Near-Optimal Parameterization of the Intersection of Quadrics: I. The Generic Algorithm

Laurent Dupont, Daniel Lazard, Sylvain Lazard, Sylvain Petitjean

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

Near-Optimal Parameterization of the Intersection of Quadrics: I. The Generic Algorithm

Laurent Dupont — Daniel Lazard — Sylvain Lazard — Sylvain Petitjean

N° 5667

Septembre 2005

Thème SYM

Rapport de recherche

ISSN 0249-6399

Near-Optimal Parameterization of the Intersection of Quadrics: I. The Generic Algorithm

Laurent Dupont* , Daniel Lazard† , Sylvain Lazard* , Sylvain Petitjean*

Thème SYM —Systèmes symboliques
Projets Vegas et Salsa

Rapport de recherche n° 5667 —Septembre 2005 — 36 pages

Abstract: We present the first efficient algorithm for computing an exact parametric representation of the intersection of two quadrics in three-dimensional real space given by implicit equations with rational coefficients. The output functions parameterizing the intersection are rational functions whenever it is possible, which is the case when the intersection is not a smooth quartic (for example, a singular quartic, a cubic and a line, or two conics). Furthermore, the parameterization is near-optimal in the sense that the number of square roots appearing in the coefficients of these functions is minimal except in a small number of cases where there may be an extra square root. In addition, the algorithm is practical: a complete, robust and efficient C++ implementation is described in Part IV [12] of this paper.

In Part I, we present an algorithm for computing a parameterization of the intersection of two arbitrary quadrics which we prove to be near-optimal in the generic, smooth quartic, case. Parts II and III [4, 5] treat the singular cases. We present in Part II the first classification of pencils of quadrics according to the real type of the intersection and we show how this classification can be used to efficiently determine the type of the real part of the intersection of two arbitrary quadrics. This classification is at the core of the design of our algorithms for computing near-optimal parameterizations of the real part of the intersection in all singular cases. We present these algorithms in Part III and give examples covering all the possible situations in terms of both the real type of intersection and the number and depth of square roots appearing in the coefficients.

Key-words: Intersection of surfaces, quadrics, pencils of quadrics, curve parameterization.

* Project Vegas, LORIA (UMR CNRS, INPL, INRIA Lorraine, Universités Nancy 1 & 2), Nancy, France; `Firstname.Lastname@loria.fr`.

† Project Salsa and LIP6 (UMR Université Pierre et Marie Curie and CNRS), Paris, France; `Daniel.Lazard@lip6.fr`.

Unité de recherche INRIA Lorraine
LORIA, Technopôle de Nancy-Brabois, Campus scientifique,

Paramétrisation quasi-optimale de l'intersection de quadriques : I. L'algorithme générique

Résumé : Nous présentons le premier algorithme efficace pour le calcul d'une représentation paramétrique exacte de l'intersection de deux quadriques de l'espace tridimensionnel réel données par leurs équations implicites à coefficients rationnels. Le paramétrage calculé est polynomial chaque fois que cela est possible, c'est-à-dire quand l'intersection n'est pas une quartique lisse (par exemple, une quartique singulière, une cubique et une droite, ou deux coniques). De plus, le paramétrage est *quasi-optimal* dans le sens où le nombre de racines carrées apparaissant dans les coefficients des polynômes du paramétrage est minimal sauf dans un petit nombre de cas où ils contiennent une racine carrée possiblement inutile.

Dans la partie I, nous présentons un algorithme pour le calcul du paramétrage de l'intersection de deux quadriques arbitraires dont nous prouvons qu'il est quasi-optimal dans le cas générique d'une intersection constituée d'une quartique lisse.

Nous présentons dans la partie II [4] une nouvelle classification des faisceaux de quadriques selon le type réel de l'intersection et montrons comment utiliser cette classification pour déterminer efficacement le type de la partie réelle de l'intersection de deux quadriques arbitraires.

Dans la partie III [5], nous montrons comment utiliser l'information obtenue sur le type réel de l'intersection pour diriger le paramétrage de l'intersection. Dans chaque cas possible, nous donnons la trame d'un algorithme quasi-optimal pour paramétrer la partie réelle de l'intersection et donnons des exemples couvrant toutes les situations possibles en terme de nombre et de profondeur de radicaux impliqués.

Enfin, nous présentons dans la partie IV [12] une implantation complète, robuste et efficace de notre algorithme en C++.

Mots-clés : Intersection de surfaces, quadriques, faisceaux de quadriques, paramétrisation.

1 Introduction

The simplest of all the curved surfaces, quadrics (i.e., algebraic surfaces of degree two), are fundamental geometric objects, arising in such diverse contexts as geometric modeling, statistical classification, pattern recognition, and computational geometry. In geometric modeling, for instance, they play an important role in the design of mechanical parts; patches of natural quadrics (planes, cones, spheres and cylinders) and tori make up to 95 % of all mechanical pieces according to Requicha and Voelcker [22].

Computing the intersection of two general quadrics is a fundamental problem and an explicit parametric representation of the intersection is desirable for most applications. Indeed, computing intersections is at the basis of many more complex geometric operations such as computing convex hulls of quadric patches [9], arrangements of sets of quadrics [1, 18, 25, 36], and boundary representations of quadric-based solid models [10, 24].

Until recently, the only known general method for computing a parametric representation of the intersection between two arbitrary quadrics was due to J. Levin [13, 14]. It is based on an analysis of the pencil generated by the two quadrics, i.e. the set of linear combinations of the two quadrics.

Though useful, Levin's method has serious limitations. When the intersection is singular or reducible, a parameterization by rational functions is known to exist, but Levin's pencil method fails to find it and generates a parameterization that involves the square root of some polynomial. In addition, when a floating point representation of numbers is used, Levin's method sometimes outputs results that are topologically wrong and it may even fail to produce any parameterization at all and crash. On the other hand a correct implementation using exact arithmetic is essentially out of reach because Levin's method introduces algebraic numbers of fairly high degree. A good indication of this impracticality is that even for the simple generic example of Section 8.2, an exact parametric form output by Levin's algorithm (computed by hand with Maple [15]) fills up over 100 megabytes of space!

Over the years, Levin's seminal work has been extended and refined in several different directions. Wilf and Manor [35] use a classification of quadric intersections by the Segre characteristic (see [2]) to drive the parameterization of the intersection by the pencil method. Recently, Wang, Goldman and Tu [32] further improved the method by making it capable of computing structural information on the intersection and its various algebraic components and able to produce a parameterization by rational functions when it exists. Whether their refined algorithm is numerically robust is open to question.

Another method of algebraic flavor was introduced by Farouki, Neff and O'Connor [6] when the intersection is degenerate. In such cases, using a combination of classical concepts (Segre characteristic) and algebraic tools (factorization of multivariate polynomials), the authors show that explicit information on the morphological type of the intersection curve can be reliably obtained. A notable feature of this method is that it can output an exact parameterization of the intersection in simple cases, when the input quadrics have rational coefficients. No implementation is however reported.

Rather than restricting the type of the intersection, others have sought to restrict the type of the input quadrics, taking advantage of the fact that geometric insights can then help compute the intersection curve [8, 16, 17, 26, 27, 28]. Specialized routines are devised to compute the intersection curve in each particular case. Even though such geometric approaches are numerically more stable than the algebraic ones, they are essentially limited to the class of so-called natural quadrics (i.e., the planes, right cones, circular cylinders and spheres) and planar intersections.

Perhaps the most interesting of the known algorithms for computing an explicit representation of the intersection of two arbitrary quadrics is the method of Wang, Joe and Goldman [34]. This algebraic method is based on a birational mapping between the intersection curve and a plane cubic curve. The cubic curve is obtained by projection from a point lying on the intersection. Then the classification and parameterization of the intersection are obtained by invoking classical results on plane cubics. The authors claim that their algorithm is the first to produce a complete topological classification of the intersection (singularities, number and types of algebraic components, etc.). Numerical robustness issues have however not been studied and the intersection may not be correctly classified. Also, the center of projection is currently computed using Levin's (enhanced) method: with floating point arithmetic, it will in general not exactly lie on the curve, which is another source of numerical instability.

1.1 Contributions

In this paper, we present the first exact and efficient algorithm for computing a parametric representation of the intersection of two quadric surfaces in three-dimensional real space given by implicit equations with rational coefficients. (A preliminary version of this paper was presented in [3].)

Our algorithm, as well as its implementation, has the following main features:

- it computes an exact parameterization of the intersection of two quadrics with rational coefficients of arbitrary size;
- it places no restriction of any kind on the type of the intersection or the type of the input quadrics;
- it correctly identifies, separates and parameterizes all the algebraic components of the intersection and gives all the information on the incidence between the components, that is where and how (e.g., tangentially or not) two components intersect;
- the parameterization is rational when one exists; otherwise the intersection is a smooth quartic and the parameterization involves the square root of a polynomial;
- the parameterizations are either optimal in the degree of the extension of $\mathbb{Q}$ on which their coefficients are defined or, in a small number of well-identified cases, involve one extra possibly unnecessary square root.

Moreover, our implementation of this algorithm, which uses arbitrary-precision integer arithmetic, can routinely compute parameterizations of the intersection of quadrics with input integer coefficients having ten digits in less than 50 milliseconds on a mainstream PC (see Part IV [12]).

The above features imply in particular that the output parameterization of the intersection is almost as "simple" as possible, meaning that the parameterization is rational if one exists, and that the coefficients appearing in the parameterization are almost as rational as possible. This "simplicity" is, in itself, a key factor for making the parameterization process both feasible and efficient (by contrast, an implementation of Levin's method using exact arithmetic is essentially out of reach). It is also crucial for the easy and efficient processing of parameterizations in further applications.

Formally, we prove the following.

Theorem 1.1. *In three-dimensional real space, given two quadrics in implicit form with rational coefficients, our algorithm first tests if their intersection is a smooth quartic or not. If it is a smooth quartic, there does not exist any rational parameterization of the intersection and our algorithm computes a parameterization such that, in projective space, each coordinate belongs to* $\mathbb{K}[\xi, \sqrt{\Delta}]$ *(the ring of polynomials in* ξ *and* $\sqrt{\Delta}$ *with coefficients in* $\mathbb{K}$*), where* ξ *is the (real) parameter,* $\Delta \in \mathbb{K}[\xi]$ *is a polynomial in* ξ*, and* $\mathbb{K}$ *is either the field of the rationals or an extension of* $\mathbb{Q}$ *by the square root of an integer. If the intersection is not a smooth quartic, our algorithm computes a rational parameterization of each component of the intersection over a field* $\mathbb{K}$ *of coefficients which is* $\mathbb{Q}$ *or an extension of* $\mathbb{Q}$ *of degree 2 or 4; this means that each projective coordinate of the component of the intersection is a polynomial in* $\mathbb{K}[\xi]$*.*

In all cases, either $\mathbb{K}$ *is a field of smallest possible degree*[1] *over which there exists such a parameterization or* $\mathbb{K}$ *is an extension of such a smallest field by the square root of an integer. In the latter situation, testing if this extra square root is unnecessary and, if so, finding an optimal parameterization are equivalent to finding a rational point on a curve or a surface (which is computationally hard and can even be undecidable).*

Due to the number of contributions and results of this work, this paper has been broken down into four parts. In Part I, we present a first and major improvement to Levin's pencil method and the accompanying theoretical tools. This simple algorithm, referred to from now on as the "generic algorithm", outputs a near-optimal parameterization when the intersection is a smooth quartic, i.e. the generic case. However, the generic algorithm ceases to be optimal (both from the point of view of the functions used in the parameterizations and the size of their coefficient field) in several singular situations. Parts II and III [4, 5] refine the generic algorithm by considering in turn all the possible types of intersection. Part II focuses on the classification of pencils of quadrics over the reals and on the detection of each case. Part III then gives an optimal or near-optimal algorithm in each possible situation. In Part IV [12], we present a complete, robust, and efficient C++ implementation of our algorithm.

1.2 Overview of Part I

Part I is organized as follows. In Section 2, we present basic definitions, notations and useful known results. Section 3 summarizes the ideas on which the pencil method of Levin for intersecting quadrics is based and discusses its shortcomings. In Section 4 we present our generic algorithm. Among the results of independent interest presented in this section are the almost always existence

[1] Recall that, if $\mathbb{K}$ is a field extension of $\mathbb{Q}$, the *degree* of the extension is defined as the dimension of $\mathbb{K}$ as a vector space over $\mathbb{Q}$. For instance, if $\mathbb{Q}(\rho)$ is a field extension of $\mathbb{Q}$ (distinct from $\mathbb{Q}$), then its degree is 2 since there is a one-to-one correspondence between any element $x \in \mathbb{Q}(\rho)$ and $(\alpha_1, \alpha_2) \in \mathbb{Q}^2$ such that $x = \alpha_1 + \alpha_2\, \rho$. Similarly, if $\mathbb{Q}$ and two field extensions $\mathbb{Q}(\rho)$ and $\mathbb{Q}(\rho')$ are pairwise distinct, then the degree of $\mathbb{Q}(\rho, \rho')$ is 4 since there is a one-to-one correspondence between any element $x \in \mathbb{Q}(\rho)$ and $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{Q}^2$ such that $x = \alpha_1 + \alpha_2\, \rho + \alpha_3\, \rho' + \alpha_4\, \rho\, \rho'$.

of a ruled quadric with rational coefficients in a pencil (proved in Section 5) and new parameterizations of ruled projective quadrics involving an optimal number of radicals in the worst case (a fact proved in Section 6). In Section 7, we prove the near-optimality of the output parameterization in the generic case, that is when the intersection curve is a smooth quartic, and show that the parameterization is optimal in the worst case, meaning that there are examples in which the possibly extra square root is indeed needed. Then, in Section 8, we give several examples and show the result of our implementation on these examples, before concluding.

2 Notations and preliminaries

In what follows, all the matrices considered are real square matrices. Given a real symmetric matrix S of size $n+1$, the upper left submatrix of size n, denoted S_u, is called the *principal submatrix* of S and the determinant of S_u the *principal subdeterminant* of S.

We call a *quadric* associated to S the set

$$Q_S = \{\mathbf{x} \in \mathbb{P}^n \mid \mathbf{x}^T S \mathbf{x} = 0\},$$

where $\mathbb{P}^n = \mathbb{P}(\mathbb{R})^n$ denotes the real projective space of dimension n. (Note that every matrix of the form αS, where $\alpha \in \mathbb{R} \setminus \{0\}$, represents the same quadric Q_S.) When the ambient space is $\mathbb{R}^n$ instead of $\mathbb{P}(\mathbb{R})^n$, the quadric is simply Q_S minus its points at infinity.

