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Marianne Akian, Stéphane Gaubert, Asma Lakhoua

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

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A max-plus finite element method for solving finite horizon deterministic optimal control problems

Marianne Akian*[†], Stéphane Gaubert*[‡], Asma Lakhoua*[§]

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Abstract: We introduce a max-plus analogue of the Petrov-Galerkin finite element method, to solve finite horizon deterministic optimal control problems. The method relies on a max-plus variational formulation, and exploits the properties of projectors on max-plus semimodules. We obtain a nonlinear discretized semigroup, corresponding to a zero-sum two players game. We give an error estimate of order $\sqrt{\Delta t} + \Delta x(\Delta t)^{-1}$, for a subclass of problems in dimension 1. We compare our method with a max-plus based discretization method previously introduced by Fleming and McEneaney.

Key-words: Max-plus algebra, tropical semiring, Hamilton-Jacobi equation, weak formulation, residuation, projection, idempotent semimodules, finite element method.

* INRIA, Domaine de Voluceau - Rocquencourt - BP 105 - 78153 Le Chesnay Cedex (France)
† marianne.akian@inria.fr
‡ stephane.gaubert@inria.fr
§ asma.lakhoua@inria.fr

Unité de recherche INRIA Rocquencourt
Domaine de Voluceau, Rocquencourt, BP 105, 78153 Le Chesnay Cedex (France)
Téléphone : +33 1 39 63 55 11 —Télécopie : +33 1 39 63 53 30

Une méthode des éléments finis max-plus pour la résolution de problèmes de commande optimale déterministe en horizon fini

Résumé : Nous introduisons l'analogue max-plus de la méthode des éléments finis de Petrov-Galerkin pour résoudre des problèmes de commande optimale déterministe en horizon fini. La méthode s'appuie sur une formulation variationnelle max-plus et utilise les propriétés des projecteurs sur des semi-modules max-plus. Le semi-groupe discret obtenu est non-linéaire et s'interprète comme l'opérateur de la programmation dynamique d'un jeu déterministe à somme nulle. Nous obtenons une estimation d'erreur de l'ordre de $\sqrt{\Delta t} + \Delta x(\Delta t)^{-1}$ pour une classe particulière de problèmes en dimension 1. Nous comparons notre méthode avec une méthode de discrétisation max-plus introduite précedemment par Fleming et Mceneaney.

Mots-clés : Algèbre max-plus, semi-anneau tropical, équation d'Hamilton-Jacobi, formulation faible, résiduation, projection, semi-modules idempotents, méthode des éléments finis.

1 Introduction

We consider the optimal control problem:

$$\text{maximize} \int_0^T \ell(x(s), u(s))\, ds + \phi(x(T)) \tag{1a}$$

over the set of trajectories $(x(\cdot), u(\cdot))$ satisfying

$$\dot{x}(s) = f(x(s), u(s)), \quad x(0) = x, \quad x(s) \in X, \quad u(s) \in U \ , \tag{1b}$$

for all $0 \leq s \leq T$. Here, the *state space* X is a subset of $\mathbb{R}^n$, the set of *control values* U is a subset of $\mathbb{R}^m$, the *horizon* $T > 0$ and the *initial condition* $x \in X$ are given, we assume that the map $u(\cdot)$ is measurable, and that the map $x(\cdot)$ is absolutely continuous. We also assume that the *instantaneous reward* or *Lagrangian* $\ell : X \times U \to \mathbb{R}$, and the *dynamics* $f : X \times U \to \mathbb{R}^n$, are sufficiently regular maps, and that the *terminal reward* ϕ is a map $X \to \mathbb{R} \cup \{-\infty\}$. The *value function* v associates to any $(x,t) \in X \times [0,T]$ the supremum $v(x,t)$ of $\int_0^t \ell(x(s), u(s))\, ds + \phi(x(t))$, under the constraint (1b), for $0 \leq s \leq t$. Under certain regularity assumptions, it is known that v is solution of the Hamilton-Jacobi equation

$$-\frac{\partial v}{\partial t} + H(x, \frac{\partial v}{\partial x}) = 0, \quad (x,t) \in X \times (0,T] \ , \tag{2a}$$

with initial condition:

$$v(x,0) = \phi(x), \quad x \in X \ , \tag{2b}$$

where $H(x,p) = \sup_{u \in U} \ell(x,u) + p \cdot f(x,u)$ is the *Hamiltonian* of the problem (see for instance [Lio82, FS93, Bar94]). The *evolution semigroup* S^t of (2) associates to any map ϕ the function $v^t := v(\cdot, t)$, where v is the value function of the optimal control problem (1a).

Maslov [Mas73] (see also [MS92, KM97]) observed that the evolution semigroup S^t is max-plus linear. Recall that the *max-plus semiring*, $\mathbb{R}_{\max}$, is the set $\mathbb{R} \cup \{-\infty\}$, equipped with the addition $a \oplus b = \max(a,b)$ and the multiplication $a \otimes b = a + b$. By *max-plus linearity*, we mean that for all maps f, g from X to $\mathbb{R}_{\max}$, and for all $\lambda \in \mathbb{R}_{\max}$, we have

$$S^t(f \oplus g) = S^t f \oplus S^t g \ ,$$
$$S^t(\lambda f) = \lambda S^t f \ ,$$

where $f \oplus g$ denotes the map $x \mapsto f(x) \oplus g(x)$, and λf denotes the map $x \mapsto \lambda \otimes f(x)$. Linear operators over max-plus type semirings have been widely studied, see for instance [CG79, MS92, BCOQ92, KM97, GM01].

