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Abstract: While Part I [2] of this paper was devoted mainly to quadrics intersecting in a smooth quartic, we now focus on singular intersections. To produce optimal or near-optimal parameterizations in all cases, we first determine the the real type of the intersection before computing the actual parameterization.

In this second part, we present the first classification of pencils of quadrics based on the type of the real intersection and we show how this classification can be used to compute efficiently the type of the real intersection. The near-optimal parameterization algorithms in all singular cases will be given in Part III [3].

Key-words: Intersection of surfaces, quadrics, pencils of quadrics, classification, curve parameterization.

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Paramétrisation quasi-optimale de l'intersection de quadriques : II. Classification des faisceaux

Résumé : Alors que la partie I [2] de cet article s'est principalement focalisée sur les paires de quadriques dont l'intersection est une quartique lisse, nous nous concentrons maintenant sur les intersections singulières. Pour parvenir à l'obtention de paramétrages optimaux ou quasi-optimaux dans tous les cas, nous découplons la détermination du type réel de l'intersection du paramétrage proprement dit.

Dans cette seconde partie de notre article, nous présentons la première classification des faisceaux de quadriques basée sur le type de l'intersection réelle et nous montrons comment cette classification peut être utilisée pour calculer efficacement le type réel de l'intersection. Les algorithmes quasi-optimaux de paramétrage dans tous les cas sont présentés dans la partie III [3].

Mots-clés : Intersection de surfaces, quadriques, faisceaux de quadriques, classification, paramétrisation.

1 Introduction

At the end of Part I [2], we saw that the generic algorithm we introduced, while being simple and giving optimal parameterizations in some cases, fails to achieve the stated goal of computing (near-)optimal parameterizations (both in terms of functions and coefficients) of intersections of arbitrary quadrics.

Unfortunately, it turns out that achieving this goal involves more than simple adaptations to the generic algorithm. Reaching optimality implies looking carefully at *each* type of *real* intersection and designing a dedicated algorithm to handle each situation. For this, we need to understand precisely which situations can happen over the reals and thus classify real pencils of quadrics of $\mathbb{P}^3(\mathbb{R})$.

Classifying pencils of quadrics over the complexes was achieved by Segre in the nineteenth century [6]. Its practical value is however limited since its proper interpretation lies in the complex domain (i.e. points on the intersection might be real or complex), whereas our concern is with real parts of the intersection.

Accordingly, we refine the Segre classification of pencils of $\mathbb{P}^3(\mathbb{R})$ by examining the different cases occurring over the reals. This refinement is, in itself, of partial assistance for the parameterization problem: no more than the Segre classification can it be "reverse engineered" to construct explicit representations of the various intersection components. It is however mandatory for the following two reasons: it allows us to obtain structural information on the intersection curve which we use to drive the algorithm for computing a near-optimal parameterization of the intersection curve (Part III); it is also a prerequisite for proving the (near-)optimality of our parameterization algorithm.

In this second part of our paper, we present a new classification of pencils of quadrics based on the type of the real intersection. A summary of this classification is given in Tables 4 and 5. We then show how to use this classification to compute efficiently the type of the real intersection. In particular we show how computations with non-rational numbers can be avoided for detecting the type of the intersection when the input quadrics have rational coefficients.

It should be stressed that, even though the classification of pencils of the reals is presented here as an intermediate step in a more global process (i.e., parameterization of the intersection), this classification has an interest on its own. It could be used for instance in a collision detection context to predict at which time stamps a collision between two moving objects will happen.

The rest of this part is organized as follows. Section 2 reviews the classical Segre classification of pencils of quadrics over the complexes. We then refine, in Sections 3 and 4, the Segre classification over the reals with a repeated application of the Canonical Pair Form Theorem for pairs of real symmetric matrices introduced in Part I. In Section 3, we consider *regular* pencils, i.e., pencils that contain a non-singular quadric, and, in Section 4, *singular* pencils, i.e., pencils that contain only singular quadrics, or, equivalently, pencils with identically vanishing determinantal equation. In Section 5, we use the results of the classification of pencils over the reals to design an algorithm to quickly and efficiently characterize the complex and real types of the intersection given two input quadrics. Several examples are detailed in Section 6, before concluding.

2 Classification of pencils of quadrics over the complexes

In this section, we review classical material on the classification of pencils of quadrics. It will serve as the starting point for our classification of pencils over the reals in Sections 3 and 4.

In the rest of the paper all quadrics are considered in real projective space $\mathbb{P}^3(\mathbb{R})$; their coefficients as well as the coefficients of the determinantal equations of pencils are thus real. However, we consider the intersection of quadrics both in $\mathbb{P}^3(\mathbb{R})$ and in $\mathbb{P}^3(\mathbb{C})$. Accordingly, the classification of pencils is considered both over the complexes and over the reals.

In the following, the parameters of the quadric parameterizations live in real projective spaces. For simplicity, parameter spaces $\mathbb{P}^n(\mathbb{R})$ are denoted $\mathbb{P}^n$.

We start in Section 2.1 with a proof that the existence of a singularity on the intersection curve is equivalent either to the existence of a multiple root in the determinantal equation or to the fact that the determinantal equation vanishes identically. Then Section 2.2 recalls the basic tenets of the classification of pencils over the complexes. The well-known Segre characteristic is recalled in Section 2.2.1 and its relation with the Canonical Pair Form Theorem for pairs of real symmetric matrices (Part I and [8, 9]) is thoroughly explained in Section 2.2.2.

2.1 Singular intersections and multiple roots

In the ensuing sections, we use the following equivalence for classifying the singular intersections through the multiplicities of the roots of the determinantal equation $\mathcal{D}(\lambda,\mu)$ and the rank of the corresponding quadrics.

Proposition 2.1. *If the intersection of two distinct quadrics Q_S and Q_T has a singular point* $\mathbf{p}$, *then*

- *either $\mathcal{D}\equiv 0$ and Q_S and Q_T are singular at* $\mathbf{p}$,
- *or $\mathcal{D}\equiv 0$ and there is a unique quadric Q_R of the pencil that is singular at* $\mathbf{p}$,
- *or $\mathcal{D}\not\equiv 0$, there is a unique quadric $Q_R=\lambda_0 Q_S+\mu_0 Q_T$ that is singular at* $\mathbf{p}$ *and (λ_0,μ_0) is a multiple root of $\mathcal{D}$.*

In the last two cases, all the quadrics of the pencil except Q_R share a common tangent plane at $\mathbf{p}$.

Proof. First recall that a curve C defined by implicit equations $Q_S=Q_T=0$ is singular at $\mathbf{p}$ if and only if $\mathbf{p}$ is on C and the rank of the Jacobian matrix J of C is strictly less than 2 when evaluated at $\mathbf{p}$. J is the matrix of partial derivatives:

$$J=\begin{pmatrix} \frac{\partial Q_S}{\partial x} & \frac{\partial Q_S}{\partial y} & \frac{\partial Q_S}{\partial z} & \frac{\partial Q_S}{\partial w} \\ \frac{\partial Q_T}{\partial x} & \frac{\partial Q_T}{\partial y} & \frac{\partial Q_T}{\partial z} & \frac{\partial Q_T}{\partial w} \end{pmatrix}. \tag{1}$$

Let J_S and J_T be the first and second rows of J.

If all the coefficients of J vanish at $\mathbf{p}$, then $\mathbf{p}$ is a singular point of both Q_S and Q_T and thus of all quadrics of the pencil, implying that $\mathcal{D}\equiv 0$.

Otherwise, J has rank one and there exists a linear relationship between the rows of J evaluated at $\mathbf{p}$:

$$\lambda_0 J_S|_{\mathbf{p}} + \mu_0 J_T|_{\mathbf{p}} = 0, \quad (\lambda_0, \mu_0) \in \mathbb{P}^1.$$

Also, there is a $\mathbf{v} \in \mathbb{P}^3(\mathbb{R})$ such that $J|_{\mathbf{p}}\mathbf{v}$ is a non-zero multiple of $\mathbf{v}$. This exactly means that the tangent plane at $\mathbf{p}$ of all the quadrics of the pencil is the plane P of equation $\mathbf{v} \cdot \begin{pmatrix} x & y & z & w \end{pmatrix}^T = 0$, except for the quadric $\lambda_0 Q_S + \mu_0 Q_T$. For this last quadric, all the partial derivatives at $\mathbf{p}$ vanish, implying that it is singular at $\mathbf{p}$ and has rank at most 3.

Now, we may change the generators of the pencil by taking $\lambda_0 S + \mu_0 T$ as first generator in place of S. This has the effect of translating (λ_0, μ_0) to $(1, 0)$. If we change of frame in order that the coordinates of $\mathbf{p}$ become $(0,0,0,1)$ and that the equation of P becomes $x = 0$, the matrices of the generators of the pencil become

$$S' = \begin{pmatrix} * & * & * & 0 \\ * & \multicolumn{2}{c}{\multirow{2}{*}{A}} & 0 \\ * & & & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad T' = \begin{pmatrix} * & * & * & 1 \\ * & \multicolumn{2}{c}{\multirow{2}{*}{B}} & 0 \\ * & & & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

where A and B are 2×2 matrices and the stars denote any element. It follows immediately that

$$\det(\lambda S' + \mu T') = -\mu^2 \det(\lambda A + \mu B).$$

The case $\det(\lambda A + \mu B) \equiv 0$ proves the second assertion. The case $\det(\lambda A + \mu B) \not\equiv 0$ proves the last assertion. $\square$

2.2 Classifi cation of pencils by elementary divisors

For the reader's convenience, we review, in this section, the classical classification of pencils of quadrics as originally done by the Italian mathematician Corrado Segre [6]. More recent and accessible accounts can be found in [1, 5].

2.2.1 Segre characteristic

Assume we are given a pencil $R(\lambda, \mu) = \lambda S + \mu T$ of symmetric matrices of size n such that $\mathcal{D}(\lambda, \mu) = \det R(\lambda, \mu)$ is not identically zero. In general, $\mathcal{D}$ has n complex roots to which correspond n complex projective cones of the pencil. But there can be exceptions to this when a root (λ_0, μ_0) of $\mathcal{D}$ appears with multiplicity larger than 1. It can also happen that (λ_0, μ_0) makes not only the determinant $\mathcal{D}$ vanish but also all its subdeterminants of order $n - t + 1$ say, $t > 0$. This means that the corresponding quadric has as singular set a linear space of dimension $t - 1$.

Let the $(\lambda_i, \mu_i), i = 1, \ldots, p$, be the roots of $\mathcal{D}$ and the m_i their respective multiplicities. Indicate by m_i^j the minimum multiplicity with which the root (λ_i, μ_i) appears in the subdeterminants of order $n - j$ of $\mathcal{D}$. Let $t_i \geqslant 1$ be the smallest integer such that $m_i^{t_i} = 0$. We see that $m_i^j \geqslant m_i^{j+1}$ for all j. Define a sequence of indices e_i^j as follows:

$$e_i^j = m_i^{j-1} - m_i^j, \quad j = 1, \ldots, t_i,$$

with $m_i^0 = m_i$. The multiplicity m_i of (λ_i, μ_i) is the sum $e_i^1 + \cdots + e_i^{t_j}$. We have therefore:

$$\mathcal{D}(\lambda,\mu) = (\lambda\mu_i - \mu\lambda_i)^{m_i}\mathcal{D}^*(\lambda,\mu) = (\lambda\mu_i - \mu\lambda_i)^{e_i^1}\cdots(\lambda\mu_i - \mu\lambda_i)^{e_i^{t_i}}\mathcal{D}^*(\lambda,\mu),$$

where $\mathcal{D}^*(\lambda_i, \mu_i) \neq 0$.

The factors $(\lambda\mu_i - \mu\lambda_i)^{e_i^j}$ are called the *elementary divisors* and the exponents e_i^j the *characteristic numbers,* associated with the root (λ_i, μ_i). Their study goes back to Karl Weierstrass [10]. Segre introduced the following notation to denote the various characteristic numbers associated with the degenerate quadrics that appear in a pencil:

$$\sigma_n = [(e_1^1, \ldots, e_1^{t_1}), (e_2^1, \ldots, e_2^{t_2}), \ldots, (e_p^1, \ldots, e_p^{t_p})],$$

with the convention that the parentheses enclosing the characteristic numbers of (λ_i, μ_i) are dropped when $t_i = 1$. This is known as the *Segre characteristic* or *Segre symbol* of the pencil.

The following theorem, essentially due to Weierstrass [10], proves that a pencil of quadrics and the intersection it defines are uniquely and entirely characterized, over the complexes, by its Segre symbol.

Theorem 2.2 (Characterization by Segre symbol). *Consider two pencils of quadrics $R(\lambda_1, \mu_1) = \lambda_1 S_1 + \mu_1 T_1$ and $R(\lambda_2, \mu_2) = \lambda_2 S_2 + \mu_2 T_2$ in $\mathbb{P}^n(\mathbb{R})$. Suppose that $\det R(\lambda_1, \mu_1)$ and $\det R(\lambda_2, \mu_2)$ are not identically zero and let $(\lambda_{1,i}, \mu_{1,i})$ and $(\lambda_{2,i}, \mu_{2,i})$ be their respective roots. Then the two pencils are projectively equivalent if and only if they have the same Segre symbol and there is an automorphism of $\mathbb{P}^1(\mathbb{C})$ taking $(\lambda_{1,i}, \mu_{1,i})$ to $(\lambda_{2,i}, \mu_{2,i})$.*

With the above definition, we see that (λ_i, μ_i) is a root of all subdeterminants of $R(\lambda, \mu)$ of order $n - t_i + k, k > 0$, but not of any subdeterminant of order $n - t_i$. In other words, the rank r_i of $R(\lambda_i, \mu_i)$ is $n - t_i$. In addition, since $m_i = e_i^1 + \cdots + e_i^{t_i}$, we have that $n - 1 \geqslant r_i \geqslant n - m_i$. Enumerating all possible cases for the e_i^j subject to the constraints induced by its definition gives rise to all possible types of (complex) intersection and accompanying Segre symbols. Tables 1, 2, and 3 list the possible cases for pencils in $\mathbb{P}^3(\mathbb{R})$. Incidentally, we can see that the pair (m_i, r_i) is sufficient to characterize the pencil except in the case $(m_i, r_i) = (4, 2)$.

