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Abstract: A new stabilized finite element method is introduced for the linearized version of the Navier-Stokes equation (or Oseen equation), containing a dominating zeroth order term. The method consists in subtracting a mesh dependent term from the formulation without compromising consistency. The design of this mesh dependent term, as well as the stabilization parameter involved, are suggested by bubble condensation. Stability is proved for any combination of velocity and pressure spaces, under the hypotheses of continuity for the pressure space. Optimal order error estimates are derived for the velocity and the pressure, using the standard norms for these unknowns. Numerical experiments confirming these theoretical results are presented.

Key-words: Oseen equation, reaction term, stabilized method

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Une méthode d'éléments finis stabilisée pour l'équation d'Oseen avec réaction dominante

Résumé : On introduit une nouvelle méthode d'éléments finis stabilisée pour l'équation d'Oseen (Navier-Stokes linéarisée) avec un terme d'ordre zéro dominant. La méthode est obtenue après soustraction d'un terme dépendant du maillage à la formulation de Galerkin classique, sans compromettre la consistance de la méthode. L'expression de ce terme et celle du paramètre de stabilisation associé sont déterminées par une technique de type *condensation de la bulle*. La stabilité de la méthode est assurée pour tout couple d'espaces discrets, sous une hypothèse de continuité des pressions discrètes. On fourni des estimations d'erreur optimales pour la vitesse et la pression dans ses normes naturelles. Des expériences numériques confirment les résultats théoriques.

Mots-clés : Équation d'Oseen, terme de réaction, méthode stabilisée

1 Introduction

The Navier-Stokes equation constitutes a major challenge in applied mathematics. Specifically, its numerical solution presents two major difficulties, namely, the need for a compatibility condition (the inf-sup condition, see [17, 3]) relating the discrete spaces used to approximate the velocity field $\mathbf{u}$ and the pressure p, and the treatment of the spurious modes generated by the convective term. For both these aspects, several solutions have been proposed in the last two decades. The convective terms have been treated by appropriate upwinding strategies (cf. [17, 21, 11], and the references therein), or stabilized finite element methods (cf. [6, 14], among others). On the other hand, the inf-sup condition may be treated directly (cf. [3, 17], and the references therein), or circumvented via stabilized finite element methods (cf. [20, 19, 5, 4, 15] in the context of a Stokes flow).

On the other hand, if we are dealing with the time discretization of the Navier-Stokes equation, and we choose the "classical" approach (i.e., discretizing in time by time-advancing finite differences) we have different choices for the scheme (for a resume of these techniques, see [21]). A common fact of all these techniques is the presence of a zeroth order term of type $\frac{1}{\Delta t}\mathbf{u}$, where Δt is the time step (usually very small), and $\mathbf{u}$ is the unknown velocity field. In the late nineties, several works concerning stabilization procedures for problems with zeroth order terms (or reaction terms), were proposed (see, e.g., [18, 24], and the recent paper [7], where edge stabilization has been proposed for a scalar convection-diffusion-reaction problem). In particular, in [13, 16, 2], the connection between stabilized finite element methods and Galerkin methods enriched with bubble functions was used to derive a new family of stabilized finite element method, namely, the Unusual Stabilized Finite Element Method (USFEM), which are particularly suited for treating problems with dominating reaction.

The purpose of this work is to derive, analyze and test a new stabilized finite element method, analogous to the one presented in [2], for the Oseen problem with a dominating reaction term. The method is introduced in Section 2 where the stability of the method is proved and an error analysis is performed. The error estimates obtained in Section 2 are derived, using a suitable mesh dependent norm, for the standard norms for the unknowns, namely the $H^1(\Omega)$ norm for the velocity field, and the $L^2(\Omega)$ norm for the pressure. In Section 3 we report some numerical experiments that confirm our approximation results. Finally, somme concluding remarks are made in Section 4.

2 The finite element method

First, we present the problem of interest. Let Ω be a bounded open subset of $\mathbb{R}^2$, $\mathbf{f} \in [L^2(\Omega)]^2$, σ a positive real number (typically, $\sigma \approx \frac{1}{\Delta t}$ where Δt is the time step in a time discretization procedure), and $\mathbf{a} : \Omega \to \mathbb{R}^2$ a vectorial function such that $\nabla\cdot\mathbf{a} = 0$ in Ω (this function $\mathbf{a}$ may be interpreted as the velocity field in the previous time step). Then, our generalized Oseen equation reads: *Find* $(\mathbf{u}, p) \in [H^1_0(\Omega)]^2 \times L^2_0(\Omega)$ *such that*

$$
\begin{aligned}
\sigma\mathbf{u} - \nu\Delta\mathbf{u} + \mathbf{a}\cdot\nabla\mathbf{u} + \nabla p = \mathbf{f} &\qquad \text{in } \Omega\,, \\
\nabla\cdot\mathbf{u} = 0 &\qquad \text{in } \Omega\,, \\
\mathbf{u} = 0 &\qquad \text{on } \partial\Omega\,,
\end{aligned}
\tag{1}
$$

where $L^2_0(\Omega) \stackrel{\text{def}}{=} \{q \in L^2(\Omega) : (q,1)_\Omega = 0\}$, and $(\cdot,\cdot)_D$ denotes the L^2 inner product in $L^2(D)$ (or in $L^2(D)^2, L^2(D)^{2\times 2}$, when necessary). Also, by $\|\cdot\|_{l,D}$ and $|\cdot|_{l,D}$ we will denote the $H^l(D)$ norm and seminorm, respectively, with the usual convention $H^0(D)=L^2(D)$.

From now on, let us suppose that Ω is a polygonal domain in $\mathbb{R}^2$, and let $\mathcal{T}_h$ be a triangulation of Ω constituted by triangles (or quadrilaterals) which are shape regular. Let h_K be the usual element diameter, and denote $h \stackrel{\text{def}}{=} \max\{h_K : K \in \mathcal{T}_h\}$. We suppose from now on that $h \leq 1$. Now, for $k \geq 1$, let V_k be the space of piecewise polynomial functions given by

$$
V_k \stackrel{\text{def}}{=} \left\{ v \in \mathcal{C}^0(\overline{\Omega}) / v\big|_K \in R^k(K)\,, \forall K \in \mathcal{T}_h \right\}\,.
$$

Here, $R^k(K) = P^k(K)$ for triangular elements and $R^k(K) = \{p \circ F_K^{-1} \,/\, p \in Q^k(\hat{K})\}$ for quadrilateral elements, where F_K stands for the usual transformation mapping the reference element $\hat{K}$ onto K.

Our stabilized finite element method reads: *Find* $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ *such that:*

$$
\mathbf{B}((\mathbf{u}_h, p_h), (\mathbf{v}, q)) = \mathbf{F}(\mathbf{v}, q) \qquad \forall\, (\mathbf{v}, q) \in \mathbf{V}_h \times Q_h\,,
\tag{2}
$$

where $\mathbf{V}_h \stackrel{\text{def}}{=} [V_k \cap H^1_0(\Omega)]^2$ and $Q_h \stackrel{\text{def}}{=} V_l \cap L^2_0(\Omega)$, $k, l \geq 1$, $\mathbf{B}$ and $\mathbf{F}$ are given by

$$
\begin{aligned}
\mathbf{B}((\mathbf{u}_h, p_h), (\mathbf{v}, q)) &\stackrel{\text{def}}{=} \sigma(\mathbf{u}_h, \mathbf{v})_\Omega + \nu(\nabla\mathbf{u}_h, \nabla\mathbf{v})_\Omega + (\mathbf{a}\cdot\nabla\mathbf{u}_h, \mathbf{v})_\Omega \\
&\quad - (p_h, \nabla\cdot\mathbf{v})_\Omega + (q, \nabla\cdot\mathbf{u}_h)_\Omega + \sum_{K\in\mathcal{T}_h} (\nabla\cdot\mathbf{u}_h, \delta_K\, \nabla\cdot\mathbf{v})_K \\
&\quad - \sum_{K\in\mathcal{T}_h} (\sigma\mathbf{u}_h - \nu\Delta\mathbf{u}_h + \mathbf{a}\cdot\nabla\mathbf{u}_h + \nabla p_h, \tau_K(\sigma\mathbf{v} - \nu\Delta\mathbf{v} - \mathbf{a}\cdot\nabla\mathbf{v} - \nabla q))_K\,,
\end{aligned}
\tag{3}
$$

$$
\mathbf{F}(\mathbf{v}, q) \stackrel{\text{def}}{=} (\mathbf{f}, \mathbf{v})_\Omega - \sum_{K\in\mathcal{T}_h} (\mathbf{f}, \tau_K(\sigma\mathbf{v} - \nu\Delta\mathbf{v} - \mathbf{a}\cdot\nabla\mathbf{v} - \nabla q))_K\,.
\tag{4}
$$

