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Abstract: Given a smooth surface, a blue (red) ridge is a curve along which the maximum (minimum) principal curvature has an extremum along its curvature line. Ridges are curves of *extremal* curvature and therefore encode important informations used in segmentation, registration, matching and surface analysis. State of the art methods for ridge extraction either report red and blue ridges simultaneously or separately —in which case need a local orientation procedure of principal directions is needed, but no method developed so far topologically certifies the curves reported.

On the way to developing certified algorithms independent from local orientation procedures, we make the following fundamental contribution. For any smooth parametric surface, we exhibit the implicit equation $P = 0$ of the singular curve $\mathscr{P}$ encoding all ridges of the surface (blue and red), and show how to recover the colors from factors of P. Exploiting $P = 0$, we also derive a zero dimensional system coding the so-called turning points, from which elliptic and hyperbolic ridge sections of the two colors can be derived. Both contributions exploit properties of the Weingarten map of the surface and require computer algebra. Algorithms exploiting the structure of P for algebraic surfaces are developed in a companion paper.

Key-words: Ridges, Umbilics, Differential Geometry.

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La structure implicite des ridges d'une surface lisse

Résumé : Étant donnée une surface lisse, un *ridge* bleu (rouge) est une courbe le long de laquelle la courbure principale maximum (minimum) a un extremum en suivant sa ligne de courbure. Les ridges sont des lignes d'extrêmes de courbure et codent des informations importantes utilisées en segmentation, recalage, comparaison et analyse de surfaces. Les méthodes de calcul de ridges reportent les ridges des deux couleurs simultanément ou séparément, auquel cas une procédure d'orientation locale des directions principales doit être invoquée. Néanmoins, aucune de ces alternatives n'a permis le développement d'algorithmes garantissant la topologie des courbes produites.

Soit une surface lisse paramétrée. En guise de pré-requis au développement d'algorithmes certifiés de calcul de ridges, algorithmes ne nécessitant pas qui plus est de procédure d'orientation locale, nous établissons l'équation implicite $P = 0$ de la courbe singulière codant l'ensemble des ridges de la surface, et nous montrons comment colorier les ridges en rouge et bleu à partir des facteurs de cette courbe. En utilisant cette même équation, nous établissons un système zéro-dimensionnel dont les solutions sont les *turning* points des ridges, points à partir desquels les ridges d'une couleur donnée peuvent être étiquetés *elliptiques* ou *hyperboliques*. Les deux contributions exploitent des propriétés fines de l'application de Weingarten, et font appel au calcul symbolique.

Mots-clés : Ridges et Extrêmes de courbure, Ombilics, Géométrie Différentielle.

1 Introduction

1.1 Ridges

Differential properties of surfaces embedded in $\mathbb{R}^3$ are a fascinating topic per se, and have long been of interest for artists and mathematicians, as illustrated by the parabolic lines drawn by Felix Klein on the Apollo of Belvedere [HCV52], and also by the developments reported in [Koe90]. Beyond these noble considerations, the recent development of laser range scanners and medical images shed light on the importance of being able to analyze discrete datasets consisting of point clouds in 3*D* or medical images —grids of 3*D* voxels. Whenever the datasets processed model piecewise smooth surfaces, a precise description of the models naturally calls for differential properties. In particular, applications such as shape matching [HGY$^+$99], surface analysis [HGY$^+$99], or registration [PAT00] require the characterization of high order properties and in particular the characterization of curves of *extremal* curvatures, which are precisely the so-called *ridges*. Interestingly, ridges are also ubiquitous in the analysis of Delaunay based surface meshing algorithms [ABL03].

A ridge consists of the points where one of the principal curvatures has an extremum along its curvature line. Since each point which is not an umbilic has two different principal curvatures, a point potentially belongs to two different ridges. Denoting k_1 and k_2 the principal curvatures —we shall always assume that $k_1 \geq k_2$, a ridge is called blue (red) if k_1 (k_2) has an extremum. A crossing point of a red and a blue ridge curve is called a purple point. Moreover, a ridge is called *elliptic* if it corresponds to a maximum of k_1 or a minimum of k_2, and is called *hyperbolic* otherwise. Ridges witness extrema of principal curvatures and their definition involves derivatives of curvatures, whence third order differential quantities. Moreover, the classification of ridges as elliptic or hyperbolic involves fourth order differential quantities, so that the precise definition of ridges requires C^4 differentiable surfaces.

The calculation of ridges poses difficult problems, which are of three kinds.

Topological difficulties. Ridges of a smooth surface form a singular curve on the surface. Ridges of the two colors intersect at purple points, have complex interactions with umbilics and curvature lines —giving rise to turning points. A comprehensive description of ridges can be found in [Por71, Por01].

From the application standpoint, reporting ridges of a surface faithfully requires reporting umbilics, purple points and turning points.

Numerical difficulties. It is well known that parabolic curves of a smooth surface correspond to points where the Gauss curvature vanishes. Similarly, ridges are witnessed by the zero crossings of the so-called extremality coefficients, denoted b_0 (b_3) for blue (red) ridges, which are the derivatives of the principal curvatures along their respective curvature lines.

Algorithms reporting ridges need to estimate b_0 and b_3. Estimating these coefficients depends on the particular type of surface processed —implicitly defined, parameterized, discretized by a mesh— and is numerically a difficult task.

Orientation difficulties. Since coefficients b_0 and b_3 are derivatives of principal curvatures, they are third-order coefficients in the Monge form of the surface —the Monge form is the Taylor expansion of the surface expressed as a height function in the particular frame defined by the principal directions. But like all odd terms of the Monge form, their sign depends upon the orientation of the principal frame used.

Practically, tracking the sign change of functions whose sign depends on the particular orientation of the frame in which they are expressed poses a problem. In particular, tracking a zero-crossing of b_0 or b_3 along a curve segment on the surface imposes to find a coherent orientation of the principal frame at the endpoints. Given two principal directions at these endpoints, one way to find a local orientation consists of choosing two vectors so that they make an acute angle, whence the name *Acute Rule*. This rule has been used since the very beginning of computer examination of ridges [Mor90, Mor96], and is implicitly used in almost all algorithms. But the question of specifying conditions guaranteeing the decisions made are correct has only been addressed recently [CP05b].

An other approach is to extract the zero level set of the Gaussian extremality $E_g = b_0 b_3$ defined in [Thi96]. This function has a well defined sign independent from the orientation, but it is not defined at umbilics.

1.2 Contributions and paper overview

Given the previous difficulties, the ultimate wish is the development of certified algorithms reporting ridges without resorting to local orientation procedures. As a pre-requisite for such algorithms, we make the following contribution. Let $\Phi(u,v)$ be a smooth parameterized surface over a domain $\mathscr{D} \subset \mathbb{R}^2$. We exhibit the implicit equation $P = 0$ of the singular curve $\mathscr{P}$ encoding all ridges of the surface (blue and red), and show how to recover the colors from factors of P. We also derive a zero dimensional system coding the so-called turning points, from which elliptic and hyperbolic ridge sections of the two colors can be derived.

The paper is organized as follows. Notations are set in section 2, and preliminary differential lemmas are proved in section 3. The implicit equation for ridges is derived in section 4. The system for turning points and the determination of ridge types are stated in section 5. Corollaries for polynomial parametric surfaces and an illustration of the effectiveness of the main theorem on a complex Bezier surface is given in section 6. A primer on ridges is provided in appendix 8 and symbolic computations performed with Maple are provided in appendix 9.

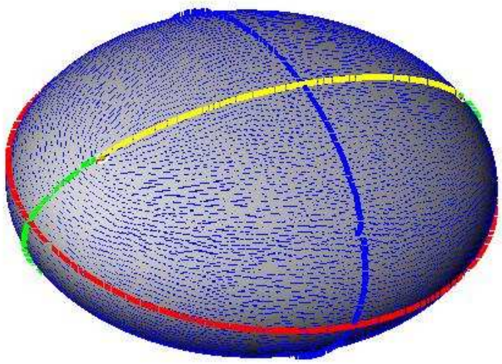

Figure 1: Umbilics, ridges, and principal blue foliation on the ellipsoid (10k points)

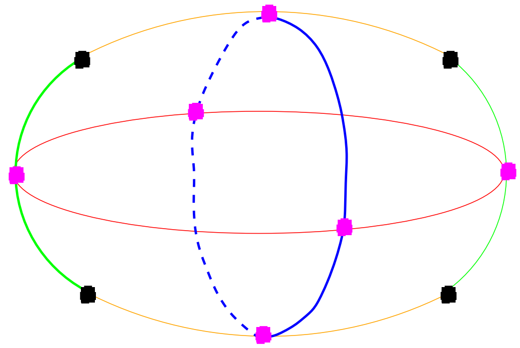

Figure 2: Schematic view of the umbilics and the ridges. Max of k_1: blue; Min of k_1: green; Min of k_2: red; Max of k_2: yellow

2 Notations

Ridges. The derivation of the implicit equation $P = 0$ of all ridges of a surface exploits properties of the Weingarten map and of the Monge form of the surface. Properties of the Monge form, i.e. of the expression of the surface as a height function in a coordinate frame associated to the principal directions are recalled in section 8. We just point out the main notations and some key conventions.

At any point of the surface, the maximal (minimal) principal curvature is denoted k_1 (k_2), and its associated direction d_1 (d_2). Anything related to the maximal (minimal) curvature is qualified blue (red), for example we shall speak of the blue curvature for k_1 or the red direction for d_2. A ridge associated with k_1 is defined by the equation $b_0 = 0$, with b_0 is the directional derivative of the principal curvature k_1 along its curvature line. Similarly, ridges associated to k_2 are defined by $b_3 = 0$. Since we shall make precise statements about ridges, it should be recalled that ridges through umbilics are open i.e. umbilics are excluded.