When the determinantal equation $\mathcal{D}(\lambda, \mu)$ vanishes identically, i.e., all the quadrics are singular (see Tables 2 and 3), the above theory does not apply directly. There are two cases, according to whether the quadrics of the pencil have singular points in common or not:

- When they do not, the pencil can be characterized by a different set of invariants the existence of which was originally proved by Kronecker. We do not detail here how this set is computed (but see [1, p. 55-60]). Suffice it to say that the cases $n = 4$ and $n = 3$ are characterized each by a single set of such invariants, designated by the strings $[1\{3\}]$ and $[\{3\}]$ respectively. In Section 4.1, we carry out the analysis of this situation when $n = 4$ without resorting to these special invariants.

- When the quadrics do have (at least one) singular point in common, say $\mathbf{p}$, we may suppose, after a change of frame, that $\mathbf{p}$ has coordinates $(0, \ldots, 0, 1)$. In the new frame, the matrices

Segre characteristic σ_4	roots of $\mathcal{D}(\lambda,\mu)$ and rank of associated quadric	complex type of intersection
$[1111]$	four simple roots	smooth quartic
$[112]$	one double root, rank 3	nodal quartic
$[11(11)]$	one double root, rank 2	two secant conics
$[13]$	triple root, rank 3	cuspidal quartic
$[1(21)]$	triple root, rank 2	two tangent conics
$[1(111)]$	triple root, rank 1	double conic
$[4]$	quadruple root, rank 3	cubic and tangent line
$[(31)]$	quadruple root, rank 2	conic and two lines crossing on the conic
$[(22)]$	quadruple root, rank 2	two lines and a double line
$[(211)]$	quadruple root, rank 1	two double lines
$[(1111)]$	quadruple root, rank 0	any non-trivial quadric of the pencil
$[22]$	two double roots, both rank 3	cubic and secant line
$[2(11)]$	two double roots, ranks 3 and 2	conic and two lines not crossing on the conic
$[(11)(11)]$	two double roots, both rank 2	four skew lines

Table 1: Classification of pencils by Segre symbol in the case where $\mathcal{D}(\lambda,\mu)$ does not identically vanish. When the determinantal equation has multiple roots, the additional simple roots are not indicated: they correspond to rank 3 quadrics.

Segre characteristic σ_3	roots of $\mathcal{D}_3(\lambda,\mu)$ and rank of associated conic	complex type of intersection
$[1\{3\}]$	no common singular point	conic and double line
$[111]$	three simple roots	four concurrent lines
$[12]$	double root, rank 2	two lines and a double line
$[1(11)]$	double root, rank 1	two double lines
$[3]$	triple root, rank 2	line and triple line
$[(21)]$	triple root, rank 1	quadruple line
$[(111)]$	triple root, rank 0	any non-trivial quadric of the pencil
$[\{3\}]$	$\mathcal{D}_3(\lambda,\mu)\equiv 0$	line and plane

Table 2: Classification of pencils by Segre symbol in the case where $\mathcal{D}(\lambda,\mu)\equiv 0$ and the quadrics of the pencil have zero (top part) or one (bottom part) singular point $\mathbf{p}$ in common. $\mathcal{D}_3(\lambda,\mu)$ is the determinant of the 3×3 upper-left matrix of $R(\lambda,\mu)$ after a congruence transformation sending $\mathbf{p}$ to $(0,0,0,1)$. The conic associated with a root of $\mathcal{D}_3(\lambda,\mu)$ corresponds to the 3×3 upper-left matrix or $R(\lambda,\mu)$.

Segre characteristic σ_2	roots of $\mathcal{D}_2(\lambda,\mu)$ and rank of associated matrix	complex type of intersection
$[11]$	two simple roots	quadruple line
$[2]$	double root, rank 1	plane
$[(11)]$	double root, rank 0	any non-trivial quadric of the pencil
	$\mathcal{D}_2(\lambda,\mu)\equiv 0$	double plane

Table 3: Classification of pencils by Segre symbol in the case where the quadrics of the pencil have (at least) two singular points $\mathbf{p}$ and $\mathbf{q}$ in common (i.e., $\mathcal{D}_3(\lambda,\mu)\equiv 0$). $\mathcal{D}_2(\lambda,\mu)$ is the determinant of the 2×2 upper-left matrix of $R(\lambda,\mu)$ after a congruence transformation sending $\mathbf{p}$ and $\mathbf{q}$ to $(0,0,0,1)$ and $(0,0,1,0)$. The matrix associated with a root of $\mathcal{D}_2(\lambda,\mu)$ corresponds to the 2×2 upper-left matrix or $R(\lambda,\mu)$.

have their last row and column filled with zeros. To sort out the different types of intersection, we may identify the quadrics with their upper left $(n-1)\times(n-1)$ matrices and classify the restricted pencils by looking at the Segre symbol σ_{n-1} of their degree $n-1$ determinantal equation. This is what we have done in Table 2 for the case of quadrics in $\mathbb{P}^3(\mathbb{R})$.

The above process can be repeated by recursing on dimension.

2.2.2 From the complexes to the reals

Theorem 2.2 can be used to find a canonical form for a pencil of quadrics when $\mathcal{D}(\lambda,\mu)$ is not identically zero (see [5]). Consider the pencil $R(\lambda)=\lambda S-T$ and its determinantal equation $\mathcal{D}(\lambda)$, with roots λ_i of multiplicity m_i. Let

$$[(e_1^1,\ldots,e_1^{t_1}),(e_2^1,\ldots,e_2^{t_2}),\ldots,(e_p^1,\ldots,e_p^{t_p})]$$

be the Segre symbol of the pencil. Then there exists a change of coordinates in $\mathbb{P}^n(\mathbb{C})$ such that, in the new frame, the pencil writes down as $R'(\lambda)=\lambda S'-T'$, where

$$S'=\operatorname{diag}(E_1^1,\ldots,E_1^{t_1},\ldots,E_p^1,\ldots,E_p^{t_p}),\quad T'=\operatorname{diag}(E_1^1J_1^1,\ldots,E_1^{t_1}J_1^{t_1},\ldots,E_p^1J_p^1,\ldots,E_p^{t_p}J_p^{t_p})$$

are block diagonal matrices with blocks:

$$E_i^k=\begin{pmatrix} 0 & & 1\\ & \iddots & \\ 1 & & 0\end{pmatrix}\quad\text{and}\quad J_i^k=\begin{pmatrix} \lambda_i & 1 & & 0\\ & \ddots & \ddots & \\ & & \ddots & 1\\ 0 & & & \lambda_i\end{pmatrix}$$

of size e_i^k. The parentheses in the Segre symbol correspond one-to-one to the singular quadrics in the pencil. The root of $\mathcal{D}$ corresponding to a singular quadric of symbol $(e_i^1,\ldots,e_i^{t_i})$ has multiplicity $m_i=\sum_{k=1}^{t_i}e_i^k$.

Segre string	roots of $\mathcal{D}(\lambda,\mu)$	rank or inertia of $R(\lambda_1,\mu_1)$	rank or inertia of $R(\lambda_2,\mu_2)$	type of (λ_2,μ_2)	s	real type of intersection
[1111]	4 simple roots					smooth quartic or $\emptyset$; see Part I
[112]	1 double root	$(3,0)$		real		point
[112]	1 double root	$(2,1)$		real	$-$	nodal quartic; isolated node
[112]	1 double root	$(2,1)$		real	$+$	nodal quartic; convex sing.
[112]	1 double root	rank 3		complex		nodal quartic; concave sing.
[11(11)]	1 double root	$(2,0)$		real	$+$	$\emptyset$
[11(11)]	1 double root	$(2,0)$		real	$-$	two points
[11(11)]	1 double root	$(1,1)$	$(2,1)$	real	$-$	two non-secant conics
[11(11)]	1 double root	$(1,1)$	$(3,0)$	real	$-$	$\emptyset$
[11(11)]	1 double root	$(1,1)$		real	$+$	two secant conics; convex sing.
[11(11)]	1 double root	rank 2		complex	$-$	conic
[11(11)]	1 double root	rank 2		complex	$+$	two secant conics; concave sing.
[13]	triple root	rank 3				cuspidal quartic
[1(21)]	triple root	$(2,0)$				double point
[1(21)]	triple root	$(1,1)$				two tangent conics
[1(111)]	triple root	rank 1	$(2,1)$			double conic
[1(111)]	triple root	rank 1	$(3,0)$			$\emptyset$
[4]	quadruple root	rank 3				cubic and tangent line
[(31)]	quadruple root	$(1,1)$			$-$	conic
[(31)]	quadruple root	$(1,1)$			$+$	conic and two lines crossing on the conic
[(22)]	quadruple root	$(2,0)$				double line
[(22)]	quadruple root	$(1,1)$			$+$	two single lines & a double line
[(211)]	quadruple root	rank 1			$-$	point
[(211)]	quadruple root	rank 1			$+$	two secant double lines
[(1111)]	quadruple root	rank 0				any smooth quadric of the pencil
[22]	2 double roots	rank 3	rank 3	real		cubic and secant line
[22]	2 double roots	rank 3	rank 3	complex		cubic and non-secant line
[2(11)]	2 double roots	$(3,0)$	rank 2	real		point
[2(11)]	2 double roots	$(2,1)$	rank 2	real	$+$	conic and two intersecting lines
[2(11)]	2 double roots	$(2,1)$	rank 2	real	$-$	conic and point
[(11)(11)]	2 double roots	$(2,0)$	$(2,0)$	real		$\emptyset$
[(11)(11)]	2 double roots	$(2,0)$	$(1,1)$	real		two points
[(11)(11)]	2 double roots	$(1,1)$	$(2,0)$	real		two points
[(11)(11)]	2 double roots	$(1,1)$	$(1,1)$	real		four skew lines
[(11)(11)]	2 double roots	rank 2	rank 2	complex		two secant lines

Table 4: Classification of pencils in the case where $\mathcal{D}(\lambda,\mu)$ does not identically vanish. (λ_1,μ_1) denotes a multiple root of $\mathcal{D}(\lambda,\mu)$ (if any) and (λ_2,μ_2) another root (not necessarily simple). If (λ_1,μ_1) is a double root then s denotes the sign of $\frac{\det(\lambda S+\mu T)}{(\mu_1\lambda-\lambda_1\mu)^2}$ at $(\lambda,\mu)=(\lambda_1,\mu_1)$; if (λ_1,μ_1) is a quadruple root then s denotes the sign of $\det(\lambda S+\mu T)$ for any $(\lambda,\mu)\neq(\lambda_1,\mu_1)$. When the determinantal equation has multiple roots, the additional simple roots are not indicated.

Segre string	roots of $\mathcal{D}_3(\lambda,\mu)$	rank or inertia of $R(\lambda_1,\mu_1)$	inertia of $R(\lambda_2,\mu_2)$	type of (λ_2,μ_2)	real type of intersection
$[1\{3\}]$	no common singular point				conic and double line
$[111]$	3 simple roots	$(1,1)$	$(1,1)$	real	four concurrent lines meeting at **p**
$[111]$	3 simple roots	$(2,0)$		real	point **p**
$[111]$	3 simple roots		$(2,0)$	real	point **p**
$[111]$	3 simple roots			complex	two lines intersecting at **p**
$[12]$	double root	$(1,1)$			2 lines and a double line meeting at **p**
$[12]$	double root	$(2,0)$			double line
$[1(11)]$	double root	rank 1	$(1,1)$		two double lines meeting at **p**
$[1(11)]$	double root	rank 1	$(2,0)$		point **p**
$[3]$	triple root	rank 2			a line and a triple line meeting at **p**
$[(21)]$	triple root	rank 1			a quadruple line
$[(111)]$	triple root	rank 0			any non-trivial quadric of the pencil
$[\{3\}]$	$\mathcal{D}_3(\lambda,\mu)\equiv 0$				the complex intersection is real; see Tables 2 and 3

Table 5: Classification of pencils in the case where $\mathcal{D}(\lambda,\mu)$ identically vanishes. In the bottom part, the quadrics of the pencil have a singular point **p** in common. $\mathcal{D}_3(\lambda,\mu)$ is the determinant of the 3×3 upper-left matrix of $R(\lambda,\mu)$ after a congruence transformation sending **p** to $(0,0,0,1)$. The conic associated with a root of $\mathcal{D}_3(\lambda,\mu)$ corresponds to the 3×3 upper-left matrix or $R(\lambda,\mu)$. (λ_1,μ_1) denotes the multiple root of $\mathcal{D}_3(\lambda,\mu)$ (if any) and (λ_2,μ_2) another root. When $\mathcal{D}_3(\lambda,\mu)$ has a multiple root, the additional simple roots are not indicated.

The parallel between the Canonical Pair Form Theorem introduced in Section I.5.1[1] and the decomposition by Segre symbol should now jump to mind: the first is in a sense a real version of the second, i.e. it gives a canonical form that is projectively equivalent by a *real* congruence transformation to the original pencil. In the real form, complex roots of the determinantal equation are somehow combined in complex Jordan blocks so that quadric pencils are equivalent by a real projective transformation.

