Optimal Power Allocation in CDMA Networks with Macrodiversity

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Abstract: This report defines and analyzes a general model of macrodiversity in CDMA networks (Code Division Multiple Access). Some differences between the uplink (from users to base stations) and downlink (from base stations to users) lead to two different models which are studied in parallel in this report. In order to satisfy the bit rate requirements of users, it is necessary to make use of a control policy of emitted powers. We prove necessary and sufficient conditions of the existence of a power allocation in the network which satisfies all users. These conditions depend on the users requirements and on the geometry of the network.

For a finite network on the downlink, we prove a necessary condition on the optimal power allocation policy. This result shows that macrodiversity has a limited impact on the downlink.

The analysis of infinite networks is led in the framework of stationary ergodic point processes. On the uplink, there is a necessary and sufficient condition of the existence of a power allocation which guarantee a given set of bit rate requirements for users. This condition is of type "$\rho < 1$", where ρ is function only of the average bits rate requirement of users and on the average number per unit of surface of users and base stations. On the contrary, on downlink, it is proven that if the location of users in space are realisations of a homogeneous Poisson point process, then almost surely there is no power allocation which satisfy all the users' demands whatever is the position and the density of base stations.

Key-words: CDMA networks, macrodiversity, point processes, power control.

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Allocation de puissance optimale dans les réseaux CDMA en macrodiverisité

Résumé : Ce rapport présente et analyse un modèle général de macrodiversité dans les réseaux CDMA (Code Division Multiple Access). Les différences entre la voie montante (de l'usager vers la station de base) et la voie descendante (de la station de base vers l'usager) aboutissent à deux modèles différents qui sont étudiés en parallèle dans le rapport. Pour satisfaire les demandes des usagers en terme de débit, il est nécessaire de mettre en oeuvre une politique de contrôle des puissances des signaux émis. On démontre des conditions nécessaires et suffisantes d'existence d'une allocation de puissance dans le réseau qui satisfasse tous les usagers. Ces conditions dépendent des demandes des usagers et de la géométrie du réseau.

Dans le cas d'un réseau fini sur la voie descendante, on démontre une condition nécessaire sur la politique optimale d'allocation des puissances. Ce résultat montre que la macrodiversité sur la voie descendante a une influence limitée.

L'étude des réseaux infinis se conduit dans le cadre des processus stationnaires ergodiques. Sur la voie montante, il existe une condition nécessaire et suffisante d'existence d'une allocation de puissance permettant de satisfaire les usagers. Cette condition est du type "$\rho < 1$", où ρ n'est fonction que de la demande moyenne de débit et du nombre moyen par unité de surface des usagers et des stations de base. Sur la voie descendante, à l'inverse, il est démontré que si les usagers sont répartis dans l'espace suivant des réalisations d'un processus de Poisson homogène, alors il n'existe presque sûrement pas de politique de contrôle de puissance qui permette de satisfaire toutes les demandes des usagers, et ce, quelque que soit la position et la densité des stations de base.

Mots-clés : contrôle de puissance, macrodiversité, processus ponctuels, réseaux CDMA.

1 Introduction

This paper concerns the characterization of capacity of CDMA networks with macrodiversity. This problem relies on finding a power allocation satisfying all users in the network. Computing the load capacity of such networks is an important issue of wireless communications.

More precisely in this paper, in section 2, for the uplink, we recall the model of Hanly in [11] and the necessary and sufficient condition for the feasibility of power control.

In Section 3 we develop a realistic and mathematically tractable model of macrodiversity for the downlink. The key components of our model are the spatial location of base stations and users, and the bit rates requirements of each user.

In Section 4, we analyze our downlink model and establish a necessary and sufficient condition for the feasibility of the power control problem with macrodiversity and find some sufficient conditions.

Section 5 gives a characterization of the optimal power allocation. This characterization establishes a bound on the increase of capacity brought by macrodiversity in a network.

Section 6 is dedicated to the extension of results on the uplink to infinite networks. In particular we find a threshold which characterizes the feasibility of power allocation in ergodic networks.