Here, the stabilization parameters τ_K and δ_K are given by

$$
\tau_K \stackrel{\text{def}}{=} \frac{h_K^2}{\sigma h_K^2 \xi(Pe_K^1) + \dfrac{4\nu}{m_k}\xi(Pe_K^2)}\,,
\tag{5}
$$

$$
\delta_K \stackrel{\text{def}}{=} \lambda\, |\mathbf{a}(x)|_p\, h_K\, \min\{1, Pe_K^2\}\,,
\tag{6}
$$

where $\lambda \geq 0$, $p \in [1, +\infty]$, and

$$Pe_K^1 = \frac{4\nu}{m_k \sigma h_K^2}\,, \tag{7}$$

$$Pe_K^2 = \frac{m_k |\mathbf{a}|_p h_K}{4\nu}\,, \tag{8}$$

$$|\mathbf{a}(x)|_p \stackrel{\text{def}}{=} (|a_1|^p + |a_2|^p)^{1/p}\,, \tag{9}$$

$$m_k = \min\{\frac{1}{3}, C_k\}\,, \tag{10}$$

$$C_k h_K^2 \, \|\Delta v\|_{0,K}^2 \leq \|\nabla v\|_{0,K}^2 \qquad \forall\, v \in V_k\,, \tag{11}$$

$$\xi(\lambda) = \max\{\lambda, 1\}\,. \tag{12}$$

Remark 1 *The design of the stabilization parameter τ_K has been suggested by bubble condensation, following very closely the arguments given in [2, 1]. The least-squares parameter δ_K is the one from [14]. In the case of a generalized Stokes problem ($\mathbf{a} = \mathbf{0}$) we recover the method from [2]. Now, in the case of a pure Oseen equation $\sigma = 0$, we recover the "plus" formulation from [14] with a stabilization parameter which satisfies $\tau_{FF} \leq \tau_K \leq 2\tau_{FF}$, where τ_{FF} denotes the stabilization parameter proposed in [14], given by*

$$\tau_{FF} \stackrel{\text{def}}{=} \frac{h_K}{2|\mathbf{a}(x)|_p} \min\{1, Pe_K^2\}\,.$$

Remark 2 *In [10], the orthogonal subscales approach was applied to a related problem containing a Coriolis terms and a zeroth order term. The resulting formulation involves stabilization parameters with free constants to be set. The performance of the method depends on how these constants are chosen.*

2.1 The stability of the method

Throughout all this section (and the following one), C will denote a positive constant independent of h (but who may depend on the physical coefficients), whose value may vary whenever it is written in two different places.

The following lemma provides the positive-definiteness of the stiffness matrix associated with our method.

Lemma 2.1 *There exists a constant C_Ω, depending only on Ω, such that*

$$\mathbf{B}((\mathbf{v}, q), (\mathbf{v}, q)) \;\geq\; C_\Omega \nu \, \|\mathbf{v}\|_{1,\Omega}^2 + \sum_{K \in \mathcal{T}_h} \left\{ \|\tau_K^{1/2}(\mathbf{a} \cdot \nabla \mathbf{v} + \nabla q)\|_{0,K}^2 + \|\delta_K^{1/2} \nabla \cdot \mathbf{v}\|_{0,K}^2 \right\},$$

for all $(\mathbf{v}, q) \in \mathbf{V}_h \times Q_h$.

Proof. - From the definition of τ_K and the fact that $\xi(Pe_K^1), \xi(Pe_K^2) \geq 1$, we observe first that

$$\tau_K \leq \overline{\tau_K} \stackrel{\text{def}}{=} \frac{m_k h_K^2}{m_k \sigma h_K^2 + 4\nu},$$

for all $K \in \mathcal{T}_h$. Now, let $(\mathbf{v}, q) \in \mathbf{V}_h \times Q_h$, then

$$\begin{aligned}\mathbf{B}((\mathbf{v},q),(\mathbf{v},q)) \geq \sigma \|\mathbf{v}\|_{0,\Omega}^2 + \nu \|\nabla \mathbf{v}\|_{0,\Omega}^2 + (\mathbf{a}\cdot \nabla \mathbf{v}, \mathbf{v})_\Omega + \sum_{K\in\mathcal{T}_h} \|\delta_K^{1/2} \nabla\cdot \mathbf{v}\|_{0,K}^2 \\ - \sum_{K\in\mathcal{T}_h} \Big\{ \overline{\tau_K} \left[\sigma^2 \|\mathbf{v}\|_{0,K}^2 - 2\sigma\nu(\mathbf{v}, \Delta\mathbf{v})_K + \nu^2 \|\Delta \mathbf{v}\|_{0,K}^2\right] \\ + \|\tau_K^{1/2} (\mathbf{a}\cdot \nabla\mathbf{v} + \nabla q)\|_{0,K}^2 \Big\}.\end{aligned}$$

Now, integrating by parts we obtain $(\mathbf{a}\cdot \nabla\mathbf{v}, \mathbf{v})_\Omega = 0$, and hence

$$\begin{aligned}\mathbf{B}((\mathbf{v},q),(\mathbf{v},q)) &\geq \sum_{K\in\mathcal{T}_h} \Big\{ (\sigma - \sigma^2 \overline{\tau_K}) \|\mathbf{v}\|_{0,K}^2 + \nu \|\nabla\mathbf{v}\|_{0,K}^2 + 2\sigma\nu\overline{\tau_K} (\mathbf{v}, \Delta\mathbf{v})_K \\ &\quad - \nu^2 \overline{\tau_K} \|\Delta\mathbf{v}\|_{0,K}^2 + \|\tau_K^{1/2}(\mathbf{a}\cdot \nabla\mathbf{v} + \nabla q)\|_{0,K}^2 + \|\delta_K^{1/2} \nabla\cdot \mathbf{v}\|_{0,K}^2 \Big\} \\ &\geq \sum_{K\in\mathcal{T}_h} \Big\{ (\sigma - \sigma^2 \overline{\tau_K}) \|\mathbf{v}\|_{0,K}^2 + \nu \|\nabla\mathbf{v}\|_{0,K}^2 - 2\sigma\nu\overline{\tau_K} \|\mathbf{v}\|_{0,K} \|\Delta\mathbf{v}\|_{0,K} \\ &\quad - \frac{\nu^2 \overline{\tau_K} h_K^{-2}}{C_k} \|\nabla\mathbf{v}\|_{0,K}^2 + \|\tau_K^{1/2}(\mathbf{a}\cdot \nabla\mathbf{v} + \nabla q)\|_{0,K}^2 + \|\delta_K^{1/2} \nabla\cdot \mathbf{v}\|_{0,K}^2 \Big\} \\ &\geq \sum_{K\in\mathcal{T}_h} \left\{ (\sigma - \sigma^2 \overline{\tau_K}) \|\mathbf{v}\|_{0,K}^2 + \left(\nu - \frac{\nu^2 \overline{\tau_K} h_K^{-2}}{C_k}\right) \|\nabla\mathbf{v}\|_{0,K}^2 - \frac{\sigma\nu\overline{\tau_K}}{\gamma} \|\mathbf{v}\|_{0,K}^2 \right. \\ &\quad \left. - \gamma\sigma\nu\overline{\tau_K} \|\Delta\mathbf{v}\|_{0,K}^2 + \|\tau_K^{1/2}(\mathbf{a}\cdot \nabla\mathbf{v} + \nabla q)\|_{0,K}^2 + \|\delta_K^{1/2} \nabla\cdot \mathbf{v}\|_{0,K}^2 \right\} \\ &\geq \sum_{K\in\mathcal{T}_h} \left\{ \left(\sigma - \sigma^2 \overline{\tau_K} - \frac{\sigma\nu\overline{\tau_K}}{\gamma}\right) \|\mathbf{v}\|_{0,K}^2 \right. \\ &\quad + \left(\nu - \frac{\nu^2 \overline{\tau_K} h_K^{-2}}{C_k} - \frac{\gamma\sigma\nu\overline{\tau_K} h_K^{-2}}{C_k}\right) \|\nabla\mathbf{v}\|_{0,K}^2 \\ &\quad \left. + \|\tau_K^{1/2}(\mathbf{a}\cdot \nabla\mathbf{v} + \nabla q)\|_{0,K}^2 + \|\delta_K^{1/2} \nabla\cdot \mathbf{v}\|_{0,K}^2 \right\}\end{aligned}$$

