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Stéphane Gaubert, Ricardo David Katz

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

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Abstract: We establish the following max-plus analogue of Minkowski's theorem. Any point of a compact max-plus convex subset of $(\mathbb{R} \cup \{-\infty\})^n$ can be written as the max-plus convex combination of at most $n + 1$ of the extreme points of this subset. We establish related results for closed max-plus convex cones and closed unbounded max-plus convex sets. In particular, we show that a closed max-plus convex set can be decomposed as a max-plus sum of its recession cone and of the max-plus convex hull of its extreme points.

Key-words: Max-plus algebra, tropical algebra, extreme points, polyhedra, polytopes, convex cones, convex sets, abstract convexity, Krein-Milman theorem, Carathéodory theorem

Théorème de Minkowski pour les ensembles convexes max-plus

Résumé : Nous établissons l'analogue max-plus suivant du théorème de Minkowski. Tout point d'un sous-ensemble convexe compact de $(\mathbb{R} \cup \{-\infty\})^n$ peut s'écrire comme barycentre d'au plus $n+1$ points extrêmes de ce sous-ensemble. Nous établissons des résultats similaires pour les cônes convexes max-plus fermés et pour les convexes max-plus fermés non bornés. Nous montrons en particulier qu'un sous-ensemble convexe max-plus fermé de $(\mathbb{R} \cup \{-\infty\})^n$ peut se décomposer comme somme de son cône de récession et de l'enveloppe convexe de ses points extrêmes.

1. INTRODUCTION

The *max-plus segment* joining two points $u, v \in (\mathbb{R} \cup \{-\infty\})^n$ is the set of vectors of the form $(\alpha + u) \vee (\beta + v)$ where α and β are elements of $\mathbb{R} \cup \{-\infty\}$ such that $\alpha \vee \beta = 0$. Here, $\vee$ denotes the maximum of scalars, or the pointwise maximum of vectors, and for all scalars $\alpha \in \mathbb{R} \cup \{-\infty\}$ and vectors $u \in (\mathbb{R} \cup \{-\infty\})^n$, $\alpha + u$ denotes the vector with entries $\alpha + u_i$.

A subset of $(\mathbb{R} \cup \{-\infty\})^n$ is *max-plus convex* if it contains any max-plus segment joining two of its points. The *max-plus convex cone* generated by u, v is the set of vectors of the form $(\alpha + u) \vee (\beta + v)$, where α and β are arbitrary elements of $\mathbb{R} \cup \{-\infty\}$. A subset of $(\mathbb{R} \cup \{-\infty\})^n$ is a *max-plus convex cone* if it contains any max-plus convex cone generated by two of its points. These definitions are natural if one considers the max-plus semiring, which is the set $\mathbb{R} \cup \{-\infty\}$ equipped with the addition $(a, b) \mapsto a \vee b$ and the multiplication $(a, b) \mapsto a + b$. Max-plus convex cones are also called *semimodules* over the max-plus semiring. An example of max-plus convex set is given in Figure 1: the convex set A is the closed grey region, together with the portion of vertical line joining the point b to it. Three max-plus segments in general position, joining the pairs of points (f, g), (h, i), and (j, k), are represented in bold. By comparing the shapes of these segments with the shape of A, one can check geometrically that A is convex.

In this paper we give representation theorems, in terms of extreme points and extreme rays, for max-plus convex sets and cones.

Motivations to study the max-plus analogues of convex cones and convex sets arise from several fields, let us review some of these motivations.

Max-plus convex sets were introduced by K. Zimmermann [Zim77]. Convexity is a powerful tool in optimization, and so, max-plus convex sets arose in the quest of solvable optimization problems [Zim84, Zim03]. See also the book of U. Zimmermann [Zim81] for an overview.

Max-plus convex cones have been studied in idempotent analysis, after the observation due to Maslov that the solutions of an Hamilton-Jacobi equation associated with a deterministic optimal control problem satisfy a "max-plus" superposition principle, and so, belong to structures similar to convex cones, which are called semimodules or idempotent linear spaces [LMS01, CGQ04]. Such structures have been used, for instance, to characterize the sets of stationary solutions of deterministic optimal control problems [AGW05], and to design numerical algorithms [FM00, AGL06], to mention a recent application.

Max-plus convex cones have also been studied in relation to discrete event systems. The reader may consult the survey papers [GP97, CGQ99] for more background. In particular, reachable and observable spaces of certain timed discrete event systems are naturally equipped with structures of max-plus polyhedral cones [Kat05]. Earlier discrete event systems motivations have been at the origin of the works [CGQ96, CGQ97, Gau98], in which the theory of max-plus polyhedral cones has been developed.

Of course, another interest in max-plus convexity stems from abstract convex analysis [Sin97]. Several recent papers in this field, in particular those of Martínez-Legaz, Rubinov, and Singer [MLRS02], and Akian and Singer [AS03], are related to max-plus algebra.

A renewed interest in max-plus convex cones, or "tropical convex sets", and specially, in tropical polyhedra, has recently arisen in relation to tropical geometry (in this context, "tropical" is essentially used as a synonym of "max-plus", or rather, of the dual term, "min-plus"). Tropical analogues of polytopes have been considered by Develin and Sturmfels [DS04], and also by Joswig [Jos05] (the tropical polytopes they consider are special finitely generated max-plus convex cones, in which the generators have finite entries). Develin and Sturmfels have also pointed out an elegant relation between tropical polytopes and phylogenetic analysis [DS04].

Some of these motivations have guided the development of max-plus analogues of classical results of convex analysis, like the Hahn-Banach theorem [Zim77, SS92, CGQ04, CGQS05].

We are interested here in the representation of convex sets in terms of extreme points or extreme rays. This problem, in the case of finitely generated max-plus convex cones, has been considered by several authors [Mol88, Wag91, Gau98, DS04, CGB04]. The general case has been less studied, with the exception of the paper [Hel88], in which Helbig established a max-plus analogue of Krein-Milman's theorem, showing that a non-empty compact convex subset of $(\mathbb{R} \cup \{-\infty\})^n$ is the closure of the convex hull of its set of extreme points, in the max-plus sense.

