Handsome Non-Commutative Proof-Nets: perfect matchings, series-parallel orders and Hamiltonian circuits

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Handsome Non-Commutative Proof-Nets: perfect matchings, series-parallel orders and Hamiltonian circuits

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Abstract: This paper provides a definition of proof-nets for non-commutative linear logic (cyclic linear logic and Lambek calculus) where there are no links, that are small graphs representing the connectives. Instead of a tree like representation with links, the formula is depicted as a graph representing the conclusion up to the algebraic properties of the connectives.

In the commutative case the formula is viewed as a cograph. In the non-commutative case it is a more complicated kind of graph which is, roughly speaking, a directed cograph. The criterion consists in the commutative condition plus a bracketing condition.

Key-words: Proof theory, linear logic. Graph theory, perfect matching, cographs, cyclic orders. Computational linguistics, Lambek calculus, categorial grammars.

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Réseaux non commutatifs sans lien : couplages parfaits, ordre séries-parallèles et circuits hamiltoniens

Résumé : Cet article donne une définition des réseaux de preuve pour la logique linéaire non commutative (logique linéaire cyclique et calcul de Lambek), dans laquelle il n'y a plus de liens, c'est-à-dire de petits graphes représentant les connecteurs. Au lieu d'une représentation avec des liens sous forme d'arbres, les formules sont décrites comme des graphes représentant les conclusions, aux propriétés algébriques des connecteurs près.

Dans le cas commutatif, la formule est vue comme un co-graphe. Dans le cas non commutatif, c'est un graphe un peu plus complexe qui est, essentiellement, un co-graphe orienté. Le critère de correction consiste en la condition commutative plus une condition de bon parenthésage.

Mots-clés : Théorie de la preuve, logique linéaire. Théorie des graphes, couplage parfait, co-graphes, ordres cycliques. Linguistique informatique, calcul de Lambek, grammaires catégorielles.

Contents

1 Presentation

Proof-nets are a graph theoretical formalism for proofs, brought to light by the invention of linear logic by Girard (1987). Basically they identify many proofs which are equivalent up to rule permutation and improve the normalization or cut-elimination.

In order to define proof-nets with standard graph-theoretical notions, a first step was to define them as edge-bicolored graphs for commutative linear logic like in (Retoré 1996*b*, Retoré 2003). The branching inside a formula tree are described by bicolored links which makes a distinction between $\otimes$ and $\wp$ links without any labels; axioms are B edges linking dual atoms a and $a^{\perp}$; the criterion consists in the absence of alternate elementary cycle plus some connectivity condition with alternate paths.

Bicolored proof-nets with links have been used for their simplicity for multiplicative calculi and non-commutative variants *à la* Lambek (1958) or Abrusci (1991) but up to now, non commutativity was described by a separate different condition for instance by Retoré (1996*a*) or Pogodalla (2001*b*). A noteworthy exception are the results of Maringelli (1996) and Abrusci & Maringelli (1998) which incorporate the non commutativity by directing the edges of the links. Bicolored proof-nets also have been used for pomset logic by Retoré (1993) which includes a non-commutative self-dual connective plus the usual ones (Retoré 1997, Schena 1996, Pogodalla 2001*a*).

The next step for including proof-nets within standard graph-theory is to leave out the links or the tree like description of the formula; indeed, it has almost no graph-theoretical structure for representing the formula.

A solution was found in (Retoré 1996*b*, Retoré 2003) in the commutative case, that is for MLL, with and without the *mix* rule. Such proof-nets consist in a graph whose vertices are atoms, and two kinds of edges: the *B* (*B*lue or *B*old) ones and the *R* ones (red or regular): the *R*-edges describe the conclusion formula as a cograph, and the *B*-edges describe the axioms between a and $a^\perp$ (not a) and consequently are a perfect matching inside the complete graph. The cograph is defined as follows: there is an *R*-edge between two atoms x and y whenever, in the formula tree, they meet on a conjunction, otherwise, that is when they meet on a disjunction, there is no *R*-edge. The condition for such an object to correspond to a proof is that any alternate elementary cycle contains a chord, plus some connectivity condition with alternate paths without chords — a chord is an edge between two vertices of a path or cycle but not in the path or cycle. The advantages of such a description is that associativity and commutativity of the connectives are interpreted by equality of the formulae and of the proofs involving them.

Proof-nets without links easily extends to pomset logic as done by Retoré (1999) and the work of Guglielmi (1999) and Guglielmi & Straßburger (2001) develops and construct a term calculus along these lines.

Up to now, regarding the non-commutative calculi like cyclic linear logic, Lambek calculus, the only specific results were the ones obtained by Abrusci & Maringelli (1998), which make use of links. So the present paper extends the results in (Retoré 1996*b*, Retoré 2003) to such non-commutative calculi. Endowing the *R*-edges with a direction is not enough. Indeed, $(a \otimes b) \wp (b^\perp \wp a^\perp)$ is provable, but not $(a \otimes b) \wp (a^\perp \wp b^\perp)$. So there ought to be something else to describe the $\wp$ connectives and the cyclic order.

This paper provides a structure for describing non-commutative formulae and cyclically ordered formulae, in such a way that associativity of the connectives yields equal graphs, as well as cyclic permutations of the conclusions, or disjunctions between conclusions. For representing the disjunctions we use a new color *N*.

In the figures that illustrate examples, colors are represented as follows:

- the *B* color is represented by undirected bold solid lines,
- the *R* color is represented by regular solid arrows,
- the *N* color is represented by regular dashed-dotted arrows.

The structure consists in *R*-directed series compositions for representing the conjunctions, with an *R*-arc from x to y whenever a formula is $X \otimes Y$ with x in X and y in Y. The *N*-directed series compositions represent the internal disjunctions, that are the ones which are inside one component of a conjunction like $\wp_0$ in $F[X \wp_0 Y] \otimes G$: in this case there is an *N*-arc from any x in X to any y in Y. The cyclic order between conclusions as well as the external disjunction are depicted by *N*-arcs as follows: whenever a conclusion X is immediately before a conclusion Y there is an *N*-arc from the last vertex in X to the first vertex in Y. The difference between internal disjunctions and external disjunctions is the following: for each external disjunction we have a single *N*-arc.

This structure defines a total cyclic order on the atoms of the sequent. The criterion consists in:

- the commutative criterion: in the underlying undirected *BR*-graph every *BR*-alternate elementary cycle contains a chord, and there is a chordless elementary path between any two vertices)
- a bracketing condition: the *B* matching fits in with the total cyclic order when the vertices are drawn on a circle: the *B*-edges, which are chords of the circle, do not cross one another.

