [4], which are statically defined on nets.

We apply the notion of branching cell to the definition and construction of concurrent probabilistic models. The probabilities we construct in this way satisfy the following essential requirement regarding concurrency: *parallel local processes are made independent in the probabilistic sense, conditionally on their common past.* Such probabilities deserve the name of *distributed* probabilities. When applied to event structure obtained by unfolding safe Petri nets, this yields *Markov nets,* a probabilistic form of Petri nets compliant with true concurrency. We prove a Markov property and a Law of Large Numbers for Markov nets, in which branching cells play the role of local states.

Our work extends [13] [1] since we do not restrict ourselves to confusion free event structures. On the other hand, for event structures obtained by unfolding confusion free nets, we show using a theorem of Thiagarajan [12] that the set of all branching cells can be made a regular labeled event structure.

The paper is organized as follows. Branching cells for prime event structures are introduced in Section 2, together with their properties. Their use for the definition and construction of concurrent probabilistic models is demonstrated in Section 3. Section 4 discusses the case of confusion free event structures and regularity. In Section 5, Markov nets are introduced and the Markov property and the Law of Large Numbers are proved.

2 Branching Cells and Their Properties

A prime event structure ([9]) is a triple $\mathcal{E} = (E, \preceq, \#)$, where $(E, \preceq)$ is a well founded, at most countable partial order, and $\#$ is a conflict relation (irreflexive and inherited by causality). We denote by $\mathcal{V}_{\mathcal{E}}$ the event structure of the finite configurations of $\mathcal{E}$, and by $\Omega_{\mathcal{E}}$ the set of all maximal configurations (finite or infinite) of $\mathcal{E}$. A subset $F \subseteq E$ implicitly defines a sub-event structure of $\mathcal{E}$ with causality and conflict relations defined by:

$$\preceq_F = \preceq \cap (F \times F), \quad \#_F = \# \cap (F \times F),$$

and we shall freely write F, $\mathcal{V}_F$, and Ω_F to denote this event structure and its set of finite and maximal configurations, respectively. For $e \in E$, $\lceil e \rceil =_{\mathrm{def}} \{e' \in \mathcal{E} : e' \preceq e\}$ denotes the minimal configuration containing e. A subset P of E is said to be a *prefix* of $\mathcal{E}$ if $\lceil e \rceil \subseteq P$ for all $e \in P$. For v a finite or infinite configuration of $\mathcal{E}$, we set $E^v =_{\mathrm{def}} \{e \in E \setminus v : \forall e' \in v, \neg(e \# e')\}$, and we denote by $\mathcal{E}^v$ the induced event structure and call it the *future* of v. Throughout the paper, we assume that $\mathcal{E}$ satisfies the following basic properties: $\lceil e \rceil$ is finite for every event e, and $\mathrm{Min}_{\preceq}(E^v)$ contains finitely many events, for every $v \in \mathcal{V}_{\mathcal{E}}$.

The *concurrency* relation on E, denoted by $\|$, is defined as the reflexive closure of $(\mathcal{E} \times \mathcal{E}) \setminus (\# \cup \preceq \cup \succeq)$. Finally, we define the *immediate conflict* relation $\#_\iota$ on E by:

$$\forall e, e' \in E, \quad e \#_\iota e' \quad \text{iff} \quad (\lceil e \rceil \times \lceil e' \rceil) \cap \# = \{(e, e')\}. \tag{1}$$

Definition 1. *(stopping prefix) A prefix $B \subseteq E$ is called a* stopping prefix *iff it is closed under immediate conflict. $\mathcal{E}$ is called* locally finite *iff for each event e of $\mathcal{E}$, there exists a finite stopping prefix B containing e.*

The following condition is assumed throughout this paper:

[1] The latter reference also discusses the relation to domain theory, a topic not considered here.

Assumption 1. *$\mathcal{E}$ is locally finite.*

Stopping prefixes B satisfy the following property:

$$\Omega_B \quad = \quad \{\omega \cap B \mid \omega \in \Omega_{\mathcal{E}}\}\,. \tag{2}$$

The set of all stopping prefixes is a complete lattice. Stopping prefix B is called *initial* iff $\emptyset$ is the only stopping prefix strictly contained in B.

A configuration v of $\mathcal{E}$ is said to be *stopped* if there is a stopping prefix B such that $v \in \Omega_B$. Call *recursively stopped* a configuration v of $\mathcal{E}$ such that there exists a finite nondecreasing sequence $(v_n)_{0\leq n\leq N}$ of configurations, where $v_0 = \emptyset$ and, for $n < N$, $v_{n+1} \setminus v_n$ is a finite stopped configuration of the future $\mathcal{E}^{v_n}$ of v_n. The set of all finite recursively stopped configurations is denoted by $\mathcal{W}_{\mathcal{E}}$, or simply $\mathcal{W}$ if no confusion can occur.

Definition 2. *(branching cell) Call* branching cell *of $\mathcal{E}$ any initial stopping prefix of $\mathcal{E}^v$, where v ranges over $\mathcal{W}$. The set of all branching cells of $\mathcal{E}$ is denoted by $X_{\mathcal{E}}$ (or simply X when no confusion can occur). Branching cells are generically denoted by the symbol x.*

For $v \in \mathcal{W}$, denote by $\delta(v)$ the set of branching cells that are initial prefixes of $\mathcal{E}^v$. Consider the following map Δ, called the *covering* map of $\mathcal{E}$:

$$\begin{aligned} \text{for } v \in \mathcal{W}:\quad & \Delta(v) =_{\text{def}} \bar{\Delta}(v) \setminus \delta(v)\,, \\ \text{where:}\quad & \bar{\Delta}(v) =_{\text{def}} \{\delta(v') \mid v' \in \mathcal{W}, v' \subseteq v\}\ . \end{aligned} \tag{3}$$

We now list some properties of branching cells. Corresponding proofs are found in [1], Chapter 3, and are not given here, due to lack of space[2].

Theorem 1. *If B is a stopping prefix of $\mathcal{E}$, then $X_B \subseteq X_{\mathcal{E}}$ and $\mathcal{W}_B \subseteq \mathcal{W}_{\mathcal{E}}$. Furthermore, the covering maps Δ and Δ_B respectively defined on $\mathcal{W}$ and $\mathcal{W}_B$ coincide on $\mathcal{W}_B$. Finally, for every $v \in \mathcal{W}$, $X_{\mathcal{E}^v} \subseteq X_{\mathcal{E}}$.*

Theorem 2. *Branching cells cover recursively stopped configurations, i.e.:*

$$\forall v \in \mathcal{W}, \quad v = \bigcup_{x\in\Delta(v)} v \cap x \tag{4}$$

and for each $x \in \Delta(v)$, $v \cap x$ is an element of Ω_x.

Theorem 3. *For $v \subseteq v'$ two finite recursively stopped configurations, $v' \setminus v$ is recursively stopped in $\mathcal{E}^v$. Denoting by Δ^v the covering map (3) defined in $\mathcal{E}^v$:*

$$\Delta(v') = \Delta(v) \cup \Delta^v(v' \setminus v),\ \textit{and}\ \Delta(v) \cap \Delta^v(v' \setminus v) = \emptyset. \tag{5}$$

[2] For the submitted version, the proofs are attached in the Appendix.

Theorem 4. *Let ξ be a subset of $\delta(\emptyset_{\mathcal{E}})$, where $\emptyset_{\mathcal{E}}$ denotes the empty configuration of $\mathcal{E}$. The formula*

$$B_\xi \quad =_{\text{def}} \quad \bigcup_{x\in\xi} x \tag{6}$$

defines a prefix of $\mathcal{E}$, whose set of finite configurations $\mathcal{V}_{B_\xi}$ and maximal configurations Ω_{B_ξ} respectively decompose as:

$$\mathcal{V}_{B_\xi} = \prod_{x\in\xi} \mathcal{V}_x \quad \text{and} \quad \Omega_{B_\xi} = \prod_{x\in\xi} \Omega_x \,.$$

Call thin *a prefix of $\mathcal{E}$ of the form (6), where $\xi \subseteq \delta(\emptyset_{\mathcal{E}})$. The complete lattice of thin prefixes has finite upper bound.*

Comment. Although Th. 4 is stated only for thin prefixes that "begin" the event structure, there is no real loss of generality since we can apply Th. 4 in the futures $\mathcal{E}^v$, for $v \in \mathcal{W}$, in which $\delta(v)$ plays the role of $\delta(\emptyset_{\mathcal{E}})$. The product forms given in Th. 4 show that branching cells are traversed by local processes that are both *concurrent* and *independent*: in the future of v, local decisions taken in a branching cell $x \in \delta(v)$ do not influence the range of possible local decisions that can be taken in other branching cells of $\delta(v)$. Section 3 adds a probabilistic interpretation to this independence.