When λ_i is real, the J_i^j are the real Jordan blocks associated with λ_i. The sum of the sizes of the blocks corresponding to λ_i is $\sum_{k=1}^{t_i} e_i^k = m_i$ and the number of those blocks is $t_i = n - \operatorname{rank} R(\lambda_i)$, as in Theorem I.5.3.

When λ_i is complex, let λ_j be its conjugate. It is intuitively clear that $t_i = t_j$ in the complex decomposition and that the associated Jordan blocks J_i^k and J_j^k have the same sizes, i.e. $e_i^k = e_j^k$. When the complex roots and their blocks are combined, they give rise to complex Jordan blocks of

[1]When reference is made to a section or result in another part of the paper, it is prefixed by the part number.

size $2e_i^k$. In the real canonical form, the number of these blocks is again t_i but the sum of their sizes is $2m_i$.

The Segre symbol can thus serve as a starting point for the study of real pencils using the Canonical Pair Form Theorem. We illustrate this with two examples concerning pencils in $\mathbb{P}^3(\mathbb{R})$. Consider first the Segre symbol $[(211)]$. The associated pencil has a quadruple root, which is necessarily real (otherwise its conjugate would also be a root of the determinantal equation of the pencil). In view of the above, the real decomposition of the pencil has three Jordan blocks, one of size 2 and two of size 1. Now consider the Segre symbol $[22]$. The associated pencil has two double roots, which can be either both real or both complex. If they are real, then each of the roots has one Jordan block of size 2. If they are complex, then the two roots appear in the same Jordan block of size 4.

3 Classification of regular pencils of $\mathbb{P}^3(\mathbb{R})$ over the reals

We now turn to the classification of pencils of quadrics of $\mathbb{P}^3(\mathbb{R})$ over the reals. In what follows, we make heavy use of the Canonical Pair Form Theorem for pairs of real symmetric matrices (Part I and [8, 9]). For each possible Segre characteristic, we examine the different cases according to whether the roots of the determinantal equation are real or not and then examine the conditions leading to different types of intersection over the reals.

In each case, we start by computing the canonical form of the pair (S,T) for a given Segre characteristic and type (real or complex) of multiple root(s) of the determinantal equation. We then deduce from this canonical form a *normal form* of the pencil over the reals by rescaling and translating the roots to particularly simple values. Recall that the congruence transformation in the Canonical Pair Form Theorem preserves the roots (values and multiplicities) of the determinantal equation of the pencil. This normal form is in a sense the "simplest pair" of quadrics having a given real intersection type. The normal pencil is equivalent by a real projective transformation to any pencil of quadrics with the same real and complex intersection type.

A word of caution: the projective transformations involved in the classification of real pencils, if they preserve the real type of the intersection, may well involve irrational numbers. This fact should be kept to mind when interpreting the results.

We treat the first case (nodal quartic) in some detail so that the reader gets accustomed to the techniques we use. For the other cases, we move directly to the normal form without first expliciting the canonical form.

Note that the case where the Segre characteristic is $[1111]$, which corresponds to a smooth quartic, has been extensively treated in Part I. Also, the $[(1111)]$ case does not necessitate any further treatment: save for the quadric corresponding to the quadruple root (which is $\mathbb{P}^3(\mathbb{R})$), all the quadrics of the pencil are equal and the intersection is thus any of those non-trivial quadrics. The case $\det R(\lambda,\mu) \equiv 0$ is treated separately in Section 4.

Here and in the ensuing sections, a singularity of the intersection will be called *convex* if the branches of the curve are on the same side of the common tangent plane to the branches at the singularity, *concave* otherwise.

An additional benefit of the classification of pencils over the reals is the ability to draw pictures of all possible situations. Such a gallery of intersection cases is given in Figure 1. The pictures were made with the *surf* visualization tool [7].

3.1 Nodal quartic in $\mathbb{P}^3(\mathbb{C})$, $\sigma_4 = [112]$

The determinantal equation has a double root λ_1, which is necessarily real (otherwise its conjugate would also be a double root of $\det R(\lambda)$). Let λ_2 and λ_3 be the other roots. The Segre characteristic implies that the three quadrics $R(\lambda_i)$ have rank 3 (equal to $n - t_i$; see Section 2.2.1). The Canonical Pair Form Theorem thus implies that to λ_1 corresponds one real Jordan block of size 2.

There are two cases.

λ_2 and λ_3 are real. $R(\lambda_2)$ and $R(\lambda_3)$ are projective cones. The Canonical Pair Form Theorem gives that S and T are simultaneously congruent to the quadrics of equations

$$\left\{\begin{array}{l} 2\varepsilon_1 xy + \varepsilon_2 z^2 + \varepsilon_3 w^2 = 0, \\ 2\varepsilon_1\lambda_1 xy + \varepsilon_1 y^2 + \varepsilon_2\lambda_2 z^2 + \varepsilon_3\lambda_3 w^2 = 0, \end{array}\right. \qquad \varepsilon_i = \pm 1, i = 1,2,3.$$

$\lambda_1 S - T$ and $\lambda_2 S - T$ are thus simultaneously congruent to the quadrics of equations

$$\left\{\begin{array}{l} -\varepsilon_1 y^2 + \varepsilon_2(\lambda_1-\lambda_2) z^2 + \varepsilon_3(\lambda_1-\lambda_3) w^2 = 0, \\ -\varepsilon_1 y^2 + 2\varepsilon_1(\lambda_2-\lambda_1) xy + \varepsilon_3(\lambda_2-\lambda_3) w^2 = 0. \end{array}\right.$$

Let $\varepsilon = \operatorname{sign}\frac{\lambda_2-\lambda_3}{\lambda_1-\lambda_3}$ (recall that $\lambda_1 \neq \lambda_3$ and $\lambda_2 \neq \lambda_3$). By multiplying the above two equations by $-\varepsilon_1\left|\frac{\lambda_2-\lambda_3}{\lambda_1-\lambda_3}\right|$ and $-\varepsilon_1$, respectively, we can rewrite them as

$$\left\{\begin{array}{l} \left|\frac{\lambda_2-\lambda_3}{\lambda_1-\lambda_3}\right| y^2 - \varepsilon\varepsilon_1\varepsilon_2 \frac{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)}{\lambda_1-\lambda_3} z^2 - \varepsilon\varepsilon_1\varepsilon_3(\lambda_2-\lambda_3) w^2 = 0, \\ \sqrt{\left|\frac{\lambda_2-\lambda_3}{\lambda_1-\lambda_3}\right|}\, y \left(\sqrt{\left|\frac{\lambda_1-\lambda_3}{\lambda_2-\lambda_3}\right|}\, y - 2(\lambda_2-\lambda_1)\sqrt{\left|\frac{\lambda_1-\lambda_3}{\lambda_2-\lambda_3}\right|}\, x\right) - \varepsilon_1\varepsilon_3(\lambda_2-\lambda_3) w^2 = 0. \end{array}\right.$$

Now, we apply the following projective transformation:

$$\sqrt{\left|\frac{\lambda_1-\lambda_3}{\lambda_2-\lambda_3}\right|}\, y - 2(\lambda_2-\lambda_1)\sqrt{\left|\frac{\lambda_1-\lambda_3}{\lambda_2-\lambda_3}\right|}\, x \mapsto x, \quad \sqrt{\left|\frac{\lambda_2-\lambda_3}{\lambda_1-\lambda_3}\right|}\, y \mapsto y,$$
$$\sqrt{\left|\frac{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)}{\lambda_1-\lambda_3}\right|}\, z \mapsto z, \quad \sqrt{|\lambda_2-\lambda_3|}\, w \mapsto w.$$

We obtain that $R(\lambda_1) = \lambda_1 S - T$ and $R(\lambda_2) = \lambda_2 S - T$ are simultaneously congruent, by a real projective transformation P, to the quadrics of equations

$$\left\{\begin{array}{l} P^T R(\lambda_1) P: \; y^2 + az^2 + bw^2 = 0, \\ P^T R(\lambda_2) P: \; xy + cw^2 = 0, \end{array}\right. \tag{2}$$

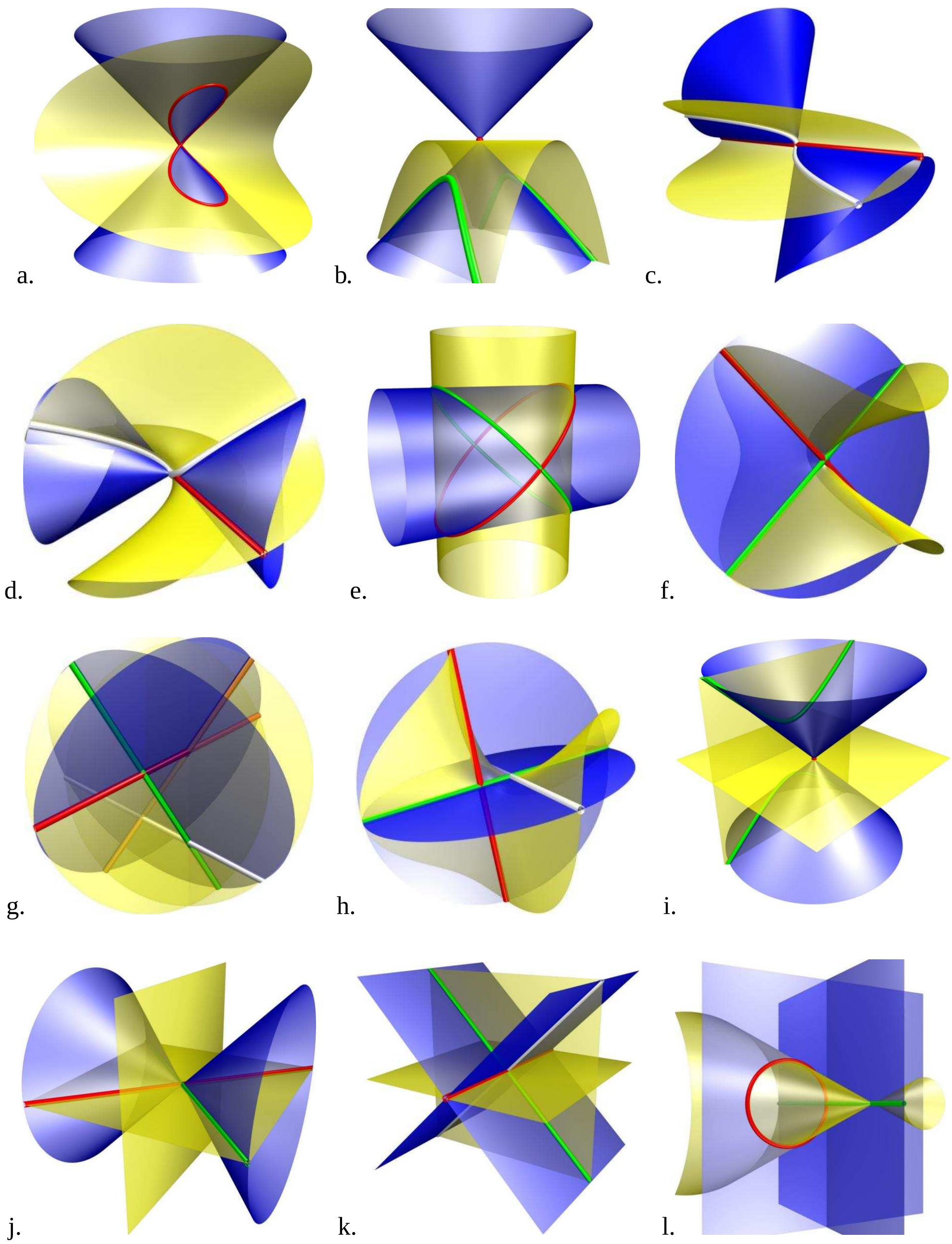

Figure 1: A gallery of intersections. a. Nodal quartic. b. Nodal quartic with isolated singular point. c. Cubic and secant line. d. Cubic and tangent line. e. Two secant conics. f. Two double lines. g. Four skew lines. h. Two lines and a double line. i. Conic and two lines not crossing on the conic, the two lines being imaginary. j. Four concurrent lines, only two of which are real. k. Two lines and a double line, the three being concurrent. l. Conic and double line.

with $a,b,c \in \{-1,1\}$. One can further assume that $c=1$ by changing x by $-x$.

From now on we forget about the transformation P and identify $R(\lambda_i)$ with $P^T R(\lambda_i)P$, but it should be kept to mind that things happen in the local frame induced by P.

If a or b is -1, the cone $R(\lambda_1)$ has inertia $(2,1)$ and thus is real. Otherwise ($a=b=1$), the cone $R(\lambda_1)$ is imaginary but for its real apex $\mathbf{p}=(1,0,0,0)$. The other cone $R(\lambda_2)$ is always real and contains the apex $\mathbf{p}$ of $R(\lambda_1)$. We distinguish the three following cases.

- $a=b=1$. The real part of the nodal quartic is reduced to its node, the apex $\mathbf{p}$ of $R(\lambda_1)$.