Differential calculus. Let $f(u,v) : \mathscr{D} \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a continuously differentiable function. The derivative of f wrt variable u denoted f_u, and the gradient of f is denoted denote $df = (f_u, f_v)$. A point in $\mathscr{D}$ is *singular* if the gradient df vanishes, else it is *regular*.

Misc. The inner product of two vectors x,y is denoted $\langle x,y \rangle$, the norm of x is $||x|| = \langle x,x \rangle^{1/2}$ and the exterior product is $x \wedge y$.

3 Manipulations involving the Weingarten map of the surface

Let Φ be the parameterization of class C^k for $k \geq 4$. Principal directions and curvatures of the surface are expressed in terms of second order derivatives of Φ. More precisely, the matrices of the first and second fundamental forms in the basis (Φ_u, Φ_v) of the tangent space are

$$I = \begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} \langle \Phi_u, \Phi_u \rangle & \langle \Phi_u, \Phi_v \rangle \\ \langle \Phi_u, \Phi_v \rangle & \langle \Phi_v, \Phi_v \rangle \end{pmatrix},$$

$$II = \begin{pmatrix} l & m \\ m & n \end{pmatrix} = \begin{pmatrix} \langle N_n, \Phi_{uu} \rangle & \langle N_n, \Phi_{uv} \rangle \\ \langle N_n, \Phi_{uv} \rangle & \langle N_n, \Phi_{vv} \rangle \end{pmatrix}, \quad \text{with} \quad N = \Phi_u \wedge \Phi_v, \quad N_n = \frac{N}{||N||}.$$

To compute the principal directions and curvatures, one resorts to the Weingarten map, whose matrix in the basis (Φ_u, Φ_v) is given by $W = (w_{ij}) = I^{-1}II$. The Weingarten map is a self-adjoint operator [1] of the tangent space [dC76]. The principal directions d_i and principal curvatures k_i are the eigenvectors and eigenvalues of the matrix W. Observing that $||N||^2 = \det I$, matrix W can also be written as follows —an expression of special interest for polynomial surfaces:

$$W = \frac{1}{(\det I)^{3/2}} \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{1}$$

Recall that a parameterized surface is called *regular* if the tangent map of the parameterization (the Jacobian) has rank two everywhere. Since the first fundamental form is the restriction of the inner product of the ambient space to the tangent space, one has:

Observation. 1 *If Φ is a parameterized surface which is regular, the quadratic form I is positive definite.*

In the following, the surface is assumed regular, thus $\det(I) \neq 0$.

3.1 Principal curvatures.

The characteristic polynomial of W is

$$P_W(k) = k^2 - tr(W)k + \det(W) = k^2 - (w_{11} + w_{22})k + w_{11}w_{22} - w_{12}w_{21}.$$

Its discriminant is

$$\Delta(k) = (tr(W))^2 - 4\det(W) = (w_{11} + w_{22})^2 - 4(w_{11}w_{22} - w_{12}w_{21}) = (w_{11} - w_{22})^2 + 4w_{12}w_{21}.$$

A simplification of this discriminant leads to the definition of the following function, denoted p_2:

$$p_2 = (\det I)^3 \Delta(k) = (A - D)^2 + 4BC$$

[1] A self-adjoint map L over a vector space V with a bilinear form $< ., . >$ is a linear map such that $\langle Lu, v \rangle = \langle u, Lv \rangle$, for all $u, v \in V$. Such a map can be diagonalized in an orthonormal basis of V.

The principal curvatures k_i, with the convention $k_1 \geq k_2$, are the eigenvalues of W, that is:

$$k_1 = \frac{A+D+\sqrt{p_2}}{2(\det I)^{3/2}} \quad k_2 = \frac{A+D-\sqrt{p_2}}{2(\det I)^{3/2}}. \tag{2}$$

A point is called an umbilic if the principal curvatures are equal. One has:

Lemma. 1 *The two following equivalent conditions characterize umbilics:*

1. $p_2 = 0$
2. $A = D$ *and* $B = C = 0$.

Proof. Since $\det(I) \neq 0$, one has $p_2 = 0 \Leftrightarrow \Delta(k) = 0$, hence 1. characterizes umbilics. Condition 2. trivially implies 1. To prove the converse, assume that $p_2 = 0$ i.e. the Weingarten map has a single eigenvalue k. This linear map is self-adjoint hence diagonalizable in an orthogonal basis, and the diagonal form is a multiple of the identity. It is easily checked that the matrix remains a multiple of the identity in any basis of the tangent space, in particular in the basis (Φ_u, Φ_v), which implies condition 2. □

3.2 Principal directions.

Let us focus on the maximum principal direction d_1. A vector of direction d_1 is an eigenvector of W for the eigenvalue k_1. Denote:

$$W - k_1 Id = \begin{pmatrix} w_{11} - k_1 & w_{12} \\ w_{21} & w_{22} - k_1 \end{pmatrix}. \tag{3}$$

At non umbilic points, the matrix $W - k_1 Id$ has rank one, hence either $(-w_{12}, w_{11} - k_1) \neq (0,0)$ or $(-w_{22} + k_1, w_{21}) \neq (0,0)$. Using the expression of W given by Eq. (1), up to a normalization factor of $(\det I)^{3/2}$, a non zero maximal principal vector can be chosen as either

$$v_1 = 2(\det I)^{3/2}(-w_{12}, w_{11} - k_1) = (-2B, A - D - \sqrt{p_2}) \text{ or } w_1 = 2(\det I)^{3/2}(-w_{22} + k_1, w_{21}) = (A - D + \sqrt{p_2}, 2C). \tag{4}$$

For the minimal principal direction d_2 one chooses $v_2 = (-2B, A - D + \sqrt{p_2})$ and $w_2 = (A - D - \sqrt{p_2}, 2C)$.

Lemma. 2 *One has the following relations:*
$v_1 = (0,0) \Leftrightarrow (B = 0 \text{ and } A \geq D)$,
$v_2 = (0,0) \Leftrightarrow (B = 0 \text{ and } A \leq D)$,
$w_1 = (0,0) \Leftrightarrow (C = 0 \text{ and } A \leq D)$,
$w_2 = (0,0) \Leftrightarrow (C = 0 \text{ and } A \geq D)$.

Proof. The proofs being equivalent, we focus on the first one. One has:

$$v_1 = (0,0) \Leftrightarrow \begin{cases} B = 0 \\ A - D = \sqrt{(A-D)^2 + 4BC} \end{cases} \Leftrightarrow \begin{cases} B = 0 \\ A - D = \sqrt{(A-D)^2} \end{cases} \Leftrightarrow \begin{cases} B = 0 \\ A - D \geq 0 \end{cases}$$

□

A direct consequence of lemma 2 is the following:

Observation. 2 *1. The two vector fields v_1 and w_1 vanish simultaneously exactly at umbilics. The same holds for v_2 and w_2.*

2. The equation $\{v_1 = (0,0)$ or $v_2 = (0,0)\}$ is equivalent to $v_1 v_2 = 0$ and eventually to $B = 0$.

4 Implicitly defining ridges

In this section, we prove the implicit expression $P = 0$ of ridges. Before diving into the technicalities, we first outline the method.

4.1 Problem

In characterizing ridges, a first difficulty comes from the fact that the sign of an extremality coefficient (b_0 or b_3) is not well defined. Away from umbilics, denoting d_1 the principal direction associated to k_1, there are two unit opposite vectors y_1 and $-y_1$ orienting d_1. That is, one can define two extremality coefficients $b_0(y_1) = \langle dk_1, y_1 \rangle$ and $b_0(-y_1) = \langle dk_1, -y_1 \rangle = -b_0(y_1)$. Therefore, the sign of b_0 is not well defined. In particular, notice that tracking the zero crossings of b_0 in-between two points of the surface requires using coherent orientation of the principal direction d_1 at these endpoints, a problem usually addressed using the acute rule [CP05b]. Notice however the equation $b_0 = 0$ is not ambiguous. A second difficulty comes from umbilics where b_0 is not defined since k_1 is not smooth —that is dk_1 is not defined.

4.2 Method outline

Principal curvatures and directions read from the Weingarten map of the surface. At each point which is not an umbilic, one can define two vector fields v_1 or w_1 which are collinear with d_1, with the additional property that one (at least) of these two vectors is non vanishing. Let z stand for one of these non vanishing vectors. The nullity of $b_0 = \langle dk_1, y_1 \rangle$ is equivalent to that of $\langle dk_1, z \rangle$ —that is the normalization of the vector along which the directional derivative is computed does not matter.

Using v_1 and w_1, the principal maximal vectors defined in the previous section, we obtain two independent equations of blue ridges. Each has the drawback of encoding, in addition to blue ridge points, the points where v_1 (or w_1) vanishes. As a consequence of observation 2, the conjunction of these two equations defines the set of blue ridges union the set of umbilics. The same holds for red ridges and the minimal principal vector fields v_2 ad w_2. One has to note the symmetry between the equations for blue and red ridges in lemma 3. Eventually, combining the equation for blue ridges with v_1 and the equation for red ridges with v_2 gives the set of blue ridges union the set of red ridges union the set of zeros of $v_1 = 0$ or $v_2 = 0$. This last set is also $B = 0$ (observation 2), hence dividing by B allows to eradicate these spurious points and yields the equation $P = 0$ of blue ridges together with red ridges. One can think of this equation as an improved version of the Gaussian extremality $E_g = b_0 b_3$ defined in [Thi96].