In this paper, we introduce a new discretization method to solve the deterministic optimal control problem (1), using the max-plus linearity of the semigroup S^t. In [FM00], Fleming and McEneaney introduced a max-plus based discretization method to solve a subclass of Hamilton-Jacobi equations (with a Lagrangian ℓ quadratic with respect to u, and a dynamics

f affine with respect to u). They approximated the evolution semigroup S^t by a max-plus linear semigroup acting on a finitely generated semimodule of functions. This work was pursued in [McE01, McE00, McE03b, McE03a]. Another max-plus based numerical work on Hamilton-Jacobi equations is due to Bacaer [Bac01, Bac02]. The different discretization that we introduce here relies on a notion of max-plus "variational formulation", which originates from the notion of generalized solution of Hamilton-Jacobi equations of Maslov and Kolokoltsov [KM88], [KM97, Section 3.2]. This discretization, which can be interpreted geometrically in terms of projections on semimodules, is similar to the classical finite element method. We shall see that the space of test functions must be different from the space in which the solution is represented, so that our discretization is indeed a max-plus analogue of the Petrov-Galerkin finite element method. We illustrate the method by numerical examples. We also give an error estimate, in dimension one, of order $\sqrt{\Delta t} + \Delta x(\Delta t)^{-1}$, which is the same as the order obtained for existing discretization methods, see [Fal87] and [BCD97, Appendix A, by M. Falcone]

The present paper is only a preliminary account: the results will be detailed elsewhere. A first presentation of the method appeared in [Lak03].

2 Preliminaries on residuation and projections over semimodules

In this section we recall some classical residuation results (see for example [DJLC53], [Bir67], [BJ72], [BCOQ92]), and their application to linear maps on idempotents semimodules (see [LMS01, CGQ04]). We also review some results of [CGQ96, CGQ04] concerning projectors over semimodules.

2.1 Residuation, semimodules, and linear maps

If $(S, \leq)$ and $(T, \leq)$ are (partially) ordered sets, we say that a map $f : S \to T$ is *monotone* if $s \leq s' \implies f(s) \leq f(s')$. We say that f is *residuated* if there exists a map $f^\sharp : T \to S$ such that

$$f(s) \leq t \iff s \leq f^\sharp(t) \ .$$

The map f is residuated if, and only if, for all $t \in T$, $\{s \in S \mid f(s) \leq t\}$ has a maximum element in S. Then,

$$f^\sharp(t) = \max\{s \in S \mid f(s) \leq t\}, \quad \forall t \in T \ .$$

Moreover, in that case, we have

$$f \circ f^\sharp \circ f = f^\sharp \text{ and } f^\sharp \circ f \circ f^\sharp = f \ . \tag{3}$$

If a set $\mathcal{K}$ is a monoid for a commutative idempotent law $\oplus$ (*idempotent* means that $a \oplus a = a$), the *natural order* on $\mathcal{K}$ is defined by $a \leq b \iff a \oplus b = b$. We say that $\mathcal{K}$ is *complete* as a naturally ordered set if any subset of $\mathcal{K}$ has a least upper bound for the natural order. If $(\mathcal{K},\oplus,\otimes)$ is an idempotent semiring, i.e., a semiring whose addition is idempotent, we say that the semiring $\mathcal{K}$ is *complete* if it is complete as a naturally ordered set, and if the left and right multiplications, $L_a^{\mathcal{K}}$, $R_a^{\mathcal{K}}:\mathcal{K}\to\mathcal{K}$, $L_a^{\mathcal{K}}(x)=ax$, $R_a^{\mathcal{K}}(x)=xa$, are residuated.

The max-plus semiring, $\mathbb{R}_{\max}$, is an idempotent semiring. It is not complete, but it can be embedded in the complete idempotent semiring $\overline{\mathbb{R}}_{\max}$ obtained by adjoining $+\infty$ to $\mathbb{R}_{\max}$, with the convention that $-\infty$ is absorbing for the multiplication $a\otimes b=a+b$. The map $x\mapsto -x$ from $\overline{\mathbb{R}}$ to itself yields an isomorphism from $\overline{\mathbb{R}}_{\max}$ to the complete idempotent semiring $\overline{\mathbb{R}}_{\min}$, obtained by replacing max by min and by exchanging the roles of $+\infty$ and $-\infty$ in the definition of $\overline{\mathbb{R}}_{\max}$.

Semimodules over semirings are defined like modules over rings, mutatis mutandis, see [LMS01, CGQ04]. When $\mathcal{K}$ is a complete idempotent semiring, we say that a (right) $\mathcal{K}$-semimodule $\mathcal{X}$ is *complete* if it is complete as a naturally ordered set, and if, for all $u\in\mathcal{X}$ and $\lambda\in\mathcal{K}$, the right and left multiplications, $R_\lambda^{\mathcal{X}}:\ \mathcal{X}\to\mathcal{X}$, $v\mapsto v\lambda$ and $L_u^{\mathcal{X}}:\ \mathcal{K}\to\mathcal{X}$, $\mu\mapsto u\mu$, are residuated. In a complete semimodule $\mathcal{X}$, we define, for all $u,v\in\mathcal{X}$,

$$u\backslash v \stackrel{\text{def}}{=} (L_u^{\mathcal{X}})^\sharp(v)=\max\{\lambda\in\mathcal{K}\mid u\lambda\leq v\}\ .$$

We shall use *semimodules of functions*: when X is a set and $(\mathcal{K},\oplus,\otimes)$ is a complete idempotent semiring, the set of functions $\mathcal{K}^X$ is a complete $\mathcal{K}$-semimodule for the componentwise addition $(u,v)\mapsto u\oplus v$ (defined by $(u\oplus v)(x)=u(x)\oplus v(x)$), and the componentwise multiplication $(\lambda,u)\mapsto u\lambda$ (defined by $(u\lambda)(x)=u(x)\otimes\lambda$).