- Only one of a and b is 1. Assume for instance that $a=1, b=-1$ (the other case is obtained by exchanging z and w). By substituting the parameterization of the cone $y^2+z^2-w^2=0$ (see Table I.3)
$$\left(s, uv, \frac{u^2-v^2}{2}, \frac{u^2+v^2}{2}\right), \quad (u,v,s) \in \mathbb{P}^{\star 2},$$
into the other cone $xy+w^2=0$, and solving in s, we get the parameterization of the nodal quartic
$$\mathbf{X}(u,v) = \left((u^2+v^2)^2, -4u^2v^2, 2uv(u^2-v^2), 2uv(u^2+v^2)\right)^T, \quad (u,v) \in \mathbb{P}^1.$$
The nodal quartic is thus real and its node, corresponding to the parameters $(1,0)$ and $(0,1)$, is at $\mathbf{p}$. The plane tangent to the quadric $Q_{R(\lambda_2)}$ at the quartic's node $\mathbf{p}$ is $y=0$. In a neighborhood of this node, $x=(u^2+v^2)^2>0$ and $y=-4u^2v^2 \leqslant 0$ (recall that $\mathbf{X}(u,v)$ is projective, so its coordinates are defined up to a non-zero scalar). We conclude that the two branches lie on the same side of the tangent plane and that the singularity is convex.

- $a=-1, b=-1$. Parameterizing the nodal quartic as above, we get the parameterization
$$\mathbf{X}(u,v) = \left(-4u^2v^2, (u^2+v^2)^2, (u^2+v^2)(u^2-v^2), 2uv(u^2+v^2)\right)^T, \quad (u,v) \in \mathbb{P}^1.$$
It can be checked that the point $\mathbf{p}=(1,0,0,0)$ which is on the intersection is not attained by any real value of the parameter (u,v) (it is only attained with the complex parameters $(1,i)$ and $(i,1)$). The nodal quartic is thus real with an isolated singular point.

We now argue that we can easily distinguish between these three cases. For this, we first prove the following lemma.

Lemma 3.1. *Given any pencil of quadrics generated by S and T whose determinantal equation $\det(\lambda S+\mu T)=0$ has a double root (λ_1,μ_1), the sign of $\frac{\det(\lambda S+\mu T))}{(\mu_1\lambda-\lambda_1\mu)^2}$ at (λ_1,μ_1) is invariant by a real projective transformation of the pencil and does not depend on the choice of S and T in the pencil.*

Proof. We suppose that $\mathcal{D}(\lambda,\mu)=\det(\lambda S+\mu T)$ has a double root (λ_1,μ_1). The lemma claims that for any real projective transformation P and any $a_1,\ldots,a_4 \in \mathbb{R}$ such that $a_1a_4-a_2a_3 \neq 0$,
$$\mathcal{D}'(\lambda',\mu') = \det(\lambda' P^T(a_1S+a_2T)P + \mu' P^T(a_3S+a_4T)P)$$

has a double root (λ'_1, μ'_1) such that $\frac{\mathcal{D}(\lambda,\mu)}{(\mu_1\lambda-\lambda_1\mu)^2}$ at (λ_1, μ_1) has same sign as $\frac{\mathcal{D}'(\lambda',\mu')}{(\mu'_1\lambda'-\lambda'_1\mu')^2}$ at (λ'_1, μ'_1). We have

$$\mathcal{D}'(\lambda', \mu') = (\det P)^2 \mathcal{D}(a_1\lambda' + a_3\mu', a_2\lambda' + a_4\mu').$$

Thus $\mathcal{D}'(\lambda', \mu') = 0$ has a double root (λ'_1, μ'_1) defined by

$$\begin{cases} a_1\lambda'_1 + a_3\mu'_1 = \lambda_1 \\ a_2\lambda'_1 + a_4\mu'_1 = \mu_1 \end{cases} \Leftrightarrow \begin{cases} \lambda'_1 = \frac{a_4\lambda_1 - a_3\mu_1}{a_1a_4 - a_2a_3} \\ \mu'_1 = \frac{-a_2\lambda_1 + a_1\mu_1}{a_1a_4 - a_2a_3}. \end{cases}$$

It follows that

$$\frac{\mathcal{D}'(\lambda', \mu')}{(\mu'_1\lambda' - \lambda'_1\mu')^2} = (\det P)^2 \frac{\mathcal{D}(a_1\lambda' + a_3\mu', a_2\lambda' + a_4\mu')}{(\mu_1(a_1\lambda' + a_3\mu') - \lambda_1(a_2\lambda' + a_4\mu'))^2}(a_1a_4 - a_2a_3)^2.$$

Hence $\frac{\mathcal{D}'(\lambda',\mu')}{(\mu'_1\lambda'-\lambda'_1\mu')^2}$ at (λ'_1, μ'_1) has same sign as $\frac{\mathcal{D}(\lambda,\mu)}{(\mu_1\lambda-\lambda_1\mu)^2}$ at (λ_1, μ_1). □

Proposition 3.2. *If the determinantal equation* $\det(\lambda S + \mu T) = 0$ *has two simple real roots and one double root* (λ_1, μ_1) *whose associated matrix* $\lambda_1 S + \mu_1 T$ *has rank three, then the intersection of* S *and* T *in* $\mathbb{C}^3$ *is a nodal quartic whose node is the apex of* $\lambda_1 S + \mu_1 T$.

Moreover, if the inertia of $\lambda_1 S + \mu_1 T$ *is* $(3,0)$ *then the real part of the nodal quartic is reduced to its node. Otherwise the nodal quartic is real; furthermore, if* $\frac{\det(\lambda S+\mu T)}{(\mu_1\lambda-\lambda_1\mu)^2}$ *is negative for* $(\lambda, \mu) = (\lambda_1, \mu_1)$, *the node is isolated and, otherwise, the singularity is convex.*

Proof. The first part of the proposition follows directly from the Segre characteristic (see Section 2.2.1 and Table 1).

If the inertia of $\lambda_1 S + \mu_1 T$ is $(3,0)$, then $a = b = 1$ in (2) and the result follows as discussed above. Otherwise, considering $S' = P^T(\lambda_1 S - T)P$ and $T' = P^T(\lambda_2 S - T)P$, (2) gives that $\det(\lambda S' + \mu T') = -a\lambda(b\lambda + \mu)\mu^2/4$. Evaluating $\frac{\det(\lambda S'+\mu T')}{\mu^2}$ at $(\lambda, \mu) = (1, 0)$, gives by Lemma 3.1 that $-ab$ has same sign as $\frac{\det(\lambda S+\mu T)}{(\mu_1\lambda-\lambda_1\mu)^2}$ at (λ_1, μ_1). The result then follows from the discussion above depending on whether $a = b = -1$ or $ab = -1$. □

λ_2 and λ_3 are complex conjugate. The reduction to normal pencil form is slightly more involved in this case. Let $\lambda_2 = \alpha + i\beta, \lambda_3 = \overline{\lambda_2}, \beta \neq 0$. The Canonical Pair Form Theorem gives that S and T are simultaneously congruent to the quadrics of equations

$$\begin{cases} 2\varepsilon xy + 2zw = 0, \\ 2\varepsilon\lambda_1 xy + \varepsilon y^2 + 2\alpha zw + \beta z^2 - \beta w^2 = 0, \end{cases} \qquad \varepsilon = \pm 1$$

Through this congruence, $S' = \lambda_1 S - T$ has equation

$$\begin{aligned} 0 &= -\varepsilon y^2 + \beta(w^2 - z^2) + 2(\lambda_1 - \alpha)zw, \\ &= -\varepsilon y^2 + \beta(w + \xi z)\left(w - \frac{1}{\xi}z\right), \\ &= -\varepsilon y^2 + \beta z'w', \end{aligned}$$

where ξ is real and positive. Through the congruence and with the above transformation $(z,w) \mapsto (z',w')$, S has equation

$$\begin{aligned} 0 &= 2\varepsilon xy + 2zw \\ &= 2\varepsilon xy + \frac{2}{(\xi+\frac{1}{\xi})^2}\left(\frac{1}{\xi}z'^2 - \xi w'^2 + \left(\xi - \frac{1}{\xi}\right)z'w'\right). \end{aligned}$$

Through the above congruence transformations, the quadric of the pencil $T' = \beta S - 2\frac{\xi-\frac{1}{\xi}}{(\xi+\frac{1}{\xi})^2}(\lambda_1 S - T)$ has equation

$$2\varepsilon y\left(\beta x + \frac{\xi-\frac{1}{\xi}}{(\xi+\frac{1}{\xi})^2}y\right) + \frac{2\beta}{(\xi+\frac{1}{\xi})^2}\left(\frac{1}{\xi}z'^2 - \xi w'^2\right) = 0.$$

Finally, by making a shift on x, rescaling on the four axes, and changing the signs of x and z, we get that the two quadrics of the pencil S' and T' are simultaneously congruent to the quadrics of equations

$$\begin{cases} y^2 + zw = 0, \\ xy + z^2 - w^2 = 0. \end{cases} \tag{3}$$

As before, we now drop reference to the accumulated congruence transformation and work in the local frame. By substituting the parameterization of the cone $y^2 + zw = 0$ (see Table I.3)

$$\left(s, uv, u^2, -v^2\right), \quad (u,v,s) \in \mathbb{P}^{\star 2},$$

into the other quadric $xy + z^2 - w^2 = 0$, and solving in s, we get the parameterization of the nodal quartic

$$\mathbf{X}(u,v) = \left(v^4 - u^4, u^2v^2, u^3v, -uv^3\right)^T, \quad (u,v) \in \mathbb{P}^1.$$

The nodal quartic is thus real and its node, corresponding to the parameters $(1,0)$ and $(0,1)$, is at $\mathbf{p} = (1,0,0,0)$, the apex of S'. The plane tangent to the quadric $xy + z^2 - w^2 = 0$ at the quartic's node $\mathbf{p}$ is $y = 0$. In a neighborhood of the quartic's node on the branch corresponding to the parameter $(0,1)$, $x = v^4 - u^4 > 0$ and $y = u^2v^2 \geqslant 0$. On the other branch corresponding to the parameter $(1,0)$, $x = v^4 - u^4 < 0$ and $y = u^2v^2 \geqslant 0$. Hence, the two branches of the quartic are on opposite sides of the tangent plane $y = 0$ in a neighborhood of the node, i.e., the singularity is concave.

We thus have the following result.

Proposition 3.3. *If the determinantal equation* $\det(\lambda S + \mu T) = 0$ *has two simple complex conjugate roots and one double root* (λ_1, μ_1) *whose associated matrix* $\lambda_1 S + \mu_1 T$ *has rank three, then the intersection of* S *and* T *is a real nodal quartic with a concave singularity at its node, the apex of* $\lambda_1 S + \mu_1 T$.

3.2 Two secant conics in $\mathbb{P}^3(\mathbb{C})$, $\sigma_4 = [11(11)]$

The determinantal equation has a double root λ_1 and the rank of $R(\lambda_1)$ is 2. λ_1 is necessarily real and there are two Jordan blocks of size 1 associated with it in the canonical form. Let λ_2 and λ_3 be the other (simple) roots, associated with quadrics of rank 3. We have two cases.

λ_2 and λ_3 are real. λ_2 and λ_3 appear in real Jordan blocks of size 1. The normal form of $R(\lambda_1)$ and $R(\lambda_2)$ is:

$$\begin{cases} z^2 + aw^2 = 0, \\ x^2 + by^2 + cw^2 = 0, \end{cases}$$

with $a, b, c \in \{-1, 1\}$.

The two planes of $R(\lambda_1)$ are real if the matrix has inertia $(1,1)$, i.e. if $a = -1$. The cone $R(\lambda_2)$ is real if its inertia is $(2,1)$, i.e. if $b = -1$ or $c = -1$. The two conics of the intersection are secant over the reals if the singular line $z = w = 0$ of the pair of planes meets the cone in real points, i.e. if $b = -1$. We have the following cases:

- $a = \pm 1, b = 1, c = 1$: The planes are real or imaginary and the cone is imaginary. The apex of the cone is not on the planes, so intersection is empty.
- $a = 1, b = 1, c = -1$: The planes are imaginary and the cone is real. Their real intersection is the intersection of the singular line $z = w = 0$ of the pair of planes with the cone. The real intersection is thus empty.
- $a = 1, b = -1, c = \pm 1$: The planes are imaginary and the cone is real. The line $z = w = 0$ intersects the cone in two points of coordinates $(1,1,0,0)$ and $(-1,1,0,0)$. The intersection is reduced to these two points.
- $a = -1, b = 1, c = -1$: The planes and the cone are real. The line $z = w = 0$ does not intersect the cone, so intersection consists of two non-secant conics.
- $a = -1, b = -1, c = \pm 1$: The planes and the cone are real. The line $z = w = 0$ intersects the conics. Intersection consists of two conics intersecting in two points $\mathbf{p}^\pm$ of coordinates $(\pm 1, 1, 0, 0)$. All the quadrics of the pencil have the same tangent plane $P^\pm : x \mp y = 0$ at $\mathbf{p}^\pm$. The two conics of the intersection are on the same side of $P^\pm$.

Computing the inertia of $R(\lambda_1)$ gives a. Also, in normal form, the determinantal equation $\det(\lambda R(\lambda_1) + \mu R(\lambda_2))$ is equal to $b\mu^2\lambda(a\lambda + c\mu)$. Thus, by Lemma 3.1, ab is equal to the sign of $\frac{\det(\lambda S + \mu T)}{(\mu_1\lambda - \lambda_1\mu)^2}$ at (λ_1, μ_1). Hence we can easily compute a and b. Finally, we need to compute c but only in the case where $a = -1$ and $b = 1$. Then $c = 1$ if the inertia of $R(\lambda_2)$ (or $R(\lambda_3)$) is $(3,0)$; otherwise $c = -1$ and the inertia of $R(\lambda_2)$ (or $R(\lambda_3)$) is $(2,1)$.

λ_2 and λ_3 are complex conjugate. There are two complex Jordan blocks of size 2 associated with the two roots. The pencil normal form is obtained as in Section 3.1. The end result is:

$$\begin{cases} zw = 0, \\ x^2 + ay^2 + z^2 - w^2 = 0, \end{cases}$$

with $a \in \{-1, 1\}$.