For conventional convex sets of finite dimension, however, a more precise result is true: the closure operator can be dispensed with, since a classical theorem of Minkowski shows that a non-empty compact convex subset of a finite dimensional space is the convex hull of its set of extreme points. One may ask whether the same is true for max-plus convex sets. We show that the answer is positive, and establish a max-plus analogue of Minkowski's theorem.

Note that the classical proof of Minkowski's theorem cannot be transposed to the max-plus case. The classical approach exploits the facial structure of convex sets. Recall that a *face* of a convex set is by definition the intersection of the convex set with a supporting hyperplane. For a conventional convex set, one can show that the extreme points of the faces are extreme points of the set, and use this observation to prove Minkowski's theorem, by induction on the dimension of the convex set. This does not work in the max-plus case, because an extreme point of a face may not be an extreme point of the set, as shown in Example 3.8 below. Hence, it does not seem possible to use Helbig's approach to derive the results of the present paper.

In fact, we give a direct proof of a Minkowski type theorem for max-plus convex cones (Theorem 3.1), from which we deduce the max-plus Minkowski theorem (Theorem 3.2), and its generalization to the case of unbounded convex sets (Theorem 3.3). Finally, we deduce as a special case a slightly more precise version of the "basis theorem" of Moller [Mol88] and Wagneur [Wag91] for finitely generated max-plus convex cones, Corollary 3.4.

Finally, we note that the main results of the present paper, Theorems 3.1–3.3, have been announced (without proof) in the survey paper [GK06].
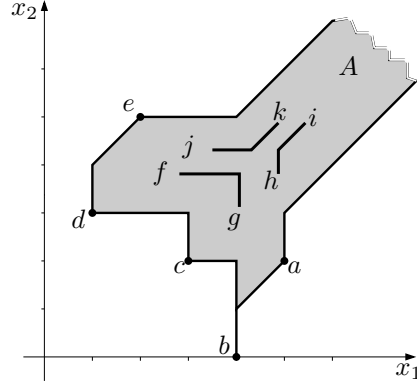
FIGURE 1. An unbounded max-plus convex set and three segments in general position contained in it.

2. PRELIMINARIES

In this section, we give basic definitions and establish some elementary lemmas. To bring to light the analogy with classical convex analysis, we shall use the following notation. We denote by $\mathbb{R}_{\max}$ the max-plus semiring. We denote by $a \oplus b := a \vee b$ the max-plus addition, and by $ab := a + b$ the max-plus multiplication. We set $\mathbb{0} := -\infty$, $\mathbb{1} := 0$. The set of vectors of size n over $\mathbb{R}_{\max}$ is denoted by $\mathbb{R}_{\max}^n$. A vector consisting only of $\mathbb{0}$ entries is denoted by $\mathbb{0}$. By scalar, we mean an element of $\mathbb{R}_{\max}$. If $u, v \in \mathbb{R}_{\max}^n$ and $\lambda \in \mathbb{R}_{\max}$, we set $u \oplus v := u \vee v$, and we denote by λu the vector with entries $\lambda + u_i$. Max-plus convex sets and cones have been defined in the introduction. In the sequel, for brevity, the term "convex" used without precisions shall always be understood in the max-plus sense. By "cone", we shall always mean a (max-plus) convex cone.

Definition 2.1. *Let A be a subset of $\mathbb{R}_{\max}^n$.*

The convex hull *of A, denoted by $\operatorname{co}(A)$, is the set of all (finite) convex combinations of elements of A. These can be written as $\oplus_{k \in K} \alpha_k u^k$, where K is a finite set, $\{u^k\}_{k \in K}$ is a family of elements of A and $\{\alpha_k\}_{k \in K}$ are scalars that satisfy $\oplus_{k \in K} \alpha_k = \mathbb{1}$.*

The cone *generated by A, denoted by $\operatorname{cone}(A)$, is the set of all (finite) linear combinations of elements of A. These can be written as $\oplus_{k \in K} \alpha_k u^k$, where K is a finite set, $\{u^k\}_{k \in K}$ is a family of elements of A and $\{\alpha_k\}_{k \in K}$ are scalars.*

The recession cone *of A at a point $v \in A$ is defined by:*

$$\operatorname{rec}_v(A) := \left\{ u \in \mathbb{R}_{\max}^n \mid v \oplus \lambda u \in A \text{ for all } \lambda \in \mathbb{R}_{\max} \right\} .$$

Example 2.2. The recession cone of the convex set A of Figure 1 at any point $v \in A$ is $\operatorname{rec}_v(A) = \operatorname{cone}(\{(0, 1), (2, 0)\})$. It is shown on the right hand side of Figure 2, below.

We equip $\mathbb{R}_{\max}$ with the usual topology, which can defined by the metric: $(x,y) \rightarrow |e^x - e^y|$. The set $\mathbb{R}_{\max}^n$ is equipped with the product topology. We denote by $\operatorname{clo}(A)$ the closure of a subset A of $\mathbb{R}_{\max}^n$.

The following lemmas give some properties of recession cones.

Lemma 2.3. *Let A be a closed subset of $\mathbb{R}_{\max}^n$. Then the recession cone of A at v is closed for all $v \in A$.*

Proof. Let $v \in A$. For all $\lambda \in \mathbb{R}_{\max}$, define the map $\varphi_\lambda : \mathbb{R}_{\max}^n \to \mathbb{R}_{\max}^n$ by $\varphi_\lambda(u) = v \oplus \lambda u$. Since φ_λ is continuous, $\operatorname{rec}_v(A) = \cap_{\lambda \in \mathbb{R}_{\max}} \varphi_\lambda^{-1}(A)$ is an intersection of closed sets, and so, it is closed. $\square$

Lemma 2.4. *Let A be a convex subset of $\mathbb{R}_{\max}^n$ and $v,w \in A$. If $\beta v \leq w$ for some $\beta \neq \mathbb{0}$, then $\operatorname{rec}_v(A) \subset \operatorname{rec}_w(A)$.*