Hence the later condition is the analogous in this setting of the necessary condition found by Roorda (1991) proved to be sufficient in Retoré (1996*a*) in proof-nets with links. Here, the main difficulty has been to represent non-commutative formulae as graphs.

A further extension would be to deal with mixed calculi introduced by de Groote (1996) in the intuitionistic case and extended to a classical setting by Ruet (1997) and Abrusci & Ruet (1999). Another direction would be to use this work for improving proof-search, e.g. by the probabilistic methods of Moot (2004).

2 Combinatorial definitions, conventions and notations

We feel obliged to provide some definitions, since graph theory is keen on multiple variants for its concepts and notations. Our sources are (Bollobás 1998, Lovàsz & Plummer 1986, Möhring 1989). A particular but simple notation that we use is the following: given a path or a totally ordered set X, the expression $\underline{\text{first}}X$ denotes the first or least vertex and $\underline{\text{last}}X$ the last or greatest vertex.

2.1 Graphs

A **graph** $G = (V,A)$ consists in a set of **vertices** V and a multiset of ordered pairs of *distinct* vertices (x,y) in $V \times V$—such a pair of vertices is called an **arc**. A graph is said to be **simple** whenever there is no multiple arc.

A **path** of length l is a sequence of $l+1$ vertices $x_0 \cdots x_l$ such that there is an arc $x_i x_{i+1}$ for $0 \leq i \leq l-1$. Such a path is said to be a path from x_0 to x_l. A **circuit** of length l is a path of length l from a vertex to itself. A path or a cycle is said to be **elementary** whenever it does not contains twice the same vertex, except, for a circuit the first and last.

Every directed graph G can be mapped into an underlying undirected graph $\text{undir}(G)$ which is obtained by adding an arc (x,y) whenever there is an arc (y,x) in G, unless there already exists one.

We say a graph is **complete** whenever its underlying graph is complete, that is when there is an edge between any two vertices.

A **chord** in a path or cycle is an arc between two vertices of the path or cycle but not in the path or cycle.

An equivalence class of the symmetric and transitive closure of the relation there exists a path from x to y is called a **connected component** — hence the connected components of G and of $\text{undir}(G)$ are the same.

2.2 Arc-colored graphs

Our graphs are **arc-colored** with colors B,R,N. Given a color C the C-subgraph with all the vertices and only the C-arcs is a simple graph, that is an anti-reflexive binary relation and a C-arc from x to y is often denoted by xCy.

The underlying undirected graph of an arc-colored graph is obtained by adding an arc (x,y) with color C whenever there is a C-arc (y,x) in G, unless there already exists one with the same color C.

Given a list of colors LC an LC-path (resp. circuit or connected component) means a path (resp. a circuit or a connected component) in the graph restricted to the arcs whose color belongs to the list of colors LC.

2.3 Series-parallel orders and cographs

Cographs are the smallest class of undirected simple graphs, containing graphs with a single vertices (hence without any edge) and closed under the following binary operations. Given two cographs (V_1,E_1) and (V_2,E_2) their disjoint union $(V_1 \uplus V_2, E_1 \uplus E_2)$ is a cograph and so is their undirected series composition $(V_1 \uplus V_2, E_1 \uplus E_2 \uplus (V_1 \times V_2) \uplus (V_2 \times V_1))$. It is also the smallest class of simple graphs closed under disjoint union and complement (G and its complement $\overline{G}$ have the same vertices and an edge is in $\overline{G}$ is and only if it is not in G). Cographs are characterized by the absence of P_4: a graph is a cograph iff whenever there is a path $x_1x_2x_3x_4$ the graph contains at least an edge x_1x_3 or x_1x_4 or x_2x_4.

Series-parallel orders are a kind of directed cographs. It is the smallest class of directed simple graphs containing graphs with a single vertices (hence without any edges) and closed under the following binary operations. Given two series-parallel orders (V_1,E_1) and (V_2,E_2) their disjoint union $(V_1 \uplus V_2, E_1 \uplus E_2)$ is a series-parallel order and so is their directed series composition $(V_1 \uplus V_2, E_1 \uplus E_2 \uplus (V_1 \times V_2)$. In a series-parallel order the relation $x < y$ whenever there is an arc from x to y defines a (partial) order, as the name suggests. Series-parallel orders are characterized by the absence of N: a graph is a series-parallel order iff, whenever there is an N (four vertices x_1,x_2,x_3 and x_4 such that there exists the following arcs $(x_1,x_2)(x_3,x_2)(x_3,x_4)$) the graph contains at least a supplementary arc (x_1,x_3) or (x_1,x_4) or (x_2,x_4) or (x_3,x_1) or (x_4,x_1) or (x_4,x_2).

2.4 Matchings and alternate paths

A **matching** in an undirected graph is a set of edges which never are adjacent one to another. A matching can be viewed as a (partial) function form vertices to vertices such that $f(f(x)) = x$. The matching is said to be perfect whenever a vertex is always incident to an edge of the matching, hence to exactly one edge of the matching. Thus a perfect matching is a bijection f with $f = f^{-1}$, or a two-permutation where all cycles are of length 2.

An **alternate elementary path** in a graph with a matching is a path with distinct vertices with arcs successively in and not in the matching. An **alternate elementary cycle** in a graph with a matching is a path of even length with distinct vertices but the first and last with arcs successively in and not in the matching.

2.5 Cyclic orders and Hamiltonian circuits

A **total cyclic order** is a ternary relation $H(\_,\_,\_)$ which describes vertices on a directed circle. $H(x,y,z)$ can be interpreted as *"y is between x and z following the circle in the correct direction"*. Formally it is a relation which, for all x, y and z, enjoys the following properties:

- $H(x,y,z) \to x \neq y \otimes y \neq z \otimes z \neq x$ (strict cyclic order)
- $H(x,y,z) \to H(z,x,y)$ (cyclicity)
- $H(x,y,z) \otimes H(y,u,z) \to H(x,y,u)$ (pseudo transitivity)
- $H(x,y,z) \wp H(x,z,y)$ (totality)

The **interval** $[x,y]$ in a total cyclic order is the set $\{u \mid H(x,u,y)\}$.

An **Hamiltonian circuit** is a circuit visiting each vertex once. In case there are just two vertices a and b, we also admit aba as an Hamiltonian circuit. An Hamiltonian circuit $x_0 \cdots x_n x_0$ in a directed graph defines a total cyclic order on the vertices by $H(x_i,x_j,x_k)$ whenever $i < j < k$ or $k < i < j$ or $j < k < i$.