Our strategy cumulates several advantages: (i)blue and red ridges are processed at once, and the information is encoded in a single equation (ii)orientation issues arising when one is tracking the zero crossings of b_0 or b_3 disappear. The only drawback is that one looses the color of the ridge. But this color is recovered with the evaluation of the sign of factors of the expression P.

4.3 Precisions of vocabulary

In the statement of the results, we shall use the following terminology. Umbilic points are points where both principal curvatures are equal. A ridge point is a point which is not an umbilic, and is an extremum of a principal curvature along its curvature line. A ridge point is further called a blue (red) ridge point is a an extrema of the blue (red) curvature along its line. A ridge point may be both blue and red, in which case it is called a purple point.

4.4 Implicit equation of ridges

Lemma. 3 *For a regular surface, there exist differentiable functions a, a', b, b' which are polynomials wrt A, B, C, D and $\det I$, as well as their first derivatives, such that:*

1. *the set of blue ridges union $\{v_1 = 0\}$ has equation $a\sqrt{p_2} + b = 0$,*
2. *the set of blue ridges union $\{w_1 = 0\}$ has equation $a'\sqrt{p_2} + b' = 0$,*
3. *the set of blue ridges union the set of umbilics has equation $\begin{cases} a\sqrt{p_2} + b = 0 \\ a'\sqrt{p_2} + b' = 0 \end{cases}$*
4. *the set of red ridges union $\{v_2 = 0\}$ has equation $a\sqrt{p_2} - b = 0$,*
5. *the set of red ridges union $\{w_2 = 0\}$ has equation $a'\sqrt{p_2} - b' = 0$,*
6. *the set of red ridges union the set of umbilics has equation $\begin{cases} a\sqrt{p_2} - b = 0 \\ a'\sqrt{p_2} - b' = 0 \end{cases}$*

Moreover, a, a', b, b' are defined by the equations:

$$a\sqrt{p_2} + b = \langle Numer(dk_1), v_1 \rangle \quad a'\sqrt{p_2} + b' = \langle Numer(dk_1), w_1 \rangle. \tag{5}$$

Proof. The principal curvatures are not differentiable at umbilics since the denominator of dk_i contains $\sqrt{p_2}$. But away from umbilics, the equation $\langle dk_1, v_1 \rangle = 0$ is equivalent to $\langle \mathrm{Numer}(dk_1), v_1 \rangle = 0$. This equation is rewritten as $a\sqrt{p_2} + b = 0$, the explicit expressions of a and b being given in appendix 9. This equation describes the set of blue ridge points union the set where v_1 vanishes. A similar derivation yields the second claim. Finally, the third claim follows from observation 2.

Results for red ridges are similar and the reader is referred to appendix 9 for the details. □

Lemma. 4 *1. If $p_2 = 0$ then $a = b = a' = b' = 0$.*

2. The set of purple points has equation $\begin{cases} a = b = a' = b' = 0 \\ p_2 \neq 0 \end{cases}$

Proof. 1. If $p_2 = 0$, one has $A = D$ and $B = C = 0$. Substituting these conditions in the expressions of a, a', b, b' gives the result, computations are sketched in appendix 9.

2. Let p be a purple point, it is a ridge point and hence not an umbilic, then $p_2 \neq 0$. The point p is a blue and a red ridge point, hence it satisfies all equations of lemma 3. If $a \neq 0$ then equations 1. and 4. imply $\sqrt{p_2} = -b/a = b/a$ hence $b = 0$ and $\sqrt{p_2} = 0$ which is a contradiction. Consequently, $a = 0$ and again equation 1. implies $b = 0$. A similar argument with equation 2. and 5. gives $a' = b' = 0$.

The converse is trivial: if $a = b = a' = b' = 0$ then equations 3. and 6. imply that the point is a purple point or an umbilic. The additional condition $p_2 \neq 0$ excludes umbilics. □

The following definition is a technical tool to state the next theorem in a simple way. The meaning of the function $Sign_{ridge}$ introduced here will be clear from the proof of the theorem. Essentially, this function describes all the possible sign configurations for ab and $a'b'$ at a ridge point.

Definition. 1 *The function $Sign_{ridge}$ takes the values*

$$\text{-1 if } \begin{cases} ab < 0 \\ a'b' \leq 0 \end{cases} \text{ or } \begin{cases} ab \leq 0 \\ a'b' < 0 \end{cases},$$

$$\text{+1 if } \begin{cases} ab > 0 \\ a'b' \geq 0 \end{cases} \text{ or } \begin{cases} ab \geq 0 \\ a'b' > 0 \end{cases},$$

$$\text{0 if } ab = a'b' = 0.$$

Theorem. 1 *The set of blue ridges union the set of red ridges union the set of umbilics has equation $P = 0$ with $P = (a^2 p_2 - b^2)/B$, and one also has $P = -(a'^2 p_2 - b'^2)/C = 2(a'b - ab')$. For a point of this set $\mathscr{P}$, one has:*

- *If $p_2 = 0$, the point is an umbilic.*
- *If $p_2 \neq 0$ then:*

– *if* $Sign_{ridge} = -1$ *then the point is a blue ridge point,*
– *if* $Sign_{ridge} = +1$ *then the point is a red ridge point,*
– *if* $Sign_{ridge} = 0$ *then the point is a purple point.*

Proof. To form the equation of $\mathscr{P}$, following the characterization of red and blue ridges in lemma 3, and the vanishing of the vector fields v_1 and v_2 in lemma 2, we take the product of equations 1. and 3. of lemma 3. The equivalence between the three equations of $\mathscr{P}$ is proven with the help of `Maple`, see appendix 9.

To qualify points on $\mathscr{P}$, first observe that the case $p_2 = 0$ has already been considered in lemma 4. Therefore, assume $p_2 \neq 0$, and first notice the following two simple facts:

- The equation $(a^2 p_2 - b^2)/B = 0$ for $\mathscr{P}$ implies that $a = 0 \Leftrightarrow b = 0 \Leftrightarrow ab = 0$. Similarly, the equation $-(a'^2 p_2 - b'^2)/C = 0$ for $\mathscr{P}$ implies that $a' = 0 \Leftrightarrow b' = 0 \Leftrightarrow a'b' = 0$.
- If $ab \neq 0$ and $a'b' \neq 0$, the equation $ab' - a'b = 0$ for $\mathscr{P}$ implies $b/a = b'/a'$, that is the signs of ab and $a'b'$ agree.

These two facts explain the introduction of the function $Sign_{ridge}$ of definition 1. This function enumerates all disjoint possible configurations of signs for ab and $a'b'$ for a point on $\mathscr{P}$. One can now study the different cases wrt the signs of ab and $a'b'$ or equivalently the values of the function $Sign_{ridge}$.

Assume $Sign_{ridge} = -1$**.**
–*First case:* $ab < 0$. The equation $(a^2 p_2 - b^2)/2B = 0$ implies that $(a\sqrt{p_2} + b)(a\sqrt{p_2} - b) = 0$. Since $\sqrt{p_2} > 0$, one must have $a\sqrt{p_2} + b = 0$ which is equation 1 of lemma 3. From the second simple fact, either $a'b' < 0$ or $a'b' = 0$.

- For the first sub-case $a'b' < 0$, equation $(a'^2 p_2 - b'^2)/C = 0$ implies $(a'\sqrt{p_2} + b')(a'\sqrt{p_2} - b') = 0$. Since $\sqrt{p_2} > 0$, one must have $a'\sqrt{p_2} + b' = 0$ which is equation 2 of lemma 3.
- For the second sub-case $a'b' = 0$, one has $a' = b' = 0$ and the equation 2 of lemma 3 is also satisfied. (Moreover, equation 5 is also satisfied which implies that $w_2 = 0$).

In both cases, equations 1 and 2 or equivalently equation 3 are satisfied. Since one has excluded umbilics, the point is a blue ridge point.
–*Second case:* $ab = 0$. One has $a'b' < 0$ the study is similar to the above. $ab = 0$ implies equation 1 and $a'b' < 0$ implies equation 2 of lemma 3. The point is a blue ridge point.

Assume $Sign_{ridge} = 1$**.**
This case is the exact symmetric of the previous, one only has to exchange the roles of a, b and a', b'.

Assume $Sign_{ridge} = 0$**.**
The first simple fact implies $a = b = a' = b' = 0$ and lemma 4 identifies a purple point. □

As shown along the proof, the conjunctions $<,=$ and $=,<$ in the definition of $Sign_{ridge} = -1$ correspond to the blue ridge points where the vector fields w_2 and v_2 vanish. The same holds for $Sign_{ridge} = 1$ and w_1 and v_1. One can also observe that the basic ingredient of the previous proof is to transform an equation with a square root into a system with an inequality. More formally:

Observation. 3 *For x,y,z real numbers and $z \geq 0$, one has:*

$$x\sqrt{z}+y=0 \Longleftrightarrow \begin{cases} x^2z-y^2=0 \\ xy \leq 0 \end{cases} \tag{6}$$

4.5 Singular points of $\mathscr{P}$

Having characterized umbilics, purple points and ridges in the domain $\mathscr{D}$ with implicit equations, an interesting question is to relate the properties of these equations to the classical differential geometric properties of these points.