If $\mathcal{K}$ is an idempotent semiring, and if $\mathcal{X}$ and $\mathcal{Y}$ are $\mathcal{K}$-semimodules, we say that a map $A:\mathcal{X}\to\mathcal{Y}$ is *additive* if for all $u,v\in\mathcal{X}$, $A(u\oplus v)=A(u)\oplus A(v)$ and that A is *homogeneous* if for all $u\in\mathcal{X}$ and $\lambda\in\mathcal{K}$, $A(u\lambda)=A(u)\lambda$. We say that A is *linear*, or is a *linear operator*, if it is additive and homogeneous. Then, as in classical algebra, we use the notation Au instead of $A(u)$. When A is residuated and $v\in\mathcal{Y}$, we use the notation $A\backslash v$ or $A^\sharp v$ instead of $A^\sharp(v)$. We denote by $L(\mathcal{X},\mathcal{Y})$ the set of linear operators from $\mathcal{X}$ to $\mathcal{Y}$. If $\mathcal{K}$ is a complete idempotent semiring, if $\mathcal{X},\mathcal{Y},\mathcal{Z}$ are complete $\mathcal{K}$-semimodules, and if $A\in L(\mathcal{Y},\mathcal{Z})$ is residuated, then the map $L_A:L(\mathcal{X},\mathcal{Y})\to L(\mathcal{X},\mathcal{Z})$, $B\mapsto A\circ B$, is residuated and we set $A\backslash C:=(L_A)^\sharp(C)$, for all $C\in L(\mathcal{X},\mathcal{Z})$.

If X and Y are two sets, $\mathcal{K}$ is a complete idempotent semiring, and $a\in\mathcal{K}^{X\times Y}$, we construct the linear operator A from $\mathcal{K}^Y$ to $\mathcal{K}^X$ which associates to any $u\in\mathcal{K}^Y$ the function $Au\in\mathcal{K}^X$ such that $Au(x)=\bigvee_{y\in Y}a(x,y)\otimes u(y)$, where $\vee$ denotes the supremum for the natural order. We say that A is the *kernel operator* with *kernel* or *matrix* a. We shall often use the same notation A for the operator and the kernel. As is well known (see for instance [BCOQ92]), the kernel operator A is residuated, and

$$(A\backslash v)(y)=\bigwedge_{x\in X}A(x,y)\backslash v(x),$$

where $\wedge$ denotes the infimum for the natural order. In particular, when $\mathcal{K}=\overline{\mathbb{R}}_{\max}$, we have

$$(A\backslash v)(y)=\bigwedge_{x\in X}(-A(x,y)+v(x))=[-A^*(-v)](y)\ ,\tag{4}$$

where A^* denotes the *transposed operator* $\mathcal{K}^X \to \mathcal{K}^Y$, which is associated to the kernel $A^*(y,x) = A(x,y)$. (In (4), we use the convention that $+\infty$ is absorbing for addition.)

2.2 Projectors on semimodules

Let $\mathcal{V}$ denote a *complete subsemimodule* of a complete semimodule $\mathcal{X}$ over a complete idempotent semiring $\mathcal{K}$, i.e., a subset of $\mathcal{X}$ that is stable by arbitrary sups and by the action of scalars. We call *canonical projector* on $\mathcal{V}$ the map

$$P_{\mathcal{V}} : \mathcal{X} \to \mathcal{X}, \quad u \mapsto P_{\mathcal{V}}(u) = \max\{v \in \mathcal{V} \mid v \leq u\}. \tag{5}$$

Let W denote a *generating family* of a complete subsemimodule $\mathcal{V}$, which means that any element $v \in \mathcal{V}$ can be written as $v = \bigvee\{w\lambda_w \mid w \in W\}$, for some $\lambda_w \in \mathcal{K}$. It is known that

$$P_{\mathcal{V}}(u) = \bigvee_{w \in W} w(w\backslash u)$$

(see for instance [CGQ04]). If $B : \mathcal{U} \to \mathcal{X}$ is a residuated linear operator, then the image $\operatorname{im} B$ of B is a complete subsemimodule of $\mathcal{X}$, and

$$P_{\operatorname{im} B} = B \circ B^\sharp. \tag{6}$$

The max-plus finite element methods relies on the notion of projection on an image, parallel to a kernel, which was introduced by Cohen, the second author, and Quadrat, in [CGQ96]. The following theorem, of which Proposition 2 below is an immediate corollary, is a variation on the results of [CGQ96, Section 6].

Theorem 1 (Projection on an image parallel to a kernel). *Let $B : \mathcal{U} \to \mathcal{X}$ and $C : \mathcal{X} \to \mathcal{Y}$ be two residuated linear operators. Let $\Pi_B^C = B \circ (C \circ B)^\sharp \circ C$. We have $\Pi_B^C = \Pi_B \circ \Pi^C$, where $\Pi_B = B \circ B^\sharp$ and $\Pi^C = C^\sharp \circ C$. Moreover, Π_B^C is a projector $\big((\Pi_B^C)^2 = \Pi_B^C\big)$, and for all $x \in \mathcal{X}$:*

$$\Pi_B^C(x) = \max\{y \in \operatorname{im} B \mid Cy \leq Cx\}.$$

The results of [CGQ96] characterize the existence and uniqueness, for all $x \in X$, of $y \in \operatorname{im} B$ such that $Cy = Cx$. In that case, $y = \Pi_B^C(x)$.

When $\mathcal{K} = \overline{\mathbb{R}}_{\max}$, and $C : \overline{\mathbb{R}}_{\max}^X \to \overline{\mathbb{R}}_{\max}^Y$ is a kernel operator, $\Pi^C = C^\sharp \circ C$ has an interpretation similar to (6):

$$\Pi^C(v) = C^\sharp \circ C(v) = -P_{\operatorname{im} C^*}(-v) = P_{-\operatorname{im} C^*}(v) \ ,$$

where $-\operatorname{im} C^*$ is thought of as a $\overline{\mathbb{R}}_{\min}$-subsemimodule of $\overline{\mathbb{R}}_{\min}^X$, so that,

$$P_{-\operatorname{im} C^*}(v) = \min\{w \in -\operatorname{im} C^* \mid w \geq v\} \ .$$

where $\leq$ denotes here the usual order on $\overline{\mathbb{R}}^X$, since the natural order of $\overline{\mathbb{R}}_{\min}^X$ is the reverse of the usual order. When $B : \overline{\mathbb{R}}_{\max}^U \to \overline{\mathbb{R}}_{\max}^X$ is also a kernel operator, we have

$$\Pi_B^C = P_{\operatorname{im} B} \circ P_{-\operatorname{im} C^*} \quad .$$

This factorization will be instrumental in the geometrical interpretation of the finite element algorithm, see Example 10 below.