Finally, in Section 7 we extend our results to infinite networks and prove a negative result for the feasibility of power control problem when the point process of users is Poisson.

The problem of power control and load constraints in CDMA networks has drawn much attention. However, most authors are only considering CDMA networks without macrodiversity. See in particular the seminal papers of Gilhousen et al. [8] and Zander [16], [17]. The model of the present paper is a generalization of the model without macrodiversity developed by Baccelli et al. [2], [3]. For macrodiversity CDMA networks, Hanly [11], [12] has solved the power control problem on the uplink for finite networks. In this paper, we will follow the approach of Hanly.

2 Uplink Macrodiversity Model

In this section we consider a finite network consisting of M users and N base stations. The users are located at points $\{X_i\}_{1\leq i\leq M} \in \mathbb{R}^2$ and the base station at points $\{Y_j\}_{1\leq j\leq N} \in \mathbb{R}^2$. We denote by $h(x,y)$ the channel gain from y to x, $x,y \in \mathbb{R}^2$. $|h(x,y)|$ represents the path loss due to shadowing, fading and distance attenuation effects.

In an uplink macrodiversity CDMA network, each user sends independently from the other a signal and the base stations are jointly decoding the received signals. This kind of channel is known as multi-receiver networks (see [10]). A base station j receives a signal equal to the sum of all the signals sent by the users plus an external white Gaussian noise. Let $w = (w_j)_{1\leq j\leq N}$ denote the power of the noise vector, $H = (h(X_i,Y_j))_{1\leq i\leq M, 1\leq j\leq N}$, the channel matrix. The user i sends a signal s_i. Let $s = (s_i)_{1\leq i\leq M}$ be the vector of the signal sent by users. Then mathematically the signals received by base stations is a $\mathbb{R}^{M\times 1}$ vector :

$$v = H's + w,$$

or, for all j :

$$v_j = \sum_i h(X_i, Y_j) s_i + w_j.$$

We set the channel bandwidth to 1Hz to simplify notations and we suppose that user i requires a rate R_i in bits per second. Let $S_i = E(|s_i|^2)$ and $W_j = E(|w_j|^2)$ the powers of the signals, some constraints require that the vector $S = (S_1, ..., S_M) \in \mathcal{S}$, a given set. This constraint is for example a power constraint. If the users are sending their signals independently, it is known (see [10]) that the rate vector $(R_1, ..., R_M)$ is achievable if and only if exists $S \in \mathcal{S}$ such that :

$$\forall i, \quad R_i \leq \frac{1}{2} \log_2 (1 + \sum_{j=1}^{N} \frac{S_i |h(X_i, Y_j)|^2}{W_j + \sum_{m \neq i} S_m |h(X_m, Y_j)|^2}).$$

We restraint the model to the case $\mathcal{S} = \mathbb{R}^M$ and :

$$\forall i, \quad R_i \leq \frac{1}{2} \log_2 (1 + \sum_{j=1}^{N} \frac{S_i |h(X_i, Y_j)|^2}{W_j + \sum_{m=1}^{M} S_m |h(X_m, Y_j)|^2}).$$

This last condition is only sufficient but when M is large it is expected not be far from being necessary.

Let $l_{ij} = L(X_i, Y_j) = |h(X_i, Y_j)|^2$ the power loss function. Thus, feasibility of a given rate vector is equivalent to a minimal requirement on the signal on interference ratio :

$$\forall i, \quad h_i \leq \sum_j \frac{L(X_i, Y_j) S_i}{W_j + \sum_m L(X_m, Y_j) S_m}. \tag{1}$$

This last equation is the uplink macrodiversity model in CDMA networks. The following theorem solves the power allocation problem :

Theorem 1 (Hanly). *Suppose for all i, j, $L(X_i, Y_j) > 0$ and $W_j > 0$.*

Then, exists a solution of (1) if and only if

$$\sum_{i=1}^{M} h_i < N.$$

This theorem is surprising because the feasibility condition does not rely on the geometry of the network (i.e. the coefficients $L(X_i, Y_j)$).

3 Downlink Macrodiversity Model

In this section we consider the same network as in the previous section, with the same notations.