$$
\begin{aligned}
&\geq \sum_{K\in\mathcal{T}_h} \left\{ \left(\sigma - \frac{\sigma^2 m_k h_K^2}{m_k\sigma h_K^2 + 4\nu} - \frac{\sigma\nu m_k h_K^2}{\gamma(m_k\sigma h_K^2+4\nu)} \right) \|\mathbf{v}\|_{0,K}^2 \right.\\
&\quad + \left(\nu - \frac{m_k\nu^2}{C_k(m_k\sigma h_K^2+4\nu)} - \frac{m_k\gamma\sigma\nu}{C_k(m_k\sigma h_K^2+4\nu)} \right) \|\nabla\mathbf{v}\|_{0,K}^2 \\
&\quad \left. + \|\tau_K^{1/2}(\mathbf{a}\cdot\nabla\mathbf{v}+\nabla q)\|_{0,K}^2 \; + \; \|\delta_K^{1/2}\nabla\cdot\mathbf{v}\|_{0,K}^2 \right\} \\
&\geq \sum_{K\in\mathcal{T}_h} \left\{ \frac{\sigma\nu(4\gamma - m_k h_K^2)}{\gamma(m_k\sigma h_K^2+4\nu)} \|\mathbf{v}\|_{0,K}^2 \; + \; \frac{3\nu^2 + \sigma\nu(m_k h_K^2 - \gamma)}{m_k\sigma h_K^2 + 4\nu} \|\nabla\mathbf{v}\|_{0,K}^2 \right. \\
&\quad \left. + \|\tau_K^{1/2}(\mathbf{a}\cdot\nabla\mathbf{v}+\nabla q)\|_{0,K}^2 \; + \; \|\delta_K^{1/2}\nabla\cdot\mathbf{v}\|_{0,K}^2 \right\} ,
\end{aligned}
$$

and it suffices to take $\gamma = \frac{m_k h_K^2}{4}$ and apply Poincaré's inequality to complete the proof. □

Now, in order to prove our main result in stability, namely the inf-sup condition for $\mathbf{B}$, we define the following mesh-dependent norm:

$$
|||(\mathbf{v},q)|||_h \stackrel{\text{def}}{=} \left\{ \|\mathbf{v}\|_{1,\Omega}^2 + \sum_{K\in\mathcal{T}_h} \left[\|\tau_K^{1/2}(\mathbf{a}\cdot\nabla\mathbf{v}+\nabla q)\|_{0,K}^2 + \|\delta_K^{1/2}\nabla\cdot\mathbf{v}\|_{0,K}^2 \right] \; + \; \|q\|_{0,\Omega}^2 \right\}^{1/2} , \tag{13}
$$

for all $(\mathbf{v},q) \in \mathbf{V}_h \times Q_h$.

We now state the main stability result.

Theorem 2.2 *There exists a constant $\beta = \beta(\sigma,\mathbf{a},\nu)$, independent of h, such that*

$$
\sup_{\theta \neq (\mathbf{w},t)\in\mathbf{V}_h\times Q_h} \frac{\mathbf{B}((\mathbf{u},p),(\mathbf{w},t))}{|||(\mathbf{w},t)|||_h} \;\geq\; \beta \, |||(\mathbf{u},p)|||_h \, ,
$$

for all $(\mathbf{u},p) \in \mathbf{V}_h \times Q_h$.

Proof. - Let $(\mathbf{u},p) \in \mathbf{V}_h \times Q_h$. Since $p \in L_0^2(\Omega)$, there exists $\mathbf{v} \in [H_0^1(\Omega)]^2$ such that $\nabla\cdot\mathbf{v} = -p$ in Ω and $\|\mathbf{v}\|_{1,\Omega} \leq C\,\|p\|_{0,\Omega}$. Now, let $\mathbf{v}_h$ be the Clément interpolate of $\mathbf{v}$ (cf. [17, 9]), which satisfies

$$
\|\mathbf{v}-\mathbf{v}_h\|_{0,K} \leq C\,h_K\,\|\mathbf{v}\|_{1,V(K)} \, , \tag{14}
$$

$$
\|\mathbf{v}_h\|_{1,\Omega} \leq C\,\|\mathbf{v}\|_{1,\Omega} \, , \tag{15}
$$

where $V(K)$ is the set of elements in $\mathcal{T}_h$ who share at least one node with K. Now, since $\mathbf{v}$ and $\mathbf{v}_h$ vanish on $\partial\Omega$ and $Q_h \subseteq C^0(\overline{\Omega})$, we can integrate by parts, use the fact that

$\nabla\cdot \mathbf{v} = -p$, and obtain

$$\begin{aligned}
\mathbf{B}((\mathbf{u},p),(\mathbf{v}_h,0)) &= \sigma\,(\mathbf{u},\mathbf{v}_h)_\Omega \;+\; \nu\,(\nabla\mathbf{u},\nabla\mathbf{v}_h)_\Omega \;+\; (\mathbf{a}\cdot\nabla\mathbf{u},\mathbf{v}_h)_\Omega \\
&\quad -\; (p,\nabla\cdot\mathbf{v}_h)_\Omega \;+\; \sum_{K\in\mathcal{T}_h} (\nabla\cdot\mathbf{u},\delta_K\nabla\cdot\mathbf{v}_h)_K \\
&\quad -\; \sum_{K\in\mathcal{T}_h} (\sigma\mathbf{u}-\nu\Delta\mathbf{u}+\mathbf{a}\cdot\nabla\mathbf{u}+\nabla p,\tau_K(\sigma\mathbf{v}_h-\nu\Delta\mathbf{v}_h-\mathbf{a}\cdot\nabla\mathbf{v}_h))_K \\
&= \sigma\,(\mathbf{u},\mathbf{v}_h)_\Omega \;+\; \nu\,(\nabla\mathbf{u},\nabla\mathbf{v}_h)_\Omega \;+\; (\mathbf{a}\cdot\nabla\mathbf{u},\mathbf{v})_\Omega \;-\; (\mathbf{a}\cdot\nabla\mathbf{u},\mathbf{v}-\mathbf{v}_h)_\Omega \\
&\quad -\; \sum_{K\in\mathcal{T}_h} (\nabla p,\mathbf{v}-\mathbf{v}_h)_K \;+\; \|p\|_{0,\Omega}^2 \;+\; \sum_{K\in\mathcal{T}_h} (\nabla\cdot\mathbf{u},\delta_K\nabla\cdot\mathbf{v}_h)_K \\
&\quad -\; \sum_{K\in\mathcal{T}_h} (\sigma\mathbf{u}-\nu\Delta\mathbf{u}+\mathbf{a}\cdot\nabla\mathbf{u}+\nabla p,\tau_K(\sigma\mathbf{v}_h-\nu\Delta\mathbf{v}_h-\mathbf{a}\cdot\nabla\mathbf{v}_h))_K .
\end{aligned}$$

Hence, using Cauchy-Schwartz one gets

$$\begin{aligned}
\mathbf{B}((\mathbf{u},p),(\mathbf{v}_h,0)) \geq &-\;\sigma\,\|\mathbf{u}\|_{0,\Omega}\|\mathbf{v}_h\|_{0,\Omega} \;-\; \nu\,|\mathbf{u}|_{1,\Omega}|\mathbf{v}_h|_{1,\Omega} \;-\; \|\mathbf{a}\cdot\nabla\mathbf{u}\|_{0,\Omega}\|\mathbf{v}\|_{0,\Omega} \\
&-\; \sum_{K\in\mathcal{T}_h} \|\tau_K^{1/2}(\mathbf{a}\cdot\nabla\mathbf{u}+\nabla p)\|_{0,K}\|\tau_K^{-1/2}(\mathbf{v}-\mathbf{v}_h)\|_{0,K} \;+\; \|p\|_{0,\Omega}^2 \\
&-\; \sum_{K\in\mathcal{T}_h} \|\delta_K^{1/2}\nabla\cdot\mathbf{u}\|_{0,K}\|\delta_K^{1/2}\nabla\cdot\mathbf{v}_h\|_{0,K} \\
&-\; \sum_{K\in\mathcal{T}_h} \|\tau_K^{1/2}(\sigma\mathbf{u}-\nu\Delta\mathbf{u}+\mathbf{a}\cdot\nabla\mathbf{u}+\nabla p)\|_{0,K} \\
&\cdot \|\tau_K^{1/2}(\sigma\mathbf{v}_h-\nu\Delta\mathbf{v}_h-\mathbf{a}\cdot\nabla\mathbf{v}_h)\|_{0,K} \,.
\end{aligned}\tag{16}$$