Proof. Let $u \in \operatorname{rec}_v(A)$. We assume, without loss of generality, that $\beta \leq \mathbb{1}$. Then, for all $\lambda \in \mathbb{R}_{\max}$,
$$w \oplus \lambda u = w \oplus \beta v \oplus \beta \beta^{-1} \lambda u = w \oplus \beta(v \oplus \beta^{-1}\lambda u) \in A$$
since $v \oplus \beta^{-1}\lambda u \in A$ and since A is convex. It follows that $u \in \operatorname{rec}_w(A)$. $\square$

Proposition-Definition 2.5. *Let A be a closed convex subset of $\mathbb{R}_{\max}^n$. Then the recession cone of A at v is independent of $v \in A$. We denote it by $\operatorname{rec}(A)$.*

Proof. Given $x \in \mathbb{R}_{\max}^n$, we define the *support* of x to be the set
$$\operatorname{supp} x := \{1 \leq i \leq n \mid x_i \neq \mathbb{0}\} \ .$$

Observe first that if $u,v \in A$, then $u \oplus v \in A$ and the support of $u \oplus v$ is the union of the supports of u and v. It follows that there is an element $w \in A$ with maximum support, meaning that $\operatorname{supp} v \subset \operatorname{supp} w$ for all $v \in A$. Hence for every $v \in A$ there exists $\lambda_v \neq \mathbb{0}$ such that $\lambda_v v \leq w$. Therefore, by Lemma 2.4, it suffices to show that $\operatorname{rec}_w(A) \subset \operatorname{rec}_v(A)$ for every $v \in A$.

Let $\{\beta_r\}_{r \in \mathbb{N}} \subset \mathbb{R}_{\max}$ be a sequence such that $\lim_{r \to \infty} \beta_r = \mathbb{0}$ and $\mathbb{0} < \beta_r \leq \mathbb{1}$ for all $r \in \mathbb{N}$. If $u \in \operatorname{rec}_w(A)$ and $\lambda \in \mathbb{R}_{\max}$, then
$$v \oplus \lambda u = \lim_{r \to \infty}(v \oplus \beta_r(w \oplus \beta_r^{-1}\lambda u))$$
is a limit of elements of A because $w \oplus \beta_r^{-1}\lambda u \in A$ for all $r \in \mathbb{N}$. Since A is closed, it follows that $v \oplus \lambda u \in A$. Therefore, $u \in \operatorname{rec}_v(A)$, and so, $\operatorname{rec}_w(A) \subset \operatorname{rec}_v(A)$ for all $v \in A$. $\square$

Remark 2.6. The closure assumption in the previous proposition cannot be dispensed with. Consider $A = ([\mathbb{0},\mathbb{1}] \times \{\mathbb{0}\}) \cup \mathbb{R}^2 = \{(x_1, \mathbb{0}) \mid x_1 \leq \mathbb{1}\} \cup \mathbb{R}^2 \subset \mathbb{R}_{\max}^2$. Then,
$$\operatorname{rec}_{(\mathbb{1},\mathbb{0})}(A) = \{(u_1,u_2) \in \mathbb{R}_{\max}^2 \mid u_2 \neq \mathbb{0}\} \cup \{(\mathbb{0},\mathbb{0})\} \text{ and } \operatorname{rec}_{(\mathbb{1},\mathbb{1})}(A) = \mathbb{R}_{\max}^2 \ .$$

The following max-plus analogue of the notion of extreme point was already used by Helbig [Hel88].

Definition 2.7 (Extreme point). *Let A be a convex subset of $\mathbb{R}^n_{\max}$. An element $x \in A$ is an* extreme point *of A if for all $y, z \in A$ and $\alpha, \beta \in \mathbb{R}_{\max}$ such that $\alpha \oplus \beta = \mathbb{1}$, the following property is satisfied*

(1) $$x = \alpha y \oplus \beta z \implies x = y \text{ or } x = z.$$

The set of extreme points of A will be denoted by $\mathrm{ext}\,(A)$.

Thus, a point of A is extreme if it cannot belong to a segment of A unless it is an end of this segment. We warn the reader that due to the idempotency of addition, the property (1), with $\alpha \oplus \beta = \mathbb{1}$ and $\alpha, \beta \neq \mathbb{0}$ is not equivalent to

$$x = \alpha y \oplus \beta z \implies x = y \text{ and } x = z.$$

Remark 2.8. If $x \in A$ is an extreme point of A, then $x = \alpha y \oplus \beta z$, with $\alpha \oplus \beta = \mathbb{1}$ and $y, z \in A$, implies:

$$(x = y, \alpha = \mathbb{1}) \text{ or } (x = z, \beta = \mathbb{1}).$$

Indeed, assume that $x = y$ but $\alpha < \mathbb{1}$. Then, $\beta = \mathbb{1}$. Assume by contradiction that $x \neq z$. Then, we have $x_i > z_i$, for some $1 \leq i \leq n$, and so, $x_i = \alpha y_i \oplus z_i = \alpha x_i \oplus z_i < x_i$, which is nonsense.

When $C \subset \mathbb{R}^n_{\max}$ is a cone, it is clear that its only extreme point is $\mathbb{0}$. In this case, the relevant notion is that of extreme generator.

Definition 2.9 (Extreme generator). *Let $C \subset \mathbb{R}^n_{\max}$ be a cone. An element $x \in C$ is an* extreme generator *of C if the following property is satisfied*

$$x = y \oplus z, \ y, z \in C \implies x = y \text{ or } x = z.$$

If x is an extreme generator of C, then the set $\mathbb{R}_{\max} x = \{\lambda x \mid \lambda \in \mathbb{R}_{\max}\}$ is an extreme ray *of C. The set of extreme generators of C will be denoted by* $\mathrm{ext\text{-}g}\,(C)$.

Extreme generators are called *join irreducible* elements in lattice theory.

Remark 2.10. It can be readily checked that every element of an extreme ray of C is an extreme generator of C.