The pair of planes $R(\lambda_1)$ is always real. The intersection consists of the two conics $z = x^2 + ay^2 - w^2 = 0$ and $w = x^2 + ay^2 + z^2 = 0$. We have two cases:

- $a = 1$: One conic is real, the other is imaginary.
- $a = -1$: The two conics are real. They intersect at the points $\mathbf{p}^{\pm}$ of coordinates $(1, \pm 1, 0, 0)$. All the quadrics of the pencil have the same tangent plane $P^{\pm} : x \mp y = 0$ at $\mathbf{p}^{\pm}$. The two conics of the intersection are on opposite sides of $P^{\pm}$.

Note finally that, in normal form, the determinantal equation $\det(\lambda R(\lambda_1) + \mu R(\lambda_2))$ is equal to $-a\mu^2(\mu^2 + \lambda^2/4)$. Hence a is opposite to the sign of $\frac{\det(\lambda S + \mu T)}{(\mu_1 \lambda - \lambda_1 \mu)^2}$ at (λ_1, μ_1) (by Lemma 3.1).

3.3 Cuspidal quartic in $\mathbb{P}^3(\mathbb{C})$, $\sigma_4 = [13]$

The determinantal equation has a triple root λ_1, which is necessarily real. To it corresponds a real Jordan block of size 3. $R(\lambda_1)$ has rank 3. Let λ_2 be the other root, necessarily real, and $R(\lambda_2)$ the associated cone. The normal form of $R(\lambda_1)$ and $R(\lambda_2)$ is:

$$\begin{cases} w^2 + yz = 0, \\ y^2 + xz = 0. \end{cases}$$

The intersection consists of a cuspidal quartic which can be parameterized (in the local frame of the normal form) by

$$\mathbf{X}(u,v) = (v^4, u^2v^2, -u^4, u^3v), \quad (u,v) \in \mathbb{P}^1.$$

The quartic has a cusp at $\mathbf{p} = (1,0,0,0)$ (the vertex of the first cone), which corresponds to $(u,v) = (0,1)$. The intersection of $R(\lambda_1)$ with the plane tangent to $R(\lambda_2)$ at $\mathbf{p}$ gives the (double) line tangent to the quartic at $\mathbf{p}$, i.e. $z = w^2 = 0$.

3.4 Two tangent conics in $\mathbb{P}^3(\mathbb{C})$, $\sigma_4 = [1(21)]$

The determinantal equation has a triple root λ_1 and the rank of $R(\lambda_1)$ is 2. λ_1 is necessarily real. Attached to λ_1 are two real Jordan blocks, one of size 2, the other of size 1. Let λ_2 be the other simple real root, with $R(\lambda_2)$ of rank 3. The normal forms of $R(\lambda_1)$ and $R(\lambda_2)$ are:

$$\begin{cases} x^2 + aw^2 = 0, \\ xy + z^2 = 0, \end{cases}$$

where $a \in \{-1, 1\}$.

The pair of planes $R(\lambda_1)$ is real when the matrix has inertia $(1,1)$, i.e. when $a = -1$. The cone $R(\lambda_2)$ is real since its inertia is $(2,1)$. So we have two cases:

- $a = 1$: The pair of planes is imaginary. Its real part is restricted to the line $x = w = 0$, which intersects the cone in the real double point $(0,1,0,0)$. The intersection is reduced to that point.
- $a = -1$: The planes are real. The intersection consists of two conics intersecting in the double point $(0,1,0,0)$ and sharing a common tangent at that point.

3.5 Double conic in $\mathbb{P}^3(\mathbb{C})$, $\sigma_4 = [1(111)]$

The determinantal equation has a real triple root λ_1 and the rank of $R(\lambda_1)$ is 1. The Jordan normal form of $S^{-1}T$ contains three blocks of size 1 for λ_1. Let λ_2 be the other real root, with $R(\lambda_2)$ of rank 3. The normal forms of $R(\lambda_1)$ and $R(\lambda_2)$ are:

$$\begin{cases} w^2 = 0, \\ x^2 + ay^2 + z^2 = 0, \end{cases}$$

where $a \in \{-1, 1\}$.

The cone $R(\lambda_2)$ is real if its inertia is $(2,1)$, i.e. if $a = -1$. We have two cases:

- $a = -1$: The cone is real. The intersection consists of a double conic lying in the plane $w = 0$.
- $a = 1$: The cone is imaginary. Its real apex does not lie on the plane $w = 0$, so the intersection is empty.

3.6 Cubic and tangent line in $\mathbb{P}^3(\mathbb{C})$, $\sigma_4 = [4]$

The determinantal equation has a quadruple root λ_1 and the rank of $R(\lambda_1)$ is 3. λ_1 is necessarily real. Associated with it is a unique real Jordan block of size 4. The normal form of $R(\lambda_1)$ and S is:

$$\begin{cases} z^2 + yw = 0, \\ xw + yz = 0. \end{cases}$$

The intersection contains the line $z = w = 0$. The cubic is parameterized by

$$\mathbf{X}(u,v) = \left(u^3, -u^2v, uv^2, v^3\right), \quad (u,v) \in \mathbb{P}^1.$$

The cubic intersects the line in the point of coordinate $(1,0,0,0)$, corresponding to the parameter $(1,0)$. The cubic and the line are tangent at that point.

3.7 Conic and two lines crossing on the conic in $\mathbb{P}^3(\mathbb{C})$, $\sigma_4 = [(31)]$

The determinantal equation has a quadruple root λ_1, with $R(\lambda_1)$ of rank 2. λ_1 is necessarily real. To it correspond two real Jordan blocks of size 3 and 1. The normal forms of $R(\lambda_1)$ and S are:

$$\begin{cases} yz = 0, \\ y^2 + xz + aw^2, \end{cases}$$

with $a \in \{-1, 1\}$. $z = 0$ gives two real or imaginary lines. $y = 0$ gives a real conic. The lines cross on the conic at the point $\mathbf{p} = (1,0,0,0)$.

Both the pair of planes and the nonsingular quadric are real. We have two cases:

- $a = 1$: The lines are imaginary. The intersection is reduced to the conic.

- $a = -1$: The lines are real. The intersection consists of a conic and two lines crossing on the conic at $\mathbf{p}$.

The determinantal equation in normal form $\det(\lambda R(\lambda_1) + \mu S) = -a\mu^4/4$ has a quadruple root and thus is always non-negative or non-positive. In this case, it is straightforward to show, similarly as in the proof of Lemma 3.1, that the sign $\geqslant 0$ or $\leqslant 0$ of $\det(\lambda S + \mu T)$ is invariant by real projective transformation and independent of the choice of S and T in the pencil. Hence a is opposite to the sign of $\det(\lambda S + \mu T)$ for any (λ, μ) that is not the quadruple root.

3.8 Two lines and a double line in $\mathbb{P}^3(\mathbb{C})$, $\sigma_4 = [(22)]$

The determinantal equation has a quadruple root λ_1, with $R(\lambda_1)$ of rank 2. λ_1 is necessarily real and there are two real Jordan blocks associated with it, both of size 2. The normal forms of $R(\lambda_1)$ and S are:

$$\begin{cases} y^2 + aw^2 = 0, \\ xy + azw = 0, \end{cases}$$

with $a \in \{-1, 1\}$. The intersection consists of the double line $y = w = 0$ and two single lines $y \pm \sqrt{-a}\, w = x \pm \sqrt{-a}\, z = 0$ cutting the double line in the points $(\mp\sqrt{-a}, 0, 1, 0)$.

The pair of planes is real if its inertia is $(1,1)$, i.e. if $a = -1$. We have two cases:

- $a = 1$: The two single lines are imaginary. The intersection is reduced to the double line $y = w = 0$.

- $a = -1$: The intersection consists of the two single lines $y \pm w = x \pm z = 0$ and the double line $y = w = 0$.

Note that the determinantal equation $\det(\lambda R(\lambda_1) + \mu S)$ is equal in normal form to $\frac{a^2\mu^4}{16}$. Thus $\mathcal{D}(\lambda, \mu)$ is positive for any (λ, μ) distinct from the quadruple root.

3.9 Two double lines in $\mathbb{P}^3(\mathbb{C})$, $\sigma_4 = [(211)]$

The determinantal equation has a quadruple root λ_1, with $R(\lambda_1)$ of rank 1. λ_1 is real and there are three real Jordan blocks associated with it, two having size 1 and the last size 2. The normal forms of $R(\lambda_1)$ and S are:

$$\begin{cases} w^2 = 0, \\ x^2 + ay^2 + zw = 0, \end{cases}$$

with $a \in \{-1, 1\}$. The intersection consists of two double lines $w^2 = x^2 + ay^2 = 0$.

There are two cases:

- $a = 1$: The two double lines are imaginary. The intersection is reduced to their real intersection point, i.e. $(0, 0, 1, 0)$.

- $a = -1$: The two double lines $w^2 = x - y = 0$ and $w^2 = x + y = 0$ are real so the intersection consists of these two lines, meeting at $(0, 0, 1, 0)$.

The determinantal equation (in normal form) is equal to $\det(\lambda R(\lambda_1)+\mu S) = -a\lambda^4/4$ thus a is opposite to the sign of $\det(\lambda S+\mu T)$ for any (λ,μ) that is not a root.

3.10 Cubic and secant line in $\mathbb{P}^3(\mathbb{C})$, $\sigma_4 = [22]$

The determinantal equation has two double roots λ_1 and λ_2. The associated quadrics both have rank 3. λ_1 and λ_2 are either both real or complex conjugate.

λ_1 and λ_2 are real. There is a real Jordan block of size 2 associated with each root. The normal form of $R(\lambda_1)$ and $R(\lambda_2)$ is:

$$\begin{cases} y^2+zw=0, \\ xy+w^2=0. \end{cases}$$

The intersection consists of the line $y=w=0$ and a cubic. The cubic is parameterized by

$$\mathbf{X}(u,v) = \left(u^3, -uv^2, -v^3, u^2v\right), \quad (u,v)\in\mathbb{P}^1.$$

The line intersects the cubic in the two points of coordinates $(1,0,0,0)$ and $(0,0,1,0)$, corresponding to the parameters $(1,0)$ and $(0,1)$.

λ_1 and λ_2 are complex conjugate. Let $\lambda_1 = \alpha+\mathrm{i}\beta, \lambda_2 = \overline{\lambda_1}, \beta\neq 0$. There is complex Jordan block of size 4 associated with the two roots. The normal form of S and $R(\alpha)$ is:

$$\begin{cases} xw+yz=0, \\ xz-yw+zw=0. \end{cases}$$

The intersection contains the line $z=w=0$. The cubic is parameterized by

$$\mathbf{X}(u,v) = \left(-u^2v, uv^2, u^3+uv^2, u^2v+v^3\right), \quad (u,v)\in\mathbb{P}^1.$$

The cubic intersects the line in the points of coordinates $(1,\mathrm{i},0,0)$ and $(1,-\mathrm{i},0,0)$. Thus, over the reals, the cubic and the line do not intersect.

3.11 Conic and two lines not crossing on the conic, $\sigma_4 = [2(11)]$

The determinantal equation has two double roots λ_1 and λ_2, with associated ranks 3 (a projective cone) and 2 (a pair of planes) respectively. The two roots are necessarily real, otherwise the ranks of the quadrics $R(\lambda_1)$ and $R(\lambda_2)$ would be the same. Associated with λ_1 and λ_2 are respectively a unique real Jordan block of size 2 and two real Jordan blocks of size 1. The pencil normal form is:

$$\begin{cases} y^2+az^2+bw^2=0, \\ xy=0, \end{cases}$$

where $a,b\in\{-1,1\}$. The plane $x=0$ contains a conic which is real when $a=-1$ or $b=-1$ and imaginary otherwise. The plane $y=0$ contains two lines which are real if $ab<0$ and imaginary otherwise. The lines cross at the point $(1,0,0,0)$, the apex of $R(\lambda_1)$, which is not on the conic.

The pair of planes $R(\lambda_2)$ is always real. The cone $R(\lambda_1)$ is real when its inertia is $(2,1)$, i.e. when $a=-1$ or $b=-1$. We have three cases:

- $a=1, b=1$: The lines and the conic are imaginary. The intersection is reduced to the real point of intersection of the two lines, i.e. $(1,0,0,0)$.
- $a=-b$: The lines and the conic are real. The intersection consists of a conic and two intersecting lines, each cutting the conic in a point (at $(0,0,1,1)$ and $(0,0,-1,1)$).
- $a=-1, b=-1$: The lines are imaginary, the conic is real. The intersection consists of a conic and the point $(1,0,0,0)$, intersection of the two lines.

To determine in which of the three situations we are, first compute the inertia of $R(\lambda_1)$. If the inertia is $(3,0)$, this implies that $a=b=1$. Otherwise, we consider as before the determinantal equation in normal form $\det(\lambda R(\lambda_1)+\mu R(\lambda_2)) = -ab\lambda^2\mu^2/4$. By Lemma 3.1, $-ab$ is equal to the sign of $\frac{\det(\lambda S+\mu T)}{(\mu_1\lambda-\lambda_1\mu)^2}$ at (λ_1,μ_1). If $ab>0$ then $a=b=-1$, otherwise $a=-b$.

3.12 Four skew lines in $\mathbb{P}^3(\mathbb{C})$, $\sigma_4 = [(11)(11)]$

The determinantal equation has two double roots λ_1 and λ_2, with associated quadrics of rank 2. λ_1 and λ_2 can be either both real or both complex conjugate.