3 The max-plus finite element method

3.1 Max-plus variational formulation

We now describe the max-plus finite element method to solve the optimal control problem (1a). Let S^t and v^t be defined as in the introduction. Using the semigroup property $S^{t+t'} = S^t \circ S^{t'}$, for $t, t' > 0$, we have the recursive equation:

$$v^{t+\Delta t} = S^{\Delta t} v^t, \quad t = 0, \Delta t, \cdots, T - \Delta t \tag{7}$$

with $v^0 = \phi$ and $\Delta t = \frac{T}{N}$, for some positive integer N. Let $\mathcal{W}$ be a $\overline{\mathbb{R}}_{\max}$-semimodule of functions from X to $\overline{\mathbb{R}}_{\max}$ such that $\phi \in \mathcal{W}$ and for all $v \in \mathcal{W}$, $t \geq 0$, $S^t v \in \mathcal{W}$. We suppose given a "dual" semimodule $\mathcal{Z}$ of "test functions" from X to $\overline{\mathbb{R}}_{\max}$. The max-plus *scalar product* is defined by $\langle u \mid v \rangle = \sup_{x \in X} u(x) + v(x)$, for all functions $u, v : X \to \overline{\mathbb{R}}$, with the convention that $-\infty$ is absorbing for the addition $+$. We replace (7) by:

$$\langle z \mid v^{t+\Delta t} \rangle = \langle z \mid S^{\Delta t} v^t \rangle, \quad \forall z \in \mathcal{Z}, \quad t = 0, \Delta t, \ldots, T - \Delta t \ , \tag{8}$$

with $v^{\Delta t}, \ldots, v^T \in \mathcal{W}$. Equation (8) can be seen as the analogue of a *variational* or*weak formulation*. Kolokoltsov and Maslov used this formulation in [KM88] and [KM97, section 3.2] to define a notion of generalized solution of Hamilton-Jacobi equations.

3.2 Ideal max-plus finite element method

We consider a semimodule $\mathcal{W}_h \subset \mathcal{W}$ with generating family $\{w_i\}_{1 \leq i \leq p}$. We call *finite elements* the functions w_i. We approximate v^t by $v_h^t \in \mathcal{W}_h$, that is:

$$v^t \simeq v_h^t = \bigvee_{1 \leq i \leq p} w_i \lambda_i^t \ ,$$

where $\lambda_i^t \in \mathbb{R}_{\max}$. We also consider a semimodule $\mathcal{Z}_h \subset \mathcal{Z}$ with generating family $\{z_j\}_{1 \leq j \leq q}$. The functions $z_1, \cdots, z_q$ will act as test functions. We replace (8) by

$$\langle z_j \mid v_h^{t+\Delta t} \rangle = \langle z_j \mid S^{\Delta t} v_h^t \rangle, \quad \forall 1 \leq j \leq q \ , \tag{9}$$

for $t = 0, \Delta t, \cdots, T - \Delta t$, with $v_h^0 = \phi_h \simeq \phi$ and $v_h^t \in \mathcal{W}_h$, $t = 0, \Delta t, \cdots, T$.

Since Equation (9) need not have a solution, we look for the maximal subsolution, i.e. the maximal solution $v_h^{t+\Delta t} \in \mathcal{W}_h$ of

$$\langle z_j \mid v_h^{t+\Delta t} \rangle \quad \leq \quad \langle z_j \mid S^{\Delta t} v_h^t \rangle \quad \forall 1 \leq j \leq q \ . \tag{10a}$$

We also take for the approximate value function v_h^0 at time 0 the maximal solution $v_h^0 \in \mathcal{W}_h$ of

$$v_h^0 \leq v^0 \ . \tag{10b}$$

Let us denote by W_h the max-plus linear operator from $\mathbb{R}_{\max}^p$ to $\mathcal{W}$ with matrix $W_h = \mathrm{col}(w_i)_{1 \leq i \leq p}$, and by Z_h^* the max-plus linear operator from $\mathcal{W}$ to $\overline{\mathbb{R}}_{\max}^q$ whose transposed matrix is $Z_h = \mathrm{col}(z_j)_{1 \leq j \leq q}$. This means that $W_h \lambda = \bigvee_{1 \leq i \leq p} w_i \lambda_i$ for all $\lambda = (\lambda_i)_{i=1,\ldots,p} \in \mathbb{R}_{\max}^p$, and $(Z_h^* v)_j = \langle z_j \mid v \rangle$ for all $v \in \mathcal{W}$ and $j = 1, \ldots, q$. Applying Theorem 1 to $B = W_h$ and $C = Z_h^*$ and using $\mathcal{W}_h = \mathrm{im}\, W_h$, we get:

Proposition 2. *The maximal solution $v_h^{t+\Delta t} \in \mathcal{W}_h$ of* (10a) *is given by $v_h^{t+\Delta t} = S_h^{\Delta t} v_h^t$, where*

$$S_h^{\Delta t} = \Pi_{W_h}^{Z_h^*} \circ S^{\Delta t} \ .$$

Proposition 3. *Let $v_h^t \in \mathcal{W}_h$ be the maximal solution of* (10)*, for $t = 0, \Delta t, \ldots, T$. Then, for every $t = 0, \Delta t, \ldots, T$, there exists $\lambda^t \in \mathbb{R}_{\max}^p$ such that $v_h^t = W_h \lambda^t$. Moreover, the maximal λ^t satisfying these conditions verifies the recursive equation*

$$\lambda^{t+\Delta t} = (Z_h^* W_h) \backslash (Z_h^* S^{\Delta t} W_h \lambda^t) \ , \tag{11a}$$

with the initial condition:

$$\lambda^0 = W_h \backslash \phi \ .$$

Proof. Since $v_h^t \in \mathcal{W}_h$, $v_h^t = W_h \lambda^t$, and the maximal λ^t satisfying this condition is $\lambda^t = W_h^\sharp(v_h^t)$, for all $t = 0, \Delta t, \ldots, T$. Since v_h^0 is the maximal solution of (10b), then by (5) and (6), $v_h^0 = P_{\mathcal{W}_h}(\phi) = W_h \circ W_h^\sharp(\phi)$, hence $\lambda^0 = W_h^\sharp \circ W_h \circ W_h^\sharp(\phi) = W_h^\sharp(\phi)$. Let $t = 0, \ldots, T_\Delta t$. Using Proposition 2, Theorem 1, (3) and the property that $(f \circ g)^\sharp = g^\sharp \circ f^\sharp$ for all residuated maps f and g, we get

$$\begin{aligned}
\lambda^{t+\Delta t} &= W_h^\sharp \circ \Pi_{W_h}^{Z_h^*} \circ S^{\Delta t}(W_h \lambda^t) \\
&= W_h^\sharp \circ W_h \circ W_h^\sharp \circ (Z_h^*)^\sharp \circ Z_h^* \circ S^{\Delta t}(W_h \lambda^t) \\
&= W_h^\sharp \circ (Z_h^*)^\sharp \circ Z_h^* \circ S^{\Delta t}(W_h \lambda^t) \\
&= (Z_h^* W_h)^\sharp (Z_h^* S^{\Delta t} W_h \lambda^t) \ .
\end{aligned}$$

which yields (11a). □

The maps $A_h := Z_h^* W_h : \mathbb{R}_{\max}^p \to \mathbb{R}_{\max}^q$ and $B_h := Z_h^* S^{\Delta t} W_h : \mathbb{R}_{\max}^p \to \mathbb{R}_{\max}^q$ are max-plus linear operators, and the entries of their corresponding matrices are given, for $1 \leq i \leq p$ and $1 \leq j \leq q$, by:

$$(A_h)_{ji} = \langle z_j \mid w_i \rangle \tag{12}$$

$$(B_h)_{ji} = \langle z_j \mid S^{\Delta t} w_i \rangle \tag{13}$$

$$= \langle (S^*)^{\Delta t} z_j \mid w_i \rangle \ , \tag{14}$$

where S^* is the *transposed semigroup* of S, which is the evolution semigroup associated to the optimal control problem in which the sign of the dynamics is changed.

The ideal max-plus finite element method can be summarized as follows:

1. Choose $\Delta t = \frac{T}{N}$ and the finite elements $(w_i)_{1 \leq i \leq p}$ and $(z_j)_{1 \leq j \leq q}$,
2. Compute the matrix A_h by (12) and the matrix B_h by (13) or by (14),
3. Compute $\lambda^0 = W_h \backslash \phi$ and $v_h^0 = W_h \lambda^0$.
4. For $t = \Delta t, 2\Delta t, \ldots, T$, compute $\lambda^t = A_h \backslash (B_h \lambda^{t-\Delta t})$ and $v_h^t = W_h \lambda^t$.

Then, v_h^t approximates the value function at time t, v^t.

The recursion $\lambda^t = A_h \backslash (B_h \lambda^{t-\Delta t})$ may be written explicitly as

$$\lambda_i^t = \min_{1 \leq j \leq q} \Big(-(A_h)_{ji} + \max_{1 \leq k \leq p} \big((B_h)_{jk} + \lambda_k^{t-\Delta t} \big) \Big), \quad \text{for } 1 \leq i \leq p \ .$$

Observe that this recursion may be interpreted as the dynamic programming equation of a deterministic zero-sum two players game, with finite action and state spaces.

In order to implement this method, we must specify how to compute the entries of A_h and B_h in (12) and (13) or (14). In some cases, these computations can be done analytically. Computing A_h from (12) is an optimization problem which may be solved by standard algorithms. We shall discuss in the following section the approximation of B_h.

3.3 Effective max-plus finite element method

We first discuss the approximation of $S^{\Delta t} w$ for every finite element w. The Hamilton-Jacobi equation (2a) suggests to approximate $S^{\Delta t} w$ by the function $[S^{\Delta t} w]^{\sim}$ such that

$$[S^{\Delta t} w]^{\sim}(x) = w(x) + \Delta t H(x, \frac{\partial w}{\partial x}), \quad \text{for all} x \in X. \tag{15}$$

Let $[S^{\Delta t} W_h]^{\sim}$ denotes the max-plus linear operator from $\mathbb{R}_{\max}^p$ to $\mathcal{W}$ with matrix $[S^{\Delta t} W_h]^{\sim} = \mathrm{col}([S^{\Delta t} w_i]^{\sim})_{1 \leq i \leq p}$, which means that $[S^{\Delta t} W_h]^{\sim} \lambda = \bigvee_{1 \leq i \leq p} [S^{\Delta t} w_i]^{\sim} \lambda_i$ for all $\lambda = (\lambda_i)_{1 \leq i \leq p} \in \mathbb{R}_{\max}^p$. The above approximation of $S^{\Delta t} w$ yields an approximation of the matrix B_h by the matrix $B_h^{\sim} := Z_h^* [S^{\Delta t} W_h]^{\sim}$, whose entries are given, for $1 \leq i \leq p$ and $1 \leq j \leq q$, by:

$$(B_h^{\sim})_{ji} \quad = \quad \sup_{x \in X} (z_j(x) + w_i(x) + \Delta t H(x, \frac{\partial w_i}{\partial x})) \ .$$

Thus, computing $B_h^{\sim}$ requires to solve an optimization problem, which is nothing but a perturbation of the optimization problem associated to the computation of A_h. We may exploit this observation by replacing $B_h^{\sim}$ by the matrix $B_h^{\sim\sim}$ with entries

$$(B_h^{\sim\sim})_{ji} \quad = \quad \langle z_j \mid w_i \rangle + \Delta t \sup_{x \in \arg\max\{z_j + w_i\}} H(x, \frac{\partial w_i}{\partial x}) \ , \tag{16}$$

for $1 \leq i \leq p$ and $1 \leq j \leq q$. Here, $\arg\max\{z_j + w_i\}$ denotes the set of x such that $z_j(x) + w_i(x) = \langle z_j \mid w_i \rangle$. When this set has only one element, (16) yields a convenient approximation of B_h.