3 Non-commutative formulae as graphs

3.1 Multiplicative linear logic formulae

Formulae of cyclic linear logic are defined from a set of propositional variables P by $F ::= A \mid F \wp F \mid F \otimes F$ where $A ::= P \mid P^\perp$ are the atoms.

Such formulae are the negative normal forms of formulae in $F' ::= P \mid F' \wp F' \mid F' \otimes F' \mid F'^\perp$ by means of de Morgan laws which are provable in full linear logic: $(X^\perp)^\perp \equiv X$, $(X \otimes Y)^\perp = Y^\perp \wp X^\perp$ and $(X \wp Y)^\perp = Y^\perp \otimes X^\perp$ see e.g. Retoré (1996*a*).

In the calculus to be defined in the next section, it is possible to prove the associativity of the conjunction and of the disjunction. Associativity means that $((X \otimes Y) \otimes Z) \equiv (X \otimes (Y \otimes Z))$ and $((X \wp Y) \wp Z) \equiv (X \wp (Y \wp Z))$. In the graphs to be defined next we want that formulae that are equal up to associativity correspond to equal graphs.

3.2 The K-graph of a formula

The K-graph $K(F)$ of a formula F is defined such that its underlying undirected graph is a complete graph. There is an R-edge aRb whenever a and b meet on a $\otimes$ in the formula tree and a is before b in the formula. There is an N-edge aNb whenever a and b meet on a $\wp$ in the formula tree and a is before b in the formula. More formally we can define $K(F)$ inductively:

$K(a) \quad a \in A:$

vertices: $\{a\}$

N-**arcs** $\emptyset$

R-**arcs** $\emptyset$

$K(X \otimes Y)$

vertices: $V(F) = V(X) \uplus V(Y)$

N-**arcs** $N(K(F)) = N(K(X)) \uplus N(K(Y))$

R-**arcs** $R(K(F)) = R(K(X)) \uplus R(K(Y)) \uplus (V(X) \times V(Y))$

$K(X \wp Y)$

vertices: $V(F) = V(X) \uplus V(Y)$

R-**arcs** $R(K(F)) = R(K(X)) \uplus R(K(Y))$

N-**arcs** $N(K(F)) = N(K(X)) \uplus N(K(Y)) \uplus (V(X) \times V(Y))$

We then have the following obvious properties:

Proposition 1 *The underlying undirected graph of a K-graph is a complete graph, whose edges are partitioned into two series-parallel orders: the R and the N subgraphs.*

Proposition 2 *If $X \equiv Y$ up to the associativity of the connectives, then $K(X) = K(Y)$*

Next comes a characterization of *K*-graphs:

Proposition 3 *An edge bicolored graph is a K-graph if and only if*

- *its underlying graph is a complete graph,*
- *the sets of N-arcs and the set of R-arcs are disjoint,*
- *the N-arcs define a series-parallel order,*
- *the R-arcs define a series-parallel order.*

PROOF : The "only if" part is an easy induction.

By induction on the number of vertices, let us show that whenever a bicolored graph satisfies all the properties it is the *K*-graph of some formula.

If it is a single vertex with no edge (simple graph) the formula is this atom.

Assume both the *N* and *R* series compositions are disjoint unions. Thus,

- R defines a partition of V as V_R^1 and V_R^2 with no *R*-arcs between them, and
- N defines a partition of V as V_N^1 and V_N^2 with no *N*-arcs between them.

A priori we thus have a partition of V into four classes:

- $V_R^1 \cap V_N^1$,

- $V_R^1 \cap V_N^2$,
- $V_R^2 \cap V_N^1$ and
- $V_R^2 \cap V_N^2$.

Let us now show that $(V_R^1 \cap V_N^2)$ or $(V_R^2 \cap V_N^1)$ is empty and that either $V_R^1 \cap V_N^1$ or $V_R^2 \cap V_N^2$ is empty. Assume that there exists $a \in (V_R^1 \cap V_N^2)$ and $b \in (V_R^2 \cap V_N^1)$: as the underlying graph is complete there is an arc between a and b but it can neither be an R-arc nor an N-arc. So either $(V_R^1 \cap V_N^2)$ or $(V_R^2 \cap V_N^1)$ is empty. A similar argument shows that either $V_R^1 \cap V_N^1$ or $V_R^2 \cap V_N^2$ is empty.

Consequently one of V_R^1, V_R^2, V_N^1 or V_N^2 is empty that is either the N or R series composition disconnect the complete graph, and given that the R and N-edges do not overlap, this series composition of type C ($C = N$ or $C = R$) is a disjoint union in the other color.

The two parts, say G_1 and G_2, enjoy the same properties: the underlying graph is a complete graph, the N and R subgraphs have no edge in common, both the R and the N subgraphs are series-parallel partial orders. Hence by induction $G_1 = K(X_1)$ and $G_2 = K(X_2)$ and the formula corresponding to G is either $X_1 \otimes X_2$ (if $C = R$) or $X_1 \wp X_2$ (if $C = N$). ⋄

Proposition 4 *An edge bicolored graph is a K-graph if and only if*

- *Forgetting the colors, we obtain the graph of a total order.*
- *There exists a series decomposition in which some series composition arcs are given the color R and the others the color N.*

Proposition 5 *It is polynomial to check whether a bicolored graph is a K-graph.*

PROOF : Here are the things to be checked.

The underlying graph should be a complete graph.

The N and R subgraphs should have no edge in common.

Both the R and the N subgraphs should be series-parallel partial orders and this is edge-linear as well as shown by Valdes, Tarjan & Lawler (1982).

All this operation are linear in the number of edges — or quadratic in the number of vertices. ⋄

Proposition 6 *A K-graph $K(F)$ induces a linear order on the vertices of F which is the one of the atoms in the formula. In particular the K-graph determines a first and a last element of a formula F respectively denoted by $\underline{\text{first}}(F)$ and $\underline{\text{last}}(F)$.*

This linear order is defined as an RN-path and contains exactly one arc of each N series composition and one arc of each R series composition. Alternatively, it is the Hasse diagram the K-graph viewed as a linear order.

The components of an R series composition and the result of an R series composition itself are intervals of the linear order.

PROOF : These properties are easily checked by induction.