In the rest of this paper, geometric objects and parameterizations are assumed to live in projective space. For instance, a point of $\mathbb{P}^3$ has four coordinates. An object (point, line, plane, cone, quadric, etc.) given by its implicit equation(s) is said to be *rational* over a field $\mathbb{K}$ if the coefficients of its equation(s) live in the field $\mathbb{K}$. Note that, when talking about parameterizations, some confusion can arise between two different notions: the rationality of the coefficients and the rationality of the defining functions (a quotient of two polynomial functions is often called a rational function). The meaning should be clear depending on the context.

Matrix S being symmetric, all of its eigenvalues are real. Let σ^+ and σ^- be the numbers of positive and negative eigenvalues of S, respectively. The *rank* of S is the sum of σ^+ and σ^-. We define the *inertia* of S and Q_S as the pair

$$(\max(\sigma^+, \sigma^-), \min(\sigma^+, \sigma^-)).$$

(Note that it is more usual to define the inertia as the pair (σ^+, σ^-), but our definition, in a sense, reflects the fact that Q_S and Q_{-S} are one and the same quadric.) A matrix of inertia $(n, 0)$ is called *definite*. It is *positive definite* if $\sigma^- = 0$, *negative definite* otherwise. Matrix S and quadric Q_S are called *singular* if the determinant of S is zero; otherwise they are called *nonsingular*.

The inertia of a quadric in $\mathbb{P}^3$ is a fundamental concept which somehow replaces the usual type of a quadric in $\mathbb{R}^3$. For the convenience of the reader we recall in Table 1 the correspondence between inertias in $\mathbb{P}^3$ and types in $\mathbb{R}^3$.

In $\mathbb{P}^3$, any quadric not of inertia $(3,1)$ is either a ruled surface or not a surface. Also, the quadrics of inertia $(3,1)$ are the only ones with a strictly negative determinant. The nonsingular quadrics are those of rank 4, i.e. those of inertia $(4,0), (3,1)$ and $(2,2)$. Quadrics of inertia $(4,0)$ are however

Inertia of Q_S	Inertia of S_u	Euclidean canonical equation	Euclidean type of Q_S
$(4,0)$	$(3,0)$	$x^2+y^2+z^2+1$	$\emptyset$ (imaginary ellipsoid)
$(3,1)$	$(3,0)$	$x^2+y^2+z^2-1$	ellipsoid
	$(2,1)$	$x^2+y^2-z^2+1$	hyperboloid of two sheets
	$(2,0)$	x^2+y^2+z	elliptic paraboloid
$(3,0)$	$(3,0)$	$x^2+y^2+z^2$	point
	$(2,0)$	x^2+y^2+1	$\emptyset$ (imaginary elliptic cylinder)
$(2,2)$	$(2,1)$	$x^2+y^2-z^2-1$	hyperboloid of one sheet
	$(1,1)$	x^2-y^2+z	hyperbolic paraboloid
$(2,1)$	$(2,1)$	$x^2+y^2-z^2$	cone
	$(2,0)$	x^2+y^2-1	elliptic cylinder
	$(1,1)$	x^2-y^2+1	hyperbolic cylinder
	$(1,0)$	x^2+y	parabolic cylinder
$(2,0)$	$(2,0)$	x^2+y^2	line
	$(1,0)$	x^2+1	$\emptyset$ (imaginary parallel planes)
$(1,1)$	$(1,1)$	x^2-y^2	intersecting planes
	$(1,0)$	x^2-1	parallel planes
	$(0,0)$	x	simple plane
$(1,0)$	$(1,0)$	x^2	double plane
	$(0,0)$	1	$\emptyset$ (double plane at infinity)

Table 1: Correspondence between quadric inertias and Euclidean types.

empty of real points. A quadric of rank 3 is called a *cone*. The cone is said to be *real* if its inertia is $(2,1)$. It is said to be *imaginary* otherwise, in which case its real projective locus is limited to its singular point. A quadric of rank 2 is a *pair of planes*. The pair of planes is real if its inertia is $(1,1)$. It is called imaginary if its inertia is $(2,0)$, in which case its real projective locus consists of its singular line, i.e. the line of intersection of the two planes. A quadric of inertia $(1,0)$ is called a *double plane* and is necessarily real.

Two real symmetric matrices S and S' of size n are said to be *similar* if and only if there exists a nonsingular matrix P such that

$$S' = P^{-1}SP.$$

Note that two similar matrices have the same characteristic polynomial, and thus the same eigenvalues. Two matrices are said to be *congruent* or *projectively equivalent* if and only if there exists a nonsingular matrix P with real coefficients such that

$$S' = P^T SP.$$

Sylvester's Inertia Law asserts that the inertia is invariant under a congruence transformation [11], i.e. S and S' have the same inertia. Note also that the determinant of S is invariant by a congruence transformation, up to a square factor (the square of the determinant of the transformation matrix).

Let S and T be two real symmetric matrices of the same size and let $R(\lambda,\mu) = \lambda S + \mu T$. The set

$$\{R(\lambda,\mu) \mid (\lambda,\mu) \in \mathbb{P}^1\}$$

is called the *pencil* of matrices generated by S and T. For the sake of simplicity, we sometimes write a member of the pencil $R(\lambda) = \lambda S - T, \quad \lambda \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. Associated to it is a pencil of quadrics $\{Q_{R(\lambda,\mu)} \mid (\lambda,\mu) \in \mathbb{P}^1\}$. Recall that the intersection of two distinct quadrics of a pencil is independent of the choice of the two quadrics. We call the binary form

$$\mathcal{D}(\lambda,\mu) = \det R(\lambda,\mu)$$

the *determinantal equation* of the pencil.

3 Levin's pencil method

Since our solution to quadric surface intersection builds upon the pencil method of J. Levin, we start by recalling the main steps of the algorithm described in [13, 14] for computing a parameterized representation of the intersection of two distinct implicit quadrics Q_S and Q_T of $\mathbb{R}^3$. Starting from this short description, we then identify where this algorithm introduces high-degree algebraic numbers and why this is a problem.

The high-level idea behind Levin's algorithm is this: if (say) Q_S is of some "good" type, then Q_S admits a parameterization which is linear in one of its parameters and plugging this parameterization in the implicit equation of Q_T yields a degree 2 equation in one of the parameters (instead of a degree 4 equation) which can be easily solved to get a parametric representation of $Q_S \cap Q_T$. When neither Q_S nor Q_T has a "good" type, then one can find a quadric Q_R of "good" type in the pencil generated by Q_S and Q_T, and we are back to the previous case replacing Q_S by Q_R.

The definition of a "good" type is embodied in Levin's notion of simple ruled quadric[2] and the existence of such a quadric Q_R is Levin's key result:

Theorem 3.1 ([13]). *The pencil generated by any two distinct quadrics contains at least one simple ruled quadric, i.e., one of the quadrics listed in Table 2, or the empty set.*

In more details, Levin's method is as follows:

1. Find a simple ruled quadric in the pencil $\{Q_{R(\lambda)=\lambda S-T} \mid \lambda \in \overline{\mathbb{R}}\}$ generated by Q_S and Q_T, or report an empty intersection. Since simple ruled quadrics have a vanishing principal subdeterminant, this is achieved by searching for a $\lambda_0 \in \overline{\mathbb{R}}$ such that $\det(R_u(\lambda_0)) = 0$ and $Q_R = Q_{R(\lambda_0)}$ is simple ruled; by Theorem 3.1, such a quadric exists or the pencil contains the empty set. Assume, for the sake of simplicity, that the intersection is not empty and that Q_R and Q_S are distinct. Then $Q_S \cap Q_T = Q_S \cap Q_R$.

[2] In [13, 14], Levin refers to these quadrics as to nonelliptic paras.

quadric	canonical equation ($a,b>0$)	parameterization $\mathbf{X}=[x,y,z]$, $u,v\in\mathbb{R}$
simple plane	$x=0$	$\mathbf{X}(u,v)=[0,u,v]$
double plane	$x^2=0$	$\mathbf{X}(u,v)=[0,u,v]$
parallel planes	$ax^2=1$	$\mathbf{X}(u,v)=[\frac{1}{\sqrt{a}},u,v]$, $\mathbf{X}(u,v)=[-\frac{1}{\sqrt{a}},u,v]$
intersecting planes	$ax^2-by^2=0$	$\mathbf{X}(u,v)=[\frac{u}{\sqrt{a}},\frac{u}{\sqrt{b}},v]$, $\mathbf{X}(u,v)=[\frac{u}{\sqrt{a}},-\frac{u}{\sqrt{b}},v]$
hyperbolic paraboloid	$ax^2-by^2-z=0$	$\mathbf{X}(u,v)=[\frac{u+v}{2\sqrt{a}},\frac{u-v}{2\sqrt{b}},uv]$
parabolic cylinder	$ax^2-y=0$	$\mathbf{X}(u,v)=[u,au^2,v]$
hyperbolic cylinder	$ax^2-by^2=1$	$\mathbf{X}(u,v)=[\frac{1}{2\sqrt{a}}(u+\frac{1}{u}),\frac{1}{2\sqrt{b}}(u+\frac{1}{u}),v]$

Table 2: Parameterizations of canonical simple ruled quadrics.

2. Determine the orthonormal transformation matrix P_u which sends R_u in diagonal form by computing the eigenvalues and the normalized eigenvectors of R_u. Deduce the transformation matrix P which sends Q_R into canonical form. In the orthonormal frame in which it is canonical, Q_R admits one of the parameterizations $\mathbf{X}$ of Table 2.

3. Compute the matrix $S'=P^TSP$ of the quadric Q_S in the canonical frame of Q_R and consider the equation
$$\mathbf{X}^T S'\mathbf{X}=a(u)v^2+b(u)v+c(u)=0, \tag{1}$$
where $\mathbf{X}$ has been augmented by a fourth coordinate set to 1. (The parameterizations of Table 2 are such that $a(u),b(u)$ and $c(u)$ are polynomials of degree two in u.)

   Solve (1) for v in terms of u and determine the corresponding domain of validity of u on which the solutions are defined, i.e., the set of u such that $\Delta(u)=b^2(u)-4a(u)c(u)\geqslant 0$. Substituting v by its expression in terms of u in $\mathbf{X}$, we have a parameterization of $Q_S\cap Q_T=Q_S\cap Q_R$ in the orthonormal coordinate system in which Q_R is canonical.

4. Output $P\mathbf{X}(u)$, the parameterized equation of $Q_S\cap Q_T$ in the global coordinate frame, and the domain of $u\in\mathbb{R}$ on which it is valid.

This method is very nice and powerful since it gives an explicit representation of the intersection of two general quadrics. However, it is far from being ideal from the point of view of precision and robustness since it introduces non-rational numbers at several different places. Thus, if a floating point representation of numbers is used, the result may be wrong (geometrically and topologically) or, worse, the program may crash (especially in Step 1 when the type of the quadrics $Q_{R(\lambda_0)}$ are incorrectly computed). In theory, exact arithmetic would do, except that it would highly slow down the computations. In practice, however, a correct implementation using exact arithmetic seems out of reach because of the high degree of the algebraic numbers involved.

Let us examine more closely the potential sources of numerical instability in Levin's algorithm.

inertia of S	canonical equation $(a,b,c,d>0)$	parameterization $\mathbf{X}=[x,y,z,w]$
$(4,0)$	$ax^2+by^2+cz^2+dw^2=0$	$Q_S=\emptyset$
$(3,0)$	$ax^2+by^2+cz^2=0$	Q_S is point $(0,0,0,1)$
$(2,2)$	$ax^2+by^2-cz^2-dw^2=0$	$\mathbf{X}=[\frac{ut+avs}{a},\frac{us-bvt}{b},\frac{ut-avs}{\sqrt{ac}},\frac{us+bvt}{\sqrt{bd}}],\ (u,v),(s,t)\in\mathbb{P}^1$
$(2,1)$	$ax^2+by^2-cz^2=0$	$\mathbf{X}=[uv,\frac{u^2-abv^2}{2b},\frac{u^2+abv^2}{2\sqrt{bc}},s],\ (u,v,s)\in\mathbb{P}^{\star 2}$
$(2,0)$	$ax^2+by^2=0$	$\mathbf{X}=[0,0,u,v],\ (u,v)\in\mathbb{P}^1$
$(1,1)$	$ax^2-by^2=0$	$\mathbf{X}_1=[u,\frac{\sqrt{ab}}{b}u,v,s],\ \mathbf{X}_2=[u,-\frac{\sqrt{ab}}{b}u,v,s],\ (u,v,s)\in\mathbb{P}^2$
$(1,0)$	$ax^2=0$	$\mathbf{X}=[0,u,v,s],\ (u,v,s)\in\mathbb{P}^2$

Table 3: Parameterization of projective quadrics of inertia different from $(3,1)$. In the parameterization of projective cones, $\mathbb{P}^{\star 2}$ stands for the 2-dimensional real quasi-projective space defined as the quotient of $\mathbb{R}^3\setminus\{0,0,0\}$ by the equivalence relation $\sim$ where $(x,y,z)\sim(y_1,y_2,y_3)$ iff $\exists\lambda\in\mathbb{R}\setminus\{0\}$ such that $(x,y,z)=(\lambda y_1,\lambda y_2,\lambda^2 y_3)$.

- Step 1: λ_0 is the root of a third degree polynomial with rational coefficients. In the worst case, it is thus expressed with nested radicals of depth two. Since determining if $Q_{R(\lambda_0)}$ is simple ruled involves computing its Euclidean type (not an easy task considering that $Q_{R(\lambda_0-\varepsilon)}$ and $Q_{R(\lambda_0+\varepsilon)}$ may be and often are of different types), this is probably the biggest source of non-robustness.

- Step 2: Since Q_R is simple ruled, the characteristic polynomial of R_u is a degree three polynomial having zero as a root and whose coefficients are in the field extension $\mathbb{Q}(\lambda_0)$. Thus, the nonzero eigenvalues of R_u may involve nested radicals of depth three. Since the corresponding eigenvectors have to be normalized, the coefficients of the transformation matrix P are expressed with radicals of nesting depth four in the worst case.

  Since the coefficients of the parameterization $\mathbf{X}$ of Q_R are expressed as square roots of the coefficients of the canonical equation Q_{P^TRP} (as in Table 2), the coefficients of the parameterization of $Q_S\cap Q_T$ can involve *nested radicals of depth five* in the worst case.