In particular, recall that generically (with the description of surfaces with Monge patches and contact theory recalled is appendix 8), umbilics of a surface are either 1-ridge umbilics or 3-ridge umbilics. This means that there are either 1 or 3 non-singular ridge branches passing through an umbilic. The later are obviously singular points of $\mathscr{P}$ since three branches of the curve are crossing at the umbilic. For the former ones, it is appealing to believe they are regular points since the tangent space to the ridge curve on the surface at such points is well defined and can be derived from the cubic of the Monge form [HGY$^+$99]. Unfortunately, one has:

Proposition. 1 *Umbilics are singular points of multiplicity at least 3 of the function P (i.e. the gradient and the Hessian of P vanish).*

Proof. Following the notations of Porteous [Por01], denote P_k, $k=1,\ldots,3$ the kth times linear form associated with P, that is $P_k = [\partial P/(\partial u^{k-i}\partial v^i)]_{i=0,\ldots,k}$. Phrased differently, P_1 is the gradient, P_2 is the vector whose three entries encodes the Hessian of P, etc. To show that the multiplicity of an umbilic of coordinates (u_0,v_0) is at least three, we need to show that $P_1(u_0,v_0) = [0,\ 0]$, $P_2(u_0,v_0) = [0,\ 0,\ 0]$. We naturally do not know the coordinates of umbilics, but lemma 1 provides the umbilical conditions. The proof consists of computing derivatives and performing the appropriate substitutions under Maple, and is given in appendix 9. □

We can go one step further so as to relate the type of the cubic P_3 —the third derivative of P— to the number of non-singular ridge branches at the umbilic.

Proposition. 2 *The classification of an umbilic as 1-ridge of 3-ridges from P_3 goes as follows:*

- *If P_3 is elliptic, that is the discriminant of P_3 is positive ($\delta(P_3) > 0$), then the umbilic is a 3-ridge umbilic and the 3 tangent lines to the ridges at the umbilic are distinct.*
- *If P_3 is hyperbolic ($\delta(P_3) < 0$) then the umbilic is a 1-ridge umbilic.*

Proof. Since the properties of interest here are local ones, studying ridges on the surface or in the parametric domain is equivalent because the parameterization is a local diffeomorphism. More precisely the parameterization Φ maps a curve passing through $(u_0, v_0) \in \mathscr{D}$ to a curve passing through the umbilic $p_0 = \Phi(u_0, v_0)$ on the surface $S = \Phi(\mathscr{D})$. Moreover, the invertible linear map $d\Phi_{(u_0,v_0)}$ maps the tangent to the curve in $\mathscr{D}$ at (u_0, v_0) to the tangent at p_0 to its image curve in the tangent space $T_{p_0}S$.

Having observed the multiplicity of umbilics is at least three, we resort to singularity theory. From [AVGZ82, Section 11.2, p157], we know that a cubic whose discriminant is non null is equivalent up to a linear transformation to the normal form $y(x^2 \pm y^2)$. Moreover, a function having a vanishing second order Taylor expansion and its third derivative of this form is diffeomorphic to the same normal form. Therefore, whenever the discriminant of P_3 is non null, up to a diffeomorphism, the umbilic is a so called $D_4^{\pm}$ singularity of P, whose normal form is $y(x^2 \pm y^2)$. It is then easily seen that the zero level set consists of three non-singular curves through the umbilic with distinct tangents which are the factor lines of the cubic. For a D_4^- singularity ($\delta(P_3) > 0$), these 3 curves are real curves and the umbilic is a 3-ridge. For a D_4^+ singularity $\delta(P_3) < 0$), only one curve is real and the umbilic is a 1-ridge. □

Note that the classifications of umbilics with the Monge cubic C_M and the cubic P_3 do not coincide. Indeed if C_M is elliptic, it may occur that two ridges have the same tangent. In such a case, the cubic P_3 is not elliptic since $\delta(P_3) = 0$.

Since purple points correspond to the intersection of two ridges, one has:

Proposition. 3 *Purple points are singular points of multiplicity at least 2 of the function P (i.e. the gradient of P vanish).*

Proof. It follows from the equation $P = 2(a'b - ab')$ that $dP = 2(d(a')b + a'd(b) - d(a)b - ad(b))$. At purple points one has $a = a' = b = b' = 0$ hence $dP = 0$. □

5 Implicit system for turning points and ridge type

In this section, we define a system of equations that encodes turning points. Once these turning points identified, we show how to retrieve the type (elliptic or hyperbolic) of a ridge from a sign evaluation.

5.1 Problem

Going one step further in the description of ridges requires distinguishing between ridges which are maxima or minima of the principal curvatures. Following the classical terminology recalled in appendix 8, a blue (red) ridge changes from a maxima to a minima at a blue (red) turning point. These turning points are witnessed by the vanishing of the second derivative of the principal curvature along its curvature line. As recalled in appendix —see [HGY $^+$99] for the details, from the parameterization of a principal curvature along its curvature line —Eq. (10), a turning points is witnessed by the vanishing of the coefficient P_1 (P_2) for blue (red) ridges. Since we are working from a parameterization, denoting *Hess* the Hessian matrix of either principal curvature, we have:

Observation. 4 *A blue turning point is a blue ridge point where $Hess(k_1)(d_1,d_1) = 0$. Similarly, a red turning point is a red ridge point with $Hess(k_2)(d_2,d_2) = 0$.*

Generically, turning points are not purple points, however we shall provide conditions identifying these cases. Even less generic is the existence of a purple point which is also a blue and a red turning point, a situation for which we also provide conditions.

Once turning points have been found, reporting elliptic and hyperbolic ridge sections is especially easy. For ridges through umbilics, since ridges at umbilics are hyperbolic, and the two types alternate at turning points, the task is immediate. For ridges avoiding umbilics, one just has to test the sign of $Hess(k_1)(d_1,d_1)$ or $Hess(k_2)(d_2,d_2)$ at a ridge points, and then propagate the alternation at turning points.

5.2 Method outline

We focus on blue turning points since the method for red turning points is similar. As already pointed out, we do not have a global expression of the blue direction d_1, but only the two blue vector fields v_1 and w_1 vanishing on some curves going through umbilics. Consequently, we have to combine equations with these two fields to get a global expression of turning points. A blue ridge point is a blue turning point iff $Hess(k_1)(d_1,d_1) = 0$. This last equation is equivalent to $Numer(Hess(k_1))(v_1,v_1) = 0$ when the vector field v_1 does not vanish. The same holds for the equation $Numer(Hess(k_1))(w_1,w_1) = 0$ and the solutions of $w_1 = (0,0)$. As a consequence of observation 2, the conjunction of these two equations defines the set of blue turning points.

The drawback of distinguishing the color of the turning points is that equations contain a square root. Combining the equations for blue and red turning points gives an equation $Q = 0$ without square roots. The intersection of the corresponding curve $\mathscr{Q}$ with the ridge curve $\mathscr{P}$ and sign evaluations allow to retrieve all turning points and their color.

5.3 System for turning points

Lemma. 5 *For a regular surface, there exist differentiable functions $\alpha,\alpha',\beta,\beta'$ which are polynomials wrt A,B,C,D and $\det I$, as well as their first and second derivatives, such that:*

1. $Numer(Hess(k_1))(v_1,v_1) = \alpha\sqrt{p_2}+\beta$.
2. $Numer(Hess(k_1))(w_1,w_1) = \alpha'\sqrt{p_2}+\beta'$.
3. *A blue ridge point is a blue turning point iff* $\begin{cases} \alpha\sqrt{p_2}+\beta = 0 \\ \alpha'\sqrt{p_2}+\beta' = 0 \end{cases}$
4. $Numer(Hess(k_2))(v_2,v_2) = \alpha\sqrt{p_2}-\beta$.

5. *Numer*$(Hess(k_2))(w_2,w_2) = \alpha'\sqrt{p_2} - \beta'$.

6. *A red ridge point is a red turning point iff* $\begin{cases} \alpha\sqrt{p_2} - \beta = 0 \\ \alpha'\sqrt{p_2} - \beta' = 0 \end{cases}$

Proof. Calculations for points 1-2-4-5 are performed with Maple cf. appendix 9. Blue turning points are blue ridge points on $\mathscr{P}$ where $Hess(k_1)(d_1,d_1) = 0$. This equation is not defined at umbilics where principal curvatures are not differentiable. Nevertheless, including umbilics and points where v_1 vanishes, this equation is equivalent to $Numer(Hess(k_1)(v_1,v_1) = 0$. This equation is rewritten as $\alpha\sqrt{p_2} + \beta = 0$ and yields point 1. The same analysis holds for w_1 and yields point 2. Point 3. is a consequence of observation 2. Results for red turning points are similar and the reader is referred to appendix 9 for the details. □

The following definition is a technical tool to state the next theorem in a simple way. As we shall see along the proof, this function describes all the possible sign configurations for $\alpha\beta$ and $\alpha'\beta'$ at a turning point.

Definition. 2 *The function $Sign_{turn}$ takes the values*

-1 if $\begin{cases} \alpha\beta < 0 \\ \alpha'\beta' \leq 0 \end{cases}$ *or* $\begin{cases} \alpha\beta \leq 0 \\ \alpha'\beta' < 0 \end{cases}$,

+1 if $\begin{cases} \alpha\beta > 0 \\ \alpha'\beta' \geq 0 \end{cases}$ *or* $\begin{cases} \alpha\beta \geq 0 \\ \alpha'\beta' > 0 \end{cases}$,

0 if $\alpha\beta = \alpha'\beta' = 0$.

Theorem. 2 *Let Q be the smooth function which is a polynomial wrt A, B, C, D and* $\det I$*, as well as their first and second derivatives defined by*

$$Q = (\alpha^2 p_2 - \beta^2)/B^2 = (\alpha'^2 p_2 - \beta'^2)/C^2 = 2(\alpha'\beta - \alpha\beta')/(D-A). \quad (7)$$

The system $\begin{cases} P = 0 \\ Q = 0 \end{cases}$ *encodes turning points in the following sense. For a point, solution of this system, one has:*

- *If $p_2 = 0$, the point is an umbilic.*
- *If $p_2 \neq 0$ then:*
  - *if $Sign_{ridge} = -1$ and $Sign_{turn} \leq 0$ then the point is a blue turning point,*
  - *if $Sign_{ridge} = +1$ and $Sign_{turn} \geq 0$ then the point is a red turning point,*
  - *if $Sign_{ridge} = 0$ then the point is purple point and in addition*
    - *if $Sign_{turn} = -1$ then the point is also a blue turning point,*

  - *if $Sign_{turn} = +1$ then the point is also a red turning point,*
  - *if $Sign_{turn} = 0$ then the point is also a blue and a red turning point.*

Proof. Following lemma 5, we form the equation of Q by taking the products of 1. and 4. in the lemma. Equalities of equation (7) are performed with Maple cf. appendix 9.