- If the formula is reduced to a single atom a, then the linear order, or Hamiltonian path, is a.
- If $F = X \otimes Y$ we have, by induction, a linear order corresponding to an Hamiltonian path P_X from $\underline{\text{first}}(X)$ to $\underline{\text{last}}(X)$ in $K(X)$, using exactly one arc by connective in X, and one Hamiltonian path P_Y in $K(Y)$, using exactly one arc by connective in Y. The linear order or Hamiltonian path we are looking for is

$$P = \underline{\text{first}}(X) P_X\, \underline{\text{last}}(X) R\, \underline{\text{first}}(Y) P_Y\, \underline{\text{last}}(Y)$$

  It uses exactly one R-arc corresponding to the conjunction between X and Y.

  The last composition includes all the vertices, therefore it is interval. By induction, the other components are intervals of P_X or of P_Y. hence they are intervals of P.
- If $F = X \wp Y$ the argument is the same *mutatis mutandis*.

◇

3.3 Cyclic sequents and cyclic K-graphs

A **cyclic K-graph** is a finite set $K_0, \ldots, K_{n-1}$ of K-graphs, *whose main series compositions are R series composition*, endowed with a cyclic order. This cyclic order is depicted by a set of N-arcs from $\underline{\text{last}} K_{i[n]}$ to $\underline{\text{first}} K_{i+1[n]}$ — we do add such arcs when there are only two or even one K-graph, although the cyclic order, as a ternary relation, is empty.

A cyclic K-graph contains an Hamiltonian circuit which consist in:

- the Hamiltonian path of each K_i (containing one R-arc of each R series composition and one N-arc of each N series composition) and
- all the N-arcs from $\underline{\text{last}} K_{i[n]}$ to $\underline{\text{first}} K_{i+1[n]}$.

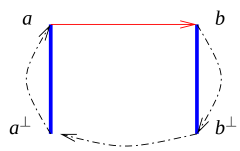
Figure 1 gives examples of cyclic K-graphs. Note that only the K-graph of figure 1(d) and 1(e) have an inner disjunction, hence a non-trivial series-parallel composition for the disjunction $a \wp b \wp c$ and for the disjunction $(a \otimes b) \wp b^{\perp} \wp a^{\perp}$.

Proposition 7 *In a cyclic K-graph whenever there are two N-arcs aNz and bNz leading to the same vertex, these arcs both belong to the same K-graph.*

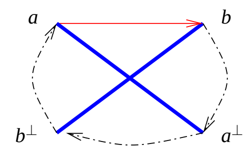
In a cyclic K-graph whenever there are two N-arcs zNa and zNb starting from the same vertex, these arcs both belong to the same K-graph.

PROOF : The two properties are clearly symmetrical.

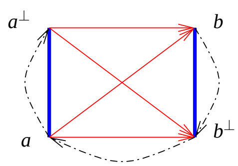
Observe that if aNz is the arc from $\underline{\text{last}} K_{i[n]}$ to $\underline{\text{first}} K_{i+1[n]}$ no N arcs inside $K_{i+1[n]}$ start with z, and that the only other N arc incident to $K_{i+1[n]}$ is the one going *from* $\underline{\text{last}} K_{i+1[n]}$ (which can be z as well) to $\underline{\text{first}} K_{i+2[n]}$. So if an external N-arc arrives to z, there cannot be any other N-arc arriving to z. ◇



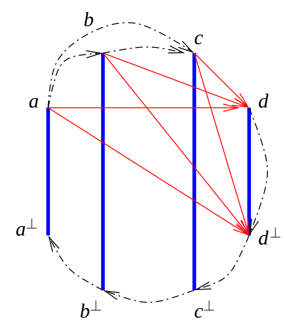
(a) Cyclic K-graph associated to the sequent $\vdash a \otimes b, b^{\perp} \parr a^{\perp}$



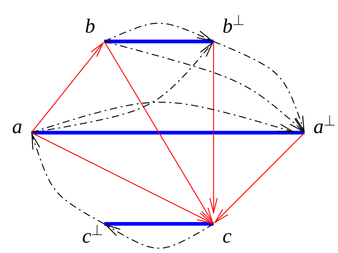
(b) Cyclic K-graph associated to the sequent $\vdash a \otimes b, a^{\perp} \parr b^{\perp}$



(c) Cyclic K-graph associated to the sequent $\vdash (a \parr a^{\perp}) \otimes (b \otimes b^{\perp})$



(d) Cyclic K-graph associated to the sequent $\vdash (a \parr b \parr c) \otimes (d \parr d^{\perp}), c^{\perp}, b^{\perp}, a^{\perp}$



(e) Cyclic K-graph associated to the sequent $\vdash\vdash ((a \otimes b) \parr b^{\perp} \parr a^{\perp}) \otimes c, c^{\perp}$

Figure 1: Examples of cyclic K-graphs

From the characterization of K-graphs we obtain:

Proposition 8 *A bicolored graph is a cyclic K-graph if and only if:*

- *The induced bicolored subgraph on every R-connected component V_i is a K-graph*
- *The other N-arcs define a cyclic order between the V_i, that is they go from the vertex $\underline{\mathrm{last}}K_{i[n]}$ to the vertex $\underline{\mathrm{first}}K_{i+1[n]}$.*

Observe that it may happen that we have only K_0: in this case we have inside the component an extra N edge from $\underline{\mathrm{last}}K_0$ to $\underline{\mathrm{first}}K_0$.

From the previous proposition, given that the computation of connected component is linear in the number of edges plus the number of vertices, hence quadratic in the number of vertices (Tarjan 1972, Gondran & Minoux 1995), we obtain:

Proposition 9 *It is polynomial to check whether a bicolored graph is a cyclic K-graph.*

A **cyclic sequent** $\vdash A_0,\ldots,A_{n-1}$ that is a sequent up to cyclic permutations of the A_i, is mapped into a cyclic K-graph as follows:

- Replace every external disjunction by a comma yielding a sequent

$$\vdash \overbrace{A_0^1,\ldots,A_0^{p_0}}^{A_0},\ldots,\overbrace{A_i^1,\ldots,A_i^{p_i}}^{A_i},\ldots,\overbrace{A_{n-1}^1,\ldots,A_{n-1}^{p_{n-1}}}^{A_{n-1}}$$

  with

  - $A_i = \wp_{k=1}^{k=p_i} A_i^k$
  - $A_k^i = X_k^i \otimes Y_k^i$ or $A_k^i = a \in A$

- the cyclic K-graph consists in

  - the disjoint union of the $K(A_k^i)$
  - N-arcs from $\underline{\mathrm{last}}(A_k^i)$ to $\underline{\mathrm{first}}(A_k^{i+1})$ and from $\underline{\mathrm{last}}(A_i^{p_i})$ to $\underline{\mathrm{first}}(A_{i+1[n]}^1)$

- its Hamiltonian cycle is made of the Hamiltonian paths of the A_k^i and the added N-arcs.