λ_1 and λ_2 are real. Each root appears in two real Jordan blocks of size 1. The normal forms of $R(\lambda_1)$ and $R(\lambda_2)$ are:

$$\begin{cases} z^2+aw^2, \\ x^2+by^2, \end{cases}$$

where $a,b\in\{-1,1\}$.

The first pair of planes is imaginary if $a=1$. The second pair of planes is imaginary if $b=1$. There are three cases:

- $a=1, b=1$: The four lines are imaginary and the intersection is empty.
- $a=-b$: One pair of planes is real, the other is imaginary. If $a=1$, the intersection consists of the points of intersection of the line $z=w=0$ with the pair of planes $x^2-y^2=0$, i.e. the points $(1,1,0,0)$ and $(-1,1,0,0)$. Similarly, if $b=1$ the intersection is reduced to the two points $(0,0,1,1)$ and $(0,0,-1,1)$.
- $a=-1, b=-1$: The four lines are real. The intersection consists of four skew lines.

The values of a and b follow from the inertia of $R(\lambda_1)$ and $R(\lambda_2)$. Note also that b directly follows from a because, the determinantal equation (in normal form) $\det(\lambda R(\lambda_1)+\mu R(\lambda_2)) = ab\lambda^2\mu^2$ and it is straightforward to show that ab is equal to the sign of $\det(\lambda S+\mu T)$ for any (λ,μ) that is not a root.

λ_1 and λ_2 are complex conjugate. Let $\lambda_1 = a + \mathrm{i}b, \lambda_2 = \overline{\lambda_1}, b \neq 0$. The roots appear in two complex Jordan blocks of size 2. The normal forms of S and $aS - T$ are:

$$\begin{cases} xy + zw = 0, \\ x^2 - y^2 + z^2 - w^2 = 0. \end{cases}$$

The intersection consists of two real lines of equations $x \pm w = y \mp z = 0$ and two imaginary lines of equations $x \pm \mathrm{i}z = y \mp \mathrm{i}w = 0$.

4 Classification of singular pencils of $\mathbb{P}^3(\mathbb{R})$ over the reals

We now examine the singular pencils of $\mathbb{P}^3(\mathbb{R})$, i.e. those whose determinantal equation vanishes identically.

There are two cases according to whether two arbitrary quadrics of the pencil have a singular point in common or not.

4.1 Q_S and Q_T have no singular point in common, $\sigma_4 = [1\{3\}]$

We first prove the following lemma.

Lemma 4.1. *If* $\det R(\lambda, \mu) \equiv 0$ *and* Q_S *and* Q_T *have no singular point in common, then every quadric of the pencil has a singular point such that all the other quadrics of the pencil contains this point and share a common tangent plane at this point.*

Proof. Let Q_R be any quadric of the pencil. First note that R has rank at most 3, otherwise the determinantal equation would not identically vanish.

If R has rank 1, it is a double plane in $\mathbb{P}^3(\mathbb{C})$ containing only singular points. Since there is no quadric of inertia $(4,0)$ in the pencil, the intersection of the double plane with every other quadric of the pencil is not empty in $\mathbb{P}^3(\mathbb{R})$ (by Theorem I.4.3). Hence Q_R contains a singular point that belongs to all the quadrics of the pencil.

If R has rank 2, it is a pair of planes in $\mathbb{P}^3(\mathbb{C})$ with a real singular line. By the Segre classification (see Table 2) we know that the intersection in $\mathbb{P}^3(\mathbb{C})$ contains a conic and a double line. Furthermore, the line is necessarily real because otherwise its conjugate would also be in the intersection. This line lies in one of the two planes of Q_R and thus cuts any other line in that plane and in particular the singular line of the pair of planes. Hence Q_R contains a singular point that belongs to all the quadrics of the pencil.

If R has rank 3, we apply a congruence transformation so that Q_R has the diagonal form $ax^2 + by^2 + cz^2 = 0$, with $abc \neq 0$. We may also change the generators of the pencil, replacing S by R. After these transformations, the determinant $\mathcal{D}(\lambda, \mu)$ becomes the sum of $\delta abc\lambda^3\mu$ and of other terms of degree at least 2 in μ, where δ is the coefficient of w^2 in the equation of Q_T. The hypothesis that $\mathcal{D}(\lambda, \mu) \equiv 0$ thus implies that $\delta = 0$. Hence the singular point $(0,0,0,1)$ of Q_R belongs to Q_T and thus to all the quadrics of the pencil.

Thus, in all cases, every quadric of the pencil has a singular point that belongs to all the quadrics of the pencil. Any such point $\mathbf{p}$ lies on the intersection of Q_S and Q_T and is a singular point of the intersection: since **p** is a singular point of a quadric of the pencil, the rank of the Jacobian matrix (1) is less than two. We conclude on the common tangent plane by applying Proposition 2.1 under the assumption that **p** is not a singular point of both Q_S and Q_T. □

By Lemma 4.1, there exist a singular point **s** of Q_S and a singular point **t** of Q_T that belong to all the quadrics $Q_{\lambda S+\mu T}$ of the pencil. Quadrics Q_S, Q_T, and Q_{S+T} have rank at most 3 since the determinantal equation identically vanishes, and they are not of inertia $(3,0)$ (see Table I.1) since they contain **s** and **t** that are distinct by assumption. Hence Q_S, Q_T, and Q_{S+T} are cones or pairs of (possibly complex) planes. Thus, since **s** and **t** are singular points of Q_S and Q_T, respectively, the line **st** is entirely contained in Q_S and Q_T, and thus is also contained in Q_{S+T}. Moreover, **s** and **t** are not singular points of Q_{S+T} because otherwise all the quadrics of the pencil would be singular at these points, contradicting the hypothesis. It now follows from the fact that Q_{S+T} is a cone or a pair of planes that its tangent planes at **s** and **t** coincide. Therefore, by Lemma 4.1, the tangent plane of Q_S at **t** is the same as the tangent plane of Q_T at **s**.

Now we change of frames in such a way that **s** and **t** have coordinates $(0,0,0,1)$ and $(0,0,1,0)$ and that the common tangent plane has equation $x=0$. Then the equations of Q_S and Q_T become $xz+q_1(x,y)=0$ and $xw+q_2(x,y)=0$, where q_1 and q_2 are binary quadratic forms. The two equations can thus be expressed in the form $ay^2+x(by+cx+z)=0$ and $a'y^2+x(b'y+c'x+w)=0$. By a new change of frame, we get equations of the form $ay^2+xz=0$ and $a'y^2+xw=0$. Replacing the second quadric by a linear combination of the two and applying the change of coordinates $a'z-aw \to w$ and a scaling on y, gives as normal form for the pencil:

$$\begin{cases} xw=0, \\ xz+ay^2=0, \end{cases}$$

with $a \in \{-1,1\}$. Furthermore, we can set $a=1$ by changing z in $-z$.

Therefore, the intersection consists of the double line $x=y^2=0$ and the conic $w=xz-y^2=0$. The line and the conic meet at $(0,0,1,0)$ in the local frame of the normal form.

4.2 Q_S and Q_T have a singular point in common

Let **p** be the common singular point. We proceed as already outlined in Section 2.2. After a rational change of frame, we may suppose that **p** has coordinates $(0,0,0,1)$. In the new frame, the equations of the quadrics are homogeneous polynomials of degree 2 in three variables. To classify the different types of intersection, we may identify the quadrics with their upper left 3×3 matrices and look at the multiple roots of the cubic determinantal equation, which we refer to as the *restricted determinantal equation*, and the ranks of the associated matrices. We thus apply the Canonical Pair Theorem to pairs of conics.

The case $[(111)]$ is trivial and left aside: in that situation, the cubic determinantal equation has a (real) triple root, the associated quadric has rank 0 and all the other quadrics of the pencil are cones. The intersection consists of any cone of the pencil, that is any quadric of the pencil distinct from $\mathbb{P}^3(\mathbb{R})$.

4.2.1 Four concurrent lines in $\mathbb{P}^3(\mathbb{C})$, $\sigma_3 = [111]$

The restricted determinantal equation has three simple roots. At least one is real: call it λ_1. Let λ_2 be another root. To these roots correspond quadrics of rank 2.

If λ_2 is real, then the three roots are real. The normal form of $R(\lambda_1)$ and $R(\lambda_2)$ is:

$$\begin{cases} ay^2 + z^2 = 0, \\ bx^2 + z^2 = 0, \end{cases}$$

with $a, b \in \{-1, 1\}$. Note that the equation of the third pair of planes of the pencil is obtained by subtracting the two equations, giving $ay^2 - bx^2 = 0$. We have two cases:

- $a = b = -1$: The intersection consists of four concurrent lines $y - \varepsilon_1 z = x - \varepsilon_2 z = 0$, with $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$, meeting at **p**.
- $a = b = 1$ or $a = -b$: When $a = b = 1$, both $R(\lambda_1)$ and $R(\lambda_2)$ represent imaginary pairs of planes. When $a = -b$, the third pair of planes is imaginary, as well as one of the first two. In both cases, the intersection is reduced to the point **p**.

Both a and b are equal to -1 if and only if $R(\lambda_1)$ and $R(\lambda_2)$ have inertia $(1, 1)$.

If $\lambda_2 = \alpha + i\beta$ is complex, $\beta \neq 0$, we obtain the following normal form (proceeding as in Section 3.1):

$$\begin{cases} x^2 + y^2 - z^2 = 0, \\ yz = 0. \end{cases}$$

The intersection consists of two real lines $y = x^2 - z^2 = 0$, intersecting at **p**, and two complex lines $z = x^2 + y^2 = 0$.

4.2.2 Two lines and a double line in $\mathbb{P}^3(\mathbb{C})$, $\sigma_3 = [12]$

The restricted determinantal equation has a double root λ_1, which is real. The Jordan normal form of $S^{-1}T$ contains one real Jordan block of size 2. Let λ_2 be the other root, also real. The normal forms of $R(\lambda_1)$ and $R(\lambda_2)$ are:

$$\begin{cases} y^2 + az^2 = 0, \\ xy = 0, \end{cases}$$

where $a \in \{-1, 1\}$. There are two cases:

- $a = -1$: The intersection consists of the double line $y = z^2 = 0$ and the two single lines $x = y - z = 0$ and $x = y + z = 0$. The three lines are concurrent at **p**.
- $a = 1$: The two single lines are imaginary. Their common point is on the double line, so the intersection consists of this double line $y = z^2 = 0$.

Note that the value of a follows from the inertia of $R(\lambda_1)$.

4.2.3 Two double lines in $\mathbb{P}^3(\mathbb{C})$, $\sigma_3 = [1(11)]$

The restricted determinantal equation has a double root λ_1, which is real. The canonical pair form has two real Jordan blocks of size 1 associated with λ_1. Let λ_2 be the other root, also real. The normal forms of $R(\lambda_1)$ and $R(\lambda_2)$ are:

$$\begin{cases} z^2 = 0, \\ x^2 + ay^2 = 0, \end{cases}$$

where $a \in \{-1, 1\}$. The pair of planes $R(\lambda_2)$ is real when its inertia is $(1,1)$, i.e. when $a = -1$. We have two cases:

- $a = 1$: The intersection is reduced to the point **p**.
- $a = -1$: The intersection consists of the two double lines $x - y = z^2 = 0$ and $x + y = z^2 = 0$, meeting at **p**.

Note that the value of a follows from the inertia of $R(\lambda_2)$.

4.2.4 Line and triple line in $\mathbb{P}^3(\mathbb{C})$, $\sigma_3 = [3]$

The restricted determinantal equation has a triple root λ_1, which is real. The Jordan normal form of $S^{-1}T$ contains one real Jordan block of size 3. The normal forms of S and $R(\lambda_1)$ are:

$$\begin{cases} xz + y^2 = 0, \\ yz = 0. \end{cases}$$

The intersection consists of the triple line $z = y^3 = 0$ and the simple line $x = y = 0$. The two lines cut at **p**, the singular point of all the quadrics of the pencil.

4.2.5 Quadruple line in $\mathbb{P}^3(\mathbb{C})$, $\sigma_3 = [(21)]$

The restricted determinantal equation has a real triple root λ_1. The canonical pair form has two real Jordan blocks of size 2 and 1. The normal form of $R(\lambda_1)$ and S is:

$$\begin{cases} y^2 = 0, \\ z^2 + xy = 0. \end{cases}$$

The intersection consists of the quadruple line $y^2 = z^2 = 0$.

4.2.6 $\sigma_3 = [\{3\}]$ and remaining cases

In this case, the restricted determinantal equation identically vanishes. One can easily prove that if the two conics S and T have no singular point in common, the pencil can be put in the normal form $\lambda xy + \mu xz$. The intersection consists of the plane $x = 0$ and the line $y = z = 0$, which meets the plane at **p**.

If the two conics have a singular point in common (call it $\mathbf{q}$), we can go from 3×3 matrices to 2×2 matrices pretty much as above by sending $\mathbf{q}$ to $(0,0,1,0)$. Consider the new determinantal equation, a quadratic equation. The cases are:

- Two simple real roots: The pencil can be put in the normal form $\lambda x^2 + \mu y^2$. The intersection consists of the quadruple line $x^2 = y^2 = 0$ which goes through $\mathbf{p}$ and $\mathbf{q}$.
- Two simple complex roots: A normal form for the pencil is $\lambda xy + \mu(x^2 - y^2)$, giving the quadruple line $x^2 = y^2 = 0$ for the intersection.
- A double real root, with a real Jordan block of size 2: The normal form is $\lambda xy + \mu y^2$. The intersection consists of the plane $y = 0$.
- Vanishing quadratic equation: The intersection consists of a double plane.