The case $p_2 = 0$ has already been considered in lemma 4. Assume that $p_2 \neq 0$, and first notice the following two simple facts:

- The equation $(\alpha^2 p_2 - \beta^2)/B^2 = 0$ for $\mathscr{Q}$ implies that $\alpha = 0 \Leftrightarrow \beta = 0 \Leftrightarrow \alpha\beta = 0$. Similarly, the equation $(\alpha'^2 p_2 - \beta'^2)/C^2 = 0$ for $\mathscr{Q}$ implies that $\alpha' = 0 \Leftrightarrow \beta' = 0 \Leftrightarrow \alpha'\beta' = 0$.
- If $\alpha\beta \neq 0$ and $\alpha'\beta' \neq 0$, the equation $2(\alpha'\beta - \alpha\beta')/(D-A) = 0$ for $\mathscr{Q}$ implies $\beta/\alpha = \beta'/\alpha'$, that is the signs of $\alpha\beta$ and $\alpha'\beta'$ agree.

These two facts explain the introduction of the function $Sign_{turn}$ of definition 2. This function enumerates all disjoint possible configurations of signs for $\alpha\beta$ and $\alpha'\beta'$ for a point on $\mathscr{Q}$. The analysis of the different cases is similar to that of the proof of theorem 1, the basic ingredient being observation 3. □

Observation. 5 *Note that in the formulation of equation (7) there are solutions of the system ($P = 0$ and $Q = 0$) which are not turning points nor umbilics. These points are characterized by ($Sign_{ridge} = -1$ and $Sign_{turn} = +1$) or ($Sign_{ridge} = +1$ and $Sign_{turn} = -1$). This drawback is unavoidable since equations avoiding the term $\sqrt{p_2}$ cannot distinguish colors.*

Observation. 6 *The following holds:*

- $p_2 = 0$ *implies* $\alpha = \alpha' = \beta = \beta' = 0$
- $\alpha = \alpha' = \beta = \beta' = 0$ *are singularities of Q of multiplicity at least 2.*

To test if a blue ridge segment between two turning points is a maxima or a minima requires the evaluation of the sign of $\alpha\sqrt{p_2} + \beta$ or $\alpha'\sqrt{p_2} + \beta'$, which cannot vanish simultaneously.

6 Polynomial surfaces

A fundamental class of surface used in Computer Aided Geometric Design consist of Bezier surfaces and splines. In this section, we state some elementary observations on the objects studied so far, for the particular case of polynomial parametric surfaces. Notice that the parameterization can be general, in which case $\Phi(u,v) = (x(u,v), y(u,v), z(u,v))$, or can be a height function $\Phi(u,v) = (u, v, z(u,v))$.

6.1 About W and the vector fields

Using Eq. (1), we first observe that if Φ is a polynomial then the coefficients A, B, C and D are also polynomials —this explains the factor $(\det I)^{3/2}$ in the denominator of W in equation (1). For example

$$A = (\det I)^{3/2} w_{11} = (\det I)^{3/2} \frac{gl - fm}{\det I} = \sqrt{\det I}(g\langle N/\sqrt{\det I}, \Phi_{uu}\rangle - f\langle N/\sqrt{\det I}, \Phi_{uv}\rangle)$$
$$= g\langle N, \Phi_{uu}\rangle - f\langle N, \Phi_{uv}\rangle.$$

Thus, in the polynomial case, the equation of ridges is algebraic. Hence the set of all ridges and umbilics is globally described by an algebraic curve. The function Q is also a polynomial so that turning points are described by a polynomial system.

An interesting corollary of lemma 2 for the case of polynomial surfaces is the following:

Observation. 7 *Given a principal vector v, denote Z_v the zero set of v i.e. the set of points where v vanishes.*

For a polynomial surface, the sets $Z_{v_1}, Z_{w_1}, Z_{v_2}, Z_{w_2}$ are semi-algebraic sets.

6.2 Degrees of expressions

As a corollary of Thm. 1 and 2, one can give upper bounds for the total degrees of expressions wrt that of the parameterization. Distinguishing the cases where Φ is a general parameterization or a height function (that is $\Phi(u,v) = (u, v, h(u,v))$) with $h(u,v)$ and denoting d the total degree of Φ, table 3 gives the total degrees of $A, B, C, D, \det I, P$ and Q.

Note that in the case of a height function, P is divided by its factor $\det I^2$, and Q is divided by its factor $\det I$ (cf. appendix 9).

Polynomials	General parameterization	Height function
A, B, C, D	$d_1 = 5d - 6$	$d_{1_h} = 3d - 4$
$\det I$	$d_2 = 4d - 4$	$d_2 = 4d - 4$
P	$5d_1 + 2d_2 - 2 = 33d - 40$	$5d_{1_h} - 2 = 15d - 22$
Q	$10d_1 + 4d_2 - 4 = 66d - 80$	$10d_{1_h} + 3d_2 - 4 = 42d - 56$

Figure 3: Total degrees of polynomials

6.3 A cultural comment

The second question of Hilbert's 16th problem, which is to count the number of closed orbit of a planar two-dimensional polynomial dynamical system, shows that the structure of objects globally defined from polynomial differential systems may be very intricate —the problem is still open after a century. In our setting, we seek ridges and not closed orbits of a system, but the nice observation is that ridges of a polynomial parametric surface are polynomial objects —and not transcendental ones.

6.4 An illustration

Theorem 1 is effective and allows one to report certified ridges of polynomial parametric surfaces without resorting to local orientation procedures.

Without engaging into the algebraic developments carried out in [CFPR05], we just provide an illustration of ridges for a degree four Bezier surface defined over the domain $\mathscr{D} = [0,1] \times [0,1]$. This surface, Figure 4, has control points

$$\begin{pmatrix} [0,0,0] & [1/4,0,0] & [2/4,0,0] & [3/4,0,0] & [4/4,0,0] \\ [0,1/4,0] & [1/4,1/4,1] & [2/4,1/4,-1] & [3/4,1/4,-1] & [4/4,1/4,0] \\ [0,2/4,0] & [1/4,2/4,-1] & [2/4,2/4,1] & [3/4,2/4,1] & [4/4,2/4,0] \\ [0,3/4,0] & [1/4,3/4,1] & [2/4,3/4,-1] & [3/4,3/4,1] & [4/4,3/4,0] \\ [0,4/4,0] & [1/4,4/4,0] & [2/4,4/4,0] & [3/4,4/4,0] & [4/4,4/4,0] \end{pmatrix}$$

Alternatively, this surface can be expressed as the graph of the total degree 8 polynomial $h(u,v)$ for $(u,v) \in [0,1]^2$:

$$\begin{aligned} h(u,v) = &116u^4v^4 - 200u^4v^3 + 108u^4v^2 - 24u^4v - 312u^3v^4 + 592u^3v^3 - 360u^3v^2 + 80u^3v + 252u^2v^4 - 504u^2v^3 \\ &+ 324u^2v^2 - 72u^2v - 56uv^4 + 112uv^3 - 72uv^2 + 16uv. \end{aligned}$$

The computation of the implicit curve has been performed using Maple 9.5 (see section 9). It is a bivariate polynomial $P(u,v)$ of total degree 84, of degree 43 in u, degree 43 in v with 1907 terms and coefficients with up to 53 digits. The surface and its ridges are displayed on Fig. 4. Figure 5 displays ridges the parametric domain $\mathscr{D}$, there are 25 purple points (black dots) and 8 umbilics (green dots), 3 of which are 3-ridge and 5 are 1-ridge.

7 Conclusion

This paper sets the implicit equation $P = 0$ of the singular curve encoding all ridges of a smooth parametric surface. From a mathematical standpoint, a corollary of this result shows that ridges of polynomial surfaces are polynomial objects. As exemplified by Hilbert's 16th problem —still open, qualitative properties of geometric objects defined by polynomial differential systems may be difficult to set. From an algorithmic perspective, this result paves the alley for the development of certified algorithms reporting ridges without resorting to local orientation procedures. For algebraic surfaces, such algorithms are developed in a companion paper.

