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Abstract: Blurred (previously named fuzzy) segments were introduced by Debled-Rennesson et al as an extension of the arithmetical approach of Reveillès on discrete lines, to take into account noise in digital images. An incremental linear-time algorithm was presented to decompose a discrete curve into blurred segments with order bounded by a parameter d. However, it fails to segment discrete curves into a minimal number of blurred segments. We identify in this paper, that this characteristic is intrinsic to the whole class of blurred segments. We thus introduce a subclass of blurred segments, based on a geometric measure of thickness. We provide a new convex hull based incremental linear time algorithm for segmenting discrete curves into minimally thin blurred segments.

Key-words: Segmentation, Blurred Segment, Convex Hull, Discrete Line, Noisy Curve

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Décomposition en segments flous optimaux en temps linéaire

Résumé : Les segments flous ont été introduits par Isabelle Debled-Rennesson et al comme extension de l'approche arithmétique des droites discrètes proposée par Reveillès, afin de tenir compte du bruit dans les images discrètes. Un algorithme incrémental et linéaire de décomposition d'une courbe discrète en segments flous d'ordre borné par un paramètre d a été présenté, mais cet algorithme ne segmente pas toujours la courbe en un nombre minimal de segments flous. Nous montrons dans ce rapport que cette caractéristique est intrinsèque à la classe complète des segments flous et nous introduisons une sous-classe des segments flous, basée sur sur une mesure géométrique de l'épaisseur. Nous présentons un nouvel algorithme incrémental et linéaire, utilisant l'enveloppe convexe, qui permet de segmenter les courbes discrètes en segments flous d'épaisseur minimale et donc en un nombre minimum de segments flous d'épaisseur bornée.

Mots-clés : Segmentation, Segment flou, Enveloppe convexe, Droite discrète, Courbe bruitée

1 Introduction

Discrete (also called digital) segments are well known objects which have been thoroughly studied for more than 30 years [12]. There are many definitions of discrete segments, all equivalent for 8-connected discrete sets and 4-connected discrete sets. Discrete segments serve as building blocks for representation [16], decomposition [4] or analysis of discrete curves and more generally shapes. For instance, polygonalizations of discrete curves are widely used in shape representation [16] and can be computed in linear time [14, 3, 7]. Moreover, the use of discrete segments permits a perfect representation or reconstruction of discrete curves. However, this might result in complicated representations when discrete curves include noise or have been distorted by an acquisition process. Many polygonal approximation methods have been proposed throughout the years using different approachs [9, 13, 15, 6]. To deal with noise and as an extension of the result presented in [3], the notion of fuzzy segments was introduced in [1, 2]. And from now on, we shall name these segments blurred segments rather than fuzzy segments in order to prevent any confusion with fuzzy logic and fuzzy geometry. The theorem of Debled-Rennesson and Reveillès [3] provides an incremental algorithm with linear-time complexity for the recognition of discrete segments using arithmetical properties and has been extended for blurred segments by Debled-Rennesson et al [1]. However, blurred segments represent supersets of the original discrete data and thus, the result obtained with the previous theorem can not be guaranteed to be optimal, in the sense that the orders of the blurred segments are not necessarily minimal. We present in this paper a study of the order of blurred segments and identify a difficulty in the minimization of the order of a blurred segment inside the recognition process. Hence, we present theoretical arguments to justify a restriction in the class of blurred segments in order to guarantee optimality in the recognition process. Moreover, our approach can deal with disconnected sets which was impossible with the theorem given in [1].

The paper is organized as follows. In section 2 we recall definitions and properties used in [1] to segment discrete curves into blurred segments. We present in section 2.3, a problem in the minimization of the order of recognized blurred segments. This problem is completely identified in theorem 4 and leads to the introduction of a subclass of blurred segments by adding a geometric characterization of blurred segments based on convex hulls. A recognition algorithm of blurred segments is described in section 3 by the way of a study of their equivalent characterizations in terms of convex hulls. An incremental linear-time recognition algorithm is presented which guarantees that the computed blurred segments are the thinnest possible ones. Experiments are given in section 4 to show the quality of the decomposition of the proposed algorithm. We end the paper with some conclusions and perspectives in section 5.

2 Blurred Segments

2.1 Definitions

The notion of blurred (also called fuzzy) segments relies on the arithmetical definition of discrete lines [11] where a line, whose slope is $\frac{a}{b}$, lower bound μ and thickness ω (with a, b, μ and ω being integer such that $gcd(a,b) = 1$) is the set of integer points (x, y) verifying $\mu \leq ax - by < \mu + \omega$. Such a line is denoted by $\mathcal{D}(a, b, \mu, \omega)$.