Example 2.11. Let us consider the closed convex set $A \subset \mathbb{R}^2_{\max}$ shown in Figure 1. It can be easily seen that its extreme points are $a = (5, 2)$, $b = (4, 0)$, $c = (3, 2)$, $d = (1, 3)$ and $e = (2, 5)$. The extreme rays of $\mathrm{rec}\,(A)$ are $\mathbb{R}_{\max}(0, 1)$ and $\mathbb{R}_{\max}(2, 0)$ (see the right hand side of Figure 2).

The following construction will allow us to derive results for convex sets as consequences of results for cones.

Lemma 2.12. *Let A be a convex subset of $\mathbb{R}^n_{\max}$. Then, the set*

$$C_A = \{(\lambda x, \lambda) \mid x \in A, \lambda \in \mathbb{R}_{\max}\} \subset \mathbb{R}^{n+1}_{\max}$$

is a cone.

Proof. Let $\beta \in \mathbb{R}_{\max}$ and $(\lambda_1 x^1, \lambda_1), (\lambda_2 x^2, \lambda_2) \in C_A$, with $x^1, x^2 \in A$ and $\lambda_1, \lambda_2 \in \mathbb{R}_{\max}$. Assume, without loss of generality, that $\lambda := \lambda_1 \oplus \lambda_2 \neq \mathbb{0}$. Since A is convex, $\lambda^{-1}(\lambda_1 x^1 \oplus \lambda_2 x^2) \in A$, and so,

$$(\lambda_1 x^1, \lambda_1) \oplus (\lambda_2 x^2, \lambda_2) = (\lambda \lambda^{-1}(\lambda_1 x^1 \oplus \lambda_2 x^2), \lambda) \in C_A.$$

Moreover, C_A is obviously preserved by the multiplication by a scalar. $\qquad \square$

Lemma 2.13. *If $A \subset \mathbb{R}_{\max}^n$ is a convex set then $\mathrm{clo}\,(A)$ is a convex set. The same is true for cones.*

Proof. This follows from the continuity of the functions $(x, y) \to x \oplus y$ and $(\lambda, x) \to \lambda x$. $\quad \square$

We next establish some properties of the cone C_A.

Proposition 2.14. *If $A \subset \mathbb{R}_{\max}^n$ is a closed convex set, then*

$$\mathrm{clo}\,(C_A) = C_A \cup (\mathrm{rec}\,(A) \times \{\mathbb{0}\}).$$

Proof. Let $(y, \alpha) \in \mathrm{clo}\,(C_A)$.

Assume first that $\alpha \neq \mathbb{0}$. Since $(y, \alpha) \in \mathrm{clo}\,(C_A)$, there exists a sequence $\{(\lambda_r x^r, \lambda_r)\}_{r \in \mathbb{N}} \subset C_A$ such that $\lim_{r \to \infty}(\lambda_r x^r, \lambda_r) = (y, \alpha)$. Then, as $\lim_{r \to \infty} \lambda_r = \alpha \neq \mathbb{0}$ and A is closed, we know that $x := \lim_{r \to \infty} x^r = \lim_{r \to \infty} \lambda_r^{-1} \lambda_r x^r = \alpha^{-1} y$ belongs to A. Therefore, $(y, \alpha) = \lim_{r \to \infty}(\lambda_r x^r, \lambda_r) = (\alpha x, \alpha) \in C_A$.

Assume now that $\alpha = \mathbb{0}$. Let $x \in A$ and $\beta \in \mathbb{R}_{\max}$. To prove that $(y, \alpha) \in (\mathrm{rec}\,(A) \times \{\mathbb{0}\})$ it suffices to show that $x \oplus \beta y \in A$. As $x \in A$ we know that $(x, \mathbb{1}) \in C_A$. Using the fact that $\mathrm{clo}\,(C_A)$ is a cone (by Lemmas 2.12 and 2.13), it follows that $(x \oplus \beta y, \mathbb{1}) = (x, \mathbb{1}) \oplus \beta(y, \mathbb{0}) \in \mathrm{clo}\,(C_A)$. Then, there exists a sequence $\{(\lambda_r x^r, \lambda_r)\}_{r \in \mathbb{N}} \subset C_A$ such that $\lim_{r \to \infty}(\lambda_r x^r, \lambda_r) = (x \oplus \beta y, \mathbb{1})$. Therefore,

$$x \oplus \beta y = \lim_{r \to \infty} \lambda_r x^r = (\lim_{r \to \infty} \lambda_r)(\lim_{r \to \infty} x^r) = \lim_{r \to \infty} x^r \in \mathrm{clo}\,(A) = A.$$

Thus, $\mathrm{clo}\,(C_A) \subset C_A \cup (\mathrm{rec}\,(A) \times \{\mathbb{0}\})$.

Obviously $C_A \subset \mathrm{clo}\,(C_A)$. Let now $(y, \mathbb{0}) \in \mathrm{rec}\,(A) \times \{\mathbb{0}\}$. Take any $x \in A$. We know that $x \oplus \lambda y \in A$ for all $\lambda \in \mathbb{R}_{\max}$. Then, if $\{\lambda_r\}_{r \in \mathbb{N}} \subset \mathbb{R}_{\max}$ is a sequence such that $\lim_{r \to \infty} \lambda_r^{-1} = \mathbb{0}$, it follows that $(y, \mathbb{0}) = \lim_{r \to \infty}(\lambda_r^{-1}(x \oplus \lambda_r y), \lambda_r^{-1})$ and therefore $(y, \mathbb{0}) \in \mathrm{clo}\,(C_A)$ since $(\lambda_r^{-1}(x \oplus \lambda_r y), \lambda_r^{-1}) \in C_A$ for all $r \in \mathbb{N}$.