In a downlink macrodiversity CDMA network, the base stations are jointly coding a signal for each user and users are decoding independently. This kind of channel is known as multiple input multiple output (MIMO) broadcast channel (see [4],[14]). A user i receives a signal equal to the sum of all the signals sent by the base stations plus an external white Gaussian noise. As above, $w = (w_i)_{1\leq i\leq M}$ denote the noise vector, $H = (h(X_i, Y_j))_{1\leq i\leq M, 1\leq j\leq N}$, the channel matrix and $H_i = (h(X_i, Y_j))_{1\leq j\leq N}$ the channel vector to i . The base station j sends a signal s_{ij} to the user i. Let $s_i = (s_{ij})_{1\leq j\leq N}$ be the vector of the signal sent to i. Then mathematically the signals received by users is a $\mathbb{R}^{N\times 1}$ vector :

$$u = H\sum_{i=1}^{M} s_i + w,$$

or, for all i :

$$u_i = \sum_j h(X_i, Y_j)s_{ij} + \sum_j h(X_i, Y_j)\sum_{m\neq i} s_{mj} + w_i.$$

Let Γ_i be the covariance matrix of $(s_{ij})_{1\leq j\leq N}$ and $W_i = E(|w_i|^2)$ the power of the noise at i. Some constraints require that $(\Gamma_1, ..., \Gamma_M) \in \mathcal{G}$. User i requires a rate R_i in bits per second (on as before a channel bandwidth of 1Hz).

If we make the assumption, that for all j, for all $m \neq i$, the signals s_{mj} are regarded as noise by the base stations in the coding of signal s_i, a classical information theory result (see [6]) implies that the rate vector $R = (R_1, ..., R_M)$ is achievable if there exists $(\Gamma_1, ..., \Gamma_M) \in \mathcal{G}$ such that :

$$\forall i, \quad R_i \leq \frac{1}{2}\log_2(1 + \frac{H_i^*\Gamma_i H_i}{W_i + H_i^*\sum_{m\neq i}\Gamma_m H_i}). \tag{2}$$

Remark it is possible to achieve better rates thanks to *dirty paper coding* (from [5]). The rate vector $R = (R_1, ..., R_M)$ is achievable if there exists a permutation σ of $\{1, ..., M\}$ and $(\Gamma_1, ..., \Gamma_M) \in \mathcal{G}$ such that :

$$\forall i, \quad R_i \leq \frac{1}{2}\log_2(1 + \frac{H_i^*\Gamma_i H_i}{W_i + H_i^*\sum_{m:\sigma(m)>\sigma(i)}\Gamma_m H_i}).$$

In this paper, we only consider achievable rates as in Equation (2) in the case where Γ_i is diagonal : the base stations are sending uncorrelated signals to each user. This is a natural assumption for an efficient coding. We note $S_{ij} = \Gamma_i(j, j)$ and $l_{ij} = L(X_i, Y_j) = |h(X_i, Y_j)|^2$

the power loss function. Then, a rate vector $R = (R_1, ..., R_M)$ is achievable if there exists a power allocation (S_{ij}) such that :

$$\forall i, \quad R_i \leq \frac{1}{2}\log_2(1 + \frac{\sum_j L(X_i, Y_j) S_{i,j}}{W_i + \sum_j L(X_i, Y_j) \sum_{m \neq i} S_{ij}}),$$

thus, feasibility of a given rate vector is equivalent to a minimal requirement on the signal on interference ratio :

$$\forall i, \quad C_i \leq \frac{\sum_j L(X_i, Y_j) S_{ij}}{W_i + \sum_j L(X_i, Y_j) \sum_{m \neq i} S_{mj}}. \tag{3}$$

The set of inequalities (3) is our macrodiversity model for CDMA downlink networks.

4 Power Allocation Problem on Downlink

In this section, we study the power allocation problem (3), following the lines of [2].

4.1 Power Allocation Algebra

The following obvious proposition restates (3).