- Step 3: Computing the domain of $\mathbf{X}$ amounts to solving the fourth degree equation $\Delta(u)=0$ whose coefficients are nested radicals of worst-case depth five in $\mathbb{Q}$.

Note that this worst-case picture is the generic case. Indeed, given two arbitrary quadrics with rational coefficients, the polynomial $\det(R_u(\lambda))$ will generically have no rational root (a consequence of Hilbert's Irreducibility Theorem).

4 Generic algorithm

We now present a first but major improvement to Levin's pencil method for computing parametric representations of the intersection of quadrics.

This so-called "generic algorithm" removes most of the sources of radicals in Levin's algorithm. We prove in Section 7 that it is near-optimal in the generic, smooth quartic case. It is however not optimal for all the possible types of intersection and will need later refinements (see the comments in Section 9, and Parts II and III). But it is sufficiently simple, robust and efficient to be of interest to many.

We start by introducing the projective framework underlying our approach and stating the main theorem on which the generic approach rests. We then outline our algorithm and detail particular steps in ensuing sections.

From now on, all the input quadrics considered have their coefficients (i.e., the entries of the corresponding matrices) in $\mathbb{Q}$.

4.1 Key ingredients

The first ingredient of our approach is to work not just over $\mathbb{R}^3$ but over the real projective space $\mathbb{P}^3$. Recall that, in projective space, quadrics are entirely characterized by their inertia (i.e., two quadrics with the same inertia are projectively equivalent), while in Euclidean space they are characterized by their inertia and the inertia of their principal submatrix.

In our algorithm, quadrics of inertia different from $(3,1)$ (i.e., ruled quadrics) play the role of simple ruled quadrics in Levin's method. In Table 3, we present a new set of parameterizations of ruled projective quadrics that are both linear in one of their parameters and involve, in the worst case, a minimal number of square roots[3], which we prove in Section 6. That these parameterizations are faithful parameterizations of the projective quadrics (i.e., there is a one-to-one correspondence between the points of the quadric and the parameters) is proved in the appendix.

Another key ingredient of our approach is encapsulated in the following theorem, which mirrors, in the projective setting, Levin's theorem on the existence of ruled quadrics in a pencil.

Theorem 4.1. *In a pencil generated by any two distinct quadrics, the set $\mathcal{S}$ of quadrics of inertia different from $(3,1)$ is not empty. Furthermore, if no quadric in $\mathcal{S}$ has rational coefficients, then the intersection of the two initial quadrics is reduced to two distinct points.*

This theorem, which is proved in Section 5.2, generalizes Theorem 3.1. Indeed, it ensures that the two quadrics we end up intersecting have rational coefficients, except in one very specific situation. This is how we remove the main source of nested radicals in Levin's algorithm.

The last basic ingredient of our approach is the use of Gauss reduction of quadratic forms for diagonalizing a symmetric matrix and computing the canonical form of the associated projective quadric, instead of the traditional eigenvalues/eigenvectors approach used by Levin. Since Gauss transformation is rational (the elements of the matrix P which sends S into canonical form are rational), this removes some layers of nested radicals from Levin's algorithm. Note, also, that there is no difficulty parameterizing the reduced quadric $S' = P^T SP$ since, by Sylvester's Inertia Law, S and S' have the same inertia.

[3]Note that there is necessarily a trade-off between the minimal degree of a parameterization in one of its parameters and the degree of its coefficient field. For instance, Wang, Joe and Goldman [33] give parameterizations of quadrics that have rational coefficients but are quadratic in all of their parameters.

4.2 Algorithm outline

Armed with these ingredients, we are now in a position to outline our generic algorithm.

Let $R(\lambda) = \lambda S - T$ be the pencil generated by the quadrics Q_S and Q_T of $\mathbb{P}^3$ and $\mathcal{D}(\lambda) = \det(R(\lambda))$ be the determinantal equation of the pencil. Recall that, although working in all cases, our generic algorithm is best designed when $\mathcal{D}(\lambda)$ is not identically zero and does not have any multiple root. In the other case, a better algorithm is described in parts II and III. The outline of our intersection algorithm is as follows (details follow in ensuing sections):

1. Find a quadric Q_R with rational coefficients in the pencil, such that $\det R > 0$ if possible or $\det R = 0$ otherwise. (If no such R exists, the intersection is reduced to two points, which we output.) If the inertia of R is $(4,0)$, output empty intersection. Otherwise, proceed.

   Assume for the sake of simplicity that $Q_S \neq Q_R$, in such a way that $Q_S \cap Q_R = Q_S \cap Q_T$.

2. If the inertia of R is not $(2,2)$, apply Gauss reduction to R and compute a frame in which $P^T RP$ is diagonal.

   If the inertia of R is $(2,2)$, find a rational point close enough to Q_R that the quadric $Q_{R'}$ through this point has the same inertia as Q_R. Replace Q_R by this quadric. Use that rational point to compute a frame in which $P^T RP$ is the diagonal matrix $\operatorname{diag}(1,1,-1,-\delta)$, with $\delta \in \mathbb{Q}$.

   In the local frame, Q_R can be described by one of the parameterizations $\mathbf{X}$ of Table 3. Compute the parameterization $P\mathbf{X}$ of Q_R in the global frame.

3. Consider the equation

   $$\Omega : (P\mathbf{X})^T S(P\mathbf{X}) = 0. \tag{2}$$

   Equation Ω is of degree at most 2 in (at least) one of the parameters. Solve it for this parameter in terms of the other(s) and compute the domain of the solution.

4. Substitute this parameter in $P\mathbf{X}$, giving a parameterization of the intersection of Q_S and Q_T.

4.3 Details of Step 1

The detailed description of Step 1 is as follows. Recall that $\mathcal{D}(\lambda) = \det(R(\lambda))$ is the determinantal equation of the pencil.

1. a. If $\mathcal{D}(\lambda) \equiv 0$, set $R = S$ and proceed.

   b. Otherwise, compute isolating intervals for the real roots of $\mathcal{D}(\lambda)$ (using for instance a variant of Uspensky's algorithm [23]). Compute a rational number λ_0 in between each of the separating intervals and, for each λ_0 such that $\mathcal{D}(\lambda_0) > 0$, compute the inertia of the corresponding quadrics using Gauss reduction. If one of the inertias is $(4,0)$, output $Q_S \cap Q_T = \emptyset$. Otherwise, one of these inertias is $(2,2)$ and we proceed with the corresponding quadric.

c. Otherwise (i.e. $\mathcal{D}(\lambda) \not\equiv 0$ and $\mathcal{D}(\lambda) \leqslant 0$ for all λ), compute the greatest common divisor $\gcd(\lambda)$ of $\mathcal{D}(\lambda)$ and its derivative with respect to λ. If $\gcd(\lambda)$ has a rational root λ_0, proceed with the corresponding quadric $Q_{R(\lambda_0)}$.

d. Otherwise (i.e. $\mathcal{D}(\lambda)$ has two non-rational double real roots), $Q_S \cap Q_T$ is reduced to two points. The quadric corresponding to one of these two roots is of inertia $(2,0)$ (an imaginary pair of planes). The singular line of this pair of planes is real and can be parameterized easily, even though it is not rational. Intersecting that line with any of the input quadrics gives the two points.

To assert the correctness of this algorithm, we have several things to prove. First, we make clear why, when looking for a quadric in the pencil (S,T) with inertia different from those of S and T, the right polynomial to consider is $\mathcal{D}(\lambda)$:

Lemma 4.2. *The inertia of $R(\lambda)$ is invariant on any interval of λ not containing a root of $\mathcal{D}(\lambda)$.*

Proof. It suffices to realize that the eigenvalues of $R(\lambda)$ are continuous functions of λ and that the characteristic polynomial of $R(\lambda)$

$$\det(R(\lambda) - lI)$$

is a polynomial in l whose constant coefficient is $\mathcal{D}(\lambda)$, where I is the identity matrix of size 4. Thus the eigenvalues of $R(\lambda)$ may change of sign only at a zero of $\det(R(\lambda))$. □

Let us now show that Step 1 of our algorithm always outputs empty intersection when $Q_S \cap Q_T = \emptyset$. This, in fact, is a direct consequence of Lemma 4.2 and of the following theorem proved in 1936/1937 by the German mathematician Paul Finsler.

Theorem 4.3 ([7]). *Assume $n \geqslant 3$ and let S,T be real symmetric matrices of size n. Then $Q_S \cap Q_T = \emptyset$ if and only if the pencil of matrices generated by S and T contains a definite matrix.*

In Step 1.d, Q_S and Q_T intersect in two points by Theorem 4.1. Furthermore, the quadric corresponding to one of two roots of $\mathcal{D}(\lambda)$ is a real line by the proof of Theorem 4.1.

Finally, note that we can further refine Step 1.b by computing the inertia of the quadrics $Q_{R(\lambda_0)}$ with positive determinant only when the determinantal equation has four real roots counted with multiplicities. Indeed, in view of the following proposition, testing for the presence of a definite matrix in the pencil needs to be done only in that case.

Proposition 4.4. *Assume $n \geqslant 3$ and let S,T be real symmetric matrices of size n. Then $Q_S \cap Q_T = \emptyset$ implies that* $\det(\lambda S + \mu T)$ *does not identically vanish and that all its roots are real.*

Proof. We use the equivalence provided by Theorem 4.3 of the emptiness of the intersection and the existence of a definite matrix in the pencil. Let U be a definite matrix of the pencil which we choose positive (a similar proof goes for negative definite).

Since U is positive definite, we can apply to it a Cholesky factorization: $U = HH^T$, where H is a lower triangular matrix. Consider the matrix $C = (H^{-1})S(H^{-1})^T$. Since C is real symmetric, it has n pairs of real eigenvalues and eigenvectors $(\nu_i, \mathbf{x}_i)$. Let $\mathbf{y}_i = (H^{-1})^T \mathbf{x}_i$. Then we have

$$H(C\mathbf{x}_i) = H(\nu_i \mathbf{x}_i) \quad \Longrightarrow \quad S\mathbf{y}_i = \nu_i U \mathbf{y}_i.$$

Hence all the roots of the characteristic polynomial of $U^{-1}S$ are real, which implies that all the roots of $\det(\lambda S+\mu U)=0$ are real. It follows that all the roots of $\det(\lambda S+\mu T)=0$ are also real. □

4.4 Details of Step 2

There are two cases, according to the inertia of R.

4.4.1 The inertia of R is not $(2,2)$

When the inertia of R is different from $(2,2)$, we use Gauss reduction of quadratic forms and parameterize the resulting quadric, whose associated matrix P^TRP is diagonal. In view of Sylvester's Inertia Law, the reduced quadric Q_{P^TRP} has the same inertia as Q_R. Thus it can be parameterized with at most one square root by one of the parameterizations $\mathbf{X}$ of Table 3. Since Gauss reduction is rational (i.e. P is a matrix with rational coefficients), the parameterization $P\mathbf{X}$ of Q_R contains at most one square root.

4.4.2 The inertia of R is $(2,2)$

When the inertia of R is $(2,2)$, the coefficients of the parameterization of Q_R can live, in the worst case, in an extension $\mathbb{Q}(\sqrt{m},\sqrt{n})$ of degree 4 of $\mathbb{Q}$ (see Table 3). We show here that there exists, in the neighborhood of Q_R, a quadric $Q_{R'}$ with rational coefficients such that

$$Q_S\cap Q_{R'}=Q_S\cap Q_R=Q_S\cap Q_T$$

and the coefficients of the parameterization of $Q_{R'}$ are in $\mathbb{Q}(\sqrt{\det R'})$.

First, apply Gauss reduction to Q_R. If any of $\sqrt{ac}$ or $\sqrt{bd}$ is rational in the parameterization of Q_R (as in Table 3), we are done. Otherwise, compute an arbitrary point $\mathbf{p}\in\mathbb{P}^3(\mathbb{R})$ on Q_R by taking any value of the parameters like, say, $(u,v)=(0,1)$ and $(s,t)=(0,1)$. Approximate $\mathbf{p}$ by a point $\mathbf{p}'\in\mathbb{P}^3(\mathbb{Q})$ not on $Q_S\cap Q_T$. Then compute $\lambda_0'\in\mathbb{Q}$ such that $\mathbf{p}'$ belongs to the quadric $Q_{R(\lambda_0')}$ of the pencil. This is easy to achieve in view of the following lemma.

Lemma 4.5. *In a pencil generated by two quadrics Q_S,Q_T with rational coefficients, there is exactly one quadric going through a given point $\mathbf{p}'$ that is not on $Q_S\cap Q_T$. If $\mathbf{p}'$ is rational, this quadric is rational.*

Proof. In the pencil generated by Q_S and Q_T, a quadric $Q_{R(\lambda,\mu)}$ contains $\mathbf{p}'$ if and only if $\mathbf{p}'^T(\lambda S+\mu T)\mathbf{p}'=0$, that is if and only if $\lambda(\mathbf{p}'^TS\mathbf{p}')+\mu(\mathbf{p}'^TT\mathbf{p}')=0$. If $\mathbf{p}'$ is not on $Q_S\cap Q_T$, this equation is linear in $(\lambda,\mu)\in\mathbb{P}^1$ and thus admits a unique solution. Moreover, if $\mathbf{p}'$ is rational, the equation has rational coefficients and thus the quadric of the pencil containing $\mathbf{p}'$ is rational. □

Note that λ_0' and the λ_0 such that $R=R(\lambda_0)$ get arbitrarily close to one another as $\mathbf{p}'$ gets close to $\mathbf{p}$. Thus if $\mathbf{p}'$ is close enough to $\mathbf{p}$, $R'=R(\lambda_0')$ has the same inertia $(2,2)$ as R, by Lemma 4.2. We refine the approximation $\mathbf{p}'$ of $\mathbf{p}$ until R' has inertia $(2,2)$.