Of course, w_i must be differentiable for the approximation (15) to make sense. When w_i is non-differentiable, but z_j is differentiable, we may approximate $(B_h)_{ji}$ by

$$\sup_{x \in X} (z_j(x) + \Delta t H(x, -\frac{\partial z_j}{\partial x}) + w_i(x)) \ ,$$

using the dual formula (14). We may also use the dual formula of (16), where $\frac{\partial w_i}{\partial x}$ is replaced by $-\frac{\partial z_j}{\partial x}$.

3.4 Comparison with the method of Fleming and McEneaney

Fleming and McEneaney proposed a max-plus based method [FM00], which also uses a space $\mathcal{W}_h$ generated by finite elements, $w_1, \ldots, w_p$, together with the linear formulation (7). Their method approaches the value function at time t, v^t, by $W_h \mu^t$, where $W_h = \mathrm{col}(w_i)_{1 \leq i \leq p}$ as above, and μ^t is defined inductively by

$$\mu^0 = W_h \backslash \phi \tag{17a}$$

$$\mu^{t+\Delta t} = \big(W_h \backslash (S^{\Delta t} W_h)\big) \mu^t \ , \tag{17b}$$

for $t = 0, \Delta t, \ldots, T - \Delta t$. This can be compared with the limit case of our finite element method, in which the space of test functions $\mathcal{Z}_h$ generates the set of all functions. This limit case corresponds to replacing Z_h^* by the identity operator in (11a), so that

$$\lambda^{t+\Delta t} = W_h \backslash (S^{\Delta t} W_h \lambda^t) \ . \tag{18}$$

Proposition 4. *Let (μ^t) be the sequence of vectors defined by the algorithm of Fleming and McEneaney, (17); let (λ^t) be the sequence of vectors defined by the max-plus finite element method, in the limit case (18); and let v^t denote the value function at time t. Then,*

$$W_h \mu^t \leq W_h \lambda^t \leq v^t \ , \quad \text{for } t = 0, \Delta t, \ldots, T \ .$$

Sketch of proof. This can be proved by induction, by using the residuation inequality $W_h^\sharp S^{\Delta t} W_h \lambda \geq \big(W_h \backslash (S^{\Delta t} W_h)\big) \lambda$, which holds for all vectors λ, together with the monotonicity of the operators arising in the construction of λ^t and μ^t. □

An approximation of (17b) using formulae of the same type as (15) is also discussed in [MH99]. An experimental comparison will appear elsewhere.

4 Error analysis

The following general lemma shows that the error of the finite element method is controlled by the projection errors, $\|\Pi_{W_h} v^t - v^t\|_\infty$ and $\|\Pi^{Z_h^*} v^t - v^t\|_\infty$, and by the approximation errors, $\|[S^{\Delta t} w_i]^\sim - S^{\Delta t} w_i\|_\infty$, and $|(B_h^{\sim\sim})_{ji} - (B_h^\sim)_{ji}|$.

Lemma 5. *For $t = 0, \Delta t, \cdots, T$, let v^t be the value function at time t, and v_h^t be its approximation given by the effective max-plus finite element method, implemented with the approximation $B_h^{\sim\sim}$ of B_h, given by* (16). *We have*

$$\|v_h^T - v^T\|_\infty \leq (1 + \frac{T}{\Delta t})\Big(\sup_{t=0,\Delta t,\ldots,T} (\|\Pi^{Z_h^*} v^t - v^t\|_\infty + \|\Pi_{W_h} v^t - v^t\|_\infty) + \max_{1\leq i\leq p} \|[S^{\Delta t} w_i]^\sim - S^{\Delta t} w_i\|_\infty + \max_{\substack{1\leq j\leq q \\ 1\leq i\leq p}} |(B_h^{\sim\sim})_{ji} - (B_h^\sim)_{ji}|\Big).$$

The proof of this lemma uses the fact that projectors over max-plus semimodules are non-expansive in the sup-norm.

To state an error estimate, we make the following assumptions:

- $(H1)$ The semigroup preserves the set of $\frac{1}{c}$-semiconvex functions, for some $c > 0$.
- $(H2)$ $f : X \times U \to \mathbb{R}^n$ is bounded and Lipschitz continuous with respect to x:

$$\begin{array}{lll} \exists L_f > 0, & \forall x, y \in X, & |f(x,u) - f(y,u)| \leq L_f |x - y| \quad \forall u \in U, \\ \exists M_f > 0, & \forall x, y \in X, & |f(x,u)| \leq M_f. \end{array}$$

- $(H3)$ $\ell : X \times U \to \mathbb{R}$ is bounded and Lipschitz continuous with respect to x:

$$\left\{ \begin{array}{ll} |\ell(x,u) - \ell(y,u)| \leq L_l |x - y| & \forall x, y \in X, u \in U, \\ |\ell(x,u)| \leq M_l, & \forall x, y \in X, u \in U. \end{array} \right.$$

- $(H4)$ $\phi : X \to \mathbb{R}$ is bounded and Lipschitz continuous:

$$|\phi(x) - \phi(y)| \leq L_\phi |x - y| \quad \forall x, y \in X.$$

Recall that a function f is $\frac{1}{c}$*-semiconvex* if $f(x) + \frac{1}{2c} x^2$ is convex. Spaces of semiconvex functions were already used by Fleming and McEneaney [FM00].

We shall use the following finite elements.

Definition 6 (Lipschitz finite elements). Assume that X is an interval of $\mathbb{R}$. We call *Lipschitz finite element* centered at point $\hat{x} \in X$, with constant $A > 0$, the function $w(x) = -A|x - \hat{x}|$.

Definition 7 (Quadratic finite elements). Assume that X is an interval of $\mathbb{R}$. We call *quadratic finite element* centered at point $\hat{x} \in X$, with Hessian $\frac{1}{c} > 0$, the function $w(x) = -\frac{1}{2c}(x - \hat{x})^2$.