Proposition 1. *An allocation $(S_{ij})_{1 \leq i \leq M, 1 \leq j \leq N}$ is a solution of (3) if and only if there exists a non-negative matrix $A = (a_{ij})_{1 \leq i \leq M, 1 \leq j \leq N}$ verifying $\forall i$, $\sum_j a_{ij} = 1$ and :*

$$\forall i, j \quad a_{ij} C_i \leq \frac{L(X_i, Y_j) S_{ij}}{W_i + \sum_j L(X_i, Y_j) \sum_{m \neq i} S_{mj}}. \tag{4}$$

For a fixed $A = (a_{ij})$, the restatement (4) reduces our problem to a CDMA network power allocation problem without macrodiversity as it is addressed in [2]. Our $M \times N$ macrodiversity network is equivalent to a $MN \times N$ fixed cell network : each user X_i is subdivided into N independent users $(X_i^j)_{1 \leq j \leq N}$, X_i^j is affiliated to base station j and has a signal to interference ratio requirements of $a_{ij} C_i$.

We note $V = \{A = (a_{ij}) \in \mathbb{R}^{M \times N}, A \geq 0, \forall i \quad \sum_j a_{ij} = 1\}$. Let $h_i = \frac{C_i}{1+C_i}$ and define the linear mapping : $\mathcal{T} : \left\{ \begin{array}{ccc} V & \rightarrow & \mathbb{R}^{N \times N} \\ A & \mapsto & T = (\sum_i a_{ij} h_i \frac{l_{ik}}{l_{ij}})_{1 \leq j,k \leq N} \end{array} \right.$.

Let $W = \mathcal{T}(V)$ and let $\rho(T)$ denotes the spectral radius of the square matrix T.

We then have the following necessary and sufficient condition :

Proposition 2. *Let,*

$$I = \min_{A \in V} \rho(\mathcal{T}(A)) = \min_{T \in W} \rho(T). \tag{5}$$

Equation (4) has a solution if and only if $I < 1$.

proof Since V is compact, $I = \rho(T^*)$, $T^* \in V$. Then, this proposition appears to be a direct consequence of Propositions 3.1 to 3.3 of [2] in the finite dimensional case. □

The following sufficient conditions are useful, since the spectral radius could be difficult to handle.

Proposition 3. *Let $\|\bullet\|$ be a matrix norm (i.e. an algebraic norm, see [13]) on $\mathbb{R}^{N\times N}$, and define :*

$$J = \min_{T\in W} \|T\|. \tag{6}$$

If $J < 1$ then (4) has a solution.

In particular, define for all i and j, $f_{ij} = \sum_k h_i \frac{l_{ik}}{l_{ij}}$ and $\rho_j(T) = \sum_k T_{jk}$. Let,

$$J_1 = \min_{A\in V} \max_j \sum_i a_{ij} f_{ij} = \min_{T\in W} \max_j \rho_j(T). \tag{7}$$

If $J_1 < 1$ then (4) has a solution.

proof This proposition follows from the general result : for all matrix norms, $\rho(T) \leq \|T\|$ (see [13]). The second part of the proposition is an application of this result to $\|T\|_1 = \max_j \sum_k |T_{jk}|$. □

We can notice that the second part of Proposition 3 is a sufficient condition of feasibility which relies only on a local condition on each base station.

Remark It is stated in [13] : $\rho(T) = \inf\{\|T\|, \|\bullet\| \text{ matrix norm }\}$, hence we also have a converse of Proposition 3.

4.2 Bounds on the Spectral Radius

Let $V^* = \{A \in V : \rho(\mathcal{T}(A)) = \min_{T\in W} \rho(T)\}$. We define $W^* = \mathcal{T}(V^*)$ similarly. In this part we establish bounds on the spectral radius on W^*.

Proposition 4. *If $T \in W^*$:*

$$\frac{1}{N}\sum_{i=1}^{M} h_i \leq \rho(T) \leq \sum_{i=1}^{M} h_i. \tag{8}$$

Moreover, if we suppose the power attenuation function is $L(x,y) = l(|x-y|)$ where $r \mapsto l(r)$ is a function tending to 0 at infinity, then there are limit configurations where the bounds in (8) are reached.