Local finiteness as a generalization of confusion freeness. Recall that event structure $\mathcal{E}$ is said to be *confusion free* [9] if $\mathcal{E}$ satisfies the Q axiom: for every finite configurations v, v' with $v \subseteq v'$, and for every event e minimal in E^v,

$$\exists y \in v' : e\#y \Rightarrow \exists! f \in \mathrm{Min}_{\preceq}(E^v) : \; f \in v', \; e\#f \,.$$

It is straightforward to establish that $\mathcal{E}$ is confusion free iff $\#_\iota$ is transitive and satisfies: for all $e, e' \in \mathcal{E}$, $e \,\#_\iota\, e'$ implies $\lceil e\rceil \setminus \{e\} = \lceil e'\rceil \setminus \{e'\}$. It follows that, setting for every event $e \in E$:

$$F(e) = \{f \in E \,:\, e \,\#_\iota\, f\}, \quad B(e) = \bigcup_{e' \in \lceil e\rceil} F(f)\,, \tag{7}$$

$B(e)$ is a finite stopping prefix that contains event e. This shows that $\mathcal{E}$ is locally finite. Moreover every finite configuration is stopped—and therefore recursively stopped. The set of branching cells is equal to $\{F(e) : e \in E\}$, which forms a partition of E. Hence, for confusion free event structures, branching cells reduce to the *cells* defined in [13]. Such simple properties fail for event structures having confusion, as shown by Fig. 1 below. To summarize, confusion free event structures are locally finite, but the converse is not true. Locally finite event structures appear as event structures with sort of "finite confusion".

3 Application to Probabilistic Event Structures

We recall that a *probabilistic event structure* is a pair $(\mathcal{E}, \mathbb{P})$ with $\mathbb{P}$ a probability measure[3] on the space Ω of maximal configurations of $\mathcal{E}$. We shall prove that a probabilistic event structure can be naturally defined from the new notion of *locally randomized* event structure.

Definition 3. *(locally randomized event structure) A* locally randomized event structure *is a pair $(\mathcal{E}, (p_x)_{x \in X})$, where X is the set of branching cells of $\mathcal{E}$, and for each $x \in X$, p_x is a probability over Ω_x.*

For $F \subseteq E$ a sub-event structure of $\mathcal{E}$, denote by X_F the set of all branching cells of F. Call F *well formed* if it is finite and such that $X_F \subseteq X_{\mathcal{E}}$. Note that finite stopping prefixes are well formed according to Th. 1. For F well formed, set:

$$\text{for } \omega_F \in \Omega_F \quad : \qquad \mathbb{P}_F(\omega_F) = \prod_{x \in \Delta(\omega_F)} p_x(\omega_F \cap x), \tag{8}$$

which is well defined since, according to Th. 2, $\omega_F \cap x \in \Omega_x$.

Lemma 1. *If $B = B_\xi$ is a thin prefix (see Th. 4), then $\mathbb{P}_B$ is the direct product of the p_x's, for x ranging over ξ. In particular, $\mathbb{P}_B$ is a probability.*

Proof. This is a direct consequence of formula (8) and Theorem 4. ◇

Lemma 2. *If $F \subseteq E$ is a well formed sub event structure, then $\mathbb{P}_F$ is a probability. In particular, for each stopping prefix B, $\mathbb{P}_B$ is a probability.*

Proof. We show that $\mathbb{P}_F$ is a probability by induction on integer $n_F = \sup_{\omega_F \in \Omega_F} \operatorname{Card}\Delta(\omega_F) < \infty$. The result is a direct consequence of Lemma 1 for $n_F \leq 1$. Assume it holds until $n \geq 1$, and let F be well formed and such that $n_F \leq n+1$. Consider the (finite) upper bound D of thin prefixes of F. Applying property (2) to D yields the following decomposition for Ω_F: $\Omega_F = \bigcup_{v \in \Omega_D} \{v\} \times \Omega_{F^v}$. Moreover, for each $v \in \Omega_D$ and $\omega' \in \Omega_{F^v}$, and setting $\omega = v \cup \omega'$, we obtain by Th. 3:

$$\Delta(\omega) = \Delta(v) \cup \Delta^v(\omega'), \quad \Delta(v) \cap \Delta^v(\omega') = \emptyset\,. \tag{9}$$

Formulas (8) and (9) together imply:

$$\sum_{\omega \in \Omega_F} \mathbb{P}_F(\omega) = \sum_{v \in \Omega_D} \mathbb{P}_D(v) \Big(\sum_{\omega' \in \Omega_{F^v}} \mathbb{P}_{F^v}(\omega') \Big). \tag{10}$$

It follows from Th. 1 that for each $v \in \Omega_D$, the future F^v of v in F satisfies $X_{F^v} \subseteq X_F \subseteq X_{\mathcal{E}}$. Formula (9) implies that $n_{F^v} \leq n$. Hence we can apply the induction hypothesis to F^v and obtain $\sum_{\omega' \in \Omega_{F^v}} \mathbb{P}_{F^v}(\omega') = 1$. From Lemma 1 we get: $\sum_{v \in \Omega_D} \mathbb{P}_D(v) = 1$. This, together with Eq. (10), implies $\sum_{\omega \in \Omega_F} \mathbb{P}_F(\omega) = 1$, which completes the induction. ◇

[3] The σ-algebra considered is the Borel σ-algebra generated by the Scott topology on Ω, see [1] for details. In the remaining of the paper, we do not mention the σ-algebras considered since they are always canonical.

Corollary 1. *Let $B \subseteq B'$ be two finite stopping prefixes of $\mathcal{E}$. The following formula holds:*

$$\forall \omega_B \in \Omega_B \ : \quad \mathbb{P}_B(\omega_B) \ = \sum_{\omega' \in \Omega_{B'},\, \omega' \supseteq \omega_B} \mathbb{P}_{B'}(\omega'). \tag{11}$$

Proof. Let ω_B be an element of Ω_B, and denote by $B'' =_{\mathrm{def}} B'^{\omega_B}$ the future of ω_B in B'. Then $\{\omega' \in \Omega_{B'} : \omega' \supseteq \omega_B\}$ is one to one with $\Omega_{B''}$. Equation (5) gives $\Delta(\omega') = \Delta(\omega_B) \cup \Delta^{\omega_B}(\omega' \setminus \omega_B)$, whence:

$$\sum_{\omega' \in \Omega_{B'},\, \omega' \supseteq \omega_B} \mathbb{P}_{B'}(\omega') = \mathbb{P}_B(\omega_B) \sum_{z \in \Omega_{B''}} \mathbb{P}_{B''}(z). \tag{12}$$

From Lemma 2 applied to finite event structure B'', the sum on the right hand side of (12) equals 1, which implies (11). ◇

Lemma 1 expresses that the family $(\Omega_B, \mathbb{P}_B)$, where B ranges over the set of finite stopping prefixes, is a *projective system* of probability spaces. It is proved in [1], Chapter 2, that, under Assumption 1, this projective system defines a unique probability $\mathbb{P}$ on $\Omega_{\mathcal{E}}$ such that for every finite stopping prefix B and every element $\omega_B \in \Omega_B$: $\mathbb{P}\big(\{\omega \in \Omega : \omega \supseteq \omega_B\}\big) = \mathbb{P}_B(\omega_B)$.

Probabilistic Future and Distributed Probabilities. So far we have shown how to construct probabilistic event structures from locally randomized event structures. Conversely, each probability $\mathbb{P}$ over $\mathcal{E}$, such that $\mathbb{P}(v) > 0$ for every finite configuration[4] v, defines a family $(p_x)_{x \in X}$ of local probabilities associated to branching cells as follows, for $x \in X$ and $\omega_x \in \Omega_x$:

$$p_x(\omega_x) \quad =_{\mathrm{def}} \quad \frac{\mathbb{P}\left(\{\omega \in \Omega_{\mathcal{E}} \ : \ x \in \bar{\Delta}(\omega), \ \omega \cap x = \omega_x\}\right)}{\mathbb{P}\left(\{\omega \in \Omega_{\mathcal{E}} \ : \ x \in \bar{\Delta}(\omega)\}\right)}. \tag{13}$$

Of course, the following natural question arises: is it true that the family $(p_x)_{x \in X}$ conversely induces $\mathbb{P}$ through formula (8)? Not in general. The following Th. 5, which proof is found in [1], Chapter 4, provides the answer.