Thus, $C_A \cup (\mathrm{rec}\,(A) \times \{\mathbb{0}\}) \subset \mathrm{clo}\,(C_A)$. $\qquad \square$

Corollary 2.15. *If $A \subset \mathbb{R}_{\max}^n$ is a compact convex set, then $C_A \subset \mathbb{R}_{\max}^{n+1}$ is a closed cone.*

Proof. If A is a compact subset of $\mathbb{R}_{\max}^n$, it must be bounded from above, and so $\mathrm{rec}\,(A) = \{\mathbb{0}\}$. By Proposition 2.14, $\mathrm{clo}\,(C_A) = C_A \cup \{\mathbb{0}\} = C_A$, and so, C_A is closed. $\qquad \square$

Lemma 2.16. *Let A be a closed convex subset of $\mathbb{R}_{\max}^n$. Then,*

$$\mathrm{ext\text{-}g}\,(\mathrm{clo}\,(C_A)) \cap (\mathrm{rec}\,(A) \times \{\mathbb{0}\}) = \mathrm{ext\text{-}g}\,(\mathrm{rec}\,(A)) \times \{\mathbb{0}\}\,.$$

Proof. Let $(x, \mathbb{0}) \in \text{ext-g}\,(\text{clo}\,(C_A)) \cap (\text{rec}\,(A) \times \{\mathbb{0}\})$. Then, in particular, $x \in \text{rec}\,(A)$. Assume that $x = y \oplus z$, with $y, z \in \text{rec}\,(A)$. As $\text{clo}\,(C_A) = C_A \cup (\text{rec}\,(A) \times \{\mathbb{0}\})$ by Proposition 2.14, we know that $(y, \mathbb{0}), (z, \mathbb{0}) \in \text{clo}\,(C_A)$. Then $x = y$ or $x = z$, since $(x, \mathbb{0}) = (y, \mathbb{0}) \oplus (z, \mathbb{0})$ and $(x, \mathbb{0}) \in \text{ext-g}\,(\text{clo}\,(C_A))$. Therefore, $x \in \text{ext-g}\,(\text{rec}\,(A))$ and $(x, \mathbb{0}) \in \text{ext-g}\,(\text{rec}\,(A)) \times \{\mathbb{0}\}$.

Let now $(x, \mathbb{0}) \in \text{ext-g}\,(\text{rec}\,(A)) \times \{\mathbb{0}\}$. Then, obviously $(x, \mathbb{0}) \in \text{rec}\,(A) \times \{\mathbb{0}\}$. Assume that $(x, \mathbb{0}) = (x^1, \lambda_1) \oplus (x^2, \lambda_2)$, with $(x^1, \lambda_1), (x^2, \lambda_2) \in \text{clo}\,(C_A)$. Then, $x = x^1 \oplus x^2$ and $\lambda_1 = \lambda_2 = \mathbb{0}$. Therefore, as $\text{clo}\,(C_A) = C_A \cup (\text{rec}\,(A) \times \{\mathbb{0}\})$ by Proposition 2.14, it follows that $x^1, x^2 \in \text{rec}\,(A)$. Finally, $x = x^1$ or $x = x^2$ since $x \in \text{ext-g}\,(\text{rec}\,(A))$. Thus, $(x, \mathbb{0}) \in \text{ext-g}\,(\text{clo}\,(C_A))$. $\qquad\square$

Lemma 2.17. *Let A be a convex subset of $\mathbb{R}_{\max}^n$. Then,*

$$\text{ext-g}\,(\text{clo}\,(C_A)) \cap C_A \subset \text{ext-g}\,(C_A).$$

Proof. Obvious since $C_A \subset \text{clo}\,(C_A)$. $\qquad\square$

The following proposition relates extreme points and extreme rays.

Proposition 2.18. *Let $C \subset \mathbb{R}_{\max}^n$ be a cone, let $\gamma \neq \mathbb{0}$ and let $\psi : \mathbb{R}_{\max}^n \to \mathbb{R}_{\max}$ be a max-plus linear form, meaning that $\psi(x) = \oplus_{i=1}^n a_i x_i$ for some $a \in \mathbb{R}_{\max}^n$. Assume that $\psi(x) \neq \mathbb{0}$ for all $x \in C \setminus \{\mathbb{0}\}$, and define the convex set:*

$$\Sigma := \{x \in C \mid \psi(x) = \gamma\}\ .$$

Then,

$$\text{ext}\,(\Sigma) = \text{ext-g}\,(C) \cap \Sigma\ .$$

Proof. Let $x, y \in \Sigma$ and $\alpha, \beta \in \mathbb{R}_{\max}$ be such that $\alpha \oplus \beta = \mathbb{1}$. Then, as

$$\psi(\alpha x \oplus \beta y) = \alpha \psi(x) \oplus \beta \psi(y) = \gamma \alpha \oplus \gamma \beta = \gamma(\alpha \oplus \beta) = \gamma$$

and obviously $\alpha x \oplus \beta y \in C$, it follows that $\alpha x \oplus \beta y \in \Sigma$. Therefore, Σ is convex.

Let $x \in \text{ext}\,(\Sigma)$. Assume that $x = y \oplus z$, for some $y, z \in C \setminus \{\mathbb{0}\}$. Then, $\psi(y) = \gamma$ or $\psi(z) = \gamma$ since $\gamma = \psi(x) = \psi(y) \oplus \psi(z)$. Suppose, without loss of generality, that $\psi(y) = \gamma$. As $x = y \oplus \psi(z)\gamma^{-1}\gamma\psi(z)^{-1}z$, where clearly $\gamma\psi(z)^{-1}z \in \Sigma$ and $\psi(z)\gamma^{-1} \leq \mathbb{1}$, we know that

$$x = y \text{ or } x = \gamma\psi(z)^{-1}z.$$

Since $x \neq y$ implies $\psi(z)\gamma^{-1} = \mathbb{1}$ (see Remark 2.8), it follows that $x = y$ or $x = z$. Then, $x \in \text{ext-g}\,(C) \cap \Sigma$.