By construction, is is clear that one has the following:

Proposition 10 *A cyclic K-graph associated with a sequent $\vdash A_0,\ldots,A_{n-1}$ is unchanged:*

1. *if a formula of the cyclic sequent A_i is replaced with an equivalent formula up to the associativity of $\otimes$ and $\wp$,*
2. *if a circular permutation is applied to the cyclic sequent $A_0,\ldots,A_{n-1}$,*
3. *if two consecutive formulae of the cyclic sequent A_i and $A_{i+1[n]}$ are replaced by their disjunction $A_i \wp A_{i+1[n]}$.*

Figure 2: Sequent calculus for CyMLL

Structural rule: cyclic exchange

$$\frac{\vdash A_0, \ldots, A_{n-2}, A_{n-1}}{\vdash A_{n-1}, A_0, \ldots, A_{n-2}}\; cx$$

Axiom

$$\frac{}{\vdash a, \perp a}\; ax$$

Logical rules

disjunction

$$\frac{\vdash A_0, \ldots, A_i, A_{i+1}, \ldots, A_{n-1}}{\vdash A_0, \ldots, (A_i \wp A_{i+1}), \ldots, A_{n-1}}\; \wp_i$$

Ê **conjunction**

$$\frac{\vdash A_0, A_1, \ldots, A_{n-2}, A_{n-1} \qquad \vdash B_0, B_1, \ldots, B_{p-2}, B_{p-1}}{\vdash A_0, A_1, \ldots, A_{n-2}, (A_{n-1} \otimes B_0), B_1, \ldots, B_{p-2}, B_{p-1}}\; \otimes_i$$

4 Cyclic proof-nets as cyclic K graph enriched with a perfect matching

A cyclic proof-structure consists in a cyclic K-graph enriched with a perfect matching of B-edges linking vertices with dual names a and $a^\perp$. It is said to be a proof-net whenever the following criterion holds:

MLL proof-net the underlying undirected BR-graph is an MLL proof-net, that is to say:

> **acyclicity** Every alternate elementary cycle contains a chord.
>
> **connectedness** There exists a chordless alternate elementary path between any two vertices.

Hamiltonian adequacy B-edges are adequate to the Hamiltonian circuit: that is whenever aBa', bBb' and $H(a,b,a')$ one has $H(a,b',a')$ as well.

Cyclic K-graphs of figure 1 are cyclic proof-structures. They also are proof-nets except the one of figure 1(b) (as expected, since it corresponds to a sequent which is not derivable in CyMLL) because it does not satisfy the Hamiltonian adequacy.

Let us inductively map a proof of $\vdash A_1, \ldots, A_p$ into an K-proof-net with the cyclic K-graph corresponding to this sequent. The rules are reminded in figure 4.

Proposition 11 *Every proof in CyMLL of* $\vdash A_0, \dots, A_{n-1}$ *is inductively mapped into a proof-net with the K-graph corresponding to* $A_0, \dots, A_{n-1}$

PROOF : We proceed by induction on the sequent calculus proof. In all of the following cases, the acyclicity and connectedness of the *BR* part does not have to be checked since this part of the proof-structure is the same as in the commutative case (Retoré 2003).

cyclic exchange By induction hypothesis we have a proof-net whose cyclic *K*-graph is the one associated with $A_0, \dots, A_{n-2}, A_{n-1}$. The proof-structure is unchanged by this rule—by proposition 10 the cyclic *K*-graph associated with $A_{n-1}, A_0, \dots, A_{n-2}$ is the same.

axiom If the proof consists in an axiom $\vdash a, a^\perp$ the *K*-graph consists in two black arcs $aNa^\perp$ and $a^\perp Na$ and the single *B*-edge is $a^\perp Ba$. The Hamiltonian adequacy holds, since there is no triple x, y, z such that $H(x, y, z)$ (figure 3(a)).

disjunction By induction hypothesis we have proof-net whose cyclic *K*-graph is the one associated with $A_0, \dots, A_i, A_{i+1}, \dots, A_{n-1}$. The proof-structure is unchanged by this rule—by proposition 10 the cyclic *K*-graph associated with $A_0, \dots, (A_i \wp A_{i+1}), \dots, A_{n-1}$ is the same.

conjunction Observe that in the application of the rule it is fairly possible that $n = 1$ or $p = 1$, and even both. The construction and argument which follows work just the same.

By induction hypothesis we have two proof-nets: π_A with the *K*-graph associated with $A_0, \dots, A_{n-1}$ and a Hamiltonian circuit H_A and π_B with the *K*-graph associated with $B_0, \dots, B_{p-1}$ and a Hamiltonian circuit H_B (figure 3(b)).

The proof-structure associated with the complete proof is obtained as follows:

- suppress the *N*-arc from A_{n-1} to A_0,
- suppress the *N*-arc from B_{p-1} to B_0,
- add *N*-arcs between vertices of A_{n-1} for *NR* to be transitive on A_{n-1} (since now the possible main disjunctions of A_{n-1} become internal, these disjunctions have to be turned into series-parallel composition),
- add *N*-arcs between vertices of B_0 for *NR* to be transitive on B_0 (idem),
- add one *N*-arc from $\underline{\text{last}}(B_{p-1})$ to $\underline{\text{first}}(A_0)$
- add an *R*-arc from each vertex of A_{n-1} to each vertex in B_0—the *R*-arc used by the Hamiltonian circuit in the compound proof-structure is the one from $\underline{\text{last}}(A_{n-1})$ to $\underline{\text{first}}(B_0)$.
- *B*-arcs are unchanged, in particular there is no *B*-edge between any atom of some A_i and any atom of some B_j.

The *K*-graph thus obtained (figure 3(c)) is the one corresponding to the sequent:

$$A_0, \dots, (A_{n-1} \otimes B_0), \dots, B_{p-1}$$

The Hamiltonian circuit (or total cyclic order $H(\_, \_, \_)$) is the linear order of the atoms in the sequent completed by the *N*-arc from the last atom in B_{p-1} to the first atom in A_0. We call H_A the restriction of H to the A_i's and H_B the restriction of H to the B_i's.

We have to check cyclic adequacy. Assume we have $H(x,y,,x')$, xBx' and yBy' (hence both x and x' are in the A_i's or both are in the B_j's). Assume that both x and x' are in the A_i's. If y is in the B_j's then we have $H(x,y',x')$ since y' is in the B_j's and all or none of the vertices of the B_j's are in between two vertices of the A_i's. If y is in the A_i's so is y'. We then have $H_A(x,y,x')$ and because π_A enjoys cyclic adequacy we have $H_A(x,y',x')$ hence $H(x,y',x')$.