For $(\mathcal{E}, \mathbb{P})$ a probabilistic event structure, consider the *likelihood* function q defined on the set of finite configurations by:

$$\text{for } v \in \mathcal{V}_{\mathcal{E}} \ , \ q(v) =_{\mathrm{def}} \mathbb{P}\big(\{\omega \in \Omega_{\mathcal{E}} \ : \omega \supseteq v\}\big)\,. \tag{14}$$

For v a finite configuration, the *probabilistic future* $(\mathcal{E}^v, \mathbb{P}^v)$ is defined by

$$\mathbb{P}^v(\,\cdot\,) =_{\mathrm{def}} \frac{1}{q(v)} \mathbb{P}(\,\cdot\,).$$

The associated likelihood q^v is given by $q^v(w) = \frac{1}{q(v)} q(v \cup w)$, for w ranging over the set of finite configurations of $\mathcal{E}^v$.

[4] This condition is stated here for simplicity, it can be removed with some more technical effort.

Definition 4. *(distributed probability) Probability* $\mathbb{P}$ *is called* distributed *iff, for each recursively stopped configuration* v*, and each thin prefix* B^v_ξ *in* $\mathcal{E}^v$*, the following holds:*

$$\forall \omega \in \Omega_{B^v_\xi}\ , \quad q^v(\omega) = \prod_{x \in \xi} p_x(\omega \cap x) \tag{15}$$

where p_x *is defined from* $\mathbb{P}$ *by using (13).*

Theorem 5. *Family* $(p_x)_{x \in X}$ *defined from* $\mathbb{P}$ *by using (13) induces again* $\mathbb{P}$ *through the constructive formula (8) iff* $\mathbb{P}$ *is distributed. In this case, the likelihood function is given on* $\mathcal{W}$ *by:* $q(v) = \prod_{x \in \Delta(v)} p_x(v \cap x)$.

Remark that the likelihood given in Th. 5 extends the original formula (8). Theorem 5 shows that the *valuations with independence* defined in [13] for confusion free event structures (see Sect. 2) are equivalently defined as likelihoods (14) associated with distributed probabilities.

Comment. Equation (15), which characterises distributed probabilities, has the following interpretation. Because of the absence conflicts, and conditionally on a partial execution $v \in \mathcal{W}$, the local choices inside the different branching cells belonging to $\delta(v)$ are performed independently from one another.

4 Confusion Free Event Structures and Choice Nets

Making the set of branching cells an event structure? Set $\mathcal{V} =_{\mathrm{def}} \Delta(\mathcal{W})$. Obviously, we make $\mathcal{V}$ a prime event structure by setting $\xi' \preceq \xi$ if $\xi' \subseteq \xi$ and $\xi' \# \xi$ if there is no $\xi'' \in \mathcal{V}$ containing both ξ and ξ'. It is tempting to regard X itself as a prime event structure, which poset of finite configurations would identify with $\mathcal{V}$. Unfortunately this interpretation is not possible in general, as shown by Fig. 1. In this figure, an event structure is depicted in the first diagram. The second diagram depicts event structure $\mathcal{V}$. In this example, X cannot be a prime event structure generating $\mathcal{V}$, since x_4 would be in self-conflict. Th. 6 below shows that the situation is better if $\mathcal{E}$ is confusion free (see Sect. 2).

Theorem 6. *If* $\mathcal{E}$ *is confusion free, there are causality and conflict relations* $\preceq$ *and* $\#$ *making* $\Xi_{\mathcal{E}} =_{\mathrm{def}} (X, \preceq, \#)$ *a prime event structure satisfying* $\mathcal{V}_{\Xi_{\mathcal{E}}} = \mathcal{V}$*. We call* $\Xi_{\mathcal{E}}$ *the* choice structure *of* $\mathcal{E}$*, and we denote it by* Ξ *if no confusion can occur.*

Proof. We sketch the proof. Define the conflict and causality relations on X as follows: $x \preceq x'$ if for all $v \in \mathcal{W}$, $x' \in \Delta(v) \Rightarrow x \in \Delta(v)$, and $x \# x'$ if there is no $v \in \mathcal{W}$ such that $x \in \Delta(v)$ and $x' \in \Delta(v)$. It is straightforward to check that these relations satisfy the requirements, using the form (7) of branching cells. ◇

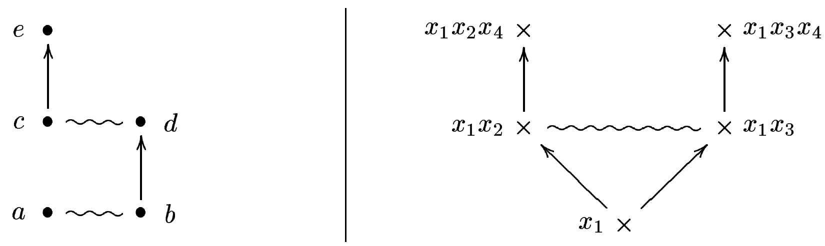

Figure 1: *An event structure with confusion (left). Its branching cells are:* $x_1 = \{a, b\}$, $x_2 = \{c\}$, $x_3 = \{c, d\}$, $x_4 = \{e\}$. *The second diagram shows event structure* $\mathcal{V}$.

Regularity of choice structures. Next, assume that $\mathcal{E}$ is still confusion free and arises from the unfolding[5] of a safe Petri net $\mathcal{N}$. Branching cells are attached to the futures $\mathcal{E}^v$, for v ranging over the set of finite configurations of $\mathcal{E}$. The $\mathcal{E}^v$s are finitely many up to an isomorphism of labelled event structures. Therefore, so are branching cells. Let

$$\Sigma_{\mathcal{N}} \text{ be the set of } \textit{dynamic clusters} \text{ of } \mathcal{N},$$

defined as the finite alphabet of isomorphism classes of branching cells of $\mathcal{E}$. We simply write Σ for short if no confusion can occur. Dynamic clusters are denoted by the symbol $\mathbf{s}$, and $\langle x \rangle$ denotes the class of branching cell x.

It is natural to ask whether the *labelled event structure* $(\Xi_{\mathcal{E}}, \langle \cdot \rangle, \Sigma)$ can be seen itself as the unfolding of a safe Petri net with transitions in Σ. Since the labelling mapping is already defined by $\langle \cdot \rangle$, we can apply Thiagarajan's results on *trace regular event structures* [12]:

Theorem 7. *Let $\mathcal{E}$ be the unfolding event structure of a confusion free net $\mathcal{N}$. We assume that $\mathcal{N}$ has a home marking*[6]*, and that every transition of the net belongs to at least one firing sequence. Then $(\Xi, \langle \cdot \rangle, \Sigma)$ is the unfolding a safe finite Petri net iff $\mathcal{N}$ satisfies the following property:*

$$\textit{for any two distinct transitions } t, t' \ : {}^\bullet t \cap {}^\bullet t' \neq \emptyset \ \Rightarrow \ t^\bullet \cap t'^\bullet = \emptyset\,. \qquad (16)$$

The so obtained safe Petri net is called the choice net *of $\mathcal{N}$, denoted by $\mathcal{X}_{\mathcal{N}}$.*

Proof. We sketch the proof. Condition (16) is clearly necessary (see Fig. 3 below). For the converse, we shall apply Thiagarajan's theorem [12], which characterises labelled event structures arising from the unfolding of a safe Petri net. For this, we need first to prove that Ξ is regular, i.e., we have to show that there is a bound K such that every future Ξ^ξ has less than K minimal events, and that the futures Ξ^ξ are finitely many up to an isomorphism of event structures, for ξ ranging over $\mathcal{V}_\Xi$. Since every $\xi \in \mathcal{V}_\Xi$ can be written as $\xi = \Delta(v)$ with

[5] Conditions on $\mathcal{N}$ for $\mathcal{E}$ to be confusion free are well known: $\mathcal{N}$ must have no symmetric nor asymmetric confusion, see [9].

[6] i.e., the initial marking is reachable from any reachable marking.