Let now $x \in \text{ext-g}\,(C) \cap \Sigma$. Suppose that $x = \alpha y \oplus \beta z$, with $y, z \in \Sigma$ and $\alpha \oplus \beta = \mathbb{1}$. Since $x \in \text{ext-g}\,(C)$, we know that $x = \alpha y$ or $x = \beta z$. Assume, without loss of generality, that $x = \alpha y$. Then, $\gamma = \psi(x) = \psi(\alpha y) = \alpha\gamma$ implies that $\alpha = \mathbb{1}$, and so $x = y$. Therefore, $x \in \text{ext}\,(\Sigma)$. $\qquad\square$

Note that the condition of the previous proposition is satisfied, in particular, when $\psi(x) = \oplus_{i=1}^n a_i x_i$ for some $a \in \mathbb{R}^n$.

Corollary 2.19. *Let A be a convex subset of $\mathbb{R}_{\max}^n$. Then,*

$$\text{ext-g}\,(C_A) \cap (A \times \{\mathbb{1}\}) = \text{ext}\,(A) \times \{\mathbb{1}\}\,.$$

Proof. Consider the max-plus linear form ψ on $\mathbb{R}_{\max}^{n+1}$ defined by $\psi(z, \lambda) = \lambda$, for all $z \in \mathbb{R}_{\max}^n$ and $\lambda \in \mathbb{R}_{\max}$, take $\gamma := \mathbb{1}$, and apply Proposition 2.18 to the cone $C_A \subset \mathbb{R}_{\max}^{n+1}$. We deduce that $\text{ext-g}\,(C_A) \cap \Sigma = \text{ext}\,(\Sigma)$, where $\Sigma := \{(x\lambda, \lambda) \in C_A \mid \lambda = \mathbb{1}\} = A \times \{\mathbb{1}\}$. Since $\text{ext}\,(A \times \{\mathbb{1}\}) = \text{ext}\,(A) \times \{\mathbb{1}\}$, the corollary is proved. $\qquad\square$

Let us recall that a cone $C \subset \mathbb{R}_{\max}^n$ is *finitely generated* if there exists a finite subset $A \subset \mathbb{R}_{\max}^n$ such that $C = \text{cone}\,(A)$.

Lemma 2.20. *A finitely generated cone of $\mathbb{R}_{\max}^n$ is closed.*

Proof. Let $A = \{u^1, \ldots, u^m\}$ and $C = \text{cone}\,(A)$. We assume, without loss of generality, that $u^k \neq \mathbb{0}$ for all $1 \leq k \leq m$. Let $\psi(x) := \bigoplus_{1 \leq i \leq n} a_i x_i$ denote a linear form, such that $a_i > \mathbb{0}$ for all $1 \leq i \leq n$. Then, $\psi(u^k) \neq \mathbb{0}$, for all $1 \leq k \leq m$. Let $\left\{x^r = \oplus_{k=1}^m \lambda_k^r u^k\right\}_{r \in \mathbb{N}}$ be a sequence of elements of C such that $\lim_{r \to \infty} x^r = x$ for some $x \in \mathbb{R}_{\max}^n$.

Since $\lambda_k^r \psi(u^k) \leq \psi(x^r)$, and since $\psi(u^k) \neq \mathbb{0}$, λ_k^r is bounded as r tends to infinity. Hence, we can assume, without loss of generality, that there exists $\lambda_k \in \mathbb{R}_{\max}$ such that $\lim_{r \to \infty} \lambda_k^r = \lambda_k$ for all $k = 1, \ldots, m$ (taking subsequences if necessary). Then,

$$\lim_{r \to \infty} x^r = \lim_{r \to \infty} \left(\bigoplus_{k=1}^m \lambda_k^r u^k \right) = \bigoplus_{k=1}^m \lambda_k u^k \in C.$$

Therefore, C is closed. $\qquad\square$

3. Representation of max-plus convex sets in terms of extreme points and extreme generators

Now we prove the main results of this paper.

Theorem 3.1. *Let $C \subset \mathbb{R}_{\max}^n$ be a non-empty closed cone. Then, every element of C is the sum of at most n extreme generators of C, and so,*

$$C = \text{cone}\,(\text{ext-g}\,(C)).$$

Proof. Let $x \in C$. For each $i \in \{1, \ldots, n\}$ define the set

$$S_i = \{u \in C \mid u \leq x, u_i = x_i\} = C \cap \{u \in \mathbb{R}_{\max}^n \mid u \leq x, u_i = x_i\}\,.$$

As $\{u \in \mathbb{R}_{\max}^n \mid u \leq x, u_i = x_i\}$ is compact and C is closed, we know that S_i is a compact subset of $\mathbb{R}_{\max}^n$ which is non-empty because $x \in S_i$. Therefore, S_i has a minimal element u^i.

We claim that u^i is an extreme generator of C. Assume that $u^i = y \oplus z$ for some $y, z \in C$. Then, $u_i^i = y_i$ or $u_i^i = z_i$. Let us assume, without loss of generality, that $u_i^i = y_i$. Therefore, $y \in S_i$ since $y \leq u^i \leq x$ and $y_i = u_i^i = x_i$. Hence, $u^i = y$ since $y \leq u^i$ and u^i is a minimal element of S_i. Thus, u^i is an extreme generator of C. It is clear that $x = \oplus_{i=1}^n u^i$, and so, $x \in \text{cone}\,(\text{ext-g}\,(C))$.

We have shown that $C \subset \text{cone}\,(\text{ext-g}\,(C))$. The other inclusion is trivial. $\qquad\square$

Theorem 3.2 (Max-Plus Minkowski Theorem). *Let A be a non-empty compact convex subset of $\mathbb{R}_{\max}^n$. Then, every element of A is the convex combination of at most $n + 1$ extreme points of A, and so,*

$$A = \mathrm{co}\,(\mathrm{ext}\,(A)).$$

Proof. Let $x \in A$. Define the cone $C_A = \{(\lambda z, \lambda) \mid z \in A, \lambda \in \mathbb{R}_{\max}\} \subset \mathbb{R}_{\max}^{n+1}$ as in Lemma 2.12. Then, by Corollary 2.15, C_A is a closed cone and thus

$$C_A = \mathrm{cone}\,(\mathrm{ext\text{-}g}\,(C_A))$$

by Theorem 3.1.