The family of Lipschitz continuous finite elements of constant A generates, in the max-plus sense, the semimodule of Lipschitz continuous functions of Lipschitz constant A. When $X = \mathbb{R}$, the family of quadratic finite elements with Hessian $\frac{1}{c}$ generates, in the max-plus sense, the semimodule of lower-semicontinuous $\frac{1}{c}$-semi-convex functions.

Theorem 8. *Let $X = [-b, b] \subset \mathbb{R}$. We make assumptions* (H1)-(H4)*, and assume that there exist $L > 0$ such that the value function at time t, v^t, is L-Lipschitz continuous and $\frac{1}{c}$-semiconvex for all $t > 0$, with the same constant c as in* (H1)*. Let us choose quadratic finite elements w_i of Hessian $\frac{1}{c}$, centered at the points of the regular grid $(\mathbb{Z}\Delta x) \cap [-(b + cL), (b + cL)]$. Let us choose, as test functions z_j, the Lipschitz finite elements with constant $A \geq L$, centered at the points of the regular grid $(\mathbb{Z}\Delta x) \cap [-b, b]$. For $t = 0, \Delta t, \ldots, T$, let v_h^t be the approximation of v^t given by the effective max-plus finite element method, implemented with the approximation $B_h^{\sim\sim}$ of B_h. Then, there exists a constant $K > 0$ such that, for Δt small enough,*

$$\|v_h^T - v^T\|_\infty \leq K(\sqrt{\Delta t} + \frac{\Delta x}{\Delta t}) \ .$$

A variant of this theorem, with a stronger assumption, is proved in [Lak03]. We shall give elsewhere the proof of Theorem 8.

5 Numerical results

Example 9 (Linear Quadratic Problem). We consider the case where $U = \mathbb{R}$, $X = \mathbb{R}$,

$$\ell(x, u) = -(\frac{a}{2}|x|^2 + \frac{|u|^2}{2}), \quad f(x, u) = u, \text{ and } \phi \equiv 0 \ .$$

We obtain $H(x, p) = -\frac{a}{2}|x|^2 + \frac{p^2}{2}$. We choose quadratic finite elements w_i and z_j of Hessian 1, centered at the points of the regular grid $(\mathbb{Z}\Delta x) \cap [-L, L]$. We represented in Figure 1 the solution given by our algorithm in the case where $T = 5$, $\Delta t = \Delta x = 0.05$, $a = 0.3$ and $L = 10$. The computations were performed using the max-plus toolbox of Scilab [Plu98].

Example 10 (Distance problem). We consider the case where $T = 1$, $\phi \equiv 0$, $X = [-1, 1]$, $U = [-1, 1]$,

$$\ell(x, u) = \begin{cases} -1 & \text{if} \quad x \in (-1, 1), \\ 0 & \text{if} \quad x \in \{-1, 1\}, \end{cases} \quad \text{and} \quad f(x, u) = \begin{cases} u & \text{if} \quad x \in (-1, 1), \\ 0 & \text{if} \quad x \in \{-1, 1\}. \end{cases}$$

Consider first quadratic finite elements w_i and z_j of Hessian $\frac{1}{c}$, centered at the points of the regular grid $(\mathbb{Z}\Delta x) \cap [-1, 1]$. In Figure 2, we represented the solution given by our algorithm in the case where $\Delta t = 0.05$, $\Delta x = 0.0125$ and $c = 1.2$. Since $\Pi^{Z_h^*}$ is a projector on a subsemimodule of the $\overline{\mathbb{R}}_{\min}$-semimodule of $-\frac{1}{c}$-semiconcave functions, and since the solution is not $-\frac{1}{c}$-semiconcave for any c, the error of projection $\|\Pi^{Z_h^*}(v^t) - v^t\|_\infty$ does not converge to zero when Δx goes to zero, which explains the magnitude of the error.

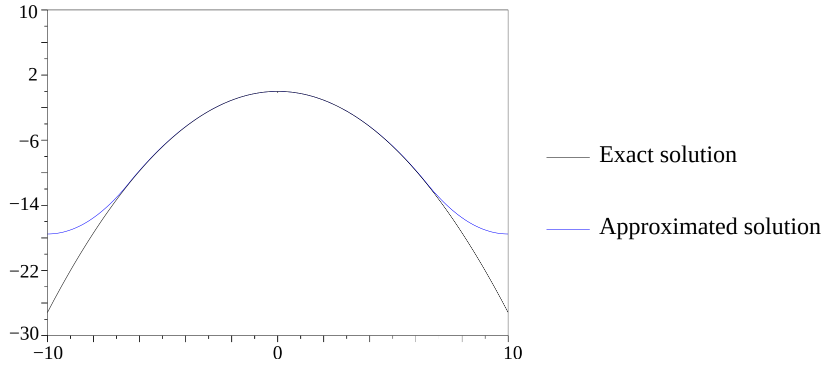

Figure 1: Max-plus approximation of a linear quadratic control problem (Example 9)

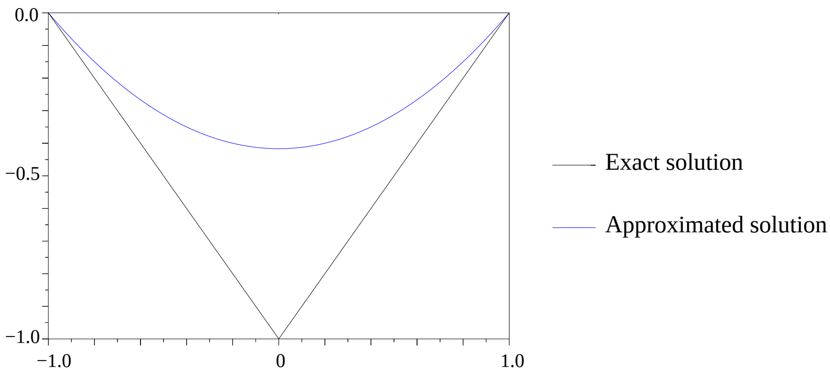

Figure 2: A bad choice of test functions for the distance problem (Example 10)

To solve this problem, it suffices to replace the test functions z_j by the Lipschitz finite elements with constant $A \geq 1$, centered at the points of the regular grid $(\mathbb{Z}\Delta x) \cap [-1, 1]$. This is illustrated in Figure 3 in the case where $\Delta t = 0.05$, $\Delta x = 0.0125$, $c = 1.2$ and $A = 1.1$.