$v \in \mathcal{W}_\Xi$, and since $\mathcal{E}$ is itself regular, both properties derive from the formula, valid for any $v \in \mathcal{W}_\Xi$: $\Xi_{\mathcal{E}^v} = \delta(v) \cup \Xi^{\Delta(v)}$, where $\Xi_{\mathcal{E}^v}$ denotes the choice structure of $\mathcal{E}^v$.

In a second step, define $\lhd$ as the immediate successor relation induced by causality on X, and define the dependence relation D on Σ by $(\mathbf{s}, \mathbf{s}') \in D$ iff there are events $x, x' \in X$ such that $\langle x \rangle = \mathbf{s}$ and $\langle x' \rangle = \mathbf{s}'$ with $x \lhd x'$ or $x \#_\iota x'$. Then we check that the following conditions are fulfilled for all $x, x' \in X$: 1. $x \#_\iota x' \Rightarrow \langle x \rangle \neq \langle x' \rangle$ and 2. $\big(\langle x \rangle, \langle x' \rangle\big) \in D \Rightarrow \neg(x \parallel x')$. Point 2 involves the home marking assumption, and point 1 derives from Eq. (16). We conclude by applying [12], Theorem 6.1. $\diamond$

Fig. 2 depicts a net satisfying the assumptions of Th. 7 (left), and its choice net (right). Remark that $\mathcal{X}_\mathcal{N}$ is *not* confusion free.

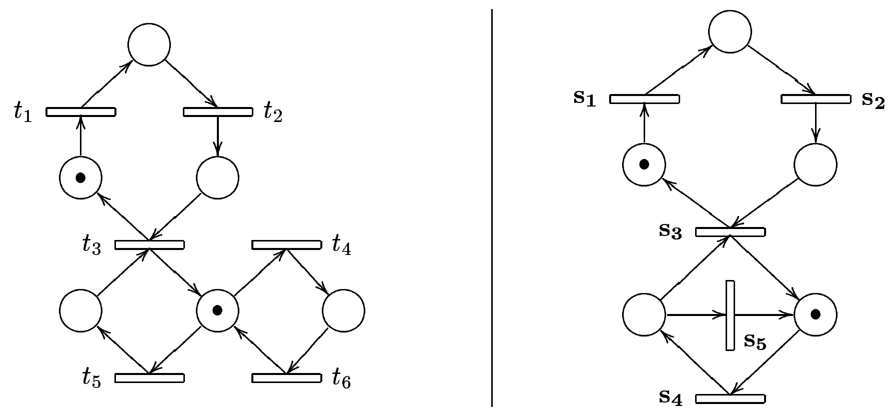

Figure 2: *A confusion free net $\mathcal{N}$ (left), and its choice net $\mathcal{X}_\mathcal{N}$ (right). Net $\mathcal{N}$ has 5 dynamic clusters:* $\mathbf{s_1} = \{t_1\}$, $\mathbf{s_2} = \{t_2\}$, $\mathbf{s_3} = \{t_3\}$, $\mathbf{s_4} = \{t_4, t_5\}$, $\mathbf{s_5} = \{t_6\}$.

To illustrate Condition (16) in Th. 7, consider the net $\mathcal{N}$ depicted in Fig. 3–left, which does not satisfy (16). A prefix of its choice structure Ξ is depicted on the right hand side, with labels in boldface. There are two classes s_1 and s_2 of branching cells: $\mathbf{s_1} = \{a, b\}$ with a and b in conflict and $\mathbf{s_2} = \{c\}$. Branching cells x and y indicated in Fig. 3 are in immediate conflict and satisfy $\langle x \rangle = \langle y \rangle = \mathbf{s_2}$. This prevents $(\Xi, \langle \cdot \rangle, \Sigma)$ from being the unfolding of a safe Petri net.

To illustrate the necessity of the home marking assumption in Th. 7, consider the net $\mathcal{N}$ depicted in Fig. 4 at left hand, which has no home marking. Its choice structure is finite, and depicted in Fig. 4–right. Consider the two pairs of events of the choice structure (x, y) (top right) and (x', y') (bottom right). These two pairs satisfy: $\langle x \rangle = \langle y \rangle$, $\langle x' \rangle = \langle y' \rangle$, $x \parallel y$ and $x' \#_\iota y'$. Again, this prevents $(\Xi, \langle \cdot \rangle, \Sigma)$ from being the unfolding of a safe Petri net.

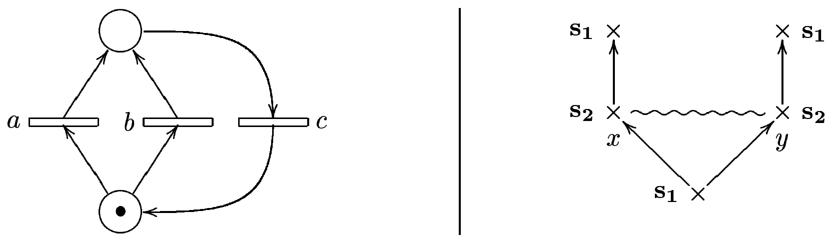

Figure 3: *A confusion free net possessing a home marking (left), and a prefix of its choice structure (right).*

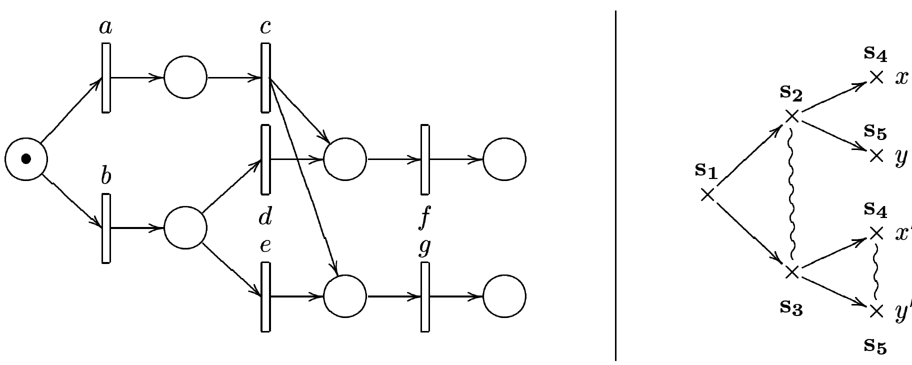

Figure 4: *A confusion free net with no home marking (left), its choice structure (right). Dynamic clusters are:* $\mathbf{s_1} = \{a, b\}$, $\mathbf{s_2} = \{c\}$, $\mathbf{s_3} = \{d, e\}$, $\mathbf{s_4} = \{f\}$ *and* $\mathbf{s_5} = \{g\}$.

5 Markov Nets

We continue the study of unfoldings of safe Petri nets, but we relax the confusion free assumption. *Markov nets* are introduced and briefly studied. Proofs of the results stated in this section as well as additional results can be found in [1], Chapters 5–7.

Markov nets: definition and basic properties. Throughout this section, we assume that $\mathcal{E}$ is a locally finite event structure arising from the unfolding $\mathcal{U}$ of a finite safe Petri net $\mathcal{N}$. Ω denotes the set of maximal configurations of $\mathcal{E}$, and $\mathcal{W}$ denotes the set of finite recursively stopped configurations of $\mathcal{E}$.

As said in Sec. 4, the set $\Sigma_{\mathcal{N}}$ of dynamic clusters—isomorphism classes of branching cells—is a finite alphabet. This, together with the local randomization introduced in Sect. 3, suggests the following definition.

Definition 5. *(Markov net) A Markov net is a pair $\big(\mathcal{N}, (p_{\mathbf{s}})_{\mathbf{s}\in\Sigma}\big)$, where $\mathcal{N}$ is a finite safe Petri net with locally finite unfolding, and $p_{\mathbf{s}}$ is a probability on the finite set $\Omega_{\mathbf{s}}$ for every $\mathbf{s} \in \Sigma$.*

Markov net $\big(\mathcal{N}, (p_{\mathbf{s}})_{\mathbf{s}\in\Sigma}\big)$ induces a locally randomized event structure $(\mathcal{E}, (p_x)_{x\in X})$ (see Def. 3) by setting $p_x = p_{\langle x\rangle}$ for every branching cell $x \in X_{\mathcal{E}}$, whence a unique distributed probability $\mathbb{P}$ on Ω. Note that, if net $\mathcal{N}$ is the product of two non interacting nets $\mathcal{N} = \mathcal{N}_1 \times \mathcal{N}_2$, then the two components $\mathcal{N}_i, i \in \{1, 2\}$ are independent in the probabilistic sense, i.e., $\mathbb{P} = \mathbb{P}_1 \otimes \mathbb{P}_2$.