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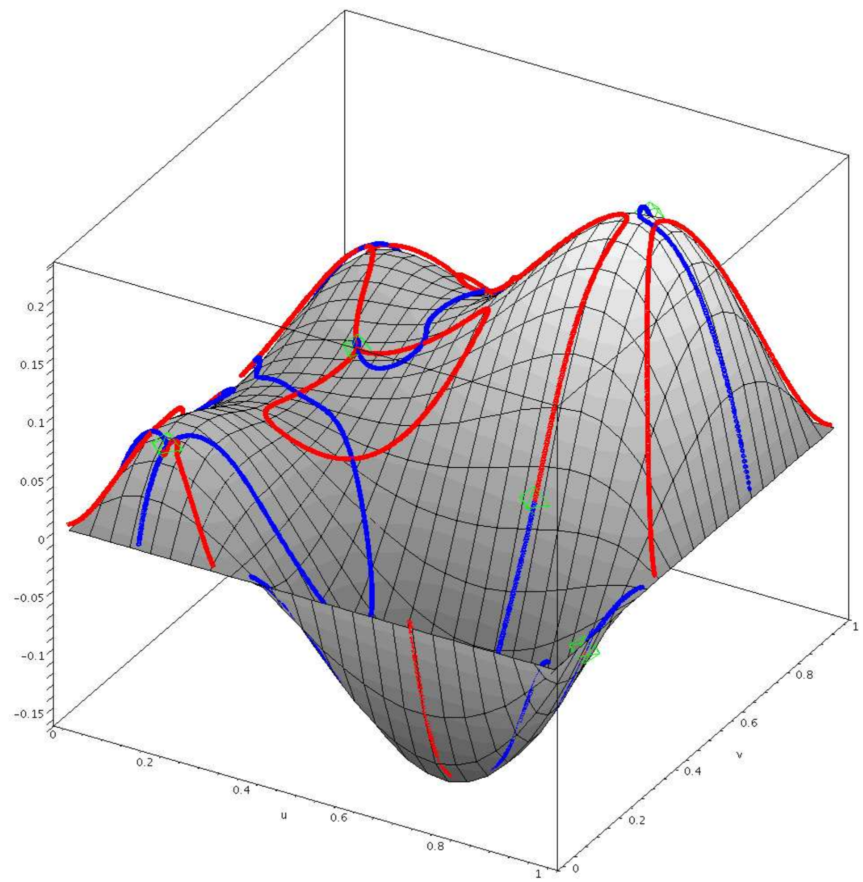

Figure 4: Plot of the degree 4 bivariate Bezier surface with ridges and umbilics

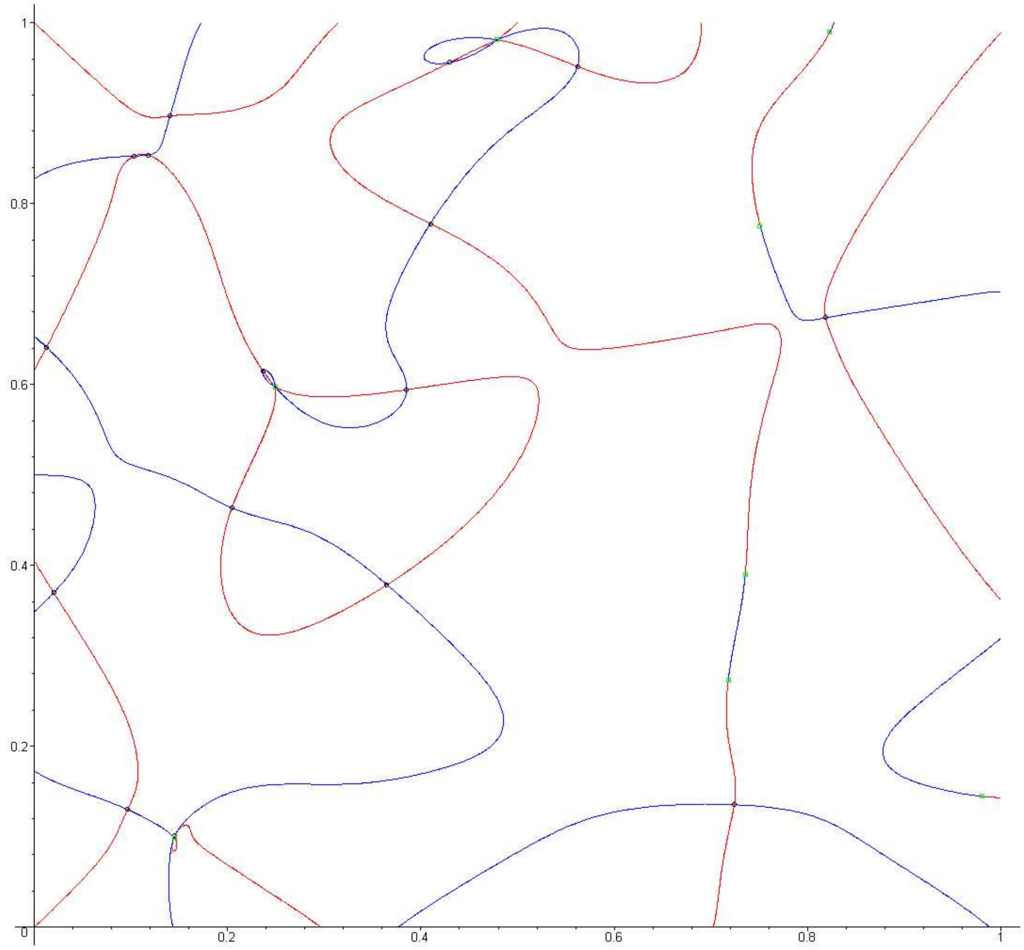

Figure 5: The certified plot of $\mathscr{P}$ (1024$\times$1024 pixels)

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8 Appendix: A primer on ridges

We consider a smooth surface S and since we want to describe local properties we can assume it is given by a parameterization from $\mathbb{R}^2$ to the Euclidean space E^3 equipped with the orientation of its world coordinate system —referred to as the *direct orientation* in the sequel.

First, recall that at each point of the surface which is not an umbilic, there are two orthogonal principal directions d_1, d_2 and two associated principal curvatures k_1 and k_2. These principal directions define two direction fields on S, one everywhere orthogonal to the other —so that it is sufficient to study only one of these. Each principal direction field defines lines of curvature which are integral curves of the corresponding principal field, and the set of all these lines defines the principal foliation. Following standard usage, we shall always sort principal curvatures, that is we will always assume $k_1 \geq k_2$. Moreover, objects related to the larger (smaller) principal curvature are painted in blue (red). For example, we shall speak of a blue curvature instead of k_1 or of the red direction for d_2. Eventually, note that if the global orientation of the surface is changed then curvatures change signs, hence the colors blue and red are swapped.

At a point of S which is not an umbilic, the non oriented principal directions d_1, d_2 together with the normal vector n define two direct orthonormal frames. If v_1 is a unit vector of direction d_1 (we call it a maximal principal vector) then there exists a unique unit minimal principal vector v_2 so that (v_1, v_2, n) is direct, and the other possible frame is $(-v_1, -v_2, n)$. (*direct* must be understood with reference with the direct orientation of the world coordinate system mentioned above.) In such a coordinate system, S can be locally described as a Monge form:

$$z = \frac{1}{2}(k_1x^2 + k_2y^2) + \frac{1}{6}(b_0x^3 + 3b_1x^2y + 3b_2xy^2 + b_3y^3) \tag{8}$$

$$+ \frac{1}{24}(c_0x^4 + 4c_1x^3y + 6c_2x^2y^2 + 4c_3xy^3 + c_4y^4) + \dots \tag{9}$$

Moreover, it should be noticed that switching from one of the two coordinate systems to the other reverts the sign of all the odd coefficients on the Monge form of the surface.

Having recalled the fundamental notions related to principal curvatures, let us get to ridges. Defining ridges precisely is a serious endeavor requiring technical notions from contact theory and singularity theory, and we refer the reader to standard textbooks [Por01, HGY$^+$99], as well as to [CP05a] for an overview. A blue (red) ridge point of a smooth surface is a non-umbilic point p on the surface such that, along the blue (red) curvature line going through p, the blue (red) principal curvature has an extremum at p. Blue (red) ridge points define curves on S called ridge curves or ridges for short. Intuitively, the essence of ridge points is best captured by looking at the Taylor expansion of a principal curvature along its corresponding line of curvature. Taking the example of the blue principal curvature, this Taylor expansion is given by [HGY$^+$99]:

$$k_1(x) = k_1 + b_0x + \frac{P_1}{2(k_1 - k_2)}x^2 + \dots, \qquad P_1 = 3b_1^2 + (k_1 - k_2)(c_0 - 3k_1^3). \tag{10}$$

A blue ridge point is characterized by $b_0 = 0$, but as illustrated on Fig. 6, the sign of b_0 depends on the orientation of the curvature line. Moreover, the value of P_1 determines the type of a ridge point:

if $P_1 < 0$ ($P_1 > 0$) the ridge point is called elliptic (hyperbolic). In between such regions, one finds isolated points called turning points characterized by $P_1 = 0$. From Eq. 10 —and its dual for k_2, it is also easily seen that an elliptic ridge point corresponds to either a maximum of k_1 or a minimum of k_2. Similarly, an hyperbolic ridge point corresponds to a minimum of k_1 or a maximum of k_2. The corresponding geometric interpretation when moving along a curvature line and crossing the ridge is recalled on Fig. 7.

To summarize, a ridge point is distinguished by its color and its type. When displaying ridge curves, we shall adopt the following conventions:

- blue elliptic (hyperbolic) ridge curves are painted in blue (green),
- red elliptic (hyperbolic) ridges curves are painted in red (yellow).
- when we do not want to distinguish the type of ridge curves, we only use blue and red regardless the type is elliptic or hyperbolic.

Umbilic points can be considered as ridge points since they are in the closure of ridge curves. We do so in the sequel because it allows to study ridges as curves passing through umbilics.

Some notions about cubics will be useful in the sequel for a classification of umbilics.