The real lines $ax - by = \mu + \omega - 1$ and $ax - by = \mu$ are respectively named the ***upper and lower leaning lines*** of $\mathcal{D}(a, b, \mu, \omega)$ [3]. The integer points (x_L, y_L) (resp. (x_U, y_U)) of the lower (resp. upper) leaning lines of $\mathcal{D}(a, b, \mu, \omega)$ are called the lower (resp. upper) leaning points of $\mathcal{D}(a, b, \mu, \omega)$. We refer to Fig. 1 for a descriptive example of those notions.

In the following, we restrict our study to points of the first octant of $\mathbb{Z}^2$, due to symmetries with respect to Ox, Oy and the real line $x = y$. We thus always have $0 \leq y \leq x$. This hypothesis can be done without loss of generality and simplifies the notations, proofs and definitions.

Definition 1 *[1] A set $\mathcal{S}_b$ of consecutive points ($|\mathcal{S}_b| \geq 2$) of an 8-connected curve is a* ***blurred segment with order d*** *if there is a discrete line $\mathcal{D}(a, b, \mu, \omega)$ such that all points of $\mathcal{S}_b$ belong to $\mathcal{D}$ and $\frac{\omega}{max(|a|,|b|)} \leq d$.*

The line $\mathcal{D}$ is said to be a ***bounding*** *line for $\mathcal{S}_b$.*

The notion of order of a blurred segment has been introduced to differentiate the thickness of bounding lines of $\mathcal{S}$, since any sufficiently thick discrete line can contain $\mathcal{S}$. Thus, two discrete lines containing $\mathcal{S}$ can be compared with respect to their orders and this gives a classification on the set of lines containing $\mathcal{S}$.

In order to be reasonably close to the points of $\mathcal{S}$, it is necessary to introduce more restrictive conditions onto the discrete lines containing $\mathcal{S}$. This leads to the notion of ***strictly*** bounding lines.

Definition 2 *[1] Let $\mathcal{S}_b$ be a blurred segment with order d whose abscissa interval is $[0, l-1]$ and let $\mathcal{D}(a, b, \mu, \omega)$ be a bounding line of $\mathcal{S}_b$. $\mathcal{D}$ is named* ***strictly bounding for*** *$\mathcal{S}_b$ if $\mathcal{D}$ possesses at least three leaning points in the interval $[0, l-1]$ and, $\mathcal{S}_b$ contains at least one lower leaning point and one upper leaning point of $\mathcal{D}$.*

In Fig. 1, $\mathcal{D}(1, 4, -4, 8)$ is a strictly bounding line of the blurred segment S_b. The leaning points of $\mathcal{D}$ are the points $(4k+3, k)$ and $(4k, k+1)$, for $k \in [0, 3]$ and S_b contains the lower leaning point (3,0) and the upper leaning point (12,4).

2.2 Segmentation algorithm into blurred segments with strictly bounding lines

We briefly recall in this paragraph the technique used in [1] to segment a discrete curve into order d blurred segments with strictly bounding lines. This segmentation relies on

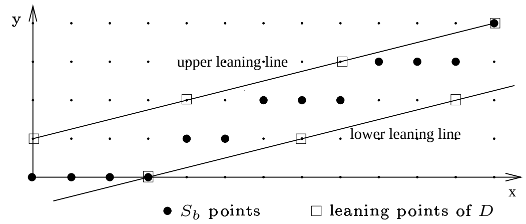

Figure 1: A strictly bounding line D of a blurred segment S_b

the following theorem, which studies the different possible cases of the growth of a blurred segment.

Theorem 3 *[1] Let us consider a blurred segment $\mathcal{S}_b$ in the first octant whose abscissa interval is $[0, l-1]$ and $\mathcal{D}(a, b, \mu, \omega)$, a strictly bounding line. In this case, the order of $\mathcal{S}_b$ is $\frac{\omega}{b}$. Let $M(x_M, y_M)$ be an integer point connected to $\mathcal{S}_b$ whose abscissa is equal to l or $l-1$. We call* ***remainder*** *at M, as a function of $\mathcal{D}$, noted $r(M)$ and defined by* $\mathbf{r(M) = ax_M - by_M}$.

(i) *If $\mu \leq r(M) < \mu + \omega$, then $M \in \mathcal{D}$;*
$\mathcal{S}_b \cup M$ is a blurred segment with order $\frac{\omega}{b}$ and $\mathcal{D}$ as strictly bounding line.