We now have a quadric $Q_{R'}$ of inertia $(2,2)$ and a rational point on $Q_{R'}$. Consider any rational line through $\mathbf{p}'$ that is not in the plane tangent to $Q_{R'}$ at $\mathbf{p}'$. This line further intersects $Q_{R'}$ in another point $\mathbf{p}''$. Point $\mathbf{p}''$ is rational because otherwise $\mathbf{p}'$ and $\mathbf{p}''$ would be conjugate in the field extension of $\mathbf{p}''$ (since $Q_{R'}$ and the line are both rational) and thus $\mathbf{p}'$ would not be rational. Compute the rational transformation P sending $\mathbf{p}',\mathbf{p}''$ onto $(1,\pm 1,0,0)$. Apply this transformation to R' and then apply Gauss reduction of quadratic forms. In the local frame, $Q_{R'}$ has equation (up to a constant factor)

$$x^2 - y^2 + \alpha z^2 + \beta w^2 = 0, \tag{3}$$

with $\alpha\beta < 0$. Now consider the linear transformation whose matrix is P'

$$P' = \frac{1}{2}\begin{pmatrix} 1+\alpha & 0 & 1-\alpha & 0 \\ 1-\alpha & 0 & 1+\alpha & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2\alpha \end{pmatrix}.$$

Applying P' to the already reduced quadric of Eq. (3) gives the equation

$$x^2 + y^2 - z^2 - \delta w^2 = 0, \tag{4}$$

where $\delta = -\alpha\beta > 0$. The quadric of Eq. (4) can be parameterized by

$$\mathbf{X}((u,v),(s,t)) = \left(ut+vs, us-vt, ut-vs, \frac{us+vt}{\sqrt{\delta}}\right),$$

with $(u,v),(s,t) \in \mathbb{P}^1$ (see Table 3).

The three consecutive transformation matrices have rational coefficients thus $\mathbb{Q}(\sqrt{\delta}) = \mathbb{Q}(\sqrt{\det R'})$ and the product of these transformation matrices with $\mathbf{X}$ is a polynomial parameterization of $Q_{R'}$ with coefficients in $\mathbb{Q}(\sqrt{\delta})$, $\delta \in \mathbb{Q}$.

4.5 Details of Step 3

Solving Equation (2) can be done as follows. Recall that the content in the variable x of a multivariate polynomial is the gcd of the coefficients of the x^i.

Equation (2) may be seen as a quadratic equation in one of the parameters. For instance, if R has inertia $(2,2)$, Eq. (2) is a homogeneous biquadratic equation in the variables $\xi = (u,v)$ and $\tau = (s,t)$. Using only gcd computations, we can factor it in its content in ξ (which is a polynomial in τ or a constant), its content in τ, and a remaining factor. If the content in ξ (or in τ) is not constant, solve it in τ (in ξ); substituting the obtained real values in $\mathbf{X}$, we have a parameterization of some components of $Q_S \cap Q_T = Q_S \cap Q_R$ in the frame in which Q_R is canonical. If the remaining factor is not constant, solve it in a parameter in which it is linear, if any, or in τ. Substituting the result in $\mathbf{X}$, we have a parameterization of the last component of the intersection. If the equation which is solved is not linear, the domain of the parameterization is the set of ξ such that the degree 4 polynomial $\Delta(\xi) = b^2(\xi) - 4a(\xi)c(\xi)$ is positive, where $a(\xi), b(\xi)$ and $c(\xi)$ are the coefficients of τ^2, τ and 1 in (2), respectively.

5 Canonical forms and proof of Theorem 4.1

We now prove Theorem 4.1, the key result stated in the previous section. We start by recalling some preliminary results.

5.1 Canonical form for a nonsingular pair of symmetric matrices

We state results, proved by Uhlig [30, 31], we need for computing the canonical form of a pair of real symmetric matrices. Though only part of this theory is required for the proof of Theorem 4.1 (Section 5.2), we will need its full power in Part II of this paper for characterizing real pencils of quadrics, so we explicit it entirely.

Let us start by defining the notion of Jordan blocks.

Definition 5.1. Let M be a square matrix of the form

$$(\ell) \quad \text{or} \quad \begin{pmatrix} \ell & e & & 0 \\ & \ddots & \ddots & \\ & & \ddots & e \\ 0 & & & \ell \end{pmatrix}.$$

If $l \in \mathbb{R}$ and $e = 1$, M is called a *real Jordan block* associated with ℓ. If

$$\ell = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad a, b \in \mathbb{R},\ b \neq 0, \qquad e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

M is called a *complex Jordan block* associated with $a + \mathrm{i}b$.

Now we can state the real Jordan normal form theorem for real square matrices.

Theorem 5.2 (Real Jordan normal form). *Every real square matrix A is similar over the reals to a block diagonal matrix* $\operatorname{diag}(A_1, \ldots, A_k)$, *called real Jordan normal form of A, in which each A_j is a (real or complex) Jordan block associated with an eigenvalue of A.*

The Canonical Pair Form Theorem then goes as follows:

Theorem 5.3 (Canonical Pair Form). *Let S and T be two real symmetric matrices of size n, with S nonsingular. Let $S^{-1}T$ have real Jordan normal form* $\operatorname{diag}(J_1, \ldots, J_r, J_{r+1}, \ldots, J_m)$, *where $J_1, \ldots, J_r$ are real Jordan blocks corresponding to real eigenvalues of $S^{-1}T$ and $J_{r+1}, \ldots, J_m$ are complex Jordan blocks corresponding to pairs of complex conjugate eigenvalues of $S^{-1}T$. Then:*

(a) The characteristic polynomial of $S^{-1}T$ and $\det(\lambda S - T)$ *have the same roots λ_j with the same (algebraic) multiplicities m_j.*

(b) S and T are simultaneously congruent by a real congruence transformation to

$$\operatorname{diag}(\varepsilon_1 E_1, \ldots, \varepsilon_r E_r, E_{r+1}, \ldots, E_m)$$

and

$$\operatorname{diag}(\varepsilon_1 E_1 J_1, \ldots, \varepsilon_r E_r J_r, E_{r+1} J_{r+1}, \ldots, E_m J_m),$$

respectively, where $\varepsilon_i = \pm 1$ *and* E_i *denotes the square matrix*

$$\begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}$$

of the same size as J_i *for* $i = 1, \ldots, m$*. The signs* ε_i *are unique (up to permutations) for each set of indices* i *that are associated with a set of identical real Jordan blocks* J_i*.*

(c) The sum of the sizes of the blocks corresponding to one of the λ_j *is the multiplicity* m_j *if* λ_j *is real or twice this multiplicity if* λ_j *is complex. The number of the corresponding blocks (the geometric multiplicity of* λ_j*) is* $t_j = n - \mathrm{rank}\,(\lambda_j S - T)$*, and* $1 \leqslant t_j \leqslant m_j$*.*

Note that the canonical pair form of Theorem 5.3 can be considered the finest simultaneous block diagonal structure that can be obtained by real congruence for a given pair of real symmetric matrices, in the sense that it maximizes the number of blocks in the diagonalization of S and T.

5.2 Proof of Theorem 4.1

To prove Theorem 4.1, we consider a pencil of real symmetric 4×4 matrices generated by two symmetric matrices S and T of inertia $(3, 1)$. We may suppose that they have the block diagonal form of the above theorem.

If all the blocks had an even size, the determinant of S would be positive, contradicting our hypothesis. Thus, there is a block of odd size in the canonical form of S. It follows that $\det(\lambda S - T)$ has at least one real root and the matrix of the pencil corresponding to this root has an inertia different from $(3, 1)$. This proves the first part.

If $\det(\lambda S - T)$ has a simple real root, there is an interval of values for λ for which $\det(\lambda S - T) > 0$, and we are done with any rational value of λ in this interval. If $\det(\lambda S - T)$ has either a double real root and two complex roots, two rational double real roots or a quadruple real root, the quadrics corresponding to the real root(s) have rational coefficients and have inertia different from $(3, 1)$.

Thus we are left with the case where $\det(\lambda S - T)$ has two non rational double real roots, which are algebraically conjugate. In other words,

$$\det(\lambda S - T) = c(\lambda - \lambda_1)^2 (\lambda - \lambda_2)^2,$$

with $\lambda_1, \lambda_2 \in \mathbb{R} \setminus \mathbb{Q}$ and $\lambda_2 = \overline{\lambda}_1$ its (real algebraic) conjugate. Following the notations of Theorem 5.3, we have $m_1 = m_2 = 2$ and $1 \leqslant t_i \leqslant 2$, for $i = 1, 2$. In other words, $(t_1, t_2) \in \{(1,1), (1,2), (2,1), (2,2)\}$.

We can quickly get rid of the case $(t_1, t_2) = (1, 1)$. Indeed, in this case the blocks have an even size and S is not of inertia $(3, 1)$. We can also eliminate the cases $(t_1, t_2) \in \{(1,2), (2,1)\}$, because the matrices $\lambda_1 S - T$ and $\lambda_2 S - T$ are algebraically conjugate, and so must have the same rank and the same number of blocks.

We are thus left with the case $(t_1, t_2) = (2, 2)$. In this situation, S and T have four blocks, i.e., they are diagonal:

$$\begin{cases} S = \mathrm{diag}\,(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4), \\ T = \mathrm{diag}\,(\varepsilon_1 \lambda_1, \varepsilon_2 \lambda_1, \varepsilon_3 \lambda_2, \varepsilon_4 \lambda_2). \end{cases}$$

The pencil $\lambda S - T$ is generated by the two quadrics of rank 2

$$\begin{cases} S' = \lambda_1 S - T = \operatorname{diag}(0, 0, \varepsilon_3(\lambda_1 - \lambda_2), \varepsilon_4(\lambda_1 - \lambda_2)), \\ T' = \lambda_2 S - T = \operatorname{diag}(\varepsilon_1(\lambda_2 - \lambda_1), \varepsilon_2(\lambda_2 - \lambda_1), 0, 0). \end{cases}$$

We have that

$$\det(S' + T') = \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4(\lambda_1 - \lambda_2)^4$$

is negative since all the quadrics of the pencil have negative determinant except $Q_{S'}$ and $Q_{T'}$. Thus $\varepsilon_1\varepsilon_2$ and $\varepsilon_3\varepsilon_4$ have opposite signs. It follows that one of S' and T' has inertia $(2,0)$ (say S') and the other has inertia $(1,1)$. Thus $Q_{S'}$ is a straight line, which intersects the real pair of planes $Q_{T'}$. Since $Q_{S'} \cap Q_{T'}$ is contained in all the quadrics of the pencil and since the pencil has quadrics of inertia $(3,1)$ (which are not ruled), the line $Q_{S'}$ is not included in $Q_{T'}$ and the intersection is reduced to two real points. Since the equations of $Q_{S'}$ and $Q_{T'}$ are $z^2 + w^2 = 0$ and $x^2 - y^2 = 0$ respectively, the two points have coordinates $(1,1,0,0)$ and $(-1,1,0,0)$. They are thus distinct. □

Remark 5.4. Pencils generated by two quadrics of inertia $(3,1)$ and having no quadric with rational coefficients of inertia different from $(3,1)$ do exist. Consider for instance

$$Q_S : 2x^2 - 2xz - 2yw + z^2 + w^2 = 0,$$
$$Q_T : 4x^2 + 2y^2 - 2yw + z^2 - 6xz + 3w^2 = 0.$$

Then, $\det(\lambda S - T) = -(\lambda^2 - 5)^2$.

6 Optimality of the parameterizations

We now prove that, among the parameterizations of projective quadrics linear in one of the parameters, the ones of Table 3 have, in the worst case, an optimal number of radicals. In other words, for each type of projective quadric, there are examples of surfaces for which the number of square roots of the parameterizations of Table 3 is required.

More precisely, we prove the following theorem, which will be crucial in asserting the near-optimality of our algorithm for parameterizing quadrics intersection.

Theorem 6.1. *In the set of parameterizations linear in one of the parameters, the parameterizations of Table 3 are worst-case optimal in the degree of the extension of $\mathbb{Q}$ on which they are defined.*

For a quadric Q of equation $ax^2 + by^2 - cz^2 - dw^2 = 0$ $(a,b,c,d > 0)$, the parameterization of Table 3 is optimal if Q has no rational point, which is the case for some quadrics. Knowing a rational point on Q (if any), we can compute a rational congruent transformation sending Q into the quadric of equation $x^2 + y^2 - z^2 - abcd\, w^2 = 0$, for which the parameterization of Table 3 is optimal.

For a quadric Q of equation $ax^2 + by^2 - cz^2 = 0$ $(a,b,c > 0)$, the parameterization of Table 3 is optimal if Q has no rational point other than its singular point $(0,0,0,1)$, which is the case for some quadrics. Knowing such a rational point on Q (if any), we can compute a rational congruent transformation sending Q into the quadric of equation $x^2 + y^2 - z^2 = 0$, for which the parameterization of Table 3 is rational (and thus optimal).

For the other types of projective quadrics, the parameterizations of Table 3 are optimal in all cases.

We prove this theorem by splitting it into four more detailed propositions: Proposition 6.2 for inertia $(1,1)$, Proposition 6.3 for inertia $(2,1)$ and Propositions 6.4 and 6.6 for inertia $(2,2)$.

Proposition 6.2. *A projective quadric Q of equation $ax^2 - by^2 = 0$ $(a,b > 0)$ admits a rational parameterization in $\mathbb{Q}$ if and only if it has a rational point outside the singular line $x = y = 0$, or equivalently iff ab is a square in $\mathbb{Q}$. If ab is a square in $\mathbb{Q}$, then the parameterization of Table 3 is rational.*

Proof. A point (x,y,z,w) on Q not on its singular line $x = y = 0$ is rational if and only if y/x, z/x, and w/x are rational. Since $(y/x)^2 = \frac{ab}{b^2}$ and z and w are not constrained, there exists such a rational point if and only if ab is a square.

If there exists a parameterization which is rational over $\mathbb{Q}$, then there exists some rational point outside the line $x = y = 0$, showing *a contrario* that there is no rational parameterization if ab is not a square.

Finally, if ab is the square of a rational number, then the parameterization of Table 3 is rational. □

Proposition 6.3. *A projective quadric Q of equation $ax^2 + by^2 - cz^2 = 0$ $(a,b,c > 0)$ admits a rational parameterization in $\mathbb{Q}$ if and only it contains a rational point other than the singular point $(0,0,0,1)$. Knowing such a rational point, we can compute a rational congruent transformation P sending Q into the quadric of equation $x^2 + y^2 - z^2 = 0$ for which the parameterization of Table 3 is rational; lifting this parameterization to the original space by multiplying by matrix P, we have a rational parameterization of Q.*

On the other hand, there are such quadrics without rational point and thus without rational parameterization, for example the quadric of equation $x^2 + y^2 - 3z^2 = 0$.