It is important to remark that the two bounds do not depend on the geometry of the network : only on the users bit rates requirements. The right hand side of (8) has by itself no interest : this bound is obtained by removing all base stations but one in the network. However, from the second part of Proposition 4, this bound cannot be improved without taking into account geometry.

It is interesting to compare this proposition with Theorem 1. On the uplink, there is a solution to the power allocation if and only if $\frac{1}{N}\sum_{i=1}^{M} h_i < 1$. In a network with macrodiversity, we thus have a strong contrast between uplink and downlink where the condition $\frac{1}{N}\sum_{i=1}^{M} h_i < 1$ is only necessary from Propositions 2 and 4. The geometry plays a key role on the downlink whereas on uplink, existence of a solution does not rely on it.

proof of proposition 4

1. *Left hand side of (8).* We have for all $T \in W$, $\text{trace}(T) = \sum_{i=1}^{M} h_i = \sum_j \lambda_j$, where $(\lambda_j)_j$ are the eigenvalues of T. Since $\rho(T)$ is the largest eigenvalue, we deduce the left hand side.

2. *Right hand side of (8).* Consider the allocation matrix $A \in V$ where the j^{th} column is 1 and all the others are set to 0. We immediately check : $\rho(\mathcal{T}(A)) = \sum_i h_i$.

3. *A limit configuration reaching left hand side of (8).* Consider a network on a line and suppose to simplify : $M = KN$, K integer. Then place the base stations Y_j at locations jr and place K users $(X_1^j, ..., X_K^j)$ at jr. At last, suppose user X_m^j has a bit rate requirement h_m. Consider now the allocation $A = (a_{ij})$, a_{ij} taking value 1 if X_i is an X_m^j and 0 otherwise. We can check directly that $\rho(\mathcal{T}(A))$ tends to $\frac{1}{N}\sum_{i=1}^{M} h_i$ as r tends toward infinity.

4. *A configuration reaching right hand side of (8).* Consider, the case where all M users are at the same location. We can define : $l_j = L(X_i, Y_j)$, for all i. Let D be the diagonal matrix whose diagonal is $(l_1, ..., l_N)$. We can suppose, $l_j > 0$, for all j. In this case, we have $T = D^{-1}SD$, with $S_{jk} = \sum_i a_{ij} h_i$. T and S have the same spectral radius. We remark : $S = U\mathbb{1}^t$, U and $\mathbb{1}$ are $\mathbb{R}^{N\times 1}$ positive vectors and it follows : $\forall T \in W, \quad \rho(T) = \rho(S) = \mathbb{1}^t U = \sum_i h_i$. □

5 Some Necessary Conditions on the Optimal Power Allocation on Downlink

In this section we find an interesting property shared by the optimal allocation matrices A of (5) and we extend this result to the sufficient condition (7).

5.1 Necessary Conditions of (5)

In this part, we will give a simple necessary condition for any $A \in V^*$. This necessary condition gives a hint to understand what macrodiversity can bring in a network.

For a discrete set K, $|K|$ is the cardinal of K. We will suppose that for all $x, y \in \mathbb{R}^2$, $l(x, y) > 0$. This is not limitative since all the result found in this section can be extended to general path-loss function by continuity arguments.

We can also suppose, for $T \in W^*$,

$$\forall j, k, \quad T_{jk} > 0. \tag{9}$$

Indeed, if $T_{jk} = 0$ for some k, then the j^{th} row is equal to 0. Thus, T and the sub-matrix of T obtained by removing the j^{th} row and the j^{th} column have the same spectral radius.

For $A \in V^*$, we define two sets : $I = \{i \in \{1, .., N\}, \exists a_{i,j} \in (0,1) \text{ for some } j\}$ and $K = \{(i, j), a_{i,j} \in (0,1)\}$. I is the set of users in the network for which two or more base stations are actively contributing to satisfy its bit rate requirement.