(ii) *If $r(M) \leq \mu - 1$, then M is exterior to $\mathcal{D}$;*
$\mathcal{S}_b \cup M$ is a blurred segment with order $\frac{\omega'}{b'}$ and the line $\mathcal{D}'(a', b', \mu', \omega')$ is strictly bounding, with

- *b' and a' coordinates of the vector $\overrightarrow{P_{r(M)+1}M}$, $P_{r(M)+1}$ being the point whose remainder is $r(M)+1$ with regard to $\mathcal{D}$ and $x_{P_{r(M)+1}} \in [0, b-1]$,*
- *$\mu' = a'x_M - b'y_M$*
- *$\omega' = a'x_{L_L} - b'y_{L_L} - \mu' + 1$, with $L_L(x_{L_L}, y_{L_L})$ last lower leaning point of the line $\mathcal{D}$ present in $\mathcal{S}_b$.*

(iii) *If $r(M) \geq \mu + \omega$, then M is exterior to $\mathcal{D}$;*
$\mathcal{S}_b \cup \{M\}$ is a blurred segment with order $\frac{\omega'}{b'}$ and the line $\mathcal{D}'(a', b', \mu', \omega')$ is strictly bounding with

- *b' and a' coordinates of the vector $\overrightarrow{P_{r(M)-1}M}$, $P_{r(M)-1}$ being the point whose remainder is $r(M)-1$ with regard to $\mathcal{D}$ and $x_{P_{r(M)-1}} \in [0, b-1]$,*
- *$\mu' = a'x_{U_L} - b'y_{U_L}$ with $U_L(x_{U_L}, y_{U_L})$ last upper leaning point of the line $\mathcal{D}$ present in $\mathcal{S}_b$,*
- *$\omega' = a'x_M - b'y_M - \mu' + 1$.*

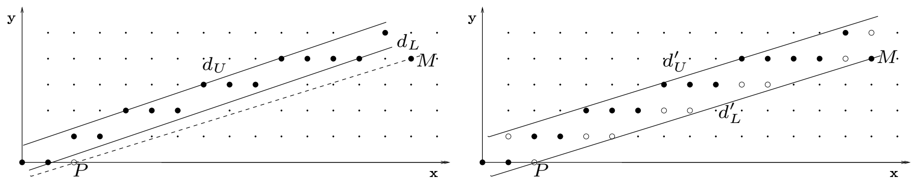

Figure 2: An example of blurred segment growth relying on Theorem 3.

An example of application of this theorem is given in Fig. 2. A blurred segment $\mathcal{S}_b$ with order 2 is depicted in Fig. 2 (left). $\mathcal{D}(1, 3, -2, 4)$ is strictly bounding for $\mathcal{S}_b$, d_U and d_L are the leaning lines of $\mathcal{D}$. The point $M(15, 4)$ is added to $\mathcal{S}_b$. Since $r_{\mathcal{D}}(M) = 3$, adding M to $\mathcal{S}_b$ corresponds to the case (iii) of the theorem: P is the point in $[0, 2]$ such that $r_{\mathcal{D}}(P) = 2$, the slope of $\mathcal{D}'$ is computed with the vector PM, therefore $\mathcal{D}'(4, 13, -12, 21)$ is strictly bounding for $\mathcal{S}_b \cup \{M\}$. On right of Fig. 2, a representation of $\mathcal{D}'$ and $\mathcal{S}_b \cup \{M\}$ (black points) is given. The points of $\mathcal{D}'$ which do not belong to $\mathcal{S}_b \cup \{M\}$ are in white, d'_U and d'_L are the leaning lines of $\mathcal{D}'$.

Thanks to this theorem, a linear time incremental algorithm of segmentation into order d blurred segments is obtained in [1, 2]; the curve $\mathcal{C}$ is incrementally scanned, each point is tested. The principle is as follows: let $\mathcal{S}_b$ be the current order d blurred segment, a point M of $\mathcal{C}$ is added to $\mathcal{S}_b$, the characteristics of a strictly bounding line of $\mathcal{S}_b \cup M$ are computed according to Theorem 3. The current segment includes the point M if the value of the obtained ratio $\omega/max(|a|, |b|)$ is lower or equal to the order d. Else, the current order d blurred segment ends at the point located before M in $\mathcal{C}$ and a new order d blurred segment starts at M.

2.3 Main drawback

Theorem 3 describes an incremental method to construct a strictly bounding line. However, the order of the associated blurred segments can not be controlled in any circumstances. Hence, the minimal ratio $w/\max(|a|, |b|)$ is sometimes not obtained. Thus given a threshold order d, the segmentation algorithm might segment a discrete curve into too many blurred segments. For instance, the curve depicted in Fig. 3 (left) might be uncorrectly segmented depending on the limiting order d chozen for the algorithm. It is uncorrectedly segmented since for $d = 1.9$, the set is decomposed into several parts but there exists a bounding line with ratio strictly lower than 1.9. Two elements might be incriminated for this fact: the algorithm and the measure. We have decided to study the measure after experimental investigations.