Proof. If Q has a rational point other than $(x = y = z = 0)$, any rational line passing through this point and not included in Q cuts Q in another rational point. Compute the rational congruent transformation sending these points onto $(\pm 1,1,0,0)$. Applying this transformation to Q gives a quadric of equation $x^2 - y^2 + r$, where r is a polynomial of degree at most one in x and y. Thus Gauss reduction algorithm leads to the form $x^2 - y^2 + dz^2 = (X^2 + Y^2 - Z^2)/d$ where $X = (1+d)x/2 + (1-d)y/2$, $Y = dz$ and $Z = (1-d)x/2 + (1+d)y/2$. The parameterization of Table 3 applied to equation $X^2 + Y^2 - Z^2$ is clearly rational. Lifting this parameterization back to the original space, we obtain a rational parameterization of Q.

Reciprocally, if Q has no rational point, then Q does not admit a rational parameterization.

Now, suppose for a contradiction that the quadric with equation $x^2 + y^2 - 3z^2 = 0$ has a rational point (x,y,z,w) different from $(0,0,0,1)$. By multiplying x,y, and z by a common denominator and dividing them by their gcd, we obtain another rational point on the quadric for which x,y and z are integers that are not all even. Note that x^2 is equal, modulo 4, to 0 if x is even and 1 otherwise (indeed, modulo 4, $0^2 = 0$, $1^2 = 1$, $2^2 = 0$ and $3^2 = 1$). Thus, $x^2 + y^2 - 3z^2 \equiv x^2 + y^2 + z^2 \pmod 4$ is equal to the number of odd numbers in x,y,z, i.e. $1,2$ or 3. Thus $x^2 + y^2 - 3z^2 \neq 0$, contradicting the hypothesis that (x,y,z,w) is a point on the quadric. □

Proposition 6.4. *Let Q be the quadric of equation $ax^2+by^2-cz^2-dw^2=0$ $(a,b,c,d>0)$. Any field $\mathbb{K}$ in which Q admits a rational parameterization, linear in one of its parameters, contains $\sqrt{abcd}$.*

Proof. Let $\mathbb{K}$ be a field in which Q admits a rational parameterization, linear in the parameter $(u,v)\in\mathbb{P}(\mathbb{R})$. Fixing the value of the other parameter $(s,t)\in\mathbb{P}(\mathbb{K})$ defines a rational line L (in $\mathbb{K}$) contained in Q. L cuts any plane (in possibly infinitely many points) in projective space. In particular, L cuts the plane of equation $z=0$. Since $L\subseteq Q$, L cuts the conic of equation $ax^2+by^2-dw^2=z=0$ in a point $\mathbf{p}=(x_0,y_0,0,1)$. Moreover, $\mathbf{p}$ is rational in $\mathbb{K}$ (i.e., $x_0,y_0\in\mathbb{K}$) because it is the intersection of a rational line and the plane $z=0$.

The plane tangent to Q at $\mathbf{p}$ has equation $ax_0x+by_0y-dw=0$. We now compute the intersection of Q with this plane. Since $ax_0^2+by_0^2=d$ and $a,b,d>0$, x_0 or y_0 is nonzero; assume for instance that $x_0\neq 0$. Squaring the equation of the tangent plane yields $(ax_0x)^2=(by_0y-dw)^2$. By eliminating x^2 between this equation and the equation of Q, we get

$$(by_0y-dw)^2+ax_0^2(by^2-cz^2-dw^2)=0$$

or

$$dw^2(d-ax_0^2)+by^2(ax_0^2+by_0^2)-2bdy_0yw-acx_0^2z^2=0.$$

It follows from $ax_0^2+by_0^2=d$ that $bd(y-y_0w)^2-acx_0^2z^2=0$ or also

$$b^2d^2(y-y_0w)^2-abcd\,x_0^2z^2=0. \tag{5}$$

The intersection of Q and its tangent plane at $\mathbf{p}$ contains the line L which is rational in $\mathbb{K}$. Thus, Equation (5) can be factored over $\mathbb{K}$ into two linear terms. Hence, $\sqrt{abcd}$ belongs to $\mathbb{K}$. ☐

Remark 6.5. *abcd* is the discriminant of the quadric, i.e., the determinant of the associated matrix, so it is invariant by a change of coordinates (up to a square factor). Thus, if R and R' are two matrices representing the same quadric in different frames, the fields $\mathbb{Q}(\sqrt{\det R})$ and $\mathbb{Q}(\sqrt{\det R'})$ are equal.

Proposition 6.6. *A projective quadric Q of equation $ax^2+by^2-cz^2-dw^2=0$ $(a,b,c,d>0)$ admits a rational parameterization in $\mathbb{Q}(\sqrt{abcd})$ if and only it contains a rational point. Knowing such a rational point, we can compute a rational congruent transformation P sending Q into the quadric of equation $x^2+y^2-z^2-abcd\,w^2=0$ for which the parameterization of Table 3 is rational over $\mathbb{Q}(\sqrt{abcd})$; lifting this parameterization to the original space by multiplying by matrix P, we have a rational parameterization of Q over $\mathbb{Q}(\sqrt{abcd})$.*

On the other hand, there are such quadrics with no rational point and thus without rational parameterization in $\mathbb{Q}(\sqrt{abcd})$, for example the quadric of equation $x^2+y^2-3z^2-11w^2=0$.

Proof. If Q admits a rational parameterization in $\mathbb{Q}(\sqrt{abcd})$, then it has infinitely many rational points over this field. If Q has a point (x,y,z,w) that is rational over $\mathbb{Q}(\sqrt{abcd})$, but not rational over $\mathbb{Q}$, we may suppose without loss of generality that $x=1$, by permuting the variables in order that $x\neq 0$ and then by dividing all coordinates by x. The conjugate point $(1,y',z',w')$ over $\mathbb{Q}(\sqrt{abcd})$ belongs also to Q. The line passing through these points is rational (over $\mathbb{Q}$), as is the point $(1,(y+y')/2,(z+$

$z')/2, (w+w')/2)$. Choose a rational frame transformation such that this line becomes the line $z = w = 0$ and this point becomes $(1,0,0,0)$. In this new frame the coordinates of the conjugate points are $(1, \pm e\sqrt{abcd}, 0, 0)$ for some rational number e, and the equation of Q is $abcd\,e^2x^2 - y^2 + r = 0$ where r is a polynomial of degree at most 1 in x and y. Gauss reduction thus provides an equation of the form $abcd\,e^2x^2 - y^2 + fz^2 - gw^2 = 0$, and the invariance of the determinant (Remark 6.5) shows that fg is the square of a rational number h. Thus $(0,0,g,h)$ is a rational point of Q over $\mathbb{Q}$.

Now, if Q has a rational point over $\mathbb{Q}$, one may get another rational point as the intersection of the quadric and any line passing through the point and not tangent to the quadric. One can compute a rational congruent transformation such that these points become $(1,\pm 1,0,0)$. In this new frame the equation of Q has the form $x^2 - y^2 - r$ where r is a polynomial of degree at most 1 in x and y. Gauss reduction provides thus an equation of the form $x^2 - y^2 + ez^2 - fw^2 = (X^2 + Y^2 - Z^2 - efw^2)/e$, with $X = (1+e)x/2 + (1-e)y/2$, $Y = ez$ and $Z = (1-e)x/2 + (1+e)y/2$. By the invariance of the determinant, $ef = g^2abcd$ for some rational number g. Putting $W = gw$, we get the equation $X^2 + Y^2 - Z^2 - abcd\,W^2 = 0$ for Q, and the parameterization of Table 3 is rational over $\mathbb{Q}(\sqrt{abcd})$.

It follows from this proof that, if a quadric of inertia $(2,2)$ has a rational point, it has a parameterization in $\mathbb{Q}(\sqrt{abcd})$, which is linear in one of the parameters. Conversely, for proving that such a parameterization does not always exist, it suffices to prove that there are quadrics of inertia $(2,2)$ having no rational point over $\mathbb{Q}$. Let us consider the quadric of equation $x^2 + y^2 - 3z^2 - 11w^2 = 0$. If it has a rational point (x,y,z,w), then by multiplying x, y, z and w by some common denominator and by dividing them by their gcd, we may suppose that x, y, z and w are integers which are not all even. As in the proof of Proposition 6.3, $x^2 + y^2 - 3z^2 - 11w^2$ is equal modulo 4 to the number of odd numbers in x,y,z,w. Thus all of them are odd. It is straightforward that the square of an odd number is equal to 1 modulo 8. It follows that $x^2 + y^2 - 3z^2 - 11w^2$ is equal to 4 modulo 8, a contradiction with $x^2 + y^2 - 3z^2 - 11w^2 = 0$. □

7 Near-optimality in the smooth quartic case

In this section, we prove that the algorithm given in Section 4 outputs, in the generic (smooth quartic) case, a parameterization of the intersection that is optimal in the number of radicals up to one possibly unnecessary square root. We also show that deciding whether this extra square root can be avoided or not is hard. Moreover, we give examples where the extra square root cannot be eliminated, for the three possible morphologies of a real smooth quartic.

7.1 Algebraic preliminaries

First recall that, as is well known from the classification of quadric pencils by invariant factors (see [2] and Part II for more), the intersection of two quadrics is a nonsingular quartic exactly when $\mathcal{D}(\lambda,\mu) = \det R(\lambda,\mu)$ has no multiple root. Otherwise the intersection is singular. Note that the intersection is nonsingular exactly when $\gcd(\frac{\partial \mathcal{D}}{\partial \lambda}, \frac{\partial \mathcal{D}}{\partial \mu}) = 1$.

Moreover, when the intersection is nonsingular, the rank of any quadric in the pencil is at least three; indeed, all the roots of $\mathcal{D}(\lambda,\mu)$ are simple and thus, in Theorem 5.3(c), $m_j = 1$, thus $t_j = 1$, hence the quadrics associated with the roots of $\mathcal{D}(\lambda,\mu)$ have rank 3.

Whether the intersection of two quadrics admits a parameterization with rational functions directly follows from classical results:

Proposition 7.1. *The intersection of two quadrics admits a parameterization with rational functions if and only if the intersection is singular.*

Proof. First recall that a curve admits a parameterization with rational functions if and only if it has zero genus [20].

Assume first that the intersection of the two quadrics is irreducible. In $\mathbb{P}^3(\mathbb{C})$, if two algebraic surfaces of degree d_1 and d_2 intersect in an irreducible curve, its genus is

$$\frac{1}{2}d_1 d_2(d_1+d_2-4)+1-\sum_{i=1}^{k}\frac{q_i(q_i-1)}{2},$$

where k is the number of singular points and $q_{i,i=1,\ldots,k}$ their respective multiplicity [19]. The intersection curve has thus genus 1 when it is smooth, 0 otherwise. The result follows.

Assume now that the intersection of the two quadrics is reducible. If the intersection contains only points, lines and conics, which are classically rational, we are done. For the remaining case (cubic and line), we use the following result. In $\mathbb{P}^3(\mathbb{C})$, if two algebraic surfaces of degree d_1 and d_2 intersect in two irreducible curves of degree d and d' and of genus g and g', then [20]

$$g'-g=\left(\frac{1}{2}(d_1+d_2)-2\right)(d'-d).$$

For quadrics, $d_1+d_2=4$, so we get $g=g'$. So the genus of the cubic is that of the line, i.e. 0. □

Finally consider the equation $\Omega : \mathbf{X}^T S' \mathbf{X}=0$, obtained in Step 3 of our algorithm, where $\mathbf{X}$ is the parameterization of Q_R and S' is the matrix of Q_S in the canonical frame of Q_R. Let C_Ω be the curve zero-set of Ω. Depending on the projective type of Q_R, C_Ω is a bidegree $(2,2)$ curve in $\mathbb{P}^1\times\mathbb{P}^1$ (inertia $(2,2)$ or $(2,0)$), a quartic curve in $\mathbb{P}^{\star 2}$ (inertia $(2,1)$) or a quartic curve in $\mathbb{P}^2$ (inertia $(1,1)$ or $(1,0)$). Let C denote the curve of intersection of the two given quadrics Q_S and Q_T. We have the following classical result.

Fact 7.2. *The parameterization of Q_R defines an isomorphism between C and C_Ω. In particular, C and C_Ω have the same genus, irreducibility, and factorization.*

7.2 Optimality

Assume the intersection is a real nonsingular quartic. Then $\mathcal{D}(\lambda,\mu)$ has no multiple root, and thus Q_R is necessarily a quadric of inertia $(2,2)$. After Step 2 of our algorithm, Q_R has a parameterization in $\mathbb{Q}(\sqrt{\delta})$ that is bilinear in $\xi=(u,v)$ and $\tau=(s,t)$. After resolution of Ω and substitution in Q_R, we get a parameterization in $\mathbb{Q}(\sqrt{\delta})[\xi,\sqrt{\Delta}]$ with $\Delta\in\mathbb{Q}(\sqrt{\delta})[\xi]$ of degree 4.

Proposition 7.1 implies that it cannot be parameterized by rational functions, so $\sqrt{\Delta}$ cannot be avoided. The question now is: can $\sqrt{\delta}$ be avoided? The answer is twofold:

1. deciding whether $\sqrt{\delta}$ can be avoided amounts, in the general case, to finding a rational point on a surface of degree 8,

2. there are cases in which $\sqrt{\delta}$ cannot be avoided.

We prove these results in the following two sections.

7.2.1 Optimality test

We first prove two preliminary lemmas.

Lemma 7.3. *If the intersection of two given quadrics has a parameterization involving only one square root, there exists a quadric with rational coefficients in the pencil that contains a rational line.*

Proof. In what follows, call *degree* of a point the degree of the smallest field extension of $\mathbb{Q}$ containing the coordinates of this point.

If the parameterization of the intersection involves only one square root, the intersection contains infinitely many points of degree at most 2, one for any rational value of the parameters. Now we have several cases according to the type of points contained in the intersection.

If the intersection contains a point $\mathbf{p}$ of degree 2, it contains also its algebraic conjugate $\overline{\mathbf{p}}$. The line passing through $\mathbf{p}$ and $\overline{\mathbf{p}}$ is invariant by conjugation, so is rational. Let $\mathbf{q}$ be a rational point on this line. The quadric of the pencil passing through $\mathbf{q}$ is rational (Lemma 4.5). Since it also contains $\mathbf{p}$ and $\overline{\mathbf{p}}$ (the intersection is contained in any quadric of the pencil), this quadric cuts the line in at least 3 points and thus contains it.