5 Classifying degenerate intersections

Our near-optimal algorithm for parameterizing intersections of quadrics works in two stages: first it determines the real type of the intersection and, second, it computes a parameterization of this intersection. The purpose of this section is to detail the first stage, called the *type-detection* phase. The second stage, which consists of case-by-case algorithms for computing (near-)optimal parameterizations of the real part of the intersection, will be presented in Part III.

The splitting in two stages reflects a key design choice of our parameterization algorithm, which sums up as: the sooner you know what is the type of the intersection, the less prone you are of making errors in later stages. Information obtained in the type-detection phase is used to drive the algorithm and make it efficiently compute precisely and uniquely what is needed.

Note however that presenting the type detection distinctly from the parameterization is quite an oversimplification. In the actual implementation, there is no clear cut separation between the two stages, which are largely intertwined. Sometimes detecting the type of the intersection is doing a very small step towards parameterization. And sometimes almost everything takes place in the type-detection phase.

We now turn to a high-level description of the detection phase, which relies heavily on the results of Sections 3 and 4 on real pencils of quadrics. We start by presenting some global tools, and then outline the type-detection algorithm for each of the possible root patterns (vanishing determinantal equation, one double root, one triple root, one quadruple root, two double roots).

5.1 Preliminaries

In what follows, we assume that the two input quadrics Q_S and Q_T have rational coefficients. We now briefly describe the basic operations needed for detecting the real type of the intersection $Q_S \cap Q_T$. They essentially fall in two categories: linear algebra routines and elementary algebraic manipulations. Most computations involve rational numbers. We give special attention to the limited number of situations where this is not the case.

Let $R(\lambda,\mu) = \lambda S + \mu T$ be the pencil generated by S and T. Computing the coefficients of the determinantal equation $\mathcal{D}(\lambda,\mu) = \det R(\lambda,\mu)$ involves nothing but computing determinants of rational matrices, so there is nothing difficult here. Next, we need to compute the inertia and rank of a matrix $R_0 = R(\lambda_0,\mu_0)$ of the pencil, where (λ_0,μ_0) is a root of $\mathcal{D}$ or a real rational projective number.

Assume first that R_0 has real and rational coefficients. Let $p(\omega) = \det(R_0 - \omega I)$, where I is the identity matrix. Since R_0 is symmetric, all its eigenvalues are real. We can thus compute the number e^+ of positive eigenvalues and the number e^- of negative eigenvalues of R_0 by applying Descartes' Sign Rule to $p(\omega)$ and $p(-\omega)$ respectively. Then the inertia I_0 of R_0 is the pair (e^+,e^-) and its rank r_0 is $e^+ + e^-$.

When the coefficients of R_0 are not rational or real, the worst situation that we have to deal with is when $\mathcal{D}$ has two real or complex conjugate double roots. In both cases, the coefficients of R_0 involve one square root. When the roots are real conjugate, we could use Descartes' Sign Rule again, except that we have to evaluate the signs of coefficients that now involve a square root. In these cases we however propose a more efficient approach based on the rank of R_0 (see Algorithm 6) which only deals with rational numbers. We show below how the rank of R_0 can be computed using only standard linear algebra operations on rational numbers.

Let $(\lambda_0,\mu_0) = (a, b \pm \sqrt{c})$, where $(a,b,c) \in \mathbb{P}^2(\mathbb{Q})$, c is not a square and c is either > 0 (real conjugate roots) or < 0 (complex conjugate roots). Form the 8×8 rational matrix

$$M_0 = \begin{pmatrix} aS + bT & cT \\ T & aS + bT \end{pmatrix}.$$

If the vector $\mathbf{k}_1 + \sqrt{c}\mathbf{k}_2$ is in the kernel of R_0, then the column vector $(\mathbf{k}_1^T, \mathbf{k}_2^T)^T$ is in the kernel of M_0. But the vector $(c\mathbf{k}_2^T, \mathbf{k}_1^T)$ also is in the kernel of M_0. It is not too difficult to realize that the kernel of M_0 has twice the number of elements of the kernel of R_0, i.e. $\dim\ker R_0 = \frac{1}{2}(\dim\ker M_0 - 1)$. We can thus conclude that

$$r_0 = 3 - \dim\ker R_0 = \frac{1}{2}(7 - \dim\ker M_0).$$

Computing the singular space of a quadric with rational coefficients is another operation we need. This only amounts to computing the kernel of the associated matrix. Also, intersecting the singular spaces of two quadrics is like computing the intersection between two linear spaces: it is a standard linear algebra operation.

In terms of algebraic computations, we need to be able to compute the gcd of polynomials of degree at most 3 and to isolate the roots of a cubic polynomial (in the four concurrent lines case). This last task can be done using Uspensky's algorithm as in Part I or using special methods for comparing the roots of low-degree polynomials (see [4]).

The top level type-detection loop is given in Algorithm 1. It does not necessitate much comment save for the fact that when the gcd $\mathcal{H}$ of the two partial derivatives of $\mathcal{D}$ has degree 2, then either its discriminant vanishes, in which case $\mathcal{D}$ has a triple root, or it does not vanish and $\mathcal{D}$ has two double roots.

Algorithm 1 Main loop for degenerate intersection classification.

Require: a pencil of quadrics $R(\lambda,\mu) = \lambda S + \mu T$

compute $\mathcal{D}(\lambda,\mu) := \det R(\lambda,\mu)$
if $\mathcal{D} \equiv 0$ **then** *// vanishing determinantal equation*
  execute Algorithm 2
else
  compute $\mathcal{H} := \gcd(\partial\mathcal{D}/\partial\lambda, \partial\mathcal{D}/\partial\mu)$ and let $d := \mathrm{degree}(\mathcal{H}, \{\lambda,\mu\})$
  if $d = 0$ **then** *// no multiple root*
    output: `smooth quartic` $(\mathbb{C})$ – see Part I
  else if $d = 1$ **then** *// double real root*
    execute Algorithm 3
  else if $d = 2$ **then**
    if discriminant$(\mathcal{H}) = 0$ **then** *// triple real root*
      execute Algorithm 4
    else *// two double roots*
      execute Algorithm 6
    end if
  else *//* $d = 3$*: quadruple real root*
    execute Algorithm 5
  end if
end if

5.2 $\mathcal{D}$ vanishes identically

The type-detection algorithm when $\mathcal{D}$ is identically zero, outlined in Algorithm 2, is little more than a reprise of the information contained in Section 4. The general idea is: determine if two arbitrary quadrics of the pencil have a singular point **p** in common. If they do, send **p** to infinity and work on the pencil of conics living in the plane $w = 0$. To actually compute the restricted pencil $R_3(\lambda, mu)$, build the matrix of a real projective transformation P obtained by putting **p** as the last column and completing P so that its columns form a basis of $\mathbb{P}^3(\mathbb{R})$. $R_3(\lambda,\mu)$ is then the principal submatrix of the matrix $P^T(\lambda S + \mu T)P$.

Two comments are in order. First, a multiple root of a cubic form – the determinantal equation of the pencil of conics – is necessarily real (otherwise its complex conjugate would also be a multiple root) and rational (otherwise its real conjugate would also be a multiple root).

So the only place where we might end up working with non-rational numbers is in the four concurrent lines case. Indeed, in this situation the restricted determinantal equation $\mathcal{D}_3$ is a cubic form with three generically non-rational simple roots. Computing the sign of the discriminant of $\mathcal{D}_3$ can help us distinguish between the cases when only two lines are real and when the four lines are either all real or all imaginary. But this is not enough to give a complete characterization over the reals. We have thus decided to apply Finsler's Theorem (Theorem I.4.3), as in the smooth quartic case, after isolating the roots of the cubic. If the restricted pencil contains a conic of inertia $(3,0)$, then the intersection of conics is empty in the plane $w = 0$ and the intersection of the two initial quadrics is reduced to **p**.

Algorithm 2 Classifying the intersection when the determinantal equation vanishes.

Require: $R(\lambda,\mu)$ (from Algorithm 1)

let $\gamma := \text{singular}(Q_S) \cap \text{singular}(Q_T)$, $\kappa := \dim\gamma$
if $\kappa = -1$ **then** *// conic and double line* $(\mathbb{C})$
    output: `conic and double line`
else *//* $\kappa \geqslant 0$*: at least one common singular point* $\gamma(1)$
    send $\gamma(1)$ to the point $[0\ \ 0\ \ 0\ \ 1]$
    compute the restricted pencil $R_3(\lambda,\mu)$ of upper left 3×3 matrices and $\mathcal{D}_3(\lambda,\mu) := \det R_3(\lambda,\mu)$
    if $\mathcal{D}_3 \equiv 0$ **then** *// vanishing restricted determinantal equation*
        either $\sigma_3 = [\{3\}]$ or repeat restriction
    else *//* $\mathcal{D}_3 \not\equiv 0$
        compute $\mathcal{H}_3 := \gcd(\partial\mathcal{D}_3/\partial\lambda, \partial\mathcal{D}_3/\partial\mu)$ and let $d_3 := \text{degree}(\mathcal{H}_3, \{\lambda,\mu\})$
        if $d_3 = 0$ **then** *// no multiple root: four concurrent lines* $(\mathbb{C})$
            if $\mathcal{D}_3$ has only one real root **then**
                output: `two concurrent lines`
            else if $R_3(\lambda,\mu)$ contains a conic of inertia $(3,0)$ **then**
                output: `point`
            else
                output: `four concurrent lines`
            end if
        else *// one multiple root*
            let (λ_0,μ_0) be the multiple root of $\mathcal{D}_3$, I_0 and r_0 the inertia and rank (resp.) of $R(\lambda_0,\mu_0)$
            if $d_3 = 1$ **then** *//* $\mathcal{D}_3$ *has one double root*
                if $r_0 = 2$ **then** *// two concurrent lines and a double line* $(\mathbb{C})$
                    if $I_0 = (1,1)$ **then** *// pair of planes* $R_3(\lambda_0,\mu_0)$ *is real*
                        output: `two concurrent lines and a double line`
                    else *// pair of planes* $R_3(\lambda_0,\mu_0)$ *is imaginary*
                        output: `double line`
                    end if
                else *//* $r_0 = 1$*: two double lines* $(\mathbb{C})$
                    let (λ_1,μ_1) be the other root of $\mathcal{D}_3$, I_1 the inertia of $R(\lambda_1,\mu_1)$
                    if $I_1 = (1,1)$ **then** *// pair of planes* $R_3(\lambda_1,\mu_1)$ *is real*
                        output: `two double lines`
                    else *// pair of planes* $R_3(\lambda_1,\mu_1)$ *is imaginary*
                        output: `point`
                    end if
                end if
            else *//* $d_3 = 2$*:* $\mathcal{D}_3$ *has one triple root*
                if $r_0 = 2$ **then** *// line and triple line* $(\mathbb{C})$
                    output: `line and triple line`
                else if $r_0 = 1$ **then** *// quadruple line* $(\mathbb{C})$
                    output: `quadruple line`
                else *//* $r_0 = 0$*: projective cone* $(\mathbb{C})$
                    if S or T (in restricted form) has inertia $(2,1)$ **then**
                        output: `cone`
                    else
                        output: `point`
                    end if
                end if
            end if
        end if
    end if
end if

5.3 $\mathcal{D}$ has a single multiple root

The type-detection algorithms when $\mathcal{D}$ has a unique multiple root are given in Algorithms 3 (double root), 4 (triple root) and 5 (quadruple root).

First note that when $\mathcal{D}$ has a single multiple root, it is necessarily real and rational, for the same reasons as above. So the singular quadrics that we deal with all have rational coefficients and their singular set can be parameterized rationally.

In the double real root case, we use the result of Lemma 3.1 and classify the intersections according (among others) the sign

$$s := \operatorname{sign} \mathcal{E}(\lambda_0, \mu_0), \quad \text{with} \quad \mathcal{E}(\lambda, \mu) := \mathcal{D}(\lambda, \mu)/(\mu_0\lambda - \lambda_0\mu)^2, (\lambda_0, \mu_0) \text{ double root of } \mathcal{D}.$$

The other slight difficulty occurs in the two secant conics case, when the pair of planes $R(\lambda_0, \mu_0)$ is real and $s = -1$. To separate the two subcases (two non-secant conics or empty set), we can compute the inertia of $R(\lambda_1, \mu_1)$, where (λ_1, μ_1) is a simple root of $\mathcal{D}$, when this root is rational. But in the general case, we use again Finsler's theorem, looking for a quadric of inertia $(4,0)$ between and outside the two simple roots of $\mathcal{D}$. If such a quadric is found, the intersection is empty.

The triple real root case does not necessitate further comment except for noticing that the additional simple root (λ_1, μ_1) of $\mathcal{D}$ is necessarily real and rational, so the associated quadric $R(\lambda_1, \mu_1)$ has rational coefficients.

The type detection in the quadruple real root case is pretty straightforward. The only subtlety is that the case of a quadruple real root (λ_0, μ_0) with associated quadric of rank 2 corresponds to two different Jordan decompositions, with Segre symbols $[(22)]$ and $[(31)]$, as already mentioned in Section 2.2. To distinguish between the two, we simply note that in the first situation, the singular line of the pair of planes $R(\lambda_0, \mu_0)$ is entirely contained in all the quadrics of the pencil.

5.4 $\mathcal{D}$ has two double roots

When $\mathcal{D}$ has two double roots, we have distinguished between two situations: the roots are either real and rational, or they are not. In the first case, computing the inertia of singular quadrics is easy since computations take place over the rationals. Also, note that if one of the singular quadrics has rank 2 and the other has rank 3, then the associated roots of the determinantal equation are necessarily rational.