As $(x, \mathbb{1}) \in C_A$, by Theorem 3.1 we know that there exist $n + 1$ extreme generators of C_A, namely $(\lambda_1 u^1, \lambda_1), \ldots, (\lambda_{n+1} u^{n+1}, \lambda_{n+1})$, such that

$$(x, \mathbb{1}) = \bigoplus_{k=1}^{n+1} (\lambda_k u^k, \lambda_k).$$

Hence,

$$x = \bigoplus_{k=1}^{n+1} \lambda_k u^k, \text{where } \bigoplus_{k=1}^{n+1} \lambda_k = \mathbb{1}.$$

By Corollary 2.19, we know that $(u^k, \mathbb{1}) \in C_A$ is an extreme generator of C_A if, and only if, $u^k \in A$ is an extreme point of A. This shows that x is the convex combination of at most $n + 1$ extreme points of A. It follows that $A \subset \mathrm{co}\,(\mathrm{ext}\,(A))$. The other inclusion is trivial. $\qquad\square$

Theorem 3.3. *Let $A \subset \mathbb{R}_{\max}^n$ be a non-empty closed convex set. Then, every element of A is the sum of the convex combination of p extreme points of A, and of q extreme generators of $\mathrm{rec}\,(A)$, with $p + q \leq n + 1$, and so:*

$$A = \mathrm{co}\,(\mathrm{ext}\,(A)) \oplus \mathrm{rec}\,(A).$$

Here, we denote by $\oplus$ the max-plus analogue of the Minkowski sum of two subsets, which is defined as the set of max-plus sums of a vector from the first set and of a vector from the second one.

Proof. Let $x \in A$. Define the cone $C_A = \{(\lambda z, \lambda) \mid z \in A, \lambda \in \mathbb{R}_{\max}\} \subset \mathbb{R}_{\max}^{n+1}$ as in Lemma 2.12. Then, by Lemma 2.13, $\mathrm{clo}\,(C_A)$ is a closed cone and thus

$$\mathrm{clo}\,(C_A) = \mathrm{cone}\,(\mathrm{ext\text{-}g}\,(\mathrm{clo}\,(C_A)))$$

by Theorem 3.1.

By Proposition 2.14 we know that

$$\mathrm{clo}\,(C_A) = C_A \cup (\mathrm{rec}\,(A) \times \{\mathbb{0}\}),$$

and then Lemmas 2.16 and 2.17 imply:

$$
\begin{aligned}
\mathrm{ext\text{-}g}\,(\mathrm{clo}\,(C_A)) &= [\mathrm{ext\text{-}g}\,(\mathrm{clo}\,(C_A)) \cap C_A] \cup [\mathrm{ext\text{-}g}\,(\mathrm{clo}\,(C_A)) \cap (\mathrm{rec}\,(A) \times \{\mathbb{0}\})] \\
&\subset \mathrm{ext\text{-}g}\,(C_A) \cup (\mathrm{ext\text{-}g}\,(\mathrm{rec}\,(A)) \times \{\mathbb{0}\}).
\end{aligned}
$$

Now, as $(x, \mathbb{1}) \in C_A \subset \text{clo}\,(C_A)$, by Theorem 3.1 we know that there exist a finite number of elements of ext-g (C_A), namely $(\lambda_k u^k, \lambda_k)$ with $1 \leq k \leq p$, and a finite number of elements of ext-g $(\text{rec}\,(A)) \times \{\mathbb{0}\}$, namely $(y^h, \mathbb{0})$ with $1 \leq h \leq q$, such that

$$(x, \mathbb{1}) = \Big(\bigoplus_{1 \leq k \leq p} (\lambda_k u^k, \lambda_k) \Big) \oplus \Big(\bigoplus_{1 \leq h \leq q} (y^h, \mathbb{0}) \Big),$$

with $p + q \leq n + 1$. Therefore,

$$x = \Big(\bigoplus_{1 \leq k \leq p} \lambda_k u^k \Big) \oplus \Big(\bigoplus_{1 \leq h \leq q} y^h \Big), \text{ where } \bigoplus_{1 \leq k \leq p} \lambda_k = \mathbb{1}$$

and $y^h \in \text{rec}\,(A)$ for all $1 \leq h \leq q$. By Corollary 2.19 we know that $(u^k, \mathbb{1}) \in C_A$ is an extreme generator of C_A if, and only if, $u^k \in A$ is an extreme point of A. This shows that x is the sum of the convex combination of p extreme points of A and of q extreme generators of $\text{rec}\,(A)$ with $p + q \leq n + 1$. Hence, $A \subset \text{co}\,(\text{ext}\,(A)) \oplus \text{rec}\,(A)$. The other inclusion is trivial. $\qquad \square$

As a corollary of Theorem 3.1 we get a precise version of the "basis theorem" for finitely generated cones. The first results of this kind were obtained by Moller [Mol88] and Wagneur [Wag91]. Several variants of this result have appeared in [Gau98, DS04, CGB04].

Corollary 3.4 (Basis theorem). *Let $C \subset \mathbb{R}^n_{\max}$ be a finitely generated cone and $A \subset C$. Then, $C = \text{cone}\,(A)$ if, and only if, A contains at least one nonzero element of each extreme ray of C.*

Proof. This follows readily from Theorem 3.1 and from Lemma 2.20. $\qquad \square$

Example 3.5. As an illustration of Theorem 3.3, let us consider once again the closed convex set $A \subset \mathbb{R}^2_{\max}$ depicted in Figure 1. We have already seen (Examples 2.2 and 2.11) that $\text{ext}\,(A) = \{a, b, c, d, e\}$ and $\text{rec}\,(A) = \text{cone}\,\{(0, 1), (2, 0)\}$. Then,

$$A = \text{co}\,\{a, b, c, d, e\} \oplus \text{cone}\,\{(0, 1), (2, 0)\}$$

by Theorem 3.3. The sets $\text{co}\,(\text{ext}\,(A))$ and $\text{rec}\,(A)$ are depicted in Figure 2.