Now, in each $K\in\mathcal{T}_h$, we have

$$\frac{1}{\tau_K} = \left\{\begin{array}{rl}
\dfrac{\frac{4\nu}{m_k}+|\mathbf{a}(\mathbf{x})|_p h_K}{h_K^2} \leq \dfrac{C\|\mathbf{a}\|_{\infty,\Omega}}{h_K} & \text{if } Pe_K^1, Pe_K^2 > 1\,, \\
\dfrac{8\nu}{m_k h_K^2} & \text{if } Pe_K^1 > 1, Pe_K^2 < 1\,, \\
\dfrac{\sigma h_K^2+|\mathbf{a}(\mathbf{x})|_p h_K}{h_K^2} \leq C\,\dfrac{\sigma h_K+\|\mathbf{a}\|_{\infty,\Omega}}{h_K} & \text{if } Pe_K^1 < 1, Pe_K^2 > 1\,, \\
\dfrac{\sigma h_K^2+\frac{4\nu}{m_k}}{h_K^2} \leq 2\sigma & \text{if } Pe_K^1, Pe_K^2 < 1\,,
\end{array}\right. . \tag{17}$$

Thus, denoting generically by $\frac{1}{\tau_K^*}$ the right hand side of (17), and applying (14) we arrive at

$$
\begin{aligned}
\mathbf{B}((\mathbf{u},p),(\mathbf{v}_h,0)) \geq & -\sigma\,\|\mathbf{u}\|_{0,\Omega}\|\mathbf{v}_h\|_{0,\Omega} - \nu\,|\mathbf{u}|_{1,\Omega}|\mathbf{v}_h|_{1,\Omega} - \|\mathbf{a}\cdot\nabla\mathbf{u}\|_{0,\Omega}\|\mathbf{v}\|_{0,\Omega} \\
& - C\sum_{K\in\mathcal{T}_h}\|\tau_K^{1/2}(\mathbf{a}\cdot\nabla\mathbf{u}+\nabla p)\|_{0,K}\,\frac{h_K}{\sqrt{\tau_K^*}}\,\|\mathbf{v}\|_{1,V(K)} + \|p\|_{0,\Omega}^2 \\
& - \sum_{K\in\mathcal{T}_h}\|\delta_K^{1/2}\nabla\cdot\mathbf{u}\|_{0,K}\|\delta_K^{1/2}\nabla\cdot\mathbf{v}_h\|_{0,K} \\
& - \sum_{K\in\mathcal{T}_h}\left[\sigma\,\overline{\tau_K}^{1/2}\|\mathbf{u}\|_{0,\Omega}+\nu\,\overline{\tau_K}^{1/2}\,\|\Delta\mathbf{u}\|_{0,\Omega}+\|\tau_K^{1/2}(\mathbf{a}\cdot\nabla\mathbf{u}+\nabla p)\|_{0,K}\right] \\
& \quad\cdot\left[\sigma\,\overline{\tau_K}^{1/2}\|\mathbf{v}_h\|_{0,\Omega}+\nu\,\overline{\tau_K}^{1/2}\,\|\Delta\mathbf{v}_h\|_{0,\Omega}+\overline{\tau_K}^{1/2}\|\mathbf{a}\cdot\nabla\mathbf{v}_h\|_{0,K}\right]
\end{aligned}
$$

$$
\begin{aligned}
\geq & -C\,\Big\{\sigma\,\|\mathbf{u}\|_{0,\Omega}^2 + \nu\,|\mathbf{u}|_{1,\Omega}^2 + \|\mathbf{a}\cdot\nabla\mathbf{u}\|_{0,\Omega}^2 \\
& \quad + \sum_{K\in\mathcal{T}_h}\|\tau_K^{1/2}(\mathbf{a}\cdot\nabla\mathbf{u}+\nabla p)\|_{0,K}^2 + \sum_{K\in\mathcal{T}_h}\|\delta_K^{1/2}\nabla\cdot\mathbf{u}\|_{0,K}^2\Big\}^{1/2} \\
& \cdot\Big\{\sigma\,\|\mathbf{v}_h\|_{0,\Omega}^2 + \nu\,|\mathbf{v}_h|_{1,\Omega}^2 + \|\mathbf{v}\|_{0,\Omega}^2 + \sum_{K\in\mathcal{T}_h}\frac{h_K^2}{\tau_K^*}\,\|\mathbf{v}\|_{1,V(K)}^2 \\
& \quad + \sum_{K\in\mathcal{T}_h}\|\delta_K^{1/2}\nabla\cdot\mathbf{v}_h\|_{0,K}^2\Big\}^{1/2} + \|p\|_{0,\Omega}^2 \\
& - C\left\{\sum_{K\in\mathcal{T}_h}\left[\sigma^2\,\overline{\tau_K}\|\mathbf{u}\|_{0,\Omega}^2+\nu^2\,\overline{\tau_K}\,\|\Delta\mathbf{u}\|_{0,\Omega}^2+\|\tau_K^{1/2}(\mathbf{a}\cdot\nabla\mathbf{u}+\nabla p)\|_{0,K}^2\right]\right\}^{1/2} \\
& \cdot\left\{\sum_{K\in\mathcal{T}_h}\left[\sigma^2\,\overline{\tau_K}\|\mathbf{v}_h\|_{0,\Omega}^2+\nu^2\,\overline{\tau_K}\,\|\Delta\mathbf{v}_h\|_{0,\Omega}^2+\overline{\tau_K}\|\mathbf{a}\cdot\nabla\mathbf{v}_h\|_{0,K}^2\right]\right\}^{1/2}.
\end{aligned}
$$

Now, from inverse inequality (11) a simple computation shows that

$$
\nu^2\overline{\tau_K}\,\|\Delta\mathbf{w}\|_{0,K}^2 \leq \nu\,|\mathbf{w}|_{1,K}^2 \qquad \forall\,\mathbf{w}\in\mathbf{V}_h\,, \tag{18}
$$

and hence, from the definition of δ_K and τ_K^*, the fact that $\sigma\overline{\tau_K} \leq 1$, and $\|\mathbf{v}_h\|_{1,\Omega} \leq C\,\|\mathbf{v}\|_{1,\Omega}$, inequality above becomes

$$
\begin{aligned}
\mathbf{B}((\mathbf{u},p),(\mathbf{v}_h,0)) \geq &- C\,\Big\{\sigma\,\|\mathbf{u}\|_{0,\Omega}^2 + \nu\,|\mathbf{u}|_{1,\Omega}^2 + \|\mathbf{a}\|_{\infty,\Omega}^2 |\mathbf{u}|_{1,\Omega}^2 \\
&\quad + \sum_{K\in\mathcal{T}_h} \|\tau_K^{1/2}(\mathbf{a}\cdot\nabla\mathbf{u} + \nabla p)\|_{0,K}^2 + \sum_{K\in\mathcal{T}_h} \|\delta_K^{1/2}\nabla\cdot\mathbf{u}\|_{0,K}^2 \Big\}^{1/2} \\
&\cdot \Big\{\sigma\,\|\mathbf{v}_h\|_{0,\Omega}^2 + \nu\,|\mathbf{v}_h|_{1,\Omega}^2 + \|\mathbf{v}\|_{0,\Omega}^2 \\
&\quad + \max\{\frac{8\nu}{m_k}, \|\mathbf{a}\|_{\infty,\Omega}h, \sigma h^2\}\ \|\mathbf{v}\|_{1,\Omega}^2 + \lambda\|\mathbf{a}\|_{\infty,\Omega}h\,|\mathbf{v}_h|_{1,\Omega}^2 \Big\}^{1/2} \\
&+ \|p\|_{0,\Omega}^2 \\
&- C\,\Big\{\sum_{K\in\mathcal{T}_h}\Big[\sigma\,\|\mathbf{u}\|_{0,K}^2 + \nu\,|\mathbf{u}|_{1,K}^2 + \|\tau_K^{1/2}(\mathbf{a}\cdot\nabla\mathbf{u} + \nabla p)\|_{0,K}^2\Big]\Big\}^{1/2} \\
&\cdot \Big\{\sum_{K\in\mathcal{T}_h}\big[\sigma\,\|\mathbf{v}_h\|_{0,K}^2 + \nu\,|\mathbf{v}_h|_{1,K}^2 + \|\mathbf{a}\|_{\infty,K}^2\overline{\tau_K}\,|\mathbf{v}_h|_{1,K}^2\big]\Big\}^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
\geq &- C_*\,\Big\{\|\mathbf{u}\|_{1,\Omega}^2 + \sum_{K\in\mathcal{T}_h}\Big[\|\tau_K^{1/2}(\mathbf{a}\cdot\nabla\mathbf{u} + \nabla p)\|_{0,K}^2 + \|\delta_K^{1/2}\nabla\cdot\mathbf{u}\|_{0,K}^2\Big]\Big\}^{1/2} \cdot \|\mathbf{v}\|_{1,\Omega} \\
&+ \|p\|_{0,\Omega}^2 \\
&- C_{**}\,\Big\{\|\mathbf{u}\|_{1,\Omega}^2 + \sum_{K\in\mathcal{T}_h}\|\tau_K^{1/2}(\mathbf{a}\cdot\nabla\mathbf{u} + \nabla p)\|_{0,K}^2\Big\}^{1/2} \cdot \|\mathbf{v}\|_{1,\Omega}\,,
\end{aligned}
$$

where C_* and C_{**} are positive constants depending on σ, ν, λ and $\|\mathbf{a}\|_{\infty,\Omega}$.