Theorem 8. *(Markov property) Let $(\mathcal{N}, (p_{\mathbf{s}})_{\mathbf{s}\in\Sigma})$ be a Markov net, and let $\mathbb{P}$ be the associated distributed probability on Ω. For v a finite recursively stopped configuration of $\mathcal{E}$, let $m(v)$ and Σ^v denote respectively the marking reached by v and the classes of branching cells of $\mathcal{E}^v$. Then for every $v \in \mathcal{W}$, the probabilistic future $(\mathcal{E}, \mathbb{P}^v)$ is associated with Markov net $(\mathcal{N}^v, (p_{\mathbf{s}})_{\mathbf{s}\in\Sigma^v})$, where $\mathcal{N}^v$ is the same net as $\mathcal{N}$, except that $\mathcal{N}^v$ has initial marking $m(v)$. Moreover we have:*

$$\forall v, v' \in \mathcal{W}, \quad m(v) = m(v') \Rightarrow \mathbb{P}^v = \mathbb{P}^{v'} . \tag{17}$$

Equation (17) expresses the memoryless nature of Markov nets: the probabilistic future of a $v \in \mathcal{W}$ only depends on the marking $m(v)$. It is the most convenient form for the Markov property for concurrent systems.

The Law of Large Numbers (LLN). Call *return* to the initial marking M_0 any pair (v, v') of recursively stopped configurations of $\mathcal{E}$ and $\mathcal{E}^v$, respectively, such that: 1/ $m(v) = m(v') = M_0$, and 2/ there exists a prefix v'' of v' such that $m(v'') \cap M_0 = \emptyset$. For our study of LLN, we restrict ourselves to *recurrent* Markov nets, i.e., Markov nets such that, with probability 1, $\omega \in \Omega$ contains infinitely many returns to M_0. If the considered net is indeed sequential, then our definition reduces to the classical notion of recurrence, for Markov chains [11].

For recurrent markov chains, the LLN states as follows. Let Σ be the finite state space of a Markov chain $(X_k)_{k\geq 1}$, and let $f:\Sigma\to\mathbb{R}$ be a test function. The sums $S_n(f)=\sum_{k=1}^{n} f(X_k)$ are called *ergodic sums*, and the LLN studies the limit, for $n\to\infty$, of the *ergodic means*: $M_n(f)=\frac{1}{n}S_n(f)$. In extending the LLN to Markov net $\mathcal{N}$, we are faced with two difficulties:

1. What is the proper concept of state?
2. What replaces counter n, since time is not totally ordered?

Corresponding answers are:

1. The set $\Sigma_{\mathcal{N}}$ of dynamic clusters of $\mathcal{N}$ is taken as the state space.
2. For v a recursively stopped configuration, the number of branching cells contained in $\Delta(v)$ is taken as the "duration" of v.

More precisely, call *distributed function* a finite family $f=(f_{\mathbf{s}})_{\mathbf{s}\in\Sigma}$ of real-valued functions $f_{\mathbf{s}}:\Omega_{\mathbf{s}}\to\mathbb{R}$. Distributed functions form a vector space of finite dimension over $\mathbb{R}$. The *concurrent ergodic sums* of f are defined as being the function:

$$S(f):\mathcal{W}\to\mathbb{R}\ ,\quad \forall v\in\mathcal{W},\ S(f)(v)=\sum_{x\in\Delta(v)} f_{\langle x\rangle}(v\cap x)\,. \tag{18}$$

For example, if $N=(N_{\mathbf{s}})_{\mathbf{s}\in\Sigma}$ is the distributed function given by $N_{\mathbf{s}}(w)=1$ for all $\mathbf{s}\in\Sigma$ and $w\in\Omega_{\mathbf{s}}$, then $S(N)(v)$ counts the number of branching cells contained in $\Delta(v)$. The *concurrent ergodic means* $M(f):\mathcal{W}\to\mathbb{R}$ associated with a distributed function f are defined as follows:

$$\forall v\in\mathcal{W},\quad M(f)(v)=\frac{1}{S(N)(v)}S(f)(v)\,. \tag{19}$$

The LLN is concerned by the limit

$$\lim_{v\subseteq\omega,\,v\to\omega} M(f)(v) \tag{20}$$

in a sense we shall make precise. The following notion of *stopping operator* will be central in this respect—stopping operators indeed generalize stopping times [11] for sequential stochastic processes:

Definition 6. *(stopping operator) A random variable $V:\Omega\to\mathcal{W}$, satisfying $V(\omega)\subseteq\omega$ for all $\omega\in\Omega$, is called a* stopping operator *if for all $\omega,\omega'\in\Omega$, we have: $\omega'\supseteq V(\omega)\Rightarrow V(\omega')=V(\omega)$. Say that a sequence $(V_n)_{n\geq 1}$ of stopping operators is* regular *if the following properties are satisfied—such sequences exist:*

1. *$V_n\subseteq V_{n+1}$ for all n, and $\bigcup_n V_n(\omega)=\omega$ for all $\omega\in\Omega$;*
2. *there are two constants $k_1,k_2>0$ such that, with N the distributed function defined above, for all $\omega\in\Omega$ and all $n\geq 1$: $k_1 n\leq S(N)\big(V_n(\omega)\big)\leq k_2 n$.*

Using this concept, (20) is re-expressed as follows:

Definition 7. *(convergence of ergodic means) For for f a distributed function, we say that the ergodic means $M(f)$* converge *to a function $\mu : \Omega \to \mathbb{R}$ if for* every *regular sequence $(V_n)_{n\geq 1}$ of stopping operators,*

$$\lim_{n\to\infty} M(f)\big(V_n(\omega)\big) = \mu(\omega) \text{ with probability } 1. \tag{21}$$

Clearly, condition (21) cannot be satisfied if net $\mathcal{N}$ decomposes as $\mathcal{N} = \mathcal{N}_1 \times \mathcal{N}_2$ and the two components $\mathcal{N}_1$ and $\mathcal{N}_2$ do not interact at all. In this case, regular sequences $V = (V_n)_{n\geq 1}$ of stopping operators decompose into pairs (V^1, V^2) of independent regular sequences, one for each component. For f and v decomposed as $f = (f_1, f_2)$ and $v = (v_1, v_2)$ respectively, we have $S(f)(v) = S(f_1)(v_1)+S(f_2)(v_2)$ and $S(N)(v) = S(N_1)(v_1)+S(N_2)(v_2)$. Since V_n^1 and V_n^2 are free to converge at their own speed, we cannot expect that convergence of ergodic means will hold for this case.

We cannot formally state the condition needed to overcome this problem, the reader is referred to [1], Chapter 7 for details. We only give an informal explanation, in terms of Petri nets and branching cells. If, in an execution $\omega \in \Omega$, we block a token represented by some condition b in the unfolding, we measure the "loss of synchronization" of the system by counting the number of branching cells that can be traversed *without moving the blocked token.* This length defines a random variable $\Omega \to \mathbb{R}$ for each condition b of the unfolding. We say that the considered Markov net has *integrable concurrency height* if all these random variables are integrable, i.e. have finite expectation w.r.t. probability $\mathbb{P}$, for b ranging over the conditions of the unfolding.

Theorem 9. *(LLN) Let $(\mathcal{N}, (p_{\mathbf{s}})_{\mathbf{s}\in\Sigma})$ be a Markov net. Assume that $\mathcal{N}$ is recurrent and has integrable concurrency height. Then*

1. *For any distributed function $f = (f_{\mathbf{s}})_{\mathbf{s}\in\Sigma}$, the ergodic means $M(f)$ converge in the sense of Def. 7 to a function $\mu(f) : \Omega \to \mathbb{R}$.*

2. *Except possibly on a set of zero probability, $\mu(f)$ is* constant *and given by:*

$$\mu(f) \quad = \quad \sum_{\mathbf{s}\in\Sigma} p_{\mathbf{s}}(f_{\mathbf{s}})\alpha(\mathbf{s})\,, \text{ where } p_{\mathbf{s}}(f_{\mathbf{s}}) = \sum_{w\in\Omega_{\mathbf{s}}} f_{\mathbf{s}}(w)p_{\mathbf{s}}(w). \tag{22}$$

3. *In formula (22), coefficients $\alpha(\mathbf{s})$ are equal to*

$$\alpha(\mathbf{s}) = \mu(N^{\mathbf{s}}), \tag{23}$$

and satisfy $\alpha(\mathbf{s}) \in [0,1]$ and $\sum_{\mathbf{s}} \alpha(\mathbf{s}) = 1$; $\alpha(\mathbf{s})$ is the asymptotic rate of occurrence of local state $\mathbf{s}$ in a typical execution $\omega \in \Omega$.