So assume that the roots are not real or not rational. The ranks of the non-rational singular quadrics are necessarily the same, so we need only compute one of them, in the way indicated above. When this rank is 2 and the roots are real conjugate, we distinguish between the remaining subcases (four real skew lines or empty set) by testing whether any quadric of the pencil between the two roots has inertia $(4,0)$ or not.

6 Examples

We now give several examples for which the type of the real part of the intersection is determined using the type-detection algorithms of the previous section.

Algorithm 3 Classifying the intersection: the double real root case, $d = 1$.

Require: $R(\lambda,\mu)$, $\mathcal{D}$ (from Algorithm 1)

Require: double real root (λ_0,μ_0), inertia I_0 and rank r_0 of $R(\lambda_0,\mu_0)$

```
let s := sign E(λ0,μ0) and δ := sign(discriminant(E)), where E(λ,μ) := D(λ,μ)/(μ0 λ − λ0 μ)^2
if r0 = 3 then    // nodal quartic (ℂ)
  if δ = −1 then    // other roots are complex
    output: nodal quartic, concave singularity
  else    // other roots are real
    if s = +1 then
      output: nodal quartic, convex singularity
    else if I0 = (2,1) then    // cone R(λ0,μ0) is real
      output: nodal quartic with isolated singular point
    else    // cone R(λ0,μ0) is imaginary
      output: point
    end if
  end if
else    // r0 = 2: two secant conics (ℂ)
  if δ = −1 then    // other roots are complex
    if s = +1 then
      output: two secant conics, concave singularities
    else
      output: one conic
    end if
  else    // other roots are real
    if I0 = (1,1) then    // pair of planes R(λ0,μ0) is real
      if s = +1 then
        output: two secant conics, convex singularities
      else
        if R(λ,μ) contains a quadric of inertia (4,0) then
          output: ∅
        else
          output: two non-secant conics
        end if
      end if
    else    // pair of planes R(λ0,μ0) is imaginary
      if s = +1 then
        output: ∅
      else
        output: two points
      end if
    end if
  end if
end if
```

Algorithm 4 Classifying the intersection: the triple real root case, $d = 2$ and discriminant$(\mathcal{H}) = 0$

Require: $R(\lambda,\mu), \mathcal{D}$ (from Algorithm 1)
Require: triple real root (λ_0,μ_0), inertia I_0 and rank r_0 of $R(\lambda_0,\mu_0)$
if $r_0 = 3$ **then** *// cuspidal quartic* $(\mathbb{C})$
  output: `cuspidal quartic`
else if $r_0 = 2$ **then** *// two tangent conics* $(\mathbb{C})$
  if $I_0 = (1,1)$ **then** *// pair of planes $R(\lambda_0,\mu_0)$ is real*
    output: `two tangent conics`
  else *// pair of planes $R(\lambda_0,\mu_0)$ is imaginary*
    output: `point`
  end if
else *// $r_0 = 1$: double conic* $(\mathbb{C})$
  let I_1 be the inertia of $R(\lambda_1,\mu_1)$, (λ_1,μ_1) the second root of $\mathcal{D}$
  if $I_1 = (2,1)$ **then** *// cone $R(\lambda_1,\mu_1)$ is real*
    output: `double conic`
  else *// cone $R(\lambda_1,\mu_1)$ is imaginary*
    output: $\emptyset$
  end if
end if

6.1 Example 1

Consider the following pair of quadrics:

$$\begin{cases} Q_S : -x^2 - 4xy + 4xz - 6y^2 + 2yz - 4yw + 2zw - 2w^2 = 0, \\ Q_T : -x^2 - 6xy + 4xz - 2xw - 6y^2 - 8yw - 6w^2 = 0. \end{cases}$$

The determinantal equation is

$$\mathcal{D}(\lambda,\mu) = \det(\lambda S + \mu T) = -16(2\lambda^4 - 10\lambda^3\mu - 19\lambda^2\mu^2 - 16\lambda\mu^3 - 5\mu^4).$$

The gcd of the partial derivatives is equal to $32(\lambda+\mu)$. So, by Algorithm 1, $\mathcal{D}$ has a double real root at $(\lambda_0,\mu_0) = (1,-1)$.

We then follow Algorithm 3. Let $R_0 = \lambda_0 S + \mu_0 T$. We have:

$$\det(R_0 - xI) = x^4 - 4x^3 - 8x^2.$$

Descartes' Sign Rule gives that the inertia of R_0 is $I_0 = (1,1)$ and the rank is $r_0 = 2$. The intersection thus consists of two secant conics over the complexes. We compute

$$\mathcal{E}(\lambda,\mu) = \frac{\mathcal{D}(\lambda,\mu)}{(\mu_0\lambda - \lambda_0\mu)^2} = -16(2\lambda^2 + 6\lambda\mu + 5\mu^2).$$

So $\delta = \operatorname{sign}(\operatorname{discriminant}(\mathcal{E})) = -1$ and $s = \operatorname{sign}\mathcal{E}(\lambda_0,\mu_0) = -1$. We conclude that the intersection consists, over the reals, of a single conic.

This conic can be parameterized by (see Part III)

$$\mathbf{X}(u,v) = (2u^2 - 12uv + 18v^2, -u^2 + 2uv + 3v^2, 8v^2, u^2 - 2uv - 3v^2), \quad (u,v) \in \mathbb{P}^1(\mathbb{R}).$$

Algorithm 5 Classifying the intersection: the quadruple real root case, $d = 3$.

Require: $R(\lambda,\mu), \mathcal{D}$ (from Algorithm 1)
Require: quadruple real root (λ_0,μ_0), inertia I_0 and rank r_0 of $R(\lambda_0,\mu_0)$

if $r_0 = 3$ **then** *// cubic and tangent line* $(\mathbb{C})$
  output: `cubic and tangent line`
else if $r_0 = 2$ **then** *// conic and two lines crossing or two skew lines and a double line* $(\mathbb{C})$
  if $I_0 = (2,0)$ **then** *// pair of planes $R(\lambda_0,\mu_0)$ is imaginary*
    output: `double line`
  else *// pair of planes $R(\lambda_0,\mu_0)$ is real*
    if $s = +1$ **then**
      let l_0 be the singular line of $R(\lambda_0,\mu_0)$
      if l_0 is contained in Q_S and Q_T **then**
        output: `two skew lines and a double line`
      else
        output: `conic and two lines crossing on conic`
      end if
    else // $s = -1$
      output: `conic`
    end if
  end if
else if $r_0 = 1$ **then** *// two double lines* $(\mathbb{C})$
  if $s = +1$ **then**
    output: `two secant double lines`
  else // $s = -1$
    output: `point`
  end if
else *// $r_0 = 0$: smooth quadric* $(\mathbb{C})$
  if S or T has inertia $(2,2)$ **then**
    output: `smooth quadric`
  else
    output: `∅`
  end if
end if

6.2 Example 2

Consider the following pair of quadrics:

$$\begin{cases} Q_S : -5x^2 - 2xy - 4y^2 - 12yz - 6yw - 8z^2 - 4zw + w^2 = 0, \\ Q_T : -2x^2 - 2xy + 3y^2 + 6yz + 4z^2 + 2zw + w^2 = 0. \end{cases}$$

The determinantal equation is

$$\mathcal{D}(\lambda,\mu) = \det(\lambda S + \mu T) = -3(16\lambda^4 - 8\lambda^2\mu^2 + \mu^4).$$

Algorithm 6 Classifying the intersection: the two double roots case, $d = 2$ and discriminant$(\mathcal{H}) \neq 0$.

Require: $R(\lambda,\mu)$, $\mathcal{D}$, $\mathcal{H}$ (from Algorithm 1)
Require: double roots (λ_0,μ_0) and (λ_1,μ_1)

let $\delta :=$ discriminant$(\mathcal{H})$ and s be the sign of $\mathcal{D}$ outside the roots
if $\delta > 0$ and δ is a square **then** *// double roots are real and rational*
let r_0 and r_1 ($r_0 \leqslant r_1$) be the ranks of $R(\lambda_0,\mu_0)$ and $R(\lambda_1,\mu_1)$, I_1 the inertia of the second
if $r_0 = 3$ and $r_1 = 3$ **then** *// cubic and secant line* $(\mathbb{C})$
output: `cubic and secant line`
else *//* $r_0 = 2$
if $r_1 = 3$ **then** *// conic and two lines not crossing on conic* $(\mathbb{C})$
if $I_1 = (3,0)$ **then** *// cone* $R(\lambda_1,\mu_1)$ *is imaginary*
output: `point`
else *// cone* $R(\lambda_1,\mu_1)$ *is real*
if $s = +1$ **then**
output: `conic and two lines`
else
output: `conic and point`
end if
end if
else *//* $r_1 = 2$*: four skew lines* $(\mathbb{C})$
if $s = -1$ **then**
output: `two points`
else if $R(\lambda,\mu)$ contains a quadric of inertia $(4,0)$ **then**
output: $\emptyset$
else
output: `four skew lines`
end if
end if
end if
else *// double roots are complex or real non-rational*
let $r_0 := 3 - \dim(\text{singular}(Q_{R(\lambda_0,\mu_0)}))$
if $r_0 = 2$ **then** *//* $R(\lambda_0,\mu_0)$ *and* $R(\lambda_1,\mu_1)$ *are pairs of planes*
if $\delta < 0$ **then** *// roots are complex conjugate*
output: `two skew lines`
else *//* $\delta > 0$*: roots are real conjugate*
if $s = -1$ **then**
output: `two points`
else if $R(\lambda,\mu)$ contains a quadric of inertia $(4,0)$ **then**
output: $\emptyset$
else
output: `four skew lines`
end if
end if
else *//* $r_0 = 3$*:* $R(\lambda_0,\mu_0)$ *and* $R(\lambda_1,\mu_1)$ *are cones*
if $\delta < 0$ **then** *// roots are complex conjugate*
output: `cubic and non-secant line`
else
output: `cubic and secant line`
end if
end if
end if

The gcd of the partial derivatives is equal to $\mathcal{H} = 12(\mu^2 - 4\lambda^2)$. Since the discriminant δ of $\mathcal{H}$ is not zero, $\mathcal{D}$ has two double roots, according to Algorithm 1. Further, δ is positive and is a square, so $\mathcal{D}$ has two real rational double roots. These roots are $(\lambda_0, \mu_0) = (-1, -2)$ and $(\lambda_1, \mu_1) = (-1, 2)$.

We now follow Algorithm 6. Let $R_0 = \lambda_0 S + \mu_0 T$ and $R_1 = \lambda_1 S + \mu_1 T$. Applying Descartes' Sign Rule, we find that $r_0 = r_1 = 2$. So the intersection, over the complexes, consists of four skew lines. Since $\mathcal{D}(1,0) < 0$, the determinantal equation is negative outside the roots, and $s = -1$. So the intersection, over the reals, consists of two points.

The two points can be computed with the algorithms of Part III:

$$(-3, -3, 3 + \sqrt{3}, -3 - 4\sqrt{3}) \quad \text{and} \quad (-3, -3, 3 - \sqrt{3}, -3 + 4\sqrt{3}).$$

6.3 Example 3

Consider the following pair of quadrics:

$$\begin{cases} Q_S : -2xy + 2xw - y^2 - z^2 + w^2, \\ Q_T : 4xy - 4xw + 2y^2 + z^2 - 2w^2. \end{cases}$$

The determinantal equation vanishes identically. We then follow Algorithm 2. Q_S has rank 3, and its singular point $\mathbf{p}$ has coordinates $(-1, 1, 0, 1)$. Q_T has rank 3, and its singular point is again $\mathbf{p}$. So the dimension κ of the intersection of the singular sets of Q_S and Q_T is 0. Let P be the transformation matrix sending $\mathbf{p}$ to $(0, 0, 0, 1)$, completed as follows:

$$P = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let $S' = P^T S P$ and $T' = P^T T P$, and remove the last line and column of the two matrices. This gives:

$$\begin{cases} Q_{S'} : -2xy - y^2 - z^2, \\ Q_{T'} : 4xy + 2y^2 + z^2. \end{cases}$$

The restricted determinantal equation is then

$$\mathcal{D}_3(\lambda, \mu) = \lambda^3 - 5\lambda^2\mu + 8\lambda\mu^2 - 4\mu^3.$$

It has a double root at $(\lambda_0, \mu_0) = (2, 1)$. The associated conic $R'_0 = \lambda_0 S' + \mu_0 T'$ has rank 1. So the intersection consists, over the complexes, of two double lines. $\mathcal{D}_3$ has a second root at $(\lambda_1, \mu_1) = (1, 1)$. The associated conic has inertia $(1, 1)$, from which we conclude that the intersection consists, over the reals, of two double lines.

The two lines can easily be parameterized as follows:

$$\mathbf{X}_1(u, v) = (v, -u - v, 0, u - v) \quad \text{and} \quad \mathbf{X}_2(u, v) = (u, v, 0, v), \quad (u, v) \in \mathbb{P}^1(\mathbb{R}).$$

The two lines meet at point $\mathbf{p}$.

7 Conclusion

In this second part of our paper, we have shown how the real type of the intersection of two quadrics can be determined by extracting simple information from the pencil of the two quadrics, and in particular its determinantal equation. Our type-detection algorithm relies on a classification of real pencils of quadrics of $\mathbb{P}^3(\mathbb{R})$, itself derived from the Canonical Pair Form Theorem for pairs of real symmetric matrices ([8, 9]).

In Part III [3], we will use the structural information gathered in the type-detection phase to drive the parameterization process. In each case, we will show that the parameterization computed is near-optimal.

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