Now, applying the fact that $\|\mathbf{v}\|_{1,\Omega} \leq C\,\|p\|_{0,\Omega}$, and using the inequality $ab \leq \frac{1}{\gamma}a^2 + \gamma b^2$ in each expression involving a product of norms above, we obtain

$$
\begin{aligned}
\mathbf{B}((\mathbf{u},p),(\mathbf{v}_h,0)) \geq &- \frac{C_*}{\gamma_1}\,\Big\{\|\mathbf{u}\|_{1,\Omega}^2 + \sum_{K\in\mathcal{T}_h}\Big[\|\tau_K^{1/2}(\mathbf{a}\cdot\nabla\mathbf{u} + \nabla p)\|_{0,K}^2 + \|\delta_K^{1/2}\nabla\cdot\mathbf{u}\|_{0,K}^2\Big]\Big\} \\
&- \frac{C_{**}}{\gamma_2}\,\Big\{\|\mathbf{u}\|_{1,\Omega}^2 + \sum_{K\in\mathcal{T}_h}\Big[\|\tau_K^{1/2}(\mathbf{a}\cdot\nabla\mathbf{u} + \nabla p)\|_{0,K}^2\Big]\Big\} \\
&+ (1 - C_*\gamma_1 - C_{**}\gamma_2)\,\|p\|_{0,\Omega}^2 \\
\geq &- C^{**}\,\Big\{\|\mathbf{u}\|_{1,\Omega}^2 + \sum_{K\in\mathcal{T}_h}\Big[\|\tau_K^{1/2}(\mathbf{a}\cdot\nabla\mathbf{u} + \nabla p)\|_{0,K}^2 + \|\delta_K^{1/2}\nabla\cdot\mathbf{u}\|_{0,K}^2\Big]\Big\} \\
&+ C^*\,\|p\|_{0,\Omega}^2\,, \qquad (19)
\end{aligned}
$$

where C^* and C^{**} are positive constants, independent of h, if γ_1 and γ_2 are chosen small enough.

In this form, if we set $(\mathbf{z}, q) \stackrel{\text{def}}{=} (\mathbf{u} + \gamma \mathbf{v}_h, p)$, $\gamma > 0$, we have by the bilinearity of $\mathbf{B}$, Lemma 2.1 and (19) that

$$\begin{aligned}
\mathbf{B}((\mathbf{u},p),(\mathbf{z},q)) &= \mathbf{B}((\mathbf{u},p),(\mathbf{u},p)) + \gamma\, \mathbf{B}((\mathbf{u},p),(\mathbf{v}_h,0)) \\
&\geq C_\Omega \nu \,\|\mathbf{u}\|_{1,\Omega}^2 + \sum_{K\in\mathcal{T}_h} \Big[\|\tau_K^{1/2}(\mathbf{a}\cdot\nabla\mathbf{u} + \nabla p)\|_{0,K}^2 + \|\delta_K^{1/2}\nabla\cdot\mathbf{u}\|_{0,K}^2\Big] \\
&\quad - \gamma C^{**}\Big\{\|\mathbf{u}\|_{1,\Omega}^2 + \sum_{K\in\mathcal{T}_h}\Big[\|\tau_K^{1/2}(\mathbf{a}\cdot\nabla\mathbf{u} + \nabla p)\|_{0,K}^2 + \|\delta_K^{1/2}\nabla\cdot\mathbf{u}\|_{0,K}^2\Big]\Big\} \\
&\quad + \gamma\, C^*\,\|p\|_{0,\Omega}^2 \\
&\geq C\,|||(\mathbf{u},p)|||_h^2\,,
\end{aligned} \tag{20}$$

where $C > 0$ is independent of h if γ is chosen small enough.

Finally, using that $h \leq 1$ and $\|\mathbf{v}\|_{1,\Omega} \leq C\,\|p\|_{0,\Omega}$, we obtain

$$\begin{aligned}
|||(\mathbf{z},q)|||_h^2 &\leq 2\,\|\mathbf{u}\|_{1,\Omega}^2 + 2\sum_{K\in\mathcal{T}_h}\|\tau_K^{1/2}(\mathbf{a}\cdot\nabla\mathbf{u} + \nabla p)\|_{0,K}^2 + 2\sum_{K\in\mathcal{T}_h}\|\delta_K^{1/2}\nabla\cdot\mathbf{u}\|_{0,K}^2 + \|p\|_{0,\Omega}^2 \\
&\quad + 2\gamma^2\Big[\|\mathbf{v}_h\|_{1,\Omega}^2 + \sum_{K\in\mathcal{T}_h}\overline{\tau_K}^{1/2}\|\mathbf{a}\cdot\nabla\mathbf{v}_h\|_{0,K}^2 + \sum_{K\in\mathcal{T}_h}\|\delta_K^{1/2}\nabla\cdot\mathbf{v}_h\|_{0,K}^2\Big] \\
&\leq C\,|||(\mathbf{u},p)|||_h^2\,,
\end{aligned}$$

which, together with (20), finish the proof. □

2.2 Error Analysis

Let k, l be integers with $k, l \geq 1$. We use the Lagrange interpolation operator $\mathbf{I}_h^k : (C^0(\overline{\Omega}))^2 \to [V_k]^2$, we denote $\tilde{\mathbf{u}}_h \stackrel{\text{def}}{=} \mathbf{I}_h^k(\mathbf{u})$, define the interpolation error $\eta^{\mathbf{u}} \stackrel{\text{def}}{=} \mathbf{u} - \tilde{\mathbf{u}}_h$, and we have (cf. [8])

$$|\eta^{\mathbf{u}}|_{m,K} \leq C h_K^{s-m}\,|\mathbf{u}|_{s,K}\,, \tag{21}$$

if $\mathbf{u} \in H^s(K)^2$ for all $K \in \mathcal{T}_h$, with $0 \leq m \leq 2$ and $\max\{m, 2\} \leq s \leq k+1$. Now, for the pressure we define $\tilde{p}_h$ as being the Clément interpolate of p. Denoting now by $\eta^p \stackrel{\text{def}}{=} p - \overline{p_h}$, where

$$\overline{p_h} \stackrel{\text{def}}{=} \tilde{p}_h - \frac{1}{|\Omega|}\,(\tilde{p}_h, 1)_\Omega \in L_0^2(\Omega)\,,$$

we have (cf. [17])

$$\|\eta^p\|_{0,\Omega} \leq C h^t\,\|p\|_{t,\Omega}\,, \tag{22}$$

$$|\eta^p|_{1,K} \leq C h^{t-1}\,\|p\|_{t,V(K)}\,, \tag{23}$$

if $p \in H^t(\Omega)$, with $1 \le t \le l+1$.

The main result concerning approximation is now stated.