◇

5 From proof-nets to sequent calculus proofs

In this section, we show that every proof-net corresponds to a proof. Firstly, observe the following:

Proposition 12 *Given an arc-colored graph is quadratic in the number of vertices (atoms) to check that it is a proof-structure and cubic in the number of vertices to check that is is a proof-net.*

PROOF : Checking that the N and R edges are a cyclic K-graph is linear in the number of edge, and checking that the $B*$edges define a perfect matching is linear as well. It is linear to compute the Hamiltonian cycle, which exists as soon as we have a cyclic K-graph.

The commutative part of the correctness is checked in cubic time in the number of vertices (Retoré 2003).

Finally the Hamiltonian adequacy is linearly checked by stack automaton. ◇

Let us remind proposition 32 from Retoré (1996*a*), with the terms of the present paper:

Proposition 13 *Consider a graph containing:*

- *an Hamiltonian circuit H*
- *B-edges enjoying cyclic adequacy.*

Assume the Hamiltonian circuit is divided into intervals $E_0,\dots,E_{N-1}$. We define $\mathscr{B}$ the relation $E_i\mathscr{B}E_j$ if and only if there exists $x\in E_i$ and $y\in E_j$ such that xBy. Let $\mathscr{B}^*$ be the transitive closure of $\mathscr{B}$.

If $\mathscr{B}^*$ exactly has two equivalent classes $\mathscr{L}$ and $\mathscr{R}$, then there exists i_0 and k such that:

- $E_{i_0},E_{i_0+1[N]},E_{i_0+2[N]},\dots,E_{i_0+k[N]}$ *all are in $\mathscr{L}$*
- $E_{i_0+k+1[N]},E_{i_0+k+2[N]},\dots,E_{i_0-1[N]}$ *all are in $\mathscr{G}$*

In other words, these two equivalent classes are intervals.

Proposition 14 *Let π be a correct proof-net associated with the following sequent: $\vdash A_0,\dots,A_{n-1}$. Then there exists a sequent calculus proof whose proof-net is π.*

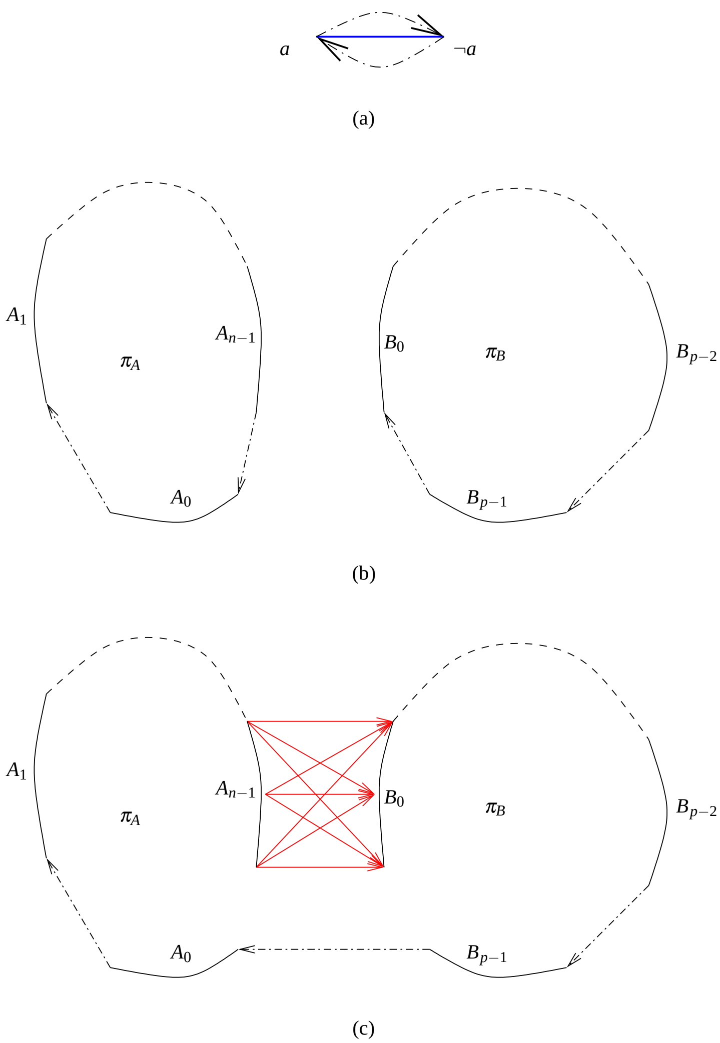


Figure 3: From sequent proofs to proof-nets

PROOF : Let π be a proof-net depicted as a K-graph enriched with a perfect matching B.

Because the underlying commutative proof-net is correct we know that either the BR-proof-net contains no R-edge but a single B-edge linking two vertices a and $a^\perp$ or it contains a splitting R series composition, yielding two smaller correct (commutative) proof-nets. In the first case, since the graph contains an Hamiltonian circuit with N and R-edges, it has to be an N Hamiltonian circuit: $aNa^\perp Na$. Given that we have $aBa^\perp$ we have the proof-net corresponding with the axiom $\vdash a, a^\perp$.

Hence, from now on we can assume that we have a splitting conjunction $\otimes_0$, and we are able to split the BR-graph into two proof-nets π^1 and π^2 enjoying acyclicity and connectedness. The only arcs of π between π^1 and π^2 are the R-arcs of $\otimes_0$ between some R connected components in π^1 and some R connected components in π^2.

Let us first show that there is no N-arc from π_1 to π_2 and exactly one from π_2 to π^1 A priori the Hamiltonian circuit can be divided into paths that are successively in π^1 and π^2:

$$p = p_1^1 p_1^2 p_2^1 p_2^2 p_3^1 p_3^2 \cdots p_K^1 p_K^2$$

As the only R-arcs between π^1 and π^2 are the ones of $\otimes_0$ and that only one of them is in H:

- all the arcs from $\underline{\text{last}}\, p_i^1$ to $\underline{\text{first}}\, p_i^2$ are N-arcs but one say $\underline{\text{last}}\, p_1^1 R\, \underline{\text{first}}\, p_1^2$
- all the arcs from $\underline{\text{last}}\, p_i^2$ to $\underline{\text{first}}\, p_{i+1[k]}^1$ are N-arcs

There cannot be an R-arc from $x \in p_j^\varepsilon$ to $y \in p_k^\varepsilon$ (with $j \neq k$) Only one R-connected component intersects both π^1 and π^2, the one of $\otimes_0$, which is included in $p_1^1 p_1^2$; we denote it by RCC_0. Assume for $\varepsilon = 1$ or $\varepsilon = 2$ that there is an R-arc between $x \in p_i^\varepsilon$ and $y \in p_j^\varepsilon$ with $j \neq i$ ($K > 1$) in either direction. By proposition 6 either $[x, y]$ or $[y, x]$ is included in this R connected component. Since both intervals contains vertices in $\pi^{\overline{\varepsilon}}$ (with $\overline{\varepsilon} = 2$ iff $\varepsilon = 1$ and $\overline{\varepsilon} = 1$ iff $\varepsilon = 2$) the R connected component of x and y has to be RCC_0, but this not possible since $(RCC_0 \cap \pi^\varepsilon) \subset p_1^\varepsilon$ and either $x \notin p_1^\varepsilon$ or $y \notin p_1^\varepsilon$ ($i \neq j$).