If the intersection contains a regular rational point (i.e. a rational point which is not a singular point of the intersection), then the line tangent to the intersection at this point is rational, and is tangent to any quadric of the pencil. The quadric of the pencil passing through a rational point of this tangent line contains the contact point; thus it contains the tangent line.

If the intersection contains a singular rational point $\mathbf{p}$, then all the quadrics of the pencil which are not singular at $\mathbf{p}$ have the same tangent plane at $\mathbf{p}$. Let us consider the quadric of the pencil passing through a rational point $\mathbf{q}$ of this tangent plane (or through any rational point, if none of the quadrics is regular at $\mathbf{p}$). Similarly as above, this quadric contains the rational line $\mathbf{pq}$. ☐

Lemma 7.4. *If a quadric contains a rational line, its discriminant is a square in* $\mathbb{Q}$.

Proof. If the quadric has rank less than 4, its discriminant is zero. We may thus suppose that the discriminant is not 0 and that the equation of the quadric is $ax^2+by^2-cz^2-dw^2=0$. Since this quadric contains a rational line L, and thus a rational point, there is a rational change of frames such that the quadric has equation $x^2+y^2-z^2-abcd\,w^2=0$, by Proposition 6.6. Cut the quadric by the plane $z=0$. Since the intersection of the plane $z=0$ and the rational L is a rational point, the cone $x^2+y^2-abcd\,w^2=0$ contains a rational point outside it singular locus. By Proposition 6.3, there is a rational congruent transformation P sending this cone into the cone of equation $x^2+y^2-w^2=0$. These two cones can be seen as conics in $\mathbb{P}^2(\mathbb{Q})$ and P can be seen as a rational transformation in $\mathbb{P}^2(\mathbb{Q})$. The discriminant $-abcd$ of the conic $x^2+y^2-abcd\,w^2=0$ is thus equal to $(\det P)^2$ times -1, the discriminant of the conic $x^2+y^2-w^2=0$. Hence $abcd$ is a square in $\mathbb{Q}$. ☐

From these two technical results and the results of Section 6, we obtain the following equivalence.

Proposition 7.5. *When the intersection is a nonsingular quartic, it can be parameterized in* $\mathbb{Q}[\xi, \sqrt{\Delta}]$ *with* $\Delta \in \mathbb{Q}[\xi]$ *if and only if there exists a quadric of the pencil with rational coefficients having a nonsingular rational point and whose discriminant is a square in* $\mathbb{Q}$*.*

Proof. If $\sqrt{\delta}$ can be avoided, there exists, by Lemma 7.3, a quadric of the pencil with rational coefficients containing a rational line. By Lemma 7.4, the discriminant of this quadric is thus a square in $\mathbb{Q}$. Moreover, since the quadrics of the pencil have rank at least three, the rational line is not the singular line of some quadric (see Table 1) and thus contains a nonsingular point.

Conversely, if there exists a quadric of the pencil with rational coefficients having a rational nonsingular point and whose discriminant is a square, then it has a rational parameterization by Theorem 6.1 and thus $\sqrt{\delta}$ can be avoided. □

Mirroring Proposition 7.5, we can devise a general test for deciding, in the smooth quartic case, whether the square root $\sqrt{\delta}$ can be avoided or not. Consider the equation

$$\sigma^2 = \det\left((\mathbf{x}^T T \mathbf{x})\, S - (\mathbf{x}^T S \mathbf{x})\, T\right), \quad \mathbf{x} = (x, y, z, c)^T, \tag{6}$$

where $c \in \mathbb{Q}$ is some constant such that plane $w = c \in \mathbb{Q}$ contains the vertex of no cone (inertia $(2,1)$) of the pencil. Note that (6) has degree 8 in the worst case.

Theorem 7.6. *When the intersection is a nonsingular quartic, it can be parameterized in* $\mathbb{Q}[\xi, \sqrt{\Delta}]$ *with* $\Delta \in \mathbb{Q}[\xi]$ *if and only Equation* (6) *has a rational solution.*

Proof. Suppose first that (6) has a rational solution $(x_0, y_0, z_0, \sigma_0)$ and let $\mathbf{x}_0 = (x_0, y_0, z_0, c)^T$ and $(\lambda_0, \mu_0) = (\mathbf{x}_0^T T \mathbf{x}_0, -\mathbf{x}_0^T S \mathbf{x}_0)$. The quadric $Q = \lambda_0 Q_S + \mu_0 Q_T$ of the pencil has rational coefficients, contains the rational point $\mathbf{x}_0 = (x_0, y_0, z_0, c)^T$ and its discriminant is a square, equal to σ_0^2. Moreover, if Q has inertia $(2,1)$, then $\mathbf{x}_0$ is not its apex because, by assumption, the plane $w = c$ contains the vertex of no cone of the pencil. It then follows from Theorem 6.1 that our algorithm produces a rational parameterization of Q, and thus a parameterization of the curve of intersection with rational coefficients.

Conversely, if the curve of intersection can be parameterized in $\mathbb{Q}[\xi, \sqrt{\Delta}]$ (with $\Delta \in \mathbb{Q}[\xi]$) there exists a quadric Q of the pencil with rational coefficients containing a rational line and whose discriminant is a square in $\mathbb{Q}$, by Lemmas 7.3 and 7.4. The quadric Q contains a line and thus intersects any plane. Consider any plane $w = c \in \mathbb{Q}$. Since the intersection of a rational line with a rational plane is (or contains) a rational point, the intersection of Q with plane $w = c$ contains a rational point $\mathbf{x} = (x, y, z, c)^T$. The quadric (Q) of the pencil containing that point has associated matrix $(\mathbf{x}^T T \mathbf{x})\, S - (\mathbf{x}^T S \mathbf{x})\, T$ and its determinant is a square. Hence Equation (6) admits a rational solution. □

Unfortunately, the question underlying the above optimality test is not within the range of problems that can currently been answered by algebraic number theory. Indeed, it is not known whether the general problem of determining if an algebraic set contains rational points (known, over $\mathbb{Z}$, as

Hilbert's 10th problem) is decidable [21]. It is known that this problem is decidable for genus zero curves and, under certain conditions, for genus one curves [21], but, for varieties of dimension two or more, very little has been proved on the problem of computing rational points.

The above theorem thus implies that computing parameterizations of the intersections of two arbitrary quadrics that are always optimal in the number of radicals is currently out of reach.

However, in some particular cases, we can use the following corollary to Theorem 7.6 to prove that $\sqrt{\delta}$ cannot be avoided.

Corollary 7.7. *If the intersection C of Q_S and Q_T is a nonsingular quartic and the rational hyperelliptic quartic curve $\sigma^2 = \det(S + \lambda T)$ has no rational point, then the parameterization of C in $\mathbb{Q}(\sqrt{\delta})[\xi, \sqrt{\Delta}]$ with $\Delta \in \mathbb{Q}(\sqrt{\delta})[\xi]$ is optimal in the number of radicals.*

We use this corollary in the next section.

7.3 Worst case examples

We prove here that there are pairs of quadrics, intersecting in the different types of real smooth quartic, such that (6) has no rational solution.

Recall that a set of points L of $\mathbb{P}^3$ is called *affinely finite* if there exists a projective plane P such that $P \cap L = \emptyset$. L is called *affinely infinite* otherwise. In [29], Tu, Wang and Wang prove that a real smooth quartic can be of three different morphologies according to the number of real roots of the determinantal equation. This result, completed with the distinction between the two possible cases when the four roots of the determinantal equation are real, is embodied in the following theorem.

Theorem 7.8. *Let Q_S and Q_T be two quadrics intersecting in $\mathbb{C}$ in a smooth quartic C. C can be classified as follows:*

- *If $\mathcal{D}(\lambda, \mu)$ has four real roots, then C has either two real affinely finite connected components or is empty.*
- *If $\mathcal{D}(\lambda, \mu)$ has two real roots and two complex roots, then C has one real affinely finite connected component.*
- *If $\mathcal{D}(\lambda, \mu)$ has four complex roots, then C has two real affinely infinite connected components.*

7.3.1 Two real affinely finite components

We first look at the case where the quartic has two real affinely finite components and start with a preliminary lemma.

Lemma 7.9. *The equation*

$$y^2 = ax^4 + bx^2 + c + d(x^3 + x) \tag{7}$$

has no rational solution if $a, c \equiv 3 \pmod 8$, $b \equiv 7 \pmod 8$ and $d \equiv 4 \pmod 8$.

Proof. Assume for a contradiction that (x,y) is a rational solution to (7). We can write $x = X/Z$ and $y = Y/Z^2$, where X,Y,Z are integers, $Z \neq 0$ and X,Z are mutually prime (so are not both even).

Consider first the reduction of Equation (7) modulo 8:

$$Y^2 \equiv 3X^4 + 7X^2Z^2 + 3Z^4 + 4XZ(X^2 + Z^2) \pmod 8.$$

If both X and Z are odd, X^2 and Z^2 are equal to $1 \pmod 8$. Thus $4(X^2 + Z^2) \equiv 0 \pmod 8$ and $Y^2 \equiv 3 + 7 + 3 \equiv 5 \pmod 8$, contradicting the fact that $Y^2 \equiv 0, 1$ or $4 \pmod 8$, for all integers Y.

If X and Z are not both odd, one of X^2 and Z^2 is equal to $0 \pmod 4$ and the other is equal to $1 \pmod 4$. The reduction of Equation (7) modulo 4 thus gives $Y^2 \equiv 3 \pmod 4$, contradicting the fact that $Y^2 \equiv 0$ or $1 \pmod 4$, for all integers Y. □

Proposition 7.10. *Consider the following pair of quadrics intersecting in a smooth quartic with two real affinely finite components:*

$$Q_S : 5y^2 + 6xy + 2z^2 - w^2 + 6zw = 0,$$
$$Q_T : 3x^2 + y^2 - z^2 - w^2 = 0.$$

Then the square root $\sqrt{\delta}$ *is necessary to parameterize the curve of intersection.*

Proof. The determinantal equation has four simple real roots and we find a quadric of inertia $(2,2)$ in each of the intervals on which it is positive (in fact Q_S and Q_T are representative quadrics in these intervals). Thus, by Theorem 7.8, the intersection of Q_S and Q_T is a real smooth quartic with two affinely finite components.

We now apply Corollary 7.7 and show that the square root $\sqrt{\delta}$ is necessary to parameterize the curve of intersection. We have:

$$\begin{aligned}\sigma^2 &= \det(S + \lambda T),\\ &= 3\lambda^4 + 12\lambda^3 - 57\lambda^2 - 156\lambda + 99,\\ &\equiv 3\lambda^4 + 7\lambda^2 + 3 + 4(\lambda^3 + \lambda) \pmod 8,\end{aligned}$$

which has no rational solution by Lemma 7.9, so $\sqrt{\delta}$ cannot be avoided. □

7.3.2 One real affinely finite component

As above, we prove a preliminary lemma.

Lemma 7.11. *The equation*

$$y^2 = ax^4 + bx^3 + cx^2 + dx + e \tag{8}$$

has no rational solution if $a, e \equiv 2 \pmod 4$*,* $b, d \equiv 0 \pmod 4$ *and* $c \equiv 3 \pmod 4$*.*

Proof. As before, we assume for a contradiction that (8) has a rational solution (x,y) and write $x = X/Z$ and $y = Y/Z^2$, where X,Y,Z are integers, $Z \neq 0$ and X,Z are mutually prime (so are not both even). We consider the reduction of Equation (8) modulo 4:

$$Y^2 = 2X^4 + 3X^2Z^2 + 2Z^4.$$

If X and Z are not both odd, then $Y^2 \equiv 2 \pmod 4$. If both X and Z are odd, then $Y^2 \equiv 3 \pmod 4$. In both cases, we have a contradiction since $Y^2 \equiv 0$ or $1 \pmod 4$, for all integers Y. □

We can now prove the following.

Proposition 7.12. *Consider the following pair of quadrics intersecting in a smooth quartic with one real affinely finite component:*

$$Q_S : 2x^2 - 2xy + 2xz - 2xw + y^2 + 4yz - 4yw + 2z^2 - 4zw = 0,$$
$$Q_T : x^2 - 2xy + 4xz + 4xw - y^2 + 2yz + 4yw + 4zw - 2w^2 = 0.$$

Then the square root $\sqrt{\delta}$ is necessary to parameterize the curve of intersection.

Proof. The determinantal equation has two simple real roots so it is immediate that the intersection of Q_S and Q_T is a real smooth quartic with one affinely finite component, by Theorem 7.8.

We again apply Corollary 7.7 and show that the square root $\sqrt{\delta}$ is necessary to parameterize the curve of intersection. We have:

$$\begin{aligned}\sigma^2 &= \det(S + \lambda T),\\ &= 22\lambda^4 + 48\lambda^3 - 9\lambda^2 + 60\lambda + 30,\\ &\equiv 2\lambda^4 + 3\lambda^2 + 2 \pmod 4,\end{aligned}$$

which has no rational solution by Lemma 7.11, so $\sqrt{\delta}$ cannot be avoided. □

7.3.3 Two real affinely infinite components

We again prove a preliminary result.

Lemma 7.13. *The equation*

$$y^2 = a(x^4 + x + 1) + bx^3 + cx^2 \tag{9}$$

has no rational solution if $a \equiv 2 \pmod 4$, $b \equiv 0 \pmod 4$ and $c \equiv 1 \pmod 4$.

Proof. We proceed as in Lemmas 7.9 and 7.11, and consider the reduction of Equation (9) modulo 4:

$$Y^2 = 2X^4 + X^2Z^2 + 2XZ^3 + 2Z^4.$$

If X is even and Z is odd, the equation reduces to $Y^2 = 2XZ + 2 \equiv 2 \pmod 4$. If X is odd and Z is even, we also have $Y^2 \equiv 2 \pmod 4$. Finally, if both X and Z are odd, (9) reduces to $Y^2 = 1 + 2XZ \equiv 3 \pmod 4$. In all cases, we have a contradiction since $Y^2 \equiv 0$ or $1 \pmod 4$, for all integers Y. □

This is enough to prove the following.

Proposition 7.14. *Consider the following pair of quadrics intersecting in a smooth quartic with two real affinely infinite components:*

$$Q_S : x^2 - 2y^2 + 4zw = 0,$$
$$Q_T : xy + z^2 + 2zw - w^2 = 0.$$

Then the square root $\sqrt{\delta}$ is necessary to parameterize the curve of intersection.