Theorem 2.3 *Let us suppose that the solution* $(\mathbf{u}, p)$ *of (1) belongs to* $(H^{k+1}(\Omega) \cap H_0^1(\Omega))^2 \times (H^l(\Omega) \cap L_0^2(\Omega))$. *Then, there exists* $C > 0$*, independent of* h*, such that the error* $(\mathbf{e}^{\mathbf{u}}, e^p) \stackrel{\text{def}}{=} (\mathbf{u} - \mathbf{u}_h, p - p_h)$ *satisfies*

$$\|\!|(\mathbf{e}^{\mathbf{u}}, e^p)|\!\|_h \le C \left[h^k \, |\mathbf{u}|_{k+1,\Omega} + h^l \, \|p\|_{l,\Omega} \right].$$

Proof. - Let $\mathbf{e}_h^{\mathbf{u}} \stackrel{\text{def}}{=} \mathbf{u}_h - \tilde{\mathbf{u}}_h$ and $e_h^p \stackrel{\text{def}}{=} p_h - \overline{p_h}$. From the proof of previous theorem, we see that the supremum is attained, and then there exists $(\mathbf{v}, q) \in \mathbf{V}_h \times Q_h$ such that

$$\|\!|(\mathbf{v}, q)|\!\|_h \le C\,,$$

and

$$\beta \, \|\!|(\mathbf{e}_h^{\mathbf{u}}, e_h^p)|\!\|_h \le \mathbf{B}((\mathbf{e}_h^{\mathbf{u}}, e_h^p), (\mathbf{v}, q))\,.$$

Using the consistency of the method we obtain

$$\mathbf{B}((\mathbf{e}_h^{\mathbf{u}}, e_h^p), (\mathbf{v}, q)) = \mathbf{B}((\eta^{\mathbf{u}}, \eta^p), (\mathbf{v}, q))\,. \tag{24}$$

Now, for the right hand side of (24) we have by using Schwartz's inequality we obtain

$$\begin{aligned}
\mathbf{B}((\eta^{\mathbf{u}}, \eta^p), (\mathbf{v}, q)) &= \sigma(\eta^{\mathbf{u}}, \mathbf{v})_\Omega + \nu(\nabla \eta^{\mathbf{u}}, \nabla \mathbf{v})_\Omega + (\mathbf{a} \cdot \nabla \eta^{\mathbf{u}}, \mathbf{v})_\Omega \\
&\quad - (\eta^p, \nabla \cdot \mathbf{v})_\Omega + (q, \nabla \cdot \eta^{\mathbf{u}})_\Omega + \sum_{K \in \mathcal{T}_h} (\nabla \cdot \eta^{\mathbf{u}}, \delta_K \nabla \cdot \mathbf{v})_K \\
&\quad - \sum_{K \in \mathcal{T}_h} (\sigma \eta^{\mathbf{u}} - \nu \Delta \eta^{\mathbf{u}} + \mathbf{a} \cdot \nabla \eta^{\mathbf{u}} + \nabla \eta^p, \tau_K(\sigma \mathbf{v} - \nu \Delta \mathbf{v} - \mathbf{a} \cdot \nabla \mathbf{v} - \nabla q))_K \\
&\le \Bigg\{ \sum_{K \in \mathcal{T}_h} \sigma \, \|\eta^{\mathbf{u}}\|_{0,K}^2 + \nu \, |\eta^{\mathbf{u}}|_{1,K}^2 + \|\mathbf{a} \cdot \nabla \eta^{\mathbf{u}}\|_{0,K}^2 + \|\eta^p\|_{0,K}^2 + \|\nabla \cdot \eta^{\mathbf{u}}\|_{0,K}^2 \\
&\qquad + \|\delta_K^{1/2} \nabla \cdot \eta^{\mathbf{u}}\|_{0,K}^2 + \|\tau_K^{1/2}(\sigma \eta^{\mathbf{u}} - \nu \Delta \eta^{\mathbf{u}} + \mathbf{a} \cdot \nabla \eta^{\mathbf{u}} + \nabla \eta^p)\|_{0,K}^2 \Bigg\}^{\frac{1}{2}} \\
&\quad \cdot \Bigg\{ \sum_{K \in \mathcal{T}_h} \sigma \, \|\mathbf{v}\|_{0,K}^2 + \nu \, |\mathbf{v}|_{1,K}^2 + \|\mathbf{v}\|_{0,K}^2 + \|\nabla \cdot \mathbf{v}\|_{0,K}^2 + \|q\|_{0,K}^2 \\
&\qquad + \|\delta_K^{1/2} \nabla \cdot \mathbf{v}\|_{0,K}^2 + \|\tau_K^{1/2}(\sigma \mathbf{v} - \nu \Delta \mathbf{v} - \mathbf{a} \cdot \nabla \mathbf{v} - \nabla q)\|_{0,K}^2 \Bigg\}^{\frac{1}{2}}.
\end{aligned}$$

Now, applying inequality (18) for the $-\nu^2\tau_K\|\Delta\mathbf{v}\|^2_{0,K}$ term and $\sigma\overline{\tau_K}\leq 1$, previous inequality becomes

$$\begin{aligned}
\mathbf{B}((\eta^{\mathbf{u}},\eta^p),(\mathbf{v},q)) \leq C\,\Big\{\sum_{K\in\mathcal{T}_h} &\sigma\,\|\eta^{\mathbf{u}}\|^2_{0,K} + \nu\,|\eta^{\mathbf{u}}|^2_{1,K} + \|\mathbf{a}\cdot\nabla\eta^{\mathbf{u}}\|^2_{0,K} + \|\eta^p\|^2_{0,K} + \|\nabla\cdot\eta^{\mathbf{u}}\|^2_{0,K} \\
&+ \|\delta_K^{1/2}\nabla\cdot\eta^{\mathbf{u}}\|^2_{0,K} + \sigma^2\overline{\tau_K}\,\|\eta^{\mathbf{u}}\|^2_{0,K} + \nu^2\overline{\tau_K}\|\Delta\eta^{\mathbf{u}}\|^2_{0,K} \\
&+ \overline{\tau_K}\|\mathbf{a}\cdot\nabla\eta^{\mathbf{u}}\|^2_{0,K} + \overline{\tau_K}\|\nabla\eta^p\|^2_{0,K}\Big\}^{\frac{1}{2}} \\
&\cdot\Big\{\sum_{K\in\mathcal{T}_h}(\sigma+1)\,\|\mathbf{v}\|^2_{0,K} + (\nu+1)\,|\mathbf{v}|^2_{1,K} + \|q\|^2_{0,K} + \|\delta_K^{1/2}\nabla\cdot\mathbf{v}\|^2_{0,K} \\
&+ \sigma\,\|\mathbf{v}\|^2_{0,K} + \nu\,|\mathbf{v}|^2_{1,K} + \|\tau_K^{1/2}(\mathbf{a}\cdot\nabla\mathbf{v}+\nabla q)\|^2_{0,K}\Big\}^{\frac{1}{2}}
\end{aligned}$$

$$\begin{aligned}
\leq C\,\Big\{&\max\{\sigma,\nu,\|\mathbf{a}\|^2_{\infty,\Omega},\|\mathbf{a}\|_{\infty,\Omega}h,\frac{1}{\nu}\}\sum_{K\in\mathcal{T}_h}\Big[\|\eta^{\mathbf{u}}\|^2_{0,K} + |\eta^{\mathbf{u}}|^2_{1,K} + \|\eta^p\|^2_{0,K} \\
&+ h_K^2\|\Delta\eta^{\mathbf{u}}\|^2_{0,K} + h_K^2|\eta^p|^2_{1,K}\Big]\Big\}^{\frac{1}{2}} \\
\cdot\Big\{&\max\{\sigma,\nu,1\}\sum_{K\in\mathcal{T}_h}\Big[\|\mathbf{v}\|^2_{0,K} + |\mathbf{v}|^2_{1,K} + \|q\|^2_{0,K} + \|\delta_K^{1/2}\nabla\cdot\mathbf{v}\|^2_{0,K} \\
&+ \|\mathbf{v}\|^2_{0,K} + |\mathbf{v}|^2_{1,K} + \|\tau_K^{1/2}(\mathbf{a}\cdot\nabla\mathbf{v}+\nabla q)\|^2_{0,K}\Big]\Big\}^{\frac{1}{2}}\,.
\end{aligned}$$

Now, the second term in the product above is smaller than $C\,|\!|\!|(\mathbf{v},q)|\!|\!|_h$ and hence it is bounded by a constant. Therefore, applying interpolation inequalities (21), (22) and (23) we arrive at

$$\begin{aligned}
\mathbf{B}((\eta^{\mathbf{u}},\eta^p),(\mathbf{v},q)) &\leq C(\sigma,\nu,\mathbf{a})\Big\{\sum_{K\in\mathcal{T}_h}\Big[\|\eta^{\mathbf{u}}\|^2_{0,K} + |\eta^{\mathbf{u}}|^2_{1,K} + h_K^2\|\Delta\eta^{\mathbf{u}}\|^2_{0,K}\Big] \\
&\quad + \|\eta^p\|^2_{0,\Omega} + h^2|\eta^p|^2_{1,\Omega}\Big\}^{\frac{1}{2}} \\
&\leq C\,(h^k\,|\mathbf{u}|_{k+1,\Omega} + h^l\,\|p\|_{l,\Omega})\,. \qquad (25)
\end{aligned}$$