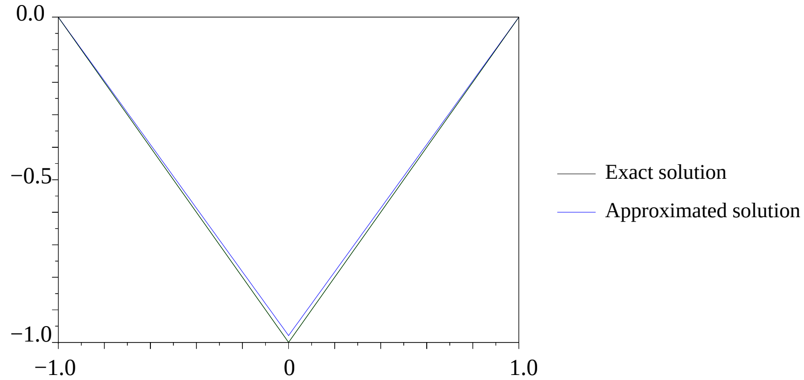

Figure 3: A good choice of test functions for the distance problem (Example 10)

The next two examples are inspired by those proposed by M. Falcone in [BCD97].

Example 11. We consider the case where $T = 1$, $\Phi \equiv 0$, $X = [-1, 1]$, $U = [0, 1]$, $\ell(x, u) = x$ and $f(x, u) = -xu$. The optimal choise is to take $u^* = 0$ whenever $x > 0$ and to move on the right with maximum speed ($u^* = 1$) whenever $x \leq 0$. For all $t \in [0, T]$, the value function is:

$$v(x,t) = \begin{cases} xt & \text{if } x > 0 \\ x(1 - e^{-t}) & \text{otherwise.} \end{cases}$$

We choose quadratic finite elements w_i of Hessian $\frac{1}{c}$ and Lipschitz finite elements z_j with constant $A \geq 1$. We represented in Figure 4 the solution given by our algorithm in the case where $T = 1$, $\Delta t = 0.05$, $\Delta x = 0.02$, $A = 1.3$ and $c = 1.4$.

Example 12. We consider the case where $T = 1$, $\Phi \equiv 0$, $X = [-1, 1]$, $U = [-1, 1]$, $\ell(x, u) = -3(1 - |x|)$ and $f(x, u) = u(1 - |x|)$. The optimal choise is to take $u^* = -1$ whenever $x > 0$ and $u^* = 1$ whenever $x < 0$. For all $t \in [0, T]$, the value function is:

$$v(x,t) = -3(1 - |x|)(1 - e^{-t})$$

We choose quadratic finite elements w_i of Hessian $\frac{1}{c}$ and Lipschitz finite elements z_j with constant A. We represented in Figure 5 the solution given by our algorithm in the case where $T = 1$, $\Delta t = 0.05$, $\Delta x = 0.02$, $A = 2$ and $c = 1.1$.

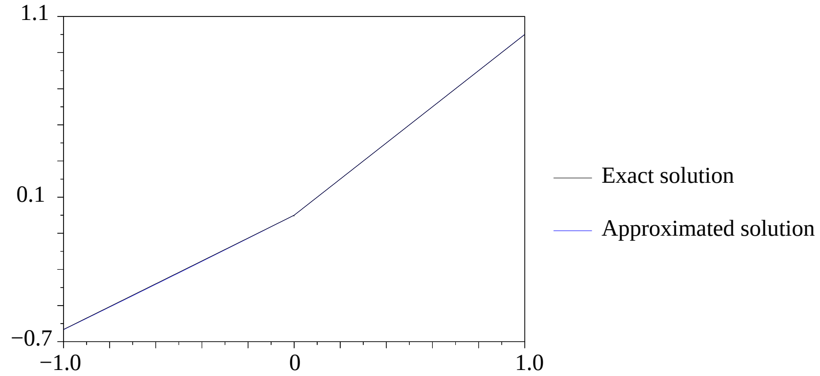

Figure 4: Value function and its max-plus approximation (Example 11)

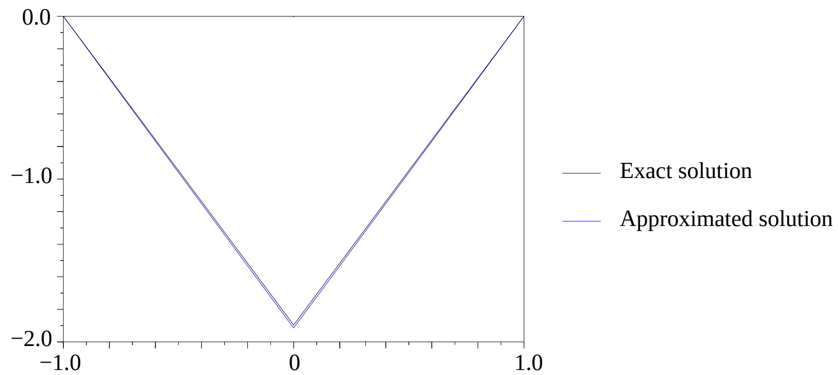

Figure 5: Value function and its max-plus approximation (Example 12)

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Unité de recherche INRIA Rocquencourt
Domaine de Voluceau - Rocquencourt - BP 105 - 78153 Le Chesnay Cedex (France)

Unité de recherche INRIA Futurs : Parc Club Orsay Université - ZAC des Vignes
4, rue Jacques Monod - 91893 ORSAY Cedex (France)
Unité de recherche INRIA Lorraine : LORIA, Technopôle de Nancy-Brabois - Campus scientifique
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