Statement 3 is a direct consequence of statements 1 and 2: Fix $\mathbf{s} \in \Sigma$, and consider the distributed function $N^{\mathbf{s}}$ defined by $N^{\mathbf{s}}_{\mathbf{s}}(w) = 1$ for all $w \in \Omega_{\mathbf{s}}$ and $N^{\mathbf{s}}_{\mathbf{s}'} = 0$ if $\mathbf{s} \neq \mathbf{s}'$. Applying statements 1 and 2 to $N^{\mathbf{s}}$ yields $\alpha(\mathbf{s}) = \mu(N^{\mathbf{s}})$. In particular, from $N = \sum_{\mathbf{s}} N^{\mathbf{s}}$ we obtain: $\sum_{\mathbf{s}} \alpha(\mathbf{s}) = 1$.

If the net is actually sequential (i.e., reduces to a recurrent finite Markov chain), then Σ is the state space of the chain and coefficients $\alpha(\mathbf{s})$ are equal to the coefficients of the invariant measure of the chain. This again reveals that dynamic clusters play the role of local states for concurrent systems.

6 Conclusion and Perspectives

We have proposed branching cells as a form of local concurrent state for prime event structures and safe Petri nets. Our study applies to so-called locally finite event structures that significantly extend the confusion free case. We have applied this to probabilistic event structures: for $\mathcal{E}$ an event structure with set of maximal configurations Ω, there is a one-to-one correspondence between local randomizations of the branching cells of $\mathcal{E}$ on the one hand, and the class of distributed probabilities on Ω on the other hand. Distributed probabilities yield concurrent systems in which locally concurrent random choices are taken independently in the probabilistic sense.

Branching cells reduce to places in the case of sequential systems. The choice net associated with a confusion free net $\mathcal{N}$ captures the local states of $\mathcal{N}$, in the form of an other (possibly confused) net. Choice nets no longer exist for confused nets, however. Nevertheless, dynamic clusters (defined as equivalence classes of branching cells up to isomorphism) still provide the right concept of local state for confused nets, as demonstrated by our study of Markov nets and their Law of Large Numbers.

Future work proceeds along two directions. First, we shall apply Markov nets to distributed probabilistic diagnosis. We shall study the statistical learning of Markov net models from partial observations. Second, we shall investigate other areas in which local states are a useful concept, e.g., control or game problems under true concurrency paradigm.

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A Appendix: proofs

This section collects the proofs of theorems of Section 2. We keep the notations of Section 2.

Throughout the section, if v is a configuration of event structure $\mathcal{E}$, $\#_\iota$ denotes the dynamic conflict relation defined in event structure $\mathcal{E}^v$.

A.1 Preliminary Lemmas

We first list some results that easily follow from the definitions of Sect. 2.

Lemma 3. *If e, f are two events in conflict, there are events e', f' with:*

$$e' \preceq e, \quad f' \preceq f, \quad e' \#_\iota f'.$$

Lemma 4. *Let v be a configuration of $\mathcal{E}$, and let $\#_\iota^v$ denote the dynamic conflict in $\mathcal{E}^v$. Then:*

$$\#_\iota^v = \#_\iota \cap (\mathcal{E}^v \times \mathcal{E}^v)$$

Lemma 5. *If v is a configuration of $\mathcal{E}$, and if B is a stopping prefix of $\mathcal{E}$, then $B \cap E^v$ is a stopping prefix of $\mathcal{E}^v$.*

A.2 Proof of Theorem 1.

We begin with a lemma.

Lemma 6. *Let B be a stopping prefix of $\mathcal{E}$ and let v be a configuration of B. Then we have:*

1. *D is a stopping prefix of $B^v \Rightarrow D$ is a stopping prefix of $\mathcal{E}^v$.*
2. *D is a stopping prefix of $\mathcal{E}^v \Rightarrow D \cap B$ is a stopping prefix of B^v.*

Proof. We obviously have: $B^v = E^v \cap B$, and it follows from Lemma 5 that B^v is thus a stopping prefix of $\mathcal{E}^v$. Hence Point 1 derives from the fact that a stopping prefix of a stopping is itself a stopping prefix. For Point 2, if D is a stopping prefix of E^v, then $C =_{\mathrm{def}} D \cap B$ is obviously a prefix of B^v. To show that C is a stopping prefix of B^v, let $e, f \in B^v$ with:

$$e \in C, \quad e \#_\iota^v f.$$

Then e, f belong to E^v. Since $e \in D$ and D is $\#_\iota^v$-closed in $\mathcal{E}^v$, it implies that $f \in D$ and thus $f \in C$. So C is a stopping prefix of B^v. $\diamond$

Proof of Theorem 1. Let B be a stopping prefix of $\mathcal{E}$. Then a configuration v of B is recursively stopped in B if and only if v is recursively stopped in $\mathcal{E}$. Indeed, it follows from Lemma 6 that the decompositions are obtained from one another by:

$$(v_n)_n \longrightarrow (v_n \cap B)_n\,, \quad (v_n)_n \longrightarrow (v_n)_n\,.$$

It implies that $X_B \subseteq X_\mathcal{E}$, and that Δ and Δ_B coincide on $\mathcal{W}_B$.

For the last point, let v be a finite recursively stopped configuration of $\mathcal{E}$, coming with finite decomposition $(v_n)_n$. Then for any recursively stopped configuration w of $\mathcal{E}^v$, coming with a finite decomposition $(w_k)_k$, the concatenation $(v_n)_n$ then $(v \cup w_k)_k$ yields a recursively stopped decomposition of $v \cup w$, which is thus a well-stopped configuration of $\mathcal{E}$. Moreover we have:

$$(\mathcal{E}^v)^w = \mathcal{E}^{v \cup w}.$$

It implies that $X_{\mathcal{E}^v} \subseteq X_\mathcal{E}$.

A.3 Proof of Theorem 4.

Theorem 4 is a direct consequence of the following lemma.

Lemma 7. *If x, y are two distinct initial stopping prefixes, then $e \parallel f$ for all pairs $(e, f) \in x \times y$.*

Proof. Let $(e, f) \in x \times y$. By minimality, it is obvious that distinct inital stopping prefixes are disjoint. As a consequence, and since x and y are prefixes, e and f are not causality related. Assume that $e\#f$. Then, according to Lemma 3 there are events e', f' with $e' \preceq e$, $f' \preceq f$ and $e' \#_\iota f'$. Since x and y are prefixes and $\#_\iota$-closed, it implies that $e' \in x$, $f' \in y$ and then, say, $f' \in x$. But this contradicts that $x \cap y = \emptyset$. This shows that e and f are not in conflict, and thus $e \parallel f$. ◇

A.4 Proofs of Theorems 2 and 3

We first state some intermediate results.

Lemma 8. *If v is a finite stopped configuration, there is a finite stopping prefix C such that $v \in \Omega_C$.*

Proof. Let v be a finite stopped configuration. Consider C the smallest stopping prefix that contains v. C exists since stopping prefixes form a complete lattice. Since $\mathcal{E}$ is locally finite, there is for each event $e \in v$ a finite stopping prefix B_e that contains e. Their union is a finite stopping prefix that contains v, and thus C is finite. Since v is maximal in B, and since $C \subseteq B$ is a stopping prefix of event structure B, property (2) of stopping prefixes, applied to event structure B, implies that v is maximal in C. ◇

Lemma 9. *Let v be a finite recursively stopped configuration, and let x be an initial stopping prefix of $\mathcal{E}$. Then $v \cap x = \emptyset$ or $v \cap x$ is maximal in x.*

Proof. We have already observed in the proof of Th. 1 that $v \cap B$ is recursively stopped in B, for any stopping prefix B. In particuliar $v \cap x$ is recursively stopped in x. But then, since x is minimal among non empty stopping prefixes, it is obvious that $v \cap x = \emptyset$ or $v \cap x \in \Omega_x$. ◇

We introduce the following definitions.