Example 3.6. The set of extreme points of a compact convex set may not be closed. In the max-plus case, there are even counter-examples in dimension 2. Such a counter-example is shown in Figure 3, where the set A is given by

$$A = ([-2, 0] \times \{0\}) \cup (\{0\} \times [0, -2]) \cup \big\{ x \in \mathbb{R}^2 \mid -1 \leq x_1 + x_2, \ x_1, x_2 \leq 0 \big\} \ ,$$

and

$$\text{ext}\,(A) = \{(-2, 0), (0, -2)\} \cup \big\{ x \in \mathbb{R}^2 \mid -1 = x_1 + x_2, \ x_1, x_2 < 0 \big\} \ .$$

Remark 3.7. As in the classical case, the set of extreme points of a compact convex set is a $\mathcal{G}_\delta$ set (a denumerable intersection of open sets). Indeed, let A be a non-empty compact convex subset of $\mathbb{R}^n_{\max}$, and let d denote any metric inducing the topology of $\mathbb{R}^n_{\max}$. For all positive integers k, let

$$F_k := \big\{ (x, y, z, \beta) \in A^3 \times \mathbb{R}_{\max} \mid d(x, y) \geq 1/k, \ d(x, z) \geq 1/k, \ \beta \leq \mathbb{1}, \ x = y \oplus \beta z \big\} \ .$$
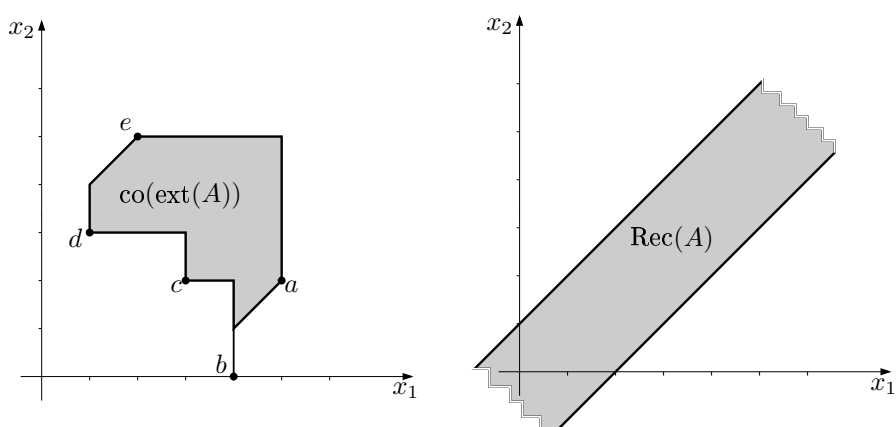
FIGURE 2. The sets $\mathrm{co}\,(\mathrm{ext}\,(A))$ and $\mathrm{rec}\,(A)$ of Theorem 3.3 for the unbounded convex set depicted in Figure 1.
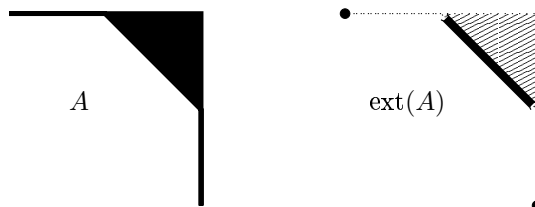

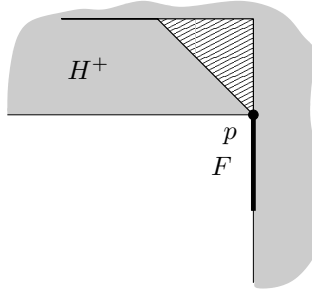FIGURE 3. A convex subset of $\mathbb{R}^2_{\max}$ and its set of extreme points.

FIGURE 4. The point p is an extreme point of the face F but it is not an extreme point of the convex set

Let π denote the projection sending (x, y, z, β) to x. Since A is compact, F_k is compact, and since π is continuous, $\pi(F_k)$ is compact. In particular, it is closed. A point x in A is not extreme if and only if there exist two points $y, z \in A$ that are both different from x, and a scalar $\beta \leq \mathbb{1}$, such that $x = y \oplus \beta z$. The latter property means that x belongs to some $\pi(F_k)$. So the set of extreme points of A, which can be written as $\cap_{k \geq 1}(\mathbb{R}^n_{\max} \setminus \pi(F_k))$, is a $\mathcal{G}_\delta$ set.

Let $a^+, a^- \in \mathbb{R}_{\max}$ and let ψ^+, ψ^- denote linear forms. We call *half-space* a set of the form
$$H^+ = \left\{ x \in \mathbb{R}^n_{\max} \mid \psi^+(x) \oplus a^+ \geq \psi^-(x) \oplus a^- \right\} \ .$$
The *opposite half-space H^-* is defined by reversing the inequality. We say that H^+ is a *minimal supporting half-space* of A if it contains A and if it contains no other half-space containing A. We define a *face* of a convex set A to be the intersection of A with an half-space opposite to a minimal supporting half-space. The following counter-example shows that unlike in classical convex analysis, the extreme points of faces are not necessarily extreme points of the set.

Example 3.8. Consider the half-space
$$H^+ = \left\{ x \in \mathbb{R}^2_{\max} \mid x_1 \oplus 1 y_1 \geq 0 \right\} \ ,$$
which is represented by the light gray region in Figure 4. One can check that this is a minimal supporting half-space of A (see [CGQS05] or [Jos05] for a description of max-plus half-spaces). Hence, $F := A \cap H^-$ is a face of A. This face is represented in bold on the figure. The point $p = (0, -1)$ is an extreme point of F, but it is not an extreme point of A.

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INRIA, Domaine de Voluceau, 78153, Le Chesnay Cédex, France. Tèl: +33 1 39 63 52 58,
Fax: +33 1 39 63 57 86
    E-mail address: Stephane.Gaubert@inria.fr

CONICET. Postal address: Dep. of Mathematics, Universidad Nacional de Rosario, Avenida
Pellegrini 250, 2000 Rosario, Argentina.
    E-mail address: rkatz@fceia.unr.edu.ar