Finally, applying triangle inequality and (25) we arrive at

$$\begin{aligned}
|\!|\!|(\mathbf{e^u}, e^p)|\!|\!|_h &\le |\!|\!|(\mathbf{e}_h^{\mathbf{u}}, e_h^p)|\!|\!|_h + |\!|\!|(\eta^{\mathbf{u}}, \eta^p)|\!|\!|_h \\
&\le \frac{C}{\beta}\left(h^k\, |\mathbf{u}|_{k+1,\Omega} + h^l\, \|p\|_{l,\Omega}\right) \\
&\quad + \left\{ \|\eta^{\mathbf{u}}\|_{1,\Omega}^2 + \|\eta^p\|_{0,\Omega}^2 + \sum_{K\in\mathcal{T}_h} \|\delta_K^{1/2}\nabla\cdot\eta^{\mathbf{u}}\|_{0,K}^2 + \|\tau_K^{1/2}(\mathbf{a}\cdot\nabla\eta^{\mathbf{u}} + \nabla\eta^p)\|_{0,K}^2 \right\}^{\frac12} \\
&\le \frac{C}{\beta}\left(h^k\, |\mathbf{u}|_{k+1,\Omega} + h^l\, \|p\|_{l,\Omega}\right) \\
&\quad + C\left\{ \|\eta^{\mathbf{u}}\|_{1,\Omega}^2 + \|\eta^p\|_{0,\Omega}^2 + \|\mathbf{a}\|_{\infty,\Omega} h\, |\eta^{\mathbf{u}}|_{1,\Omega}^2 + \|\mathbf{a}\|_{\infty,\Omega}^2 \frac{h^2}{\nu} |\eta^{\mathbf{u}}|_{1,\Omega}^2 + \frac{h^2}{\nu}\, |\eta^p|_{1,\Omega}^2 \right\}^{\frac12} \\
&\le \frac{C}{\beta}\left(h^k\, |\mathbf{u}|_{k+1,\Omega} + h^l\, \|p\|_{l,\Omega}\right) + C\left\{ \|\eta^{\mathbf{u}}\|_{1,\Omega}^2 + \|\eta^p\|_{0,\Omega}^2 + h^2\, |\eta^p|_{1,\Omega}^2 \right\}^{\frac12},
\end{aligned}$$

and the proof follows by applying interpolation inequalities (21), (22) and (23) once again.□

Remark 3 *We observe that the estimate given by previous theorem implies an error estimate in the natural norms of* $\mathbf{u}$ *and* p, *(the* $H^1(\Omega)^2$*-norm for the velocity and the* $L^2(\Omega)$*-norm for the pressure), that are optimal in order and regularity. In next section we will show that the errors are not too much affected by the physical coefficients.* □

3 Numerical experiments

We use as domain the square $(0,1)\times(0,1)$, and we set $\mathbf{f}$ to be such as the exact solution of our problem (1) is given by

$$\begin{aligned}
u_1(x_1,x_2) &= -256\, x_1^2(x_1-1)^2 x_2(x_2-1)(2x_2-1)\,, \\
u_2(x_1,x_2) &= -u_1(x_2,x_1)\,, \\
p(x_1,x_2) &= 150\,(x_1-0.5)(x_2-0.5)\,.
\end{aligned}$$

First, we report the diffusive dominated case with $\sigma = 1, \nu = 1$, $\mathbf{a} = (1,1)$ and $\lambda = 0$. The results are depicted in Figures 1 and 2, using P^1/P^1 and Q^1/Q^1 finite elements. We recover optimal orders of convergence for velocity and pressure.

Next, we have considered the reaction-convection dominated case with $\sigma = 10^2, \nu = 10^{-3}$, $\mathbf{a} = (1,1)$ and $\lambda = 0$ and $\lambda = 0.5$. Figures 3 to 6 show the convergence history, and they agree with the theoretical results. We recover an optimal order of convergence for the pressure. However, a quasi-optimal $h^{3/2}$ order of convergence is observed for $\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}$. This well known fact in SDFEM-like stabilized methods (cf. [22, 12, 25], specially refer to the introduction in [23]) is recovered by our method. On the other hand, we notice that the *div-div* term, $\sum_{K\in\mathcal{T}_h}(\nabla\cdot\mathbf{u}, \delta_K\nabla\cdot\mathbf{v})_K$, does not provide a significant improvement of

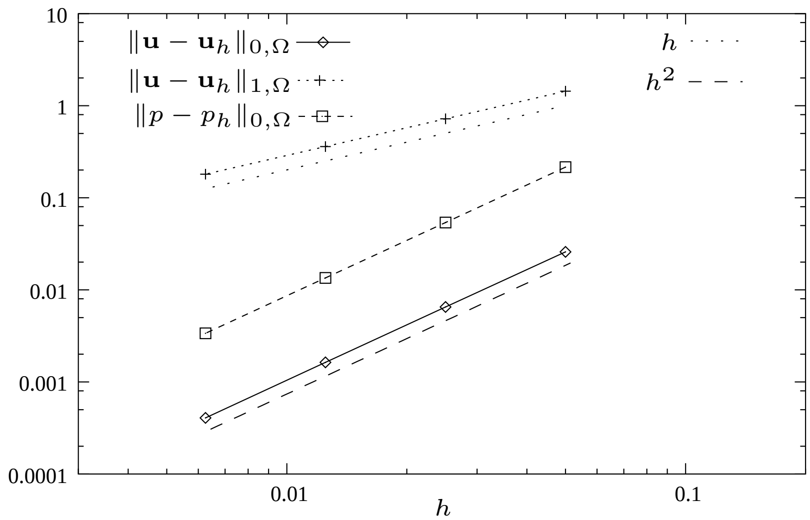

Figure 1: P^1 convergence history: $\sigma = 1$, $\nu = 1$, $\mathbf{a} = (1, 1)$, $\lambda = 0$

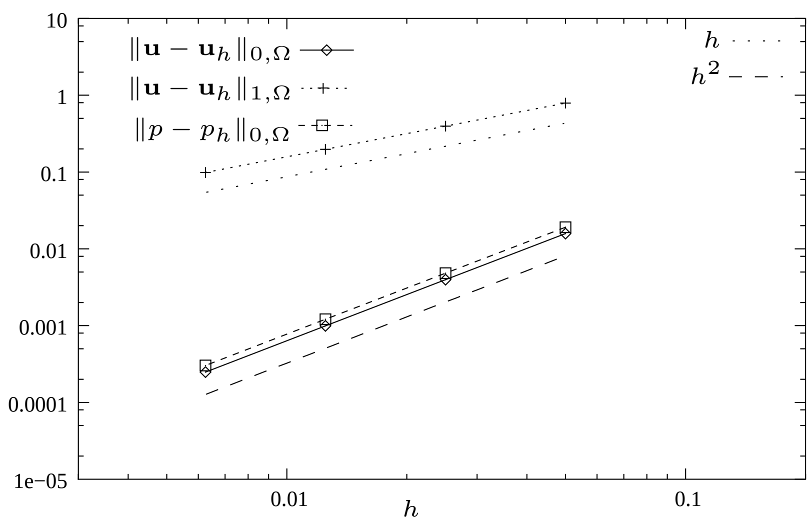

Figure 2: Q^1 convergence history: $\sigma = 1$, $\nu = 1$, $\mathbf{a} = (1, 1)$, $\lambda = 0$

the convergence rate. Because of this, in the sequel we will consider only $\lambda = 0$ in our computations.