$K = 1$, that is $p = p_1^1 p_1^2$ Let us define $\mathscr{B}^*$ such as in proposition 13 with the E_i's being the p_i^1's and the p_i^2's, and let us count its equivalent classes. Since $\otimes_0$ splits π into two parts, we know that these equivalent classes are at least two (for any i and any j, p_i^1 and p_j^2 cannot belong to the same equivalent class).

Can there be more than two equivalent classes? No, because it would mean that inside the p_k^i of the same π_i, for at least one i, $\mathscr{B}^*$ has at least two equivalent classes. That means that between any two representatives of these equivalent classes, there is no B-edge. Since the former paragraph shows there is no R-edge either, it would mean that π_i is not connected. Hence $\mathscr{B}^*$ exactly has two equivalent classes. From proposition 13, these equivalent classes are intervals. And since for any i, $p^1_{i[n]}$ and $p^2_{i+1[n]}$ cannot belong to the same class, p is reduced to $p = p^1_1 p^2_1$.

Splitting the cyclic K-graph All the N-arcs between π^1 and π^2 belong to the Hamiltonian circuit $p = p^1_1 p^2_1$. As the path uses one arc from π^1 to π^2 there is only one N-arc from π^2 to π^1 say from x^2_K to x^1_0 and the Hamiltonian circuit p looks like:

$$p = \overbrace{x^1_0 \cdots x^1_I \ N \ \overbrace{a^1_0 \cdots a^1_L}^{X}}^{\pi^1} \ R \ \overbrace{\overbrace{a^2_0 \cdots a^2_J}^{Y} \ N \ x^2_0 \cdots x^2_K}^{\pi^2} \ N$$

with X and Y the components of $\otimes_0$ – we have indicated the color of the arc on p when it is determined by what we have proved up to now.

Observe that (figure 4(a)):

- There is no N-arc starting from a^1_L:
  - There is no internal N-arc from a^1_L: this vertex is maximal in its R connected component hence there is no N-arc from it to another vertex in the same R connected component.
  - There is no external N-arc from a^1_L: indeed, the Hamiltonian circuit must use such an N-arc while it is already using an R-arc starting from a^1_L namely $a^1_L R a^2_0$.
- There is no N-arc arriving to a^2_0:
  - There is no internal N-arc to a^2_0: this vertex is minimal in its R connected component hence there is no N-arc from it to another vertex in the same R connected component.
  - There is no external N-arc to a^2_0: indeed, the Hamiltonian circuit must use such an N-arc while it is already using an R-arc to a^2_0 namely $a^1_L R a^2_0$.
- Since x^2_K and x^1_0 do not belong to the same K-graph, by proposition 7, $x^2_K N x^1_0$ is the only incident arc to both x^1_0 and x^2_K.

Assume that we:

- suppress all the R-arcs from $\otimes_0$,
- suppress the N-arc from x^2_K to x^1_0
- add an N-arc from a^1_L to x^1_0
- add an N-arc from x^2_K to a^2_0.
- compute the R connected components inside $a^1_0, \dots, a^1_L$
  - delete all the N-arcs between different R connected components
  - add one N-arc from a^1_i to a^1_{i+1} if they are in different R connected components

- compute the R connected components inside $a^2_0, \ldots, a^2_J$
  - delete all the N-arcs between different R connected components
  - add one N-arc from a^2_i to a^2_{i+1} if they are in different R connected components

We get two totally disconnected subgraphs π^1 and π^2, with Hamiltonian paths $x^1_0 \cdots x^1_I a^1_0 \cdots a^1_L$ and $a^2_0 \cdots a^2_J x^2_0 \cdots x^2_K$. As BR-graphs they obviously satisfy acyclicity and connectedness. We thus get K-graphs (the R connected components), enriched with a cyclic order defined by the N-arcs between them, that are cyclic K-graphs.

So we just have to check whether the Hamiltonian adequacy is met. Assume that we have $H^1(x,y,x')$ and xBx' and yBy'—because $\otimes_0$ is splitting, $z \in \pi^1$ and zBz' entails $z' \in \pi^1$. Because H^1 is the restriction of H to π^1 we have $H(x,y,x')$ hence $H(x,y',x')$ and finally $H^1(x,y',x')$.

So we have two smaller proof-nets, and by induction hypothesis two sequent proofs corresponding to the K-graphs of both parts, and a $\otimes$ rule between the part of the sequents corresponding to $a^1_0, \ldots, a^1_L$ and $a^2_0, \ldots, a^2_J$ yields the result.

$\diamond$

6 A more economical presentation: L-graphs

Instead of using K-graphs (for *K*omplet) we can use L-graphs (for *L*eight) which only contains one N arc per $\wp$. A given formula, in a non-commutative setting has a first atom $\underline{\text{first}}(F)$ and a last atom $\underline{\text{last}}(F)$. Every formula F is mapped onto an L-graph $L(F)$ as follows:

$L(a) \quad a \in A:$

vertices: $\{a\}$

N**-arcs** $\emptyset$

R**-arcs** $\emptyset$

$L(X \otimes Y)$

vertices: $V(F) = V(X) \uplus V(Y)$

N**-arcs** $N(K(F)) = N(K(X)) \uplus N(K(Y))$

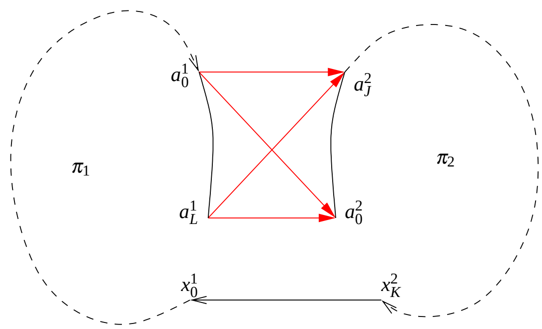
R**-arcs** $R(K(F)) = R(K(X)) \uplus R(K(Y)) \uplus V(X) \times V(Y)$