Proof. The determinantal equation has four simple complex roots so it is immediate that the intersection of Q_S and Q_T is a real smooth quartic with two affinely infinite components, by Theorem 7.8.

We again apply Corollary 7.7. We have:

$$\begin{aligned}\sigma^2 &= \det(S + \lambda T),\\ &= 2\lambda^4 + 4\lambda^3 + 5\lambda^2 + 2\lambda + 2,\\ &\equiv 2\lambda^4 + \lambda^2 + 2\lambda + 2 \pmod 4,\end{aligned}$$

which has no rational solution by Lemma 7.13, so $\sqrt{\delta}$ cannot be avoided. □

8 Examples

We now give several examples of computing a parameterization of the intersection in case the intersection of two quadrics is a smooth quartic. The examples presented cover the range of morphologies discussed in the previous section and illustrate all aspects of optimality and near-optimality. For more examples, see Part IV [12]. All parameterizations have been computed with a C++ implementation of our intersection software (see Part IV).

8.1 Example 1

Our first example consists of the quadrics given in Output 1. The gcd of the partial derivatives of the determinantal equation is 1, so the intersection consists of a (possibly complex) smooth quartic. Since the determinantal equation is found to have four real roots, the intersection, over the reals, is either empty or made of two real affinely finite components (Theorem 7.8). We find a sample quadric in each of the intervals on which $\mathcal{D}(\lambda, \mu)$ is positive and compute its inertia. In the first interval, we find a quadric of inertia $(2,2)$ so we proceed. In the second interval, we find a quadric of inertia $(4,0)$. By Theorem 4.3, we conclude the intersection is empty of real points.

8.2 Example 2

Our second example is as in Output 2. The gcd of the two partial derivatives of the determinantal equation is 1, so the intersection (over $\mathbb{C}$) is a smooth quartic. The fact that the determinantal

Output 1 Execution trace for Example 1.

```
>> quadric 1: 6*x*y + 5*y^2 + 2*z^2 + 6*z*w - w^2
>> quadric 2: 3*x^2 + y^2 - z^2 + 11*w^2

>> launching intersection
>> determinantal equation: 33*l^4 - 124*l^3*m + 137*l^2*m^2 - 32*l*m^3 - 11*m^4
>> gcd of derivatives of determinantal equation: 1
>> number of real roots: 4
>> intervals: ]-4, 0[, ]0, 1[, ]2/2^1, 3/2^1[, ]3/2^1, 4/2^1[
>> picked test point 1 at [ -4 1 ], sign > 0 -- inertia [ 2 2 ] found
>> picked test point 2 at [ 1 1 ], sign > 0 -- inertia [ 4 0 ] found
>> complex intersection: smooth quartic
>> real intersection: empty
>> end of intersection

>> time spent: 10 ms
```

Output 2 Execution trace for Example 2.

```
>> quadric 1: x^2 - x*y - y^2 - y*w + z^2 + w^2
>> quadric 2: 2*x^2 - x*y + y^2 - y*z + y*w + z^2

>> launching intersection
>> determinantal equation: - 6*l^4 - 12*l^3*m + 3*l^2*m^2 + 6*l*m^3 - 2*m^4
>> gcd of derivatives of determinantal equation: 1
>> complex intersection: smooth quartic
>> real intersection: smooth quartic, one real affinely finite component
>> number of real roots: 2
>> intervals: ]-2, -1[, ]-1, 0[
>> picked test point 1 at [ -1 1 ], sign > 0 -- inertia [ 2 2 ] found
>> quadric (2,2) found: x^2 + 2*y^2 - y*z + 2*y*w - w^2
>> decomposition of its determinant [a,b] (det = a^2*b): [ 2 1 ]
>> a point on the quadric: [ 0 0 1 0 ]
>> param of quadric (2,2): [- s*u + t*v, - 2*s*v, (2*s + 2*t)*u + (- 4*s + 2*t)*v, s*u + t*v]
>> status of smooth quartic param: optimal
>> end of intersection

>> parameterization of smooth quartic, branch 1:
[- 4*u^3 + u^2*v + 6*u*v^2 + 2*v^3 - u*sqrt(Delta), - 6*u^3 - 8*u^2*v - 4*u*v^2, - 4*u^3
 + 2*u^2*v + (2*u + 2*v)*sqrt(Delta), 4*u^3 + 5*u^2*v + 2*u*v^2 + 2*v^3 + u*sqrt(Delta)]
>> parameterization of smooth quartic, branch 2:
[- 4*u^3 + u^2*v + 6*u*v^2 + 2*v^3 + u*sqrt(Delta), - 6*u^3 - 8*u^2*v - 4*u*v^2, - 4*u^3
 + 2*u^2*v + (- 2*u - 2*v)*sqrt(Delta), 4*u^3 + 5*u^2*v + 2*u*v^2 + 2*v^3 - u*sqrt(Delta)]
Delta = - 2*u^4 + 10*u^3*v - 9*u^2*v^2 - 8*u*v^3 - 2*v^4

>> time spent: 10 ms
```

equation has two real roots implies that the smooth quartic is real and that it consists of one affinely finite component (Theorem 7.8). Here, the two input quadrics have inertia $(3,1)$ and a first quadric Q_R of inertia $(2,2)$ is found in the pencil between the two roots of $\mathcal{D}$. A point is taken on Q_R and then approximated by a point with integer coordinates. It turns out that the approximation, i.e. $(0,0,1,0)$, also lies on Q_R. We thus use this quadric to parameterize the intersection. Since the determinant of Q_R is a square, it can be rationally parameterized (Proposition 6.6). The end of the calculation is as in Section 4.

Output 3 Execution trace for Example 3.

```
>> quadric 1: 19*x^2 + 22*y^2 + 21*z^2 - 20*w^2
>> quadric 2: x^2 + y^2 + z^2 - w^2

>> launching intersection
>> determinantal equation: - 175560*l^4 - 34358*l^3*m - 2519*l^2*m^2 - 82*l*m^3 - m^4
>> gcd of derivatives of determinantal equation: 1
>> number of real roots: 4
>> intervals: ]-14/2^8, -13/2^8[, ]-26/2^9, -25/2^9[, ]-25/2^9, -24/2^9[, ]-3/2^6, -2/2^6[
>> picked test point 1 at [ -13 256 ], sign > 0 -- inertia [ 2 2 ] found
>> picked test point 2 at [ -3 64 ], sign > 0 -- inertia [ 2 2 ] found
>> complex intersection: smooth quartic
>> real intersection: smooth quartic, two real affinely finite components
>> quadric (2,2) found: - 16*x^2 + 5*y^2 - 2*z^2 + 9*w^2
>> decomposition of its determinant [a,b] (det = a^2*b): [ 12 10 ]
>> a point on the quadric: [ 3 0 0 4 ]
>> param of quadric (2,2): [0, - 24*s*u - 24*t*v, 0, 0] + sqrt(10)*[3*t*u + 6*s*v, 0,
   12*s*u - 12*t*v, - 4*t*u + 8*s*v]
>> status of smooth quartic param: near-optimal
>> end of intersection

>> parameterization of smooth quartic, branch 1:
[(72*u^3 + 4*u*v^2)*sqrt(10) + 3*v*sqrt(10)*sqrt(Delta), - 340*u^2*v + 10*v^3
 - 24*u*sqrt(Delta), (- 118*u^2*v + 5*v^3)*sqrt(10) + 12*u*sqrt(10)*sqrt(Delta),
 (96*u^3 - 12*u*v^2)*sqrt(10) - 4*v*sqrt(10)*sqrt(Delta)]
>> parameterization of smooth quartic, branch 2:
[(72*u^3 + 4*u*v^2)*sqrt(10) - 3*v*sqrt(10)*sqrt(Delta), - 340*u^2*v + 10*v^3
 + 24*u*sqrt(Delta), (- 118*u^2*v + 5*v^3)*sqrt(10) - 12*u*sqrt(10)*sqrt(Delta),
 (96*u^3 - 12*u*v^2)*sqrt(10) + 4*v*sqrt(10)*sqrt(Delta)]
Delta = 20*u^4 - 140*u^2*v^2 + 5*v^4

>> time spent: 10 ms
```

8.3 Example 3

Our third example is Example 5 from [34]. It is the intersection of a sphere and an ellipsoid that are very close to one another. The output of our implementation on that example is shown in Output 3. Since the determinantal equation has four simple real roots, the intersection is either empty or made of two real affinely finite components (Theorem 7.8). Picking a sample quadric in each of the intervals on which $\det R(\lambda,\mu)$ is positive shows that the pencil contains no quadric of inertia $(4,0)$, so the quartic is real. Here, the determinant of the quadric of inertia $(2,2)$ used to parameterize the intersection is not a square, so the parameterization of the quartic contains the square root of some integer. It is thus only near-optimal in the sense that this square root can possibly be avoided.

It turns out that in this particular example it can be avoided. Consider the cone Q_R corresponding to the rational root $(\lambda_0,\mu_0)=(-1,21)$ of the determinantal equation:

$$Q_R : -Q_S + 21\,Q_T = 2x^2 - y^2 - w^2.$$

Q_R contains the obvious rational point $(1,1,0,1)$, which is not its singular point. It implies that it can be rationally parameterized by Proposition 6.3. Plugging this parameterization in the equation

Output 4 Execution trace for Example 4.

```
>> quadric 1: x^2 - 2*y^2 + 4*z*w
>> quadric 2: x*y + z^2 + 2*z*w - w^2

>> launching intersection
>> determinantal equation: 2*l^4 + 4*l^3*m + 5*l^2*m^2 + 2*l*m^3 + 2*m^4
>> gcd of derivatives of determinantal equation: 1
>> number of real roots: 0
>> complex intersection: smooth quartic
>> real intersection: smooth quartic, two real affinely infinite components
>> quadric (2,2) found: x*y + z^2 + 2*z*w - w^2
>> decomposition of its determinant [a,b] (det = a^2*b): [ 2 2 ]
>> a point on the quadric: [ 1 0 0 0 ]
>> param of quadric (2,2): [4*t*u, - 2*s*v, s*u + t*v, s*u + t*v]
   + sqrt(2)*[0, 0, 0, - s*u + t*v]
>> status of smooth quartic param: near-optimal
>> end of intersection

>> parameterization of smooth quartic, branch 1:
[- 4*u*v^2 + 4*v*sqrt(Delta), - 2*u^3 - 8*u*v^2 + 2*u^3*sqrt(2), 4*v^3 - u^2*v*sqrt(2)
 + u*sqrt(Delta), - 2*u^2*v + 4*v^3 + (u^2*v + 4*v^3)*sqrt(2) + (u - u*sqrt(2))*sqrt(Delta)]
>> parameterization of smooth quartic, branch 2:
[- 4*u*v^2 - 4*v*sqrt(Delta), - 2*u^3 - 8*u*v^2 + 2*u^3*sqrt(2), 4*v^3 - u^2*v*sqrt(2)
 - u*sqrt(Delta), - 2*u^2*v + 4*v^3 + (u^2*v + 4*v^3)*sqrt(2) - (u - u*sqrt(2))*sqrt(Delta)]
Delta = 2*u^4 + 10*u^2*v^2 - 4*v^4 + (- 2*u^4 - 4*v^4)*sqrt(2)

>> time spent: 10 ms
```

of Q_S or Q_T gives a simple parameterization for the smooth quartic:

$$\mathbf{X}(u,v) = \begin{pmatrix} u^2+2v^2 \\ 2uv \\ u^2-2v^2 \\ 0 \end{pmatrix} \pm \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \sqrt{2u^4+4u^2v^2+8v^4}.$$

8.4 Example 4

Our last example is the one of Proposition 7.14. The result in shown in Output 4. Here, again, the gcd of the partial derivatives of the determinantal equation is 1, so the intersection curve is, over $\mathbb{C}$, a smooth quartic. But since $\mathcal{D}(\lambda,\mu)$ has in fact no real root, we know by Theorem 7.8 that the smooth quartic is real and has two affinely infinite components. Here, the intermediate quadric Q_R of inertia $(2,2)$ found (which is in fact Q_T) is such that its determinant is not a square. So the parameterization of the quartic contains a square root. Our implementation cannot decide whether this square root is needed or not, so outputs that the parameterization is near-optimal. In this particular example, we know in fact that the parameterization is optimal, by Proposition 7.14.

9 Conclusion

The generic algorithm introduced in Section 4 already represents a substantial improvement over Levin's pencil method and its subsequent refinements. Indeed, we proved that, when the intersection is a smooth quartic (the generic case) our algorithm computes a parameterization which is optimal in the number of radicals involved up to one possibly unnecessary square root. We also showed that deciding (in all cases) whether this extra square root can be avoided is out of reach, and that the parameterization is optimal in some cases. Moreover, for the first time, our algorithms enable to compute in practice an exact form of the parameterization of two arbitrary quadrics with rational coefficients.

Even though this first part of our paper has focused on the generic, smooth quartic case, this algorithm can also be used when the intersection is singular. Assume the intermediate quadric Q_R has inertia $(2,2)$. When the curve of intersection consists of a cubic and a line, the equation Ω in the parameters has a cubic factor of bidegree $(2,1)$ and a linear factor of bidegree $(0,1)$, in view of Fact 7.2. Similarly, when the curve of intersection consists of a conic and two lines, Ω factors in a quadratic factor of bidegree $(1,1)$ and two linear factors of bidegree $(1,0)$ and $(0,1)$. Thus, assuming we know how to factor Ω, we have a way to parameterize each component of the intersection.

Unfortunately, this does not always lead to a parameterization of the intersection that involves only rational functions. When the intersection C is a singular quartic, Ω is irreducible since C itself is, and solving Ω for s in terms of u (or the converse) introduces the square root of a polynomial, while we know that there exists a parameterization of C with rational functions (the genus of the curve is 0).

Always computing parameterizations with rational functions when such parameterizations are known to exist will necessitate rethinking the basic philosophy of our algorithm. Essentially, while the idea of the generic algorithm is to use the rational quadric with *largest* rank as intermediate quadric for parameterizing the intersection, the refined method will instead use the rational quadric with *smallest* rank as intermediate quadric.

Proceeding that way will have the double benefit of always computing the simplest possible parameterizations and much better controlling the size of their coefficients. The price to pay is a multiplication of the cases and the writing of a dedicated piece of software for each (real projective) type of intersection. This is the subject of Parts II and III of this paper.