Definition. 3 *A real cubic $C(x,y)$ is a bivariate homogeneous polynomial of degree three, that is $C(x,y) = b_0x^3 + 3b_1x^2y + 3b_2xy^2 + b_3y^3$. Its discriminant is defined by $\delta(C) = 4(b_1^2 - b_0b_2)(b_2^2 - b_1b_3) - (b_0b_3 - b_1b_2)^2$.*

A cubic factorizes as a product of three polynomials of degree one with complex coefficients, called its factor lines. In the (x,y) plane, a real factor line defines a direction along which C vanishes. The number of real factor lines depends on the discriminant of the cubic and we have

Proposition. 4 *Let C be a real cubic and δ its discriminant. If $\delta > 0$, C is called elliptic and there are 3 distinct real factors. If $\delta < 0$, C is called hyperbolic and there is only one real factor (and two complex conjugate factors).*

In the particular description of surfaces as Monge patches, we have a family of Monge patches with two degrees of freedom —the dimension of the manifold. A property requiring 1 (resp. 2) condition(s) on this family is expected to appear on lines (resp. isolated points) of the surface —a condition being an equation involving the Monge coefficients. A property requiring at least three conditions is not generic. As an example, ridge points (characterized by the condition $b_0 = 0$ or $b_3 = 0$) appears on lines and umbilics (the two conditions are $k_1 = k_2$ and the coefficient of the xy term vanishes) are isolated points. A flat umbilic, requiring the additional condition $k_1 = 0$, is not generic.

The classification of umbilics given by Porteous [Por01, Chap 11.6, p.191] is achieved by the classification of singular points of the contact function. This function expresses how strong the contact is between the surface and its sphere of curvature. At an umbilic and in the Monge coordinate system, this function is the Monge form of the surface without its quadratic part. The singularities of this contact function are naturally related to its cubic part which one calls the Monge cubic C_M and one proves that:

- if C_M is elliptic, there are 3 non-singular ridges passing through to the umbilic;
- if C_M is hyperbolic, there is only 1 non-singular ridge passing through to the umbilic;
- in both cases, at the umbilic, each ridge curve is smooth and changes from a minimum of k_1 to a maximum of k_2.

Definition. 4 *An umbilic is called a 1-ridge (resp. 3-ridge) umbilic if there is 1 (resp. 3) non-singular ridge curve going through it.*

Generically, the discriminant $\delta(C_M)$ of C_M does not vanish, so that C_M is either elliptic or hyperbolic. Therefore, a generic umbilic is either 1-ridge or 3-ridge.

To finish up this review, let us recall the following generic properties.

- A ridge curve contains an even number of turning points at which the ridge changes from elliptic to hyperbolic.
- Near an umbilic, ridge curves are hyperbolic, that is correspond to a minimum of k_1 or maximum of k_2.
- Ridges of the same color do not cross.
- Two ridges of different colors may cross at a so-called *purple point*.

These notions are illustrated on the famous example of the ellipsoid on Figs. 1 and 2.

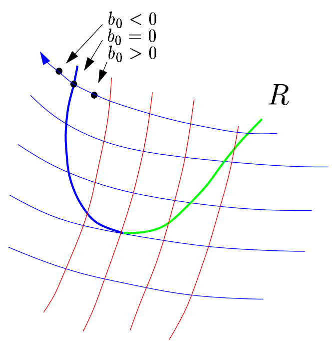

Figure 6: Variation of the b_0 coefficient and turning point of a ridge

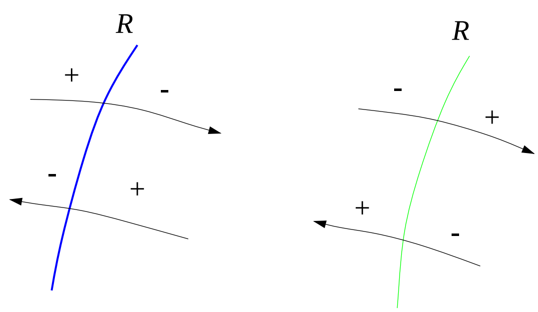

Figure 7: Classification of a blue ridge as elliptic (max of k_1, left), and hyperbolic (min of k_1, right) from the sign change of b_0

9 Appendix: Maple computations

The Maple computations are provided for convenience. The corresponding Maple worksheet is available from the authors' web pages.

9.1 Principal directions, curvatures and derivatives

```
> v1:=[-2*B(u,v),A(u,v)-DD(u,v)-sqrt(p2(u,v))];
> w1:=[A(u,v)-DD(u,v)+sqrt(p2(u,v)),2*C(u,v)];
> v2:=[-2*B(u,v),A(u,v)-DD(u,v)+sqrt(p2(u,v))];
> w2:=[A(u,v)-DD(u,v)-sqrt(p2(u,v)),2*C(u,v)];
> k1:=(A(u,v)+DD(u,v)+sqrt(p2(u,v)))/(2*detI(u,v)^(3/2));
> k2:=(A(u,v)+DD(u,v)-sqrt(p2(u,v)))/(2*detI(u,v)^(3/2));
```

$$v1 := \left[-2B(u,v), A(u,v) - DD(u,v) - \sqrt{p2(u,v)}\right]$$
$$w1 := \left[A(u,v) - DD(u,v) + \sqrt{p2(u,v)}, 2C(u,v)\right]$$
$$v2 := \left[-2B(u,v), A(u,v) - DD(u,v) + \sqrt{p2(u,v)}\right]$$
$$w2 := \left[A(u,v) - DD(u,v) - \sqrt{p2(u,v)}, 2C(u,v)\right]$$
$$k1 := 1/2\, \frac{A(u,v)+DD(u,v)+\sqrt{p2(u,v)}}{(detI(u,v))^{3/2}}$$
$$k2 := 1/2\, \frac{A(u,v)+DD(u,v)-\sqrt{p2(u,v)}}{(detI(u,v))^{3/2}}$$

First derivatives

```
> k1u:=diff(k1,u);k1un:=numer(k1u):
> k1v:=diff(k1,v):k1vn:=numer(k1v):
> k2u:=diff(k2,u):k2un:=numer(k2u):
> k2v:=diff(k2,v):k2vn:=numer(k2v):
> dk1n:=[k1un, k1vn]:
> dk2n:=[k2un, k2vn]:
```

$$k1u := 1/2\left(\frac{\partial}{\partial u}A(u,v) + \frac{\partial}{\partial u}DD(u,v) + 1/2\,\frac{\frac{\partial}{\partial u}p2(u,v)}{\sqrt{p2(u,v)}}\right)(detI(u,v))^{-3/2} - 3/4\,\frac{\left(A(u,v)+DD(u,v)+\sqrt{p2(u,v)}\right)\frac{\partial}{\partial u}detI(u,v)}{(detI(u,v))^{5/2}}$$

Second derivatives

```
> k1uu:=diff(k1,u$2):k1uun:=numer(k1uu):
> k1uv:=diff(k1,u,v):k1uvn:=numer(k1uv):
> k1vv:=diff(k1,v$2):k1vvn:=numer(k1vv):
> k2uu:=diff(k2,u$2):k2uun:=numer(k2uu):
> k2uv:=diff(k2,u,v):k2uvn:=numer(k2uv):
> k2vv:=diff(k2,v$2):k2vvn:=numer(k2vv):
```

9.2 Ridges

Blue and red equations wrt the vector fi elds v1, v2, w1, w2.

```
> subs_sqrtp2:=sqrt(p2(u,v)) = sqrtp2, p2(u,v)^(3/2)= p2(u,v)*sqrtp2;
> b0v1:=subs( subs_sqrtp2, expand(linalg[dotprod](dk1n,v1, 'orthogonal')
)):
> b0w1:=subs( subs_sqrtp2, expand(linalg[dotprod](dk1n,w1, 'orthogonal')
)):
> b3v2:=subs( subs_sqrtp2, expand(linalg[dotprod](dk2n,v2, 'orthogonal')
)):
> b3w2:=subs( subs_sqrtp2, expand(linalg[dotprod](dk2n,w2, 'orthogonal')
)):
> b0v1a:=coeff(b0v1, sqrtp2, 1):b0v1b:=coeff(b0v1, sqrtp2, 0):
> b0w1a:=coeff(b0w1, sqrtp2, 1):b0w1b:=coeff(b0w1, sqrtp2, 0):
> b3v2a:=coeff(b3v2, sqrtp2, 1):b3v2b:=coeff(b3v2, sqrtp2, 0):
> b3w2a:=coeff(b3w2, sqrtp2, 1):b3w2b:=coeff(b3w2, sqrtp2, 0):
```

$$subs\_sqrtp2 := \left\{(p2(u,v))^{3/2} = p2(u,v)\,sqrtp2, \sqrt{p2(u,v)} = sqrtp2\right\}$$