In order to understand the meaning of the ratio $w/\max(|a|, |b|)$, we decided to study it in the first octant for a not correctly segmented discrete curve (see Fig. 3). For any a and b values, with $a \leq b$, we have computed the remainder $r(M)$. The value of ω was deduced simply by adding 1 to the difference between the maximum and the minimum of

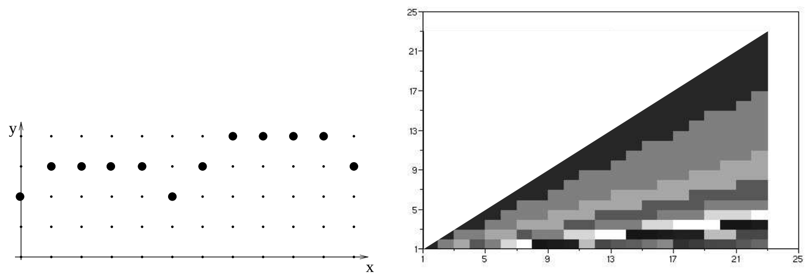

Figure 3: (left) a sample discrete curve (right) gray scale plot of ω/b with b in x-axis and a in y-axis

the remainders. We denote by $\omega(a,b)$ this value. The result is depicted on Fig. 3 (right) where light grey values correspond to low values of the ratio ω/b.

a	b	$\omega(a,b)/b$	a	b	$\omega(a,b)/b$	a	b	$\omega(a,b)/b$
1	5	2.2	3	17	1.82	4	22	1.86
1	6	1.83	3	18	1.72	4	23	1.78
1	7	1.85	3	19	1.74	4	24	1.70

Table 1: ω/b values for some couple (a,b)

As shown in Fig. 3 (right), the function $\omega(a,b)/b$ has a lot of local extrema. The minimalization of $\omega(a,b)/b$ seems to be a hard combinatorial problem. To get a precise idea, let us consider some example values given in Table 1. Several elements can be extracted from those values. First, local minima have comparable values so finding the global minima is probably a hard task. Second, $\omega(a,b)/b$ is a measure sensitive to multiplication since $(1,6)$, $(3,18)$ and $(4,24)$ do not produce the same value. Hence, a and b must be kept relatively prime. More surprisingly, the ratio $\omega(a,b)/b$ has an asymptotic behaviour which can be sometimes controlled as described by the following theorem given for the first octant without loss of generality. This theorem explains why the minimization of the ratio might be intractable.

Theorem 4 *For any finite set $\mathcal{S}_b$, let us denote by $\mathcal{W}$ the series $(\omega(a_k,b_k)/b_k)$ where k is a positive integer, $a_k = k$ and $b_k = kb_0 + \lambda$, with b_0 and λ positive integers. Then $\mathcal{W}$ is decreasing and has a limit equals to $\frac{\omega(1,b_0)-1}{b_0}$.*

Proof
We introduce the remainder $r_{(a,b)}(M) = ax_M - by_M$. It is easy to see that $r_{(a_k,b_k)}(M) = kr_{(1,b_0)}(M) - \lambda y_M$. We now introduce $\Delta_{(a,b)}(M,M')$ as follows,

$$\Delta_{(a_k,b_k)}(M,M') = r_{(a,b)}(M) - r_{(a,b)}(M') = k\Delta_{(1,b_0)}(M,M') + \lambda\,(y_{M'} - y_M) \qquad (1)$$

We first suppose that $\Delta_{(1,b_0)}(M, M') = 0$. Hence,

$$\Delta_{(a_k,b_k)}(M, M') = \lambda\,(y_{M'} - y_M) \tag{2}$$

Since $(y_{M'} - y_M)$ is bounded on any finite set and since λ is constant, we deduce that the previous value is bounded above by a constant δ.

We now suppose that $\Delta_{(1,b_0)}(M, M') \neq 0$. By using the same boundedness argument, we see that there exists a value k_0 such that

$$\Delta_{(a_k,b_k)}(M, M') > 0 \text{ (resp. } < 0) \iff \Delta_{(1,b_0)}(M, M') > 0 \text{ (resp. } < 0) \tag{3}$$

for any $k \geq k_0$. Hence asymptoticaly, the remainders $r_{(a_k,b_k)}(.)$ and the remainders $r_{(1,b_0)}(.)$ have the same ordering. However, the remainders $r_{(a_k,b_k)}(.)$ are diverging when k tends to infinity. Such that, for sufficiently large k, the minimum and the maximum of the remainders $r_{(a_k,b_k)}(.)$ are obtained exactly for the same points $M_{\min}$ and $M_{\max}$ than for the remainders $r_{(1,b_0)}(.)$. This permits us to deduce that for sufficiently large k,

$$\omega_{(a_k,b_k)} = k\left(\omega_{(1,b_0)} - 1\right) + \lambda\,(y_{M_{\min}} - y_{M_{\max}}) + 1 \tag{4}$$