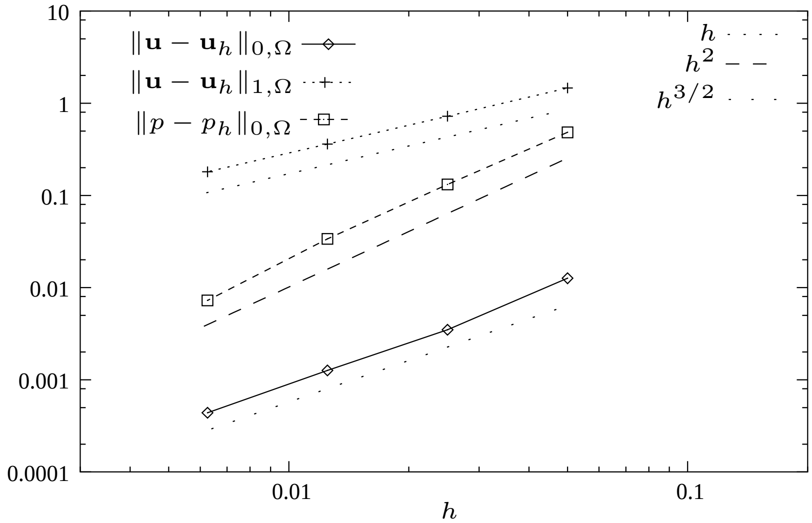

Figure 3: P^1 convergence history: $\sigma = 100$, $\nu = 10^{-3}$, $\mathbf{a} = (1, 1)$, $\lambda = 0$

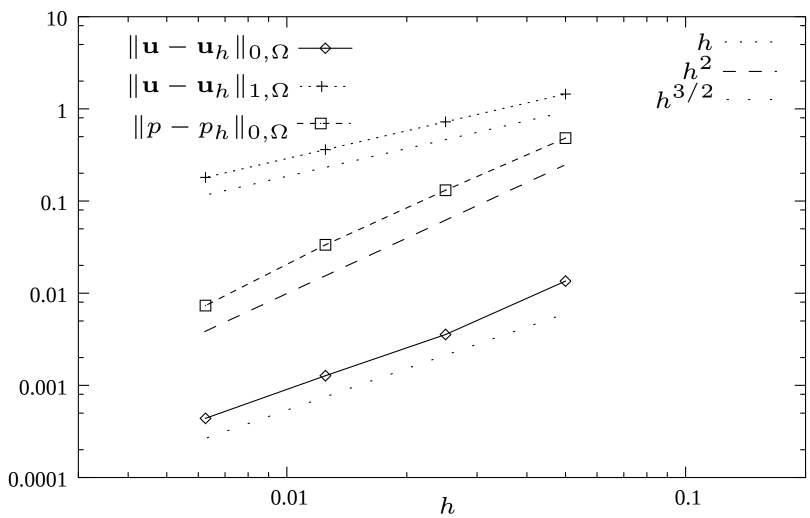

Figure 4: P^1 convergence history: $\sigma = 100$, $\nu = 10^{-3}$, $\mathbf{a} = (1, 1)$, $\lambda = 0.5$

Now, we address the study of the sensitivity of the error to the physical coefficients. To this purpose, we use a uniform 40×40 mesh ($= 3200$ P^1/P^1 elements), and we measure the errors in velocity and pressure.

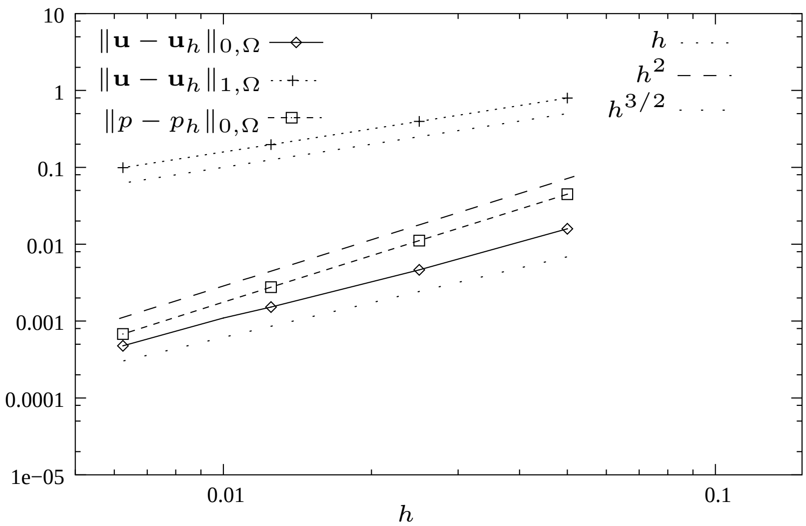

Figure 5: Q^1 convergence history: $\sigma = 100$, $\nu = 10^{-3}$, $\mathbf{a} = (1,1)$, $\lambda = 0$

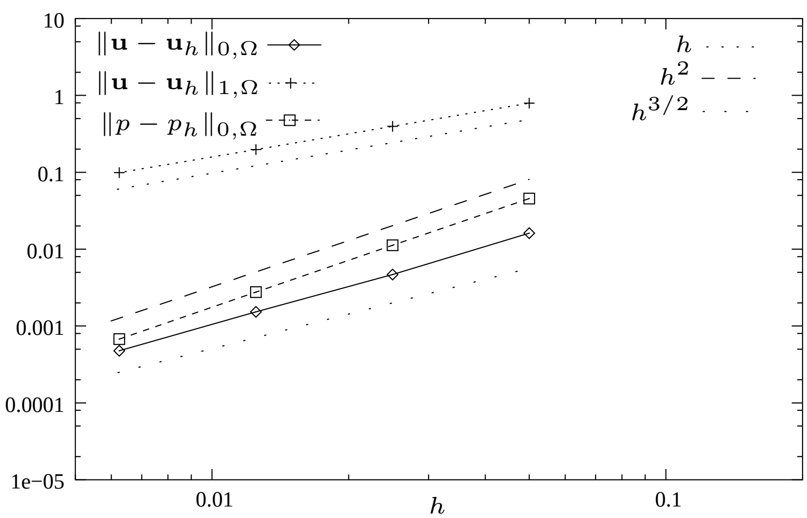

Figure 6: Q^1 convergence history: $\sigma = 100$, $\nu = 10^{-3}$, $\mathbf{a} = (1,1)$, $\lambda = 0.5$

First, we fix $\nu = 10^{-3}$ and $\mathbf{a} = (1,1)$ and make σ grow. The results are shown in Table 1, where we see that the velocity error remains bounded while σ grows and that the pressure error presents a good behavior even for very large values of σ.

σ	$\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}$	$\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega}$	$\|p-p_h\|_{0,\Omega}$
0.1	5.5269×10^{-3}	0.7425	9.9611×10^{-2}
1	5.1536×10^{-3}	0.7397	9.5769×10^{-2}
10	4.4729×10^{-3}	0.7322	0.1079
100	3.4800×10^{-3}	0.7238	0.1316
10^3	3.3633×10^{-3}	0.7245	0.1427
10^4	3.3666×10^{-3}	0.7252	0.1443

Table 1: Behavior of the Finite Element error when σ grows

ν	$\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}$	$\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega}$	$\|p-p_h\|_{0,\Omega}$
1	5.7078×10^{-3}	0.7204	5.3583×10^{-2}
0.1	5.4621×10^{-3}	0.7207	5.8640×10^{-2}
0.01	9.2784×10^{-3}	0.7234	0.1095
10^{-3}	3.4800×10^{-3}	0.7238	0.1316
10^{-4}	2.6934×10^{-3}	0.7278	0.1320
10^{-5}	2.6781×10^{-3}	0.7285	0.1320
10^{-6}	2.6774×10^{-3}	0.7286	0.1320

Table 2: Behavior of the Finite Element error when ν decreases

$\mathbf{a}$	$\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}$	$\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega}$	$\|p-p_h\|_{0,\Omega}$
(0.1,0.1)	9.3554×10^{-3}	0.7248	0.1383
(1,1)	3.4800×10^{-3}	0.7238	0.1316
(5,5)	2.8497×10^{-3}	0.7253	0.1150
(10,10)	3.0116×10^{-3}	0.7261	0.1108
(20,20)	3.2770×10^{-3}	0.7274	0.1190
(40,40)	3.5450×10^{-3}	0.7289	0.1737

Table 3: Behavior of the Finite Element error when $|\mathbf{a}|$ grows

Next, we fix $\sigma = 100$ and $\mathbf{a} = (1,1)$, and make the viscosity ν decrease. The results are shown in Table 2, where we see that both errors are not singnificantly affected by the viscosity.

Finally, we fix $\sigma = 100$ and $\nu = 10^{-3}$, and let $|\mathbf{a}|$ grow. We observe, in Table 3, that the error in velocity remains bounded, while the error in pressure remains bounded for a quite large range of local Péclet numbers $\frac{m_k|\mathbf{a}|h_K}{4\nu}$. Numerical experiences beyond that range of Péclet numbers, have shown that the pressure error grows. This is reasonable since we are already dealing with relatively high Reynolds number.

4 Conclusion

In this paper we have introduced and analyzed a new stabilized finite element method for the Oseen's equations containing dominating reaction terms. The method is obtained by subtracting a mesh dependent term to the standard Galerkin formulation without compromising consistency. The stabilization parameter is suggested by bubble condensation and it is completely determined from the data of the problem. The error estimates provide optimal convergence rate for velocity and pressure in its natural norms. The reported numerical experiments confirm the approximation results and point out that the performance of the method is not dramatically affected by the physical coefficients.

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