Identities between b0 with (v1,w1) and b3 with (v2,w2).

```
> [b3v2a-b0v1a, b0v1b+b3v2b, b3w2a-b0w1a, b0w1b+b3w2b];
```

$$[0,0,0,0]$$

Definition of a,b,abis,bbis

```
> subs_p2:= p2(u,v)=(A(u,v)-DD(u,v))^2+4*B(u,v)*C(u,v);
> a:=expand(subs( subs_p2, b0v1a));
> b:=expand(subs( subs_p2, b0v1b));
> abis:=expand(subs( subs_p2, b0w1a)):
> bbis:=expand(subs( subs_p2, b0w1b)):
```

$$subs\_p2 := \left\{p2(u,v) = (A(u,v) - DD(u,v))^2 + 4B(u,v)C(u,v)\right\}$$

$$a := -4B(u,v)\,detI(u,v)\,\frac{\partial}{\partial u}A(u,v) - 4B(u,v)\,detI(u,v)\,\frac{\partial}{\partial u}DD(u,v) + 6B(u,v)\left(\frac{\partial}{\partial u}detI(u,v)\right)A(u,v) + 6B(u,v)\left(\frac{\partial}{\partial u}detI(u,v)\right)DD(u,v) + 4\,detI(u,v)\left(\frac{\partial}{\partial v}DD(u,v)\right)A(u,v) - 4\,detI(u,v)\left(\frac{\partial}{\partial v}DD(u,v)\right)DD(u,v) - 4\,detI(u,v)\left(\frac{\partial}{\partial v}B(u,v)\right)C(u,v) - 4\,detI(u,v)\,B(u,v)\,\frac{\partial}{\partial v}C(u,v) + 6\left(\frac{\partial}{\partial v}detI(u,v)\right)(DD(u,v))^2 - 6\left(\frac{\partial}{\partial v}detI(u,v)\right)A(u,v)\,DD(u,v) + 12\left(\frac{\partial}{\partial v}detI(u,v)\right)B(u,v)\,C(u,v)$$

$$b := 6B(u,v)\left(\frac{\partial}{\partial u}detI(u,v)\right)(DD(u,v))^2 + 24\,(B(u,v))^2\left(\frac{\partial}{\partial u}detI(u,v)\right)C(u,v) - 8\,(B(u,v))^2\,detI(u,v)\,\frac{\partial}{\partial u}C(u,v) - 4\,detI(u,v)\left(\frac{\partial}{\partial v}DD(u,v)\right)(A(u,v))^2 + 6B(u,v)\left(\frac{\partial}{\partial u}detI(u,v)\right)(A(u,v))^2 - 12\left(\frac{\partial}{\partial v}detI(u,v)\right)(DD(u,v))^2A(u,v) - 4\,detI(u,v)\left(\frac{\partial}{\partial v}DD(u,v)\right)(DD(u,v))^2 + 6\left(\frac{\partial}{\partial v}detI(u,v)\right)DD(u,v)\,(A(u,v))^2 - 4B(u,v)\,detI(u,v)\,A(u,v)\,\frac{\partial}{\partial u}A(u,v) + 4B(u,v)\,detI(u,v)\left(\frac{\partial}{\partial u}A(u,v)\right)DD(u,v) + 4B(u,v)\,detI(u,v)\,A(u,v)\,\frac{\partial}{\partial u}DD(u,v) - 4B(u,v)\,detI(u,v)\,DD(u,v)\,\frac{\partial}{\partial u}DD(u,v) - 8B(u,v)\,detI(u,v)\left(\frac{\partial}{\partial u}B(u,v)\right)C(u,v) - 12B(u,v)\left(\frac{\partial}{\partial u}detI(u,v)\right)A(u,v)\,DD(u,v) - 8\,detI(u,v)\left(\frac{\partial}{\partial v}A(u,v)\right)B(u,v)\,C(u,v) + 8\,detI(u,v)\left(\frac{\partial}{\partial v}DD(u,v)\right)A(u,v)\,DD(u,v) - 8\,detI(u,v)\left(\frac{\partial}{\partial v}DD(u,v)\right)B(u,v)\,C(u,v) + 4\,detI(u,v)\,A(u,v)\left(\frac{\partial}{\partial v}B(u,v)\right)C(u,v) + 4\,detI(u,v)\,A(u,v)\,B(u,v)\,\frac{\partial}{\partial v}C(u,v) - 4\,detI(u,v)\,DD(u,v)\left(\frac{\partial}{\partial v}B(u,v)\right)C(u,v) - 4\,detI(u,v)\,DD(u,v)\,B(u,v)\,\frac{\partial}{\partial v}C(u,v) + 24\left(\frac{\partial}{\partial v}detI(u,v)\right)DD(u,v)\,B(u,v)\,C(u,v) + 6\left(\frac{\partial}{\partial v}detI(u,v)\right)(DD(u,v))^3$$

Ridge equation, identies

```
> curveb0b3v:=simplify( subs( subs_p2, (a^2*p2(u,v)-b^2)/B(u,v) )):
> curveb0b3w:=simplify( subs( subs_p2, (abis^2*p2(u,v)-bbis^2)/(-C(u,v))
)):
> curveb0b3vw:=simplify( 2*(abis*b-a*bbis) ):
> [curveb0b3v-curveb0b3w,curveb0b3v-curveb0b3vw];
```

$$[0,0]$$

Final result: ridge has 170 terms of the form 5 times a term amongst A,B,C,DD and twice detI and derivatives

```
> ridge:=simplify(curveb0b3vw/content(curveb0b3vw)):
> [whattype(ridge),nops(ridge),op(1,ridge)];
```

$$[\text{`+`}, 170, -36\,(B(u,v))^3 \left(\frac{\partial}{\partial u} detI(u,v)\right)^2 (C(u,v))^2]$$

Umbilics are points on P=0 of multiplicity t least 3

```
> list_diff:=  diff(B(u, v), u)=B[u](u,v), diff(C(u, v), u)=C[u](u,v),
diff(B(u, v), v)=B[v](u,v), diff(C(u, v), v)=C[v](u,v) ,diff(A(u, v), u)=A[u](u,v),
diff(DD(u, v), u)=DD[u](u,v), diff(A(u, v), v)=A[v](u,v), diff(DD(u, v),
v)=DD[v](u,v) :
> umb_cond:=A(u,v)=DD(u,v), B(u,v)=0, C(u,v)=0 ;
> ridge_sub:=subs( list_diff, courbe):
> ridge_gradient:=[diff(ridge_sub, u), diff(ridge_sub, v)]:
> ridge_gradient_sub:=simplify(subs( list_diff, ridge_gradient)):
> ridge_gradient_umb:=simplify(subs(umb_cond, ridge_gradient_sub));
> ridge_hessien:=[ diff(op(1,ridge_gradient_sub), u), diff(op(1,ridge_gradient_sub),
v), diff(op(2,ridge_gradient_sub), v)]:
> ridge_hessien_sub:=simplify(subs(list_diff, ridge_hessien)):
> ridge_hessien_umb:=simplify(subs( umb_cond, ridge_hessien_sub));
```

$$umb\_cond := \{A(u,v) = DD(u,v), C(u,v) = 0, B(u,v) = 0\}$$
$$ridge\_gradient\_umb := [0,0]$$

$$ridge\_hessien\_umb := [0,0,0]$$

9.3 Turning points

```
> subs_sqrtp2bis:=sqrt(p2(u,v)) =sqrtp2, p2(u,v)^(3/2)= p2(u,v)*sqrtp2,p2(u,v)^(5/2)=
p2(u,v)^2*sqrtp2:
> hessk1v1:=subs( subs_sqrtp2bis, expand(k1uun*v1[1]^2+k1uvn*v1[1]*v1[2]+k1vvn*v1[2]^2)):
> hessk1w1:=subs( subs_sqrtp2bis, expand(k1uun*w1[1]^2+k1uvn*w1[1]*w1[2]+k1vvn*w1[2]^2)):
> hessk2v2:=subs( subs_sqrtp2bis, expand(k2uun*v2[1]^2+k2uvn*v2[1]*v2[2]+k2vvn*v2[2]^2)):
> hessk2w2:=subs( subs_sqrtp2bis, expand(k2uun*w2[1]^2+k2uvn*w2[1]*w2[2]+k2vvn*w2[2]^2)):
> hessk1v1a:=coeff(hessk1v1, sqrtp2, 1):hessk1v1b:=coeff(hessk1v1, sqrtp2,
0):
> hessk1w1a:=coeff(hessk1w1, sqrtp2, 1):hessk1w1b:=coeff(hessk1w1, sqrtp2,
0):
> hessk2v2a:=coeff(hessk2v2, sqrtp2, 1):hessk2v2b:=coeff(hessk2v2, sqrtp2,
0):
> hessk2w2a:=coeff(hessk2w2, sqrtp2, 1):hessk2w2b:=coeff(hessk2w2, sqrtp2,
0):
```

Identities

```
> [hessk1v1a-hessk2v2a, hessk1v1b+hessk2v2b, hessk1w1a-hessk2w2a, hessk1w1b+hessk2w2b];
```

$$[0,0,0,0]$$

Definition of alpha, beta, alphabis, betabis: one has hessk1v1= a*sqrt(p2(u,v)) +b ; hessk2v2= a*sqrt(p2(u,v)) -b

alpha, beta, alphabis, betabis are fct of A,B,C,DD, detI and first and second derivatives

```
> alpha:=simplify( subs( subs_p2, hessk1v1a )):
> alphabis:=simplify( subs( subs_p2, hessk1w1a )):
> beta:=simplify( subs( subs_p2, hessk1v1b )):
> betabis:=simplify( subs( subs_p2, hessk1w1b )):
> [nops(alpha),nops(beta)];
```

$$[216, 371]$$

turn , turn-B and turn-C are fct of A,B,C,D,DetI and first and second derivatives

```
> turn_B:=expand( subs( subs_p2, alpha^2*p2(u,v)-beta^2 ) /B(u,v)^2 ):
> turn_C:=expand( subs( subs_p2, alphabis^2*p2(u,v)-betabis^2 ) /C(u,v)^2
):
> turn_AD:= simplify(  2*(alphabis*beta-alpha*betabis)/(-A(u, v)+DD(u,
v)) ):
```

Equivalence of equations

```
> [turn_B-turn_C,turn_B-turn_AD];
```

$$[0, 0]$$

Final equation, in each term of turn, there are 10 terms amongst A,B,C,D and 4 times DetI and 4first derivatives (2*first derivative=2nd derivative)

```
> turn:=simplify( turn_AD/ content(turn_AD) ):
> [whattype(turn), nops(turn), op(1, turn)];
```

$$[\texttt{`+`}, 17302, -96\,(detI\,(u,v))^4\,(A\,(u,v))^2\left(\frac{\partial}{\partial\,\texttt{`\$`}\,(v,2)}B\,(u,v)\right)(C\,(u,v))^3\left(\frac{\partial}{\partial u}DD\,(u,v)\right)\left(\frac{\partial}{\partial v}B\,(u,v)\right)(DD\,(u,v))^2]$$

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