So,

$$\frac{\omega_{(a_k,b_k)}}{b_k} = \frac{k\left(\omega_{(1,b_0)} - 1\right)}{kb_0 + \lambda} + \frac{y_{M_{\min}} - y_{M_{\max}} + 1}{kb_0 + \lambda} \tag{5}$$

The limits of the previous expression is given by the limit of the first term and the limit of the first term is

$$\lim_{k \to +\infty} \frac{\omega_{(a_k,b_k)}}{b_k} = \frac{\left(\omega_{(1,b_0)} - 1\right)}{b_0} \tag{6}$$

We conclude the proof by a study of a specific case obtained when the remainders $r(1, b_0)$ are all equals. In such a case, $\omega(1, b_0) = 1$ and $\omega(a_k, b_k) = 1 + \lambda\,(\max_M y_M - \min_M y_M)$. Thus by dividing by b_k and taking the limit, we obtain 0. So the result still hold and this concludes the proof. □

So in order to have an optimal algorithm, we slightly modify the subclass of considered blurred segments and based on this modification, we propose an optimal segmentation algorithm. The modification consists in taking the limit measure of the previous theorem as a measure for comparing blurred segments. We start by giving a geometric description of the limit measure.

The ***vertical distance of a discrete line*** $\mathcal{D}(a, b, \mu, \omega)$ is the vertical distance (ordinate difference) between the leaning lines of $\mathcal{D}$ and is equal to $\frac{\omega-1}{b}$. We now recall the notion of ***supporting lines*** [5]. A ***supporting line*** of a convex set C is a line l such that the intersection of l with C is not empty and such that C is entirely either below or above l. The ***vertical distance of a convex set*** C, is the minimal vertical distance of any pair of parallel supporting lines.

Definition 5 *Let us consider a set of 8-connected points $\mathcal{S}_b$. A bounding line of $\mathcal{S}_b$ is said* ***optimal*** *if its vertical distance is minimal, i.e. if its vertical distance is equal to the vertical distance of $conv(\mathcal{S}_b)$ the convex hull of $\mathcal{S}_b$.*

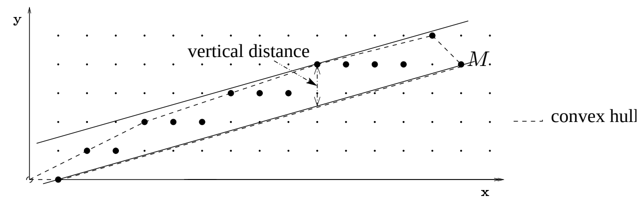

Figure 4: An optimal bounding line

This definition is illustrated in Fig. 4 and leads to the following new definition concerning blurred segments.

Definition 6 *A set $\mathcal{S}_b$ is a* ***blurred segment of width*** *v if and only if its optimal bounding line has a vertical distance less or equal to v.*

The recognition of blurred segments with width v is thus equivalent to the computation of the vertical distance of the convex set $conv(\mathcal{S}_b)$.

3 Convex Hulls and Blurred Segments

3.1 Characterization of the vertical distance of a finite convex set

To be precise, the vertical distance of a finite convex set C is the maximal length of the intersection of C with a vertical line. It is clear that by convexity, such intersections are vertical segments, so we only have to characterize the positions of extremality. But, let us denote by L_i and U_i respectively the lowest and highest intersection point of C with the vertical line given by $x = i$ and by $\mathcal{V}(i)$ the distance between L_i and U_i, then for any $i \neq j$, the quadrangle $(L_jU_jU_iL_i)$ is located inside C. So it is straightforward to see that the function $\mathcal{V}(.)$ is a concave function. Hence, every local extremum of the function is also a global extremum of the function. The following proposition characterizes local extremality of the function $\mathcal{V}(.)$.

Proposition 7 *For any finite convex set C, the function $\mathcal{V}(.)$ has an extremum value at a position i where L_i or U_i can be chosen to be a vertex of the border, Bd C, of C.*

Proof Let us consider a position i such that neither L_i nor U_i are vertices of Bd C. We consider the edges of Bd C containing L_i and U_i and we denote by $=$, $+$ or $-$ the fact that the slopes of the edges are either 0, strictly positive or strictly negative. There are 9 cases to consider which are reduced by symmetries to only 4 cases:

- $(=,=)$: it is clear that a move on the edges does not change the value of $\mathcal{V}(.)$ hence it is sufficient to move until one of the two points becomes a vertex of Bd C.

- $(+,-)$: by moving backwards $\mathcal{V}(.)$ increases.
- $(+,+)$: by comparing the slopes, it is easy to decide whether we have to move forward or backward to get an increase in the value of $\mathcal{V}(.)$.
- $(+,=)$: by moving backwards $\mathcal{V}(.)$ increases.