Definition 8. *A configuration z of $\mathcal{E}$ is a* germ *of $\mathcal{E}$ if there is an initial stopping prefix x of $\mathcal{E}$ such that $z \in \Omega_x$. A finite configuration v of $\mathcal{E}$ is said to have a* germ decomposition *if there is a sequence $(v_n)_{0 \leq n \leq N}$ of configurations of $\mathcal{E}$ such that:*

1. *$v_0 = \emptyset$, $v_n \subseteq v_{n+1}$ for all $0 \leq n \leq N-1$*
2. *$v_{n+1} \setminus v_n$ is a germ of $\mathcal{E}^{v_n}$ for all $0 \leq n \leq N-1$.*

Obviously, if v admits a germ decomposition, then v is recursively stopped. The converse also holds, as stated by the following result.

Lemma 10. *Every finite recursively stopped configuration has a germ decomposition.*

Proof. The proof involves two steps.

Step 1. *The lemma holds if $\mathcal{E}$ is finite, and for* maximal *configurations of $\mathcal{E}$.*

We prove this claim. Assume that $\mathcal{E}$ is finite, and let $\omega \in \Omega_{\mathcal{E}}$. Consider the following construction. Set $v_0 = \emptyset$, and assume that $v_0 \subsetneq \cdots \subsetneq v_n$ have been constructed until $n \geq 0$, with $v_n \subseteq \omega$. If $v_n = \omega$, stop the construction. Otherwise, $\mathcal{E}^{v_n}$ is non empty and has thus at least an initial stopping prefix. Choose an initial stopping prefix x of $\mathcal{E}^{v_n}$, and set $z = \omega \cap x$ and $v_{n+1} = v_n \cup z$. Repeating the construction is allowed, since $v_{n+1} \subseteq \omega$.

Before the construction ends, each step adds at least an event. Therefore, since ω is finite, the construction must eventually end, say at step N, and then $v_N = \omega$. This construction is moreover a germ decomposition. Indeed, at step $n \geq 0$, and with the above notation z and x, we have $z = \omega \cap x = (\omega \setminus v_n) \cap x$. As ω is maximal in $\mathcal{E}$, $\omega \setminus v_n$ is maximal in $\mathcal{E}^{v_n}$. Since x is a stopping prefix of $\mathcal{E}^{v_n}$, and from property (2) of stopping prefixes, it implies that $z \in \Omega_x$. This shows that ω admits a germ decomposition.

Step 2. *The lemma holds in all generality.*

Let v be a finite recursively stopped configuration of $\mathcal{E}$, coming with finite decomposition $(v_n)_{0 \leq n \leq N}$. For each integer $1 \leq n \leq N$, $v_n \setminus v_{n-1}$ is a finite stopped configuration of $\mathcal{E}^{v_{n-1}}$. According to Lemma 8, there is finite stopping prefix C_n of $\mathcal{E}^{v_{n-1}}$ such that $v_n \setminus v_{n-1} \in \Omega_{C_n}$. We apply Step 1 to $v_n \setminus v_{n-1}$ in finite event structure C_n to obtain a germ decomposition $(w_{n,k})_k$ of v_n. Then the concatenation $(w_{1,k})_k$, then $(v_1 \cup w_{2,k})_k$, then $\ldots$, then $(v_{N-1} \cup w_{N,k})_k$, is a germ decomposition of v. This completes the proof. ◇

Lemma 11. *Let u be a finite configuration of $\mathcal{E}$, and let x be an initial stopping prefix of $\mathcal{E}$. If $u \cap x = \emptyset$, then x is an initial stopping prefix of $\mathcal{E}^u$.*

Proof. We first prove that $x \subseteq \mathcal{E}^u$. Let e be an event of x, and assume that $e \notin \mathcal{E}^u$. Since $e \notin u$, it implies that e is in conflict with an event $g \in u$. Applying Lemma 3, there are events $e' \preceq e$ and $g' \preceq g$ with $e' \#_\iota g'$. Since x is prefix and $\#_\iota$-closed, this implies that $g' \in x$, contracting that $u \cap x = \emptyset$. We have thus shown that $x \subseteq \mathcal{E}^u$.

Since x is a stopping prefix of $\mathcal{E}$, and according to Lemma 5, this implies that x is a stopping prefix of $\mathcal{E}^u$. Since $x \neq \emptyset$, to show that x is an initial stopping prefix of $\mathcal{E}^u$, it remains only to show that x does not contain non empty stopping prefix of $\mathcal{E}^u$, excepted x itself. For this, let γ be a stopping prefix of $\mathcal{E}^u$, and assume that γ contains an event $e \in x$. We set the following subset of $\mathcal{E}$:

$$B = \gamma \cap x.$$

Then B can be written: $B = (u \cup \gamma) \cap x$, which shows that B is a prefix of $\mathcal{E}$. We show that B is $\#_\iota$-closed in E. Let $f \in B$, and let $f' \in E$ such that $f \#_\iota f'$. Then $f' \in x$ since x is $\#_\iota$-closed. Since $x \subseteq \mathcal{E}^u$, it follows that $f' \in \mathcal{E}^u$. Using Lemma 4, we get that f and f' are in minimal conflict in event structure $\mathcal{E}^u$. Since γ is $\#_\iota^v$-closed, it implies that $f' \in \gamma$, and finally $f' \in \gamma \cap x = B$. Hence B is a stopping prefix of $\mathcal{E}$. As B contains event e, and since x is initial in $\mathcal{E}$, it follows that $B \supseteq x$, and therefore $\gamma \supseteq x$. This shows that x is initial in $\mathcal{E}^u$. ◇

Lemma 12. *(First exchange lemma) Let v_0 be a finite recursively stopped configuration of $\mathcal{E}$, let ζ be a germ of $\mathcal{E}^{v_0}$, and let ξ be a germ of $\mathcal{E}$. We assume that $v_0 \cup \zeta$ and ξ are compatible. We set:*

$$v = v_0 \cup \xi, \quad v' = v \cup \zeta, \quad \zeta' = v' \setminus v\,.$$

Then ζ' is stopped in $\mathcal{E}^v$.

Proof. Let x denote the initial stopping prefix such that $\xi \in \Omega_x$. We distinguish two cases.

First case: $v_0 \cap \xi \neq \emptyset$. Then $v_0 \cap x \neq \emptyset$, and by Lemma 9, it implies that $v_0 \cap x$ is maximal in x. Hence $v_0 \cap x$ and ξ are two compatible maximal configurations of x, so they coincide. Therefore, $\xi \subseteq v_0$, $v = v_0$ and $\zeta' = \emptyset$: the result is trivial.

Second case: $v_0 \cap \xi = \emptyset$. We claim that it implies: $v_0 \cap x = \emptyset$. Indeed, $v_0 \cap x$ is empty or maximal in x according to Lemma 9. In the latter case, since ξ is also maximal in x, and compatible with v_0, both coincide, which contradicts $v_0 \cap \xi = \emptyset$. Hence, we have $v_0 \cap x = \emptyset$.

According to Lemma 11 applied with $u = v_0$, it implies that x is an initial stopping prefix of $\mathcal{E}^{v_0}$. Hence ζ and ξ are two compatible germs of $\mathcal{E}^{v_0}$. Let y be the initial stopping prefix of $\mathcal{E}^{v_0}$ such that $\zeta \in \Omega_y$. According to Th. 4, we either have $x = y$ or $x \cap y = \emptyset$. We examine both cases:

a) $x = y$. Since ξ and ζ are two compatible maximal configurations of x, $\zeta = \xi$. Then $\zeta' = \emptyset$, which is trivially stopped in $\mathcal{E}^v$.

b) $x \cap y = \emptyset$. In particular, $\xi \cap y = \emptyset$. According to Lemma 11 it implies that y is an initial stopping prefix of $(\mathcal{E}^{v_0})^{\xi} = \mathcal{E}^{v_0 \cup \xi} = \mathcal{E}^v$. We also have $\zeta \cap \xi = \emptyset$, whence $\zeta' = \zeta$. Therefore, $\zeta' \in \Omega_y$, i.e. ζ' is a germ of $\mathcal{E}^v$. In particular, ζ' is stopped in $\mathcal{E}^v$, what was to be shown.