A The parameterizations of Table 3 are proper

We prove in this section that the parameterizations of Table 3 are not only proper parameterizations of the projective quadrics (in the sense that they define one-to-one correspondences between a dense open subset of the space of the parameters and a dense open subset of the quadric) but they are bijections between the space of the parameters and the quadric. The following two lemmas deal with the parameterizations of quadrics of inertia $(2,2)$ and $(2,1)$. For other types of quadrics, it is straightforward to show that the parameterizations of Table 3 are bijections.

Lemma A.1. $(u,v),(s,t) \mapsto (\frac{ut+a_1vs}{a_1}, \frac{us-a_2vt}{a_2}, \frac{ut-a_1vs}{\sqrt{a_1a_3}}, \frac{us+a_2vt}{\sqrt{a_2a_4}})$ *is a bijection from* $\mathbb{P}^1 \times \mathbb{P}^1$ *onto the surface* $\{(x_1,x_2,x_3,x_4) \in \mathbb{P}^3 \mid a_1x_1^2 + a_2x_2^2 - a_3x_3^2 - a_4x_4^2 = 0\}$*, where* a_1,a_2,a_3,a_4 *are positive.*

Proof. To prove this lemma, we apply the change of coordinates in $\mathbb{P}^3$

$$X = \frac{a_1x_1 + \sqrt{a_1a_3}x_3}{2},\ Y = \frac{a_1x_1 - \sqrt{a_1a_3}x_3}{2a_1},\ Z = \frac{a_2x_2 + \sqrt{a_2a_4}x_4}{2},\ W = \frac{-a_2x_2 + \sqrt{a_2a_4}x_4}{2a_2},$$

or equivalently

$$x_1 = \frac{X + a_1 Y}{a_1},\ x_3 = \frac{X - a_1 Y}{\sqrt{a_1 a_3}},\ x_2 = \frac{Z - a_2 W}{a_2},\ x_4 = \frac{Z + a_2 W}{\sqrt{a_2 a_4}}.$$

In the new frame, the equation of the surface is $XY - ZW = 0$ and the map becomes

$$\Phi : (u,v),(s,t) \mapsto (X,Y,Z,W) = (ut, vs, us, vt).$$

The map Φ is clearly a map from $\mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^3$ because $\Phi((\lambda u, \lambda v),(\mu s, \mu t)) = \lambda\mu\Phi((u,v),(s,t))$ and $\Phi((u,v),(s,t)) = (0,0,0,0)$ if and only if $(u,v) = (0,0)$ or $(s,t) = (0,0)$. Moreover, the image of Φ is clearly included in the surface of equation $XY - ZW = 0$. Conversely, if (X,Y,Z,W) is a point of this surface, at least one of its coordinates is non zero (we are in a projective space), and by symmetry we may suppose that $X \neq 0$. Considering $(X,Z,W) = (ut, us, vt)$, we have $ut \neq 0$, $\frac{Z}{X} = \frac{s}{t}$, and $\frac{W}{X} = \frac{v}{u}$. Thus $\frac{Z}{X}$ uniquely defines (s,t) up to a constant factor and similarly for $\frac{W}{X}$ and (u,v), which shows the injectivity of Φ. Furthermore, $XY - ZW = 0$ implies $Y = \frac{ZW}{X} = \frac{us\ vt}{ut} = vs$ which shows that Φ is surjective. ☐

Recall that $\mathbb{P}^{\star 2}$ denotes the quasi-projective space defined as the quotient of $\mathbb{R}^3 \setminus \{0,0,0\}$ by the equivalence relation $\sim$ where $(x_1,x_2,x_3) \sim (y_1,y_2,y_3)$ if and only if $\exists \lambda \in \mathbb{R} \setminus \{0\}$ such that $(x_1,x_2,x_3) = (\lambda y_1, \lambda y_2, \lambda^2 y_3)$.

Lemma A.2. *$(u,v,s) \mapsto (uv, \frac{u^2 - a_1 a_2 v^2}{2a_2}, \frac{u^2 + a_1 a_2 v^2}{2\sqrt{a_2 a_3}}, s)$ is a bijection from $\mathbb{P}^{\star 2}$ onto the surface $\{(x_1,x_2,x_3,x_4) \in \mathbb{P}^3 \mid a_1 x_1^2 + a_2 x_2^2 - a_3 x_3^2 = 0\}$, where a_1, a_2, a_3 are positive.*

Proof. For this lemma, we consider the change of coordinates in $\mathbb{P}^3$

$$X = x_1,\ Y = \sqrt{a_2 a_3} x_3 + a_2 x_2,\ Z = \frac{\sqrt{a_2 a_3} x_3 - a_2 x_2}{a_1 a_2},\ W = x_4,$$

or equivalently

$$x_1 = X,\ x_2 = \frac{Y - a_1 a_2 Z}{2a_2},\ x_3 = \frac{Y + a_1 a_2 Z}{2\sqrt{a_2 a_3}},\ x_4 = W.$$

In the new frame, the equation of the surface is $X^2 - YZ = 0$ and the map becomes

$$\Psi : (u,v,s) \mapsto (X,Y,Z,W) = (uv, u^2, v^2, s).$$

The map Ψ is clearly a map from $\mathbb{P}^{\star 2}$ into $\mathbb{P}^3$ because $\Psi(\lambda u, \lambda v, \lambda^2 s) = \lambda^2 \Psi(u,v,s)$ and $\Psi(u,v,s) = (0,0,0,0)$ if and only if $(u,v,s) = (0,0,0)$. Moreover, the image of Ψ is clearly included in the surface of equation $X^2 - YZ = 0$. Conversely, if (X,Y,Z,W) is a point of this surface, then we have to prove that its preimage consists in exactly one point of $\mathbb{P}^{\star 2}$. If $Y = Z = 0$, we have also $X = 0$ and a point of the preimage should satisfy $u = v = 0$; it is therefore unique (in $\mathbb{P}^{\star 2}$) an it exists by $\Psi(0,0,W) = (0,0,0,W)$.

If Y or Z is nonzero, we may suppose by symmetry that $Y \neq 0$. Considering $(X,Y,W) = (uv, u^2, s)$ we have $u \neq 0$, $\frac{X}{Y} = \frac{v}{u}$ and $\frac{W}{Y} = \frac{s}{u^2}$. Thus $\frac{X}{Y}$ and $\frac{W}{Y}$ uniquely define $(u,v,s) \in \mathbb{P}^{\star 2}$ which implies that Ψ is injective. Furthermore, $YZ = X^2$ implies $Z = \frac{X^2}{Y} = \frac{(uv)^2}{u^2} = v^2$ which shows that Ψ is surjective. ☐

Remark A.3. Although the statements and the proofs of Lemma A.1 and A.2 are very similar, there is a big difference between the two bijections: the bijection is an isomorphism and a diffeomorpism in Lemma A.1 but not in Lemma A.2 where the space of the parameters is smooth while the surface is singular at (0,0,0,1).

References

[1] E. Berberich, M. Hemmer, L. Kettner, E. Schömer, and N. Wolpert. An exact, complete and efficient implementation for computing planar maps of quadric intersection curves. In *Proc. of SoCG (ACM Symposium on Computational Geometry), San Diego*, pages 99–115, 2005.

[2] T. Bromwich. *Quadratic Forms and Their Classification by Means of Invariant Factors*. Cambridge Tracts in Mathematics and Mathematical Physics, 1906.

[3] L. Dupont, D. Lazard, S. Lazard, and S. Petitjean. Near-optimal parameterization of the intersection of quadrics. In *Proc. of SoCG (ACM Symposium on Computational Geometry), San Diego*, pages 246–255, 2003.

[4] L. Dupont, D. Lazard, S. Lazard, and S. Petitjean. Near-optimal parameterization of the intersection of quadrics: II. A classification of pencils. Research Report n^{o} 5668, INRIA, September 2005.

[5] L. Dupont, D. Lazard, S. Lazard, and S. Petitjean. Near-optimal parameterization of the intersection of quadrics: III. Parameterizing singular intersections. Research Report n^{o} 5669, INRIA, September 2005.

[6] R. Farouki, C. Neff, and M. O'Connor. Automatic parsing of degenerate quadric-surface intersections. *ACM Transactions on Graphics*, 8(3):174–203, 1989.

[7] P. Finsler. Über das Vorkommen definiter und semidefiniter Formen in Scharen quadratischer Formen. *Comment. Math. Helv.*, 9:188–192, 1936/1937.

[8] R. Goldman and J. Miller. Combining algebraic rigor with geometric robustness for the detection and calculation of conic sections in the intersection of two natural quadric surfaces. In *Proc. of ACM Symposium on Solid Modeling Foundations and CAD/CAM Applications*, pages 221–231, 1991.

[9] C.-K. Hung and D. Ierardi. Constructing convex hulls of quadratic surface patches. In *Proceedings of 7th CCCG (Canadian Conference on Computational Geometry), Québec, Canada*, pages 255–260, 1995.

[10] J. Keyser, T. Culver, M. Foskey, S. Krishnan, and D. Manocha. ESOLID: A system for exact boundary evaluation. *Computer-Aided Design*, 36(2):175–193, 2004.

[11] T. Lam. *The Algebraic Theory of Quadratic Forms*. W.A. Benjamin, Reading, MA, 1973.

[12] S. Lazard, L. M. Peñaranda, and S. Petitjean. Near-optimal parameterization of the intersection of quadrics: IV. An efficient and exact implementation. Research Report n^o 5670, INRIA, September 2005.

[13] J. Levin. A parametric algorithm for drawing pictures of solid objects composed of quadric surfaces. *Communications of the ACM*, 19(10):555–563, 1976.

[14] J. Levin. Mathematical models for determining the intersections of quadric surfaces. *Computer Graphics and Image Processing*, 11(1):73–87, 1979.

[15] The Maple System. Waterloo Maple Software. `http://www.maplesoft.com`.

[16] J. Miller. Geometric approaches to nonplanar quadric surface intersection curves. *ACM Transactions on Graphics*, 6(4):274–307, 1987.

[17] J. Miller and R. Goldman. Geometric algorithms for detecting and calculating all conic sections in the intersection of any two natural quadric surfaces. *Graphical Models and Image Processing*, 57(1):55–66, 1995.

[18] B. Mourrain, J.-P. Técourt, and M. Teillaud. Sweeping of an arrangement of quadrics in 3D. *Computational Geometry: Theory and Applications*, 30:145–164, 2005. Special issue, 19th European Workshop on Computational Geometry.

[19] M. Namba. *Geometry of Projective Algebraic Curves*. Marcel Dekker, Inc., 1984.

[20] D. Perrin. *Géométrie algébrique : une introduction*. InterEditions, 1995.

[21] B. Poonen. Computing rational points on curves. In *Number Theory for the Millenium*. A. K. Peters, Boston, 2001. Proc. of Millennial Conference on Number Theory, 2000.

[22] A. Requicha and H. Voelcker. Solid modeling: a historical summary and contemporary assessment. *IEEE Computer Graphics and Applications*, 2(1):9–24, 1982.

[23] F. Rouillier and P. Zimmermann. Efficient isolation of polynomial's real roots. *Journal of Computational and Applied Mathematics*, 162(1):33–50, 2004.

[24] R. Sarraga. Algebraic methods for intersections of quadric surfaces in GMSOLID. *Computer Vision, Graphics, and Image Processing*, 22:222–238, 1983.

[25] E. Schömer and N. Wolpert. An exact and efficient approach for computing a cell in an arrangement of quadrics. *Computational Geometry: Theory and Applications*, 2005. Special Issue on Robust Geometric Algorithms and their Implementations, to appear.

[26] C.-K. Shene and J. Johnstone. Computing the intersection of a plane and a natural quadric. *Computer & Graphics*, 12(2):179–186, 1992.

[27] C.-K. Shene and J. Johnstone. On the lower degree intersections of two natural quadrics. *ACM Transactions on Graphics*, 13(4):400–424, 1994.

[28] C.-K. Shene and J. Johnstone. On the lower degree intersections of two natural quadrics. *ACM Transactions on Graphics*, 13(4):400–424, 1994.

[29] C. Tu, W. Wang, and J. Wang. Classifying the nonsingular intersection curve of two quadric surfaces. In *Proc. of GMP'02 (Geometric Modeling and Processing)*, pages 23–32, 2002.

[30] F. Uhlig. Simultaneous block diagonalization of two real symmetric matrices. *Linear Algebra and Its Applications*, 7:281–289, 1973.

[31] F. Uhlig. A canonical form for a pair of real symmetric matrices that generate a nonsingular pencil. *Linear Algebra and Its Applications*, 14:189–209, 1976.

[32] W. Wang, R. Goldman, and C. Tu. Enhancing Levin's method for computing quadric-surface intersections. *Computer-Aided Geometric Design*, 20(7):401–422, 2003.

[33] W. Wang, B. Joe, and R. Goldman. Rational quadratic parameterizations of quadrics. *International Journal of Computational Geometry and Applications*, 7(6):599–619, 1997.

[34] W. Wang, B. Joe, and R. Goldman. Computing quadric surface intersections based on an analysis of plane cubic curves. *Graphical Models*, 64(6):335–367, 2002.

[35] I. Wilf and Y. Manor. Quadric-surface intersection curves: shape and structure. *Computer-Aided Design*, 25(10):633–643, 1993.

[36] N. Wolpert. *An Exact and Efficient Algorithm for Computing a Cell in an Arrangement of Quadrics*. PhD thesis, Universität des Saarlandes, Saarbrücken, 2002.

Unité de recherche INRIA Lorraine
LORIA, Technopôle de Nancy-Brabois - Campus scientifique
615, rue du Jardin Botanique - BP 101 - 54602 Villers-lès-Nancy Cedex (France)

Unité de recherche INRIA Futurs : Parc Club Orsay Université - ZAC des Vignes
4, rue Jacques Monod - 91893 ORSAY Cedex (France)
Unité de recherche INRIA Rennes : IRISA, Campus universitaire de Beaulieu - 35042 Rennes Cedex (France)
Unité de recherche INRIA Rhône-Alpes : 655, avenue de l'Europe - 38334 Montbonnot Saint-Ismier (France)
Unité de recherche INRIA Rocquencourt : Domaine de Voluceau - Rocquencourt - BP 105 - 78153 Le Chesnay Cedex (France)
Unité de recherche INRIA Sophia Antipolis : 2004, route des Lucioles - BP 93 - 06902 Sophia Antipolis Cedex (France)

Éditeur
INRIA - Domaine de Voluceau - Rocquencourt, BP 105 - 78153 Le Chesnay Cedex (France)
http://www.inria.fr
ISSN 0249-6399