In the four cases, the position is not a local extremum and thus not a global extremum. □

From this proposition, we can deduce two facts if we are looking for the vertical distance of a convex set: first, we only have to consider points on the border of C and second, extremum values are obtained for some x positions of the vertices of Bd C. This leads to three cases: either we have a couple (edge,vertex), or (vertex,edge) or (vertex,vertex) where the first element represents the lower part of the convex set and the second element represents the upper part. It must be noticed that the previous proposition is identical to the one characterizing the width of a convex set [10].

It is clear that the edges corresponding to the position of extremum of $\mathcal{V}(.)$ are supporting lines, taking horizontal lines for a couple (vertex,vertex). Moreover, the edge and its parallel passing through the vertex define exactly the lower and upper leaning lines of an optimal bounding line for C. Hence, optimal lines are deduced from the positions of optimality.

3.2 New recognition algorithm

At this point, we consider a set of points $\mathcal{S}_b = \{(x_i, y_i),\ 0 \leq i < n\}$, a blurred segment in the first octant with $\mathcal{D}(a, b, \mu, \omega)$ as optimal bounding line. The vertical distance of the convex hull of $\mathcal{S}_b$, $conv(\mathcal{S}_b)$, is equal to the vertical distance of $\mathcal{D}$. We suppose that $\mathcal{S}_b$ contains two upper leaning points, U_F and U_L, and one lower leaning point, L_L. U_F, U_L and L_L are vertices of $conv(\mathcal{S}_b)$. Moreover, the vertical distance of $\mathcal{D}$ can be calculated at the point L_L. To compute and maintain convex hulls, we use Melkman's algorithm [8]. Let us recall that its algorithm incrementally computes the convex hull of n points forming a simple polygonal line in time $O(n)$ based on a double ended queue (deque) list. Since points are added with increasing x, we are guaranteed to have a simple polygonal line.

Suppose that we add a new point M to $\mathcal{S}_b$, $\mathcal{S}'_b = \mathcal{S}_b \cup \{M\}$. There are three cases (see Fig. 5).

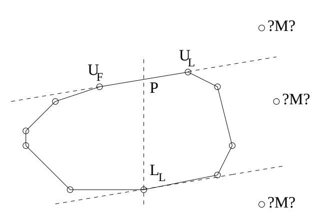

Figure 5: The three cases when adding a point

If M belongs to $\mathcal{D}$ then, after the application of Melkman algorithm [8] the vertical distance remains the same. So the new set of points remains a blurred segment with the same width and with the same optimal bounding line $\mathcal{D}$. In the other cases, M is above or below $\mathcal{D}$, the convex set is modified and so the vertical distance must be recomputed.

We first study the case where P, intersection point between the vertical line from L_L and $U_F U_L$, is strictly inside $[U_F U_L]$.

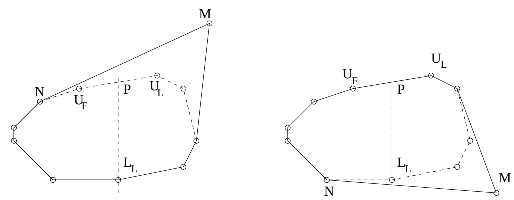

Figure 6: Adding a new point either above (left) or below (right)

Suppose that M is added above $\mathcal{D}$. Let us apply Melkman algorithm. The convex set if modified and we call N the point before M in the upper part of the resulting convex hull, see Fig. 6 (left). N is necessarily before U_F or is U_F. As a consequence, the vertical projection of L_L is inside $[NM]$. So, the vertical distance of the new convex set strictly increases. The key point is thus to locate the new position of extremality. It is clear that N cannot represent a position of extremality since $\mathcal{V}(N) \leq \mathcal{V}(L_L)$. Moreover, M does not project vertically strictly inside an edge of the lower part of $conv(\mathcal{S}'_b)$. Hence, the new position of extremality if necessarily obtained for one point at the right of L_L in the lower part of $conv(\mathcal{S}'_b)$. Let us recall now that local extrema are global extrema for the function $\mathcal{V}(.)$ such that we only have to test the candidate points in sequence and stops at the first local extremum, called C.

Suppose now that M is added below $\mathcal{D}$. Let us apply Melkman algorithm. The lower part of the convex set is modified and we still call N the point before M in the lower part of the new convex hull, see Fig. 6 (right). N is necessary at the left of L_L. Hence, N cannot be the new position of extremality since $\mathcal{V}(N) \leq \mathcal{V}(L_L)$. We do not know precisely where N is located in comparison with $[U_F U_L]$. It might be either on the left of U_F or inside $[U_F U_L]$. In both cases, it is however straightforward to see that no position strictly at the left of U_L can be a position of extremality. As in the previous case, neither N nor M can be positions of extremality. So, the new position of extremality is given by one point situated at the right of U_L on the upper part of the new convex hull and we only have to test the candidate points in sequence and stops at the first local extremum, called C.