Lemma 13. *(Second exchange lemma) Let u, u' be two finite recursively stopped configurations. Assume that u and u' are compatible. Then $(u \cup u') \setminus u'$ is recursively stoppped in $\mathcal{E}^{u'}$.*

Proof. Assume first that u' is a germ of $\mathcal{E}$. Let $(u_n)_{0 \leq n \leq N}$ be a germ decomposition of u—such a decomposition exists according to Lemma 10. Set $u'_0 = \emptyset$, and for each integer $1 \leq n \leq N$:

$$u'_n = u' \cup u_n, \quad z_n = u_n \setminus u_{n-1}, \quad z'_n = u'_n \setminus u'_{n-1}\,.$$

Then we have, for all integers $1 \leq n \leq N$:

$$z'_n \quad = (u_{n-1} \cup u' \cup z_n) \setminus (u_{n-1} \cup u').$$

We apply the exchange lemma 12 with $v_0 = u_{n-1}$, $\zeta = z_n$ and $\xi = u'$ to obtain that z'_n is stopped in $\mathcal{E}^{u_{n-1} \cup u'} = \mathcal{E}^{u'_{n-1}}$.

Hence, $(u'_n)_n$ is a recursively stopped decomposition of $u_N = u \cup u'$ in $\mathcal{E}$, such that $u'_n \supseteq u'$ for all $n \geq 1$. Therefore, $(u'_n \setminus u')_n$ is a recursively stopped decomposition of $(u \cup u') \setminus u'$ in $\mathcal{E}^{u'}$. This completes the proof for the case where u' is a germ of $\mathcal{E}$.

For the general case, let $(v_n)_{0 \leq n \leq K}$ be a germ decomposition of u'. Then, according to the first case, $(u \cup v'_1) \setminus v'_1$ is recursively stopped in $\mathcal{E}^{v'_1}$. Applying again the first case, $(u \cup v'_2) \setminus v'_2$ is recursively stopped in $\mathcal{E}^{v'_2}$, and so on. After K steps, we obtain that $(u \cup u') \setminus u'$ is recursively stopped in $\mathcal{E}^{u'}$. ◇

Recall the definition given in Sect. 2 of maps δ, Δ and $\bar{\Delta}$.

Lemma 14. *Let v, v' be two compatible finite recursively stopped configurations. Let $x \in \delta(v)$ and $x' \in \delta(v')$. If $x \cap x' \neq \emptyset$, then $x = x'$.*

Proof. According to Lemma 13, it is enough to show the result for $v' = \emptyset$. Assume that $x \cap x' \neq \emptyset$. According to Lemma 9, $v \cap x'$ is either empty or maximal in x'. The latter case cannot occur, since otherwise $x \subseteq \mathcal{E}^{v \cap x'}$, and then $x \cap x' = \emptyset$. Hence $v \cap x' = \emptyset$. According to Lemma 11 it implies that $x' \in \delta(v)$. x and x' are two initial stopping prefixes of $\mathcal{E}^v$ satisfying $x \cap x' \neq \emptyset$, hence $x = x'$. ◇

Lemma 15. *Let v be a finite recursively stopped configuration, and let $(v_n)_n$ be a germ decomposition of v. Then:*

$$\bar{\Delta}(v) = \bigcup_n \delta(v_n).$$

Proof. Obviously, $\bigcup_n \delta(v_n) \subseteq \bar{\Delta}(v)$. Conversely, let $x \in \bar{\Delta}(v)$: there is a finite recursively stopped configuration $w \subseteq v$ such that $x \in \delta(w)$. x is an initial stopping prefix of $\mathcal{E}^w$, and, according to Lemma 13, $v \setminus w$ is recursively stopped in $\mathcal{E}^w$. Therefore, applying Lemma 9 in event structure $\mathcal{E}^w$, $(v \setminus w) \cap x$ is either empty or maximal in x. We analyse both cases.

a) $(v \setminus w) \cap x = \emptyset$.

Then, applying Lemma 11 in event structure $\mathcal{E}^w$, x is an initial stopping prefix of $(\mathcal{E}^w)^{v \setminus w} = \mathcal{E}^v$. Therefore $x \in \delta(v) = \delta(v_N)$, where N is the length of decomposition $(v_n)_n$. Hence $x \in \bigcup_n \delta(v_n)$.

b) $(v \setminus w) \cap x \in \Omega_x$.

Let k be the greatest integer such that $v_k \cap x = \emptyset$. Then $k < N$ since $v \cap x \neq \emptyset$, and thus v_{k+1} is defined. Let y be the initial stopping prefix of $\mathcal{E}^{v_k}$ such that $v_{k+1} \setminus v_k \in \Omega_y$. Then by construction, $y \cap x \neq \emptyset$. It follows from Lemma 14 that $x = y$. Hence $x \in \bigcup_n \delta(v_n)$.

◇

Proof of Theorem 3. That $v' \setminus v$ is recursively stopped in $\mathcal{E}^v$ follows from Lemma 13. Let $(v_n)_n$ be a germ decomposition of v—given by Lemma 10, and let $(w_k)_k$ be a germ decomposition of $v' \setminus v$ in $\mathcal{E}^v$. Denote by δ^v and $\bar{\Delta}^v$ the maps δ and $\bar{\Delta}$ defined in event structure $\mathcal{E}^v$. Then we have, according to Lemma 15:

$$\bar{\Delta}(v) \cup \bar{\Delta}^v(v' \setminus v) = \bigcup_n \delta(v_n) \cup \bigcup_k \delta^v(w_k)\,. \tag{24}$$

On the one hand, the concatenation of $(v_n)_n$ and $(v \cup w_k)_k$ is a germ decomposition of v'. On the other hand, we have obviously for any k: $\delta^v(w_k) = \delta(v \cup w_k)$. Applying again Lemma 15, we get thus:

$$\bar{\Delta}(v') = \bigcup_n \delta(v_n) \cup \bigcup_k \delta^v(w_k)\,. \tag{25}$$

Equations (24) and (25) together yield: $\bar{\Delta}(v') = \bar{\Delta}(v) \cup \bar{\Delta}^v(v' \setminus v)$. Observe that an element $x \in \bar{\Delta}(v')$ satisfies: $x \in \Delta(v') \Leftrightarrow x \cap v' \neq \emptyset$, and similarly for $\bar{\Delta}(v)$ and $\bar{\Delta}^v(v' \setminus v)$. Therefore we obtain:

$$\Delta(v') = \Delta(v) \cup \Delta^v(v' \setminus v)\,.$$

Since $v' \setminus v$ is a recursively stopped configuration of $\mathcal{E}^v$, the branching cells in $\Delta^v(v' \setminus v)$ are subsets of $\mathcal{E}^v$, from which follows:

$$\Delta(v) \cap \Delta^v(v' \setminus v) = \emptyset\,.$$

◇

Proof of Theorem 4. Let v be a finite recursively stopped configuration. Let $(v_n)_{0\leq n\leq N}$ be a germ decomposition of v—given by Lemma 10. For each $0 \leq n \leq N-1$ let $x_n \in \delta(v_n)$ such that $v_{n+1} \setminus v_n \in \Omega_{x_n}$. We claim that $v \cap x_n = (v_{n+1} \setminus v_n) \cap x_n$ for each $0 \leq n \leq N-1$. Indeed, x_n is an initial stopping prefix of v_n. Moreover, applying Lemma 13, $v \setminus v_n$ is recursively stopped in $\mathcal{E}^{v_n}$ and $(v \setminus v_n) \cap x \neq \emptyset$. According to Lemma 9, it implies that $v \setminus v_n$ is maximal in x. Therefore $(v \setminus v_n) \cap x$ and $(v_{n+1} \setminus v_n) \cap x$ are compatible and maximal in x, so they coincide, prooving the above claim. We have thus:

$$\begin{aligned} v &= v_N \\ &= (v_1 \setminus v_0) \cup (v_2 \setminus v_1) \cup \ldots \cup (v_N \setminus v_{N-1}) \\ &= \bigcup_{n=1}^{N-1} v \cap x_n \\ &\subseteq \bigcup_{x \in \bar{\Delta}(v)} v \cap x \,. \end{aligned} \tag{26}$$

For each $y \in \delta(v)$, we have $y \cap v = \emptyset$. Therefore:

$$\bigcup_{x \in \Delta(v)} x \cap v = \bigcup_{x \in \bar{\Delta}(v)} x \cap v \,. \tag{27}$$

Equations (26) and (27) together imply:

$$v \subseteq \bigcup_{x \in \Delta(v)} x \cap v \,,$$

and the converse inclusion is obvious, whence the equality that was to be shown. We have seen that for every $x \in \Delta(v)$, there is a n such that $x = x_n$, and then $v \cap x \in \Omega_x$. This completes the proof. ◇

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Éditeur
INRIA, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)
http://www.inria.fr
ISSN 0249-6399