$L(X \wp Y)$

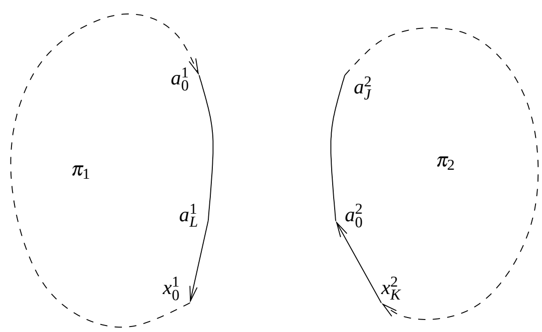
vertices: $V(F) = V(X) \uplus V(Y)$

R**-arcs** $R(K(F)) = R(K(X)) \uplus R(K(Y))$

N**-arcs** $N(K(F)) = N(K(X)) \uplus N(K(Y)) \uplus (\underline{\text{last}}(X), \underline{\text{first}}(Y))$

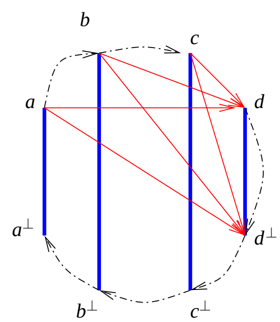


(a)

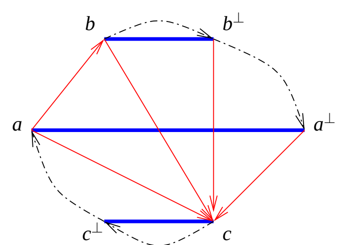


(b)

Figure 4: Pictures for sequentialisation



(a) L-graph corresponding to the sequent $\vdash (a \wp b \wp c) \otimes (d \wp d^\perp), c^\perp, b^\perp, a^\perp$



(b) L-graph corresponding to the sequent $\vdash ((a \otimes b) \wp b^\perp \wp a^\perp) \otimes c, c^\perp$

Figure 5: Examples of L-graphs

The L-graph of figure 5(a) corresponds to the same cyclic sequent as the K-graph of figure 1(d), but the inner disjunction is not yet depicted as a series-parallel composition but as a path abc. And the L-graph of figure 5(b) corresponds to the same cyclic sequent as the K-graph of figure 1(e), but the inner disjunction is not yet depicted as a series-parallel composition but as a path $bb^{\perp}a^{\perp}$.

Proposition 15 *An L-graph contains an Hamiltonian path which corresponds to the linear order of the atoms in the formula. This path contains all the N-arcs and one R-arc of each R series composition.*

Proposition 16 *A bicolored graph is an L-graph if and only if*

- *N-arcs are a set of disjoint paths*
- *R-arcs are a series-parallel order*
- *there exists an Hamiltonian path using all the N-arcs and exactly one R-arc of each R series composition.*
- *each connected component of the R series-parallel order is an interval of the linear order.*

PROOF : Let G be an L-graph. We consider the connected components of the R-subgraph.

If it includes all the vertices, then the main series composition $\otimes_0$ splits the vertices into two kinds, which are two segments of the linear order: $a_1,\ldots,a_n$ and $a_{n+1},\ldots,a_N$. Observe that all N-arcs are in one segment, or in the other or between a_n and a_{n+1}. Moreover, there is an R-arc (a_n,a_{n+1}) in $\otimes_0$, hence there is no N-arc (a_n,a_{n+1}) (NR is a simple graph), and this arc belongs to the linear order. G minus the R-arcs of $\otimes_0$ has two RN connected components. Indeed no arcs of $a_1,\ldots,a_n$ or $a_{n+1},\ldots,a_N$ belong to $\otimes_0$. The R-part of both is a directed cograph, the N-part is a set of disjoint paths, $N\cap R=\emptyset$ and the two linear orders are $a_1,\ldots,a_n$ and $a_{n+1},\ldots,a_N$. Letting A and B be formulae for these two smaller formula graphs, G is $G(A\otimes B)$.

Consequently there are several R-connected components, which are intervals of the linear order. In between two consecutive such intervals, the linear order contains an N-arc (a_n,a_{n+1}). If we suppress (a_n,a_{n+1}) we end up with two RN connected components, one with vertices $a1,\ldots,a_n$ and the other with vertices $a_{n+1},\ldots,a_N$. There is no R-arc between these two RN connected components. Indeed R-connected components are intervals of the linear order. Hence the R-subgraph on each part is a directed cograph, the N-part is a finite set of paths, there are no R and N arcs with the same endings. The restrictions of the linear order to both parts are linear orders, and the components of the series composition on each of them define an interval of the linear orders (as in the original graph and linear order). If the two formulae associated with the two formula graphs are A and B, then $G(A\wp B)=G$.

◇

Thus we can define proof-structures and nets, including the criterion, just as we did with K-graphs.

- Cyclic *L*-graphs are defined as a cyclically ordered set of *L*-graph, and they correspond to cyclic sequents.
- Proof-structures are defined as cyclic *L*-graphs together with a *B* perfect matching.
- The criterion is that the *BR*-graph should satisfy acyclicity and connectedness, and that the *B* edge must enjoy Hamiltonian adequacy wrt. the Hamiltonian circuit.

There are nevertheless two reasons to prefer the *K*-graphs:

- The *L*-graph of a formula is not a complete graph divided into two series-parallel orders, that is to say it has less combinatorial properties.
- In its characterization as graph it is unpleasant to have a condition for a graph to be an *L*-graph namely that components of conjunctions should be intervals.

7 Conclusion

In this paper we show how we can define non-commutative proof-nets within standard graph theory. Up to now we have no specific formulation of extra properties in this setting. For instance, it is easy to define Lambek calculus proof-nets as correct proof-nets which furthermore have an intuitionistic conclusion. Can the later property be formulated with standard graph theoretical notions?

Another extension would be to define this kind of proof-net for mixed calculi, such as introduced by de Groote (1996) in the intuitionistic case and extended to a classical setting by Ruet (1997) and Abrusci & Ruet (1999). The intuitionistic version is of special interest since it is possible to freely use the inclusion of series-parallel partial order as a proof rule (Bechet, de Groote & Retoré 1997).

Another direction would be to use this work for improving proof-search, e.g. by the probabilistic methods of Moot (2004).

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