In these both cases, $\mathcal{S}'_b$ is a blurred segment with optimal bounding line $\mathcal{D}'$ for which the points M, N, and C are leaning points. Moreover, the vertical distance of $conv(\mathcal{S}'_b)$ is equal to the vertical distance of $\mathcal{D}'$ and can be calculated at the point C.

We have neglected the case where L_L and P are vertices of $conv(\mathcal{S}_b)$. This is simply because this case is equivalent to the other ones. Indeed if we choose any edge containing either L_L or P and we keep the other point as a vertex, we obtain a couple (vertex,edge) or (edge,vertex) with the same vertical distance. This result is of course different from the one of the width of a convex set [5], but applies perfectly in our context.

Thus, we have obtained the following theorem where points formed a simple polygonal chain and are taken in increasing x ordering.

Theorem 8 *There is an incremental algorithm which maintains the convex hull of a finite set of points as well as the position of the vertical distance or, equivalently, there is an incremental algorithm which computes an optimal bounding line of a finite set of points.*

Hereafter, to resume the incremental process described in the proof of the theorem, a recognition algorithm of width ν blurred segments is presented (see Algorithm 1) which gives as result a boolean value equal to true if a sequence of points S (input) is a width ν blurred segment. Morever the last calculated values of a, b, μ and ω are the characteristics of an optimal bounding line of S.

Algorithm 1: Incremental recognition of blurred segment with width ν

Input : S an 8-connected sequence of integer points, ν a real value
Output : $isSegment$ a boolean value, a, b, μ, ω integers
Initialization: $isSegment = true$, $a = 0$, $b = 1$, $\omega = b$, $\mu = 0$, M = first point of S.
while *S is not entirely scanned and $isSegment$* **do**
- M = next point of S;
- add M to the upper and lower convex hulls of the scanned part of S;
- $r = ax_M - by_M$;
- **if** $r = \mu$ **then** $U_L = M$;
- **if** $r = \mu + \omega - 1$ **then** $L_L = M$;
- **if** $r \leq \mu - 1$ **then**
  - $U_L = M$;
  - Let N the point before M in the upper convex hull, $a_0 = y_M - y_N$, $b_0 = x_M - x_N$, then $a = \frac{a_0}{gcd(a_0,b_0)}$, $b = \frac{b_0}{gcd(a_0,b_0)}$, $\mu = ax_M - by_M$;
  - Find the first point C in the lower part of the convex hull starting at L_L, such that : slope of $[C, Cnext] > \frac{a}{b}$;
  - $L_L = C$;
- **else**
  - **if** $r \geq \mu + \omega - 1$ **then** *symmetrical case*
- **end**
- $isSegment = \frac{\omega-1}{b} \leq \nu$;

end

We now study the complexity of the application of the algorithm 1 on a set $\mathcal{S}$ of n points. The first part of the algorithm is the update of the convex hull after the insertion of the new point M. However, Melkman's algorithm [8] has a linear-time complexity for the whole set of insertions. Since, a point is used only for one convex hull, we deduce easily that this part still have a linear-time complexity. The second part of the algorithm concerns the updates of the positions of extremality. Each time a point is added, part of the previous convex hull is examined to detect the new position of extremality. However, the main property of the function $\mathcal{V}(.)$ is that it is a concave function. Thus as previously mentionned, we only test points for which $\mathcal{V}(.)$ is not locally extremal and stops at the first local extremum. So we obtain that in the previous algorithm, this part also has a linear-time complexity.

This implies that when using the new algorithm instead of the old one in the decomposition algorithm of discrete curves, the whole complexity does not change and hence, the new decomposition can also be done incrementally in linear-time along the whole curve.

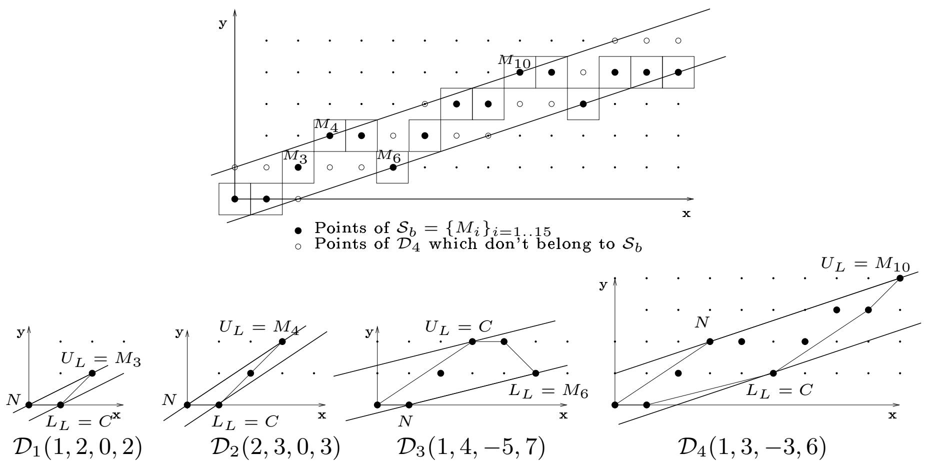

Figure 7: An example of width 2 blurred segment incremental recognition

An example of the algorithm processing is depicted in Fig. 7. Part of a discrete curve, $\mathcal{S}_b$, is drawn at the top of the figure and the different optimal bounding lines, obtained during the incremental recognition, are given. As it can be seen, there are only four steps in the recognition process and the slopes of the supporting lines decrease or increase with respect to the added points. $\mathcal{S}_b$ is a blurred segment of width 2 with $\mathcal{D}_4(1,3,-3,6)$ as optimal bounding line.

4 Experiments

The segmentation of a curve into blurred segments of width ν is done incrementally as described in [1, 2], and as recalled in section 2.2. When the width of the current segment becomes strictly greater than ν, a blurred segment ends at the previous point, and a new segment is initialised. In Fig. 8 and 9, we show how the curves on the left are segmented into blurred segments with width 2. The curve in Fig. 8 is segmented into 18 segments. The curve in Fig. 9 is a noisy version of the one in Fig. 8, so the segmentation gives more segments: 21. If we increase the width of the blurred segments, the number of segments decreases: 19 segments for the width 2.5, 18 segments for the width 3.

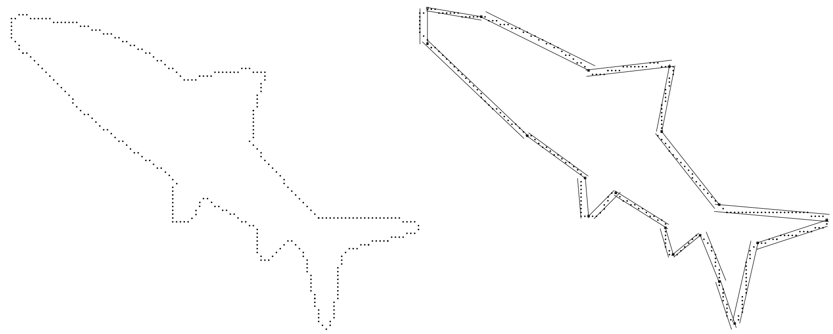

Figure 8: Decomposition of a curve into blurred segments of width 2

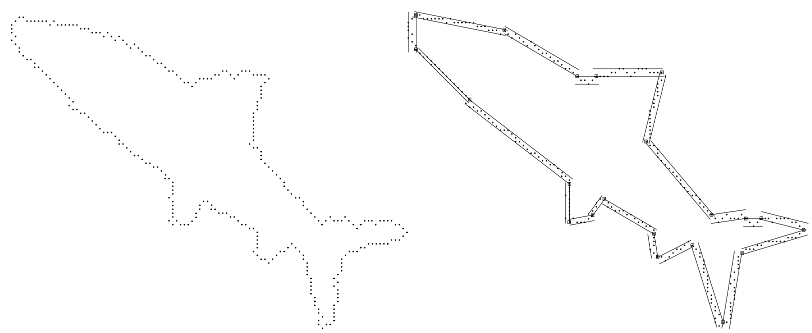

Figure 9: Shark curves with noise still decomposed into blurred segments of width 2

5 Conclusion

We have pursued in this paper the study of blurred segments and their applications to discrete noisy curves. Previous works of Debled-Rennesson et al [1] were not able to decompose discrete noisy curves with minimally thin blurred segments. We have identified the origin of this defect and based on our theoretical study, we have proposed to restrict the class of blurred segments by adding a geometrical bound on their thickness. The good measure to control the thickness of blurred segments bounded by a discrete line $\mathcal{D}(a, b, \mu, \omega)$ was demonstrated to be $(\omega - 1)/b$. Based on this modification, the recognition of blurred segments was shown to be equivalent to the computation of the vertical distance of the convex set of points of the discrete curves. We have presented an incremental linear time algorithm which solves this problem with minimality in the thickness of constructed blurred segments. Moreover, our approach also applies to disconnected sets which opens perspectives in the study of discrete curves with holes. The tools we used can be extended to 3D and this might lead to decompositions of discrete surfaces in blurred linear patches.

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