The main result of this section is the following theorem :

Theorem 2. *We suppose : for all integer n, for all sequences $i_1, ..., i_n$ of $\{1, ..., M\}$ and for all sequences of distinct integers $j_1, ..., j_n$ of $\{1, ..., N\}$, we have (with $j_{n+1} = j_1$) :*

$$\prod_{k=1}^{n} \frac{l_{i_k, j_k}}{l_{i_k, j_{k+1}}} \neq 1. \tag{10}$$

Then :

$$|K| - |I| \leq N - 1. \tag{11}$$

Corollary 1. *For $A \in V^*$, we have :*

$$|I| \leq N - 1.$$

This theorem gives an upper bound to the number of users which are really in macrodiversity ; i.e. to the number of users which are receiving a signal from more than two different base stations. Provided Hypothesis (10) is satisfied, this upper bound does not depend on the geometry. This bound is also surprisingly small : on a typical CDMA network, $M \gg N$, so the proportion of users in macrodiversity is small. Thus, we expect that the minimum of (5) computed only on allocation matrices A in V satisfying $a_{ij} \in \{0, 1\}$ is not far from the minimum over all $A \in V$. In fact, in the special case, $N = 2$ (two base stations) we can show that the two minima are equal.

Hypothesis (10) is not very restrictive in our context. It would be for example almost surely satisfied, if, in a probabilistic setting, we suppose the finite point processes $\{Y_j\}_j$ and $\{X_i\}_i$ admit a density with respect to the Lebesgue measure and $L^{-1}(x, y)$ has almost everywhere zero Lebesgue measure for all x, y.

The end of this section is dedicated to the proof of the theorem.

In the following, $\|\bullet\|$ is any given norm on $\mathbb{R}^{N\times N}$ and $\langle \bullet, \bullet \rangle$ is the usual scalar product on $\mathbb{R}^N$. Id is the identity matrix. Two lemmas are necessary before turning to the proof.

Lemma 1. *Let $\Phi_T(x)$ be the characteristic polynomial of T and $Adj(T)$ its adjoint; for all $H \in \mathbb{R}^{N\times N}$ we have :*

$$\Phi_{T+H}(x) = \Phi_T(x) + \sum_{j,k} H_{jk} Adj(xId - T)_{jk} + o(\|H\|). \tag{12}$$

proof This lemma is simply an expansion of order 1 of $T \mapsto det(xId - T)$ in the neighborhood of T. □

For $T \in W$, we define : $\mathcal{H}^T = \{H \in \mathbb{R}^{N\times N}, T + H \in W\}$.

Lemma 2. *If $T \in W^*$ then on a neighborhood $\mathcal{V}_0$ of 0 :*

$$\forall H \in \mathcal{H}^T \cap \mathcal{V}_0, \quad \langle H v_T, w_T \rangle \leq 0, \tag{13}$$

where, v_T and w_T are respectively the left and right eigenvectors of T associated to eigenvalue $\rho(T)$.

proof From hypothesis (9), T is primitive, hence (from [15]) : $Adj(\rho(T)Id - T) = \Phi'_T(\rho(T)) w_T v'_T$ and $\Phi'_T(\rho(T)) > 0$.

For $x = \rho(T)$, equation (12) reduces to :

$$\Phi_{T+H}(\rho(T)) = \Phi'_T(\rho(T)) \langle H v_T, w_T \rangle + o(\|H\|). \tag{14}$$

If $T \in W^*$, then $\rho(T + H) \geq \rho(T)$ for all $H \in \mathcal{H}^T$. This implies $\Phi_{T+H}(\rho(T)) \leq 0$ for H sufficiently small. (13) follows from (14) and $\Phi'_T(\rho(T)) > 0$. □

We can now prove Theorem 2.

proof of Theorem 2 Let $A = (a_{ij}) \in V^*$ and $T = \mathcal{T}(A) \in W^*$. w and v are the right and left eigenvectors of T associated to $\rho(T)$. For each $i_0 \in I$, we can find $j_1 \neq j_2$ such that $a_{i_0,j_1} > 0$ and $a_{i_0,j_1} > 0$, we define two matrices A_1 and A_2 by :

- $(A_1)_{ij} = A_{ij} + \epsilon\delta_{i,i_0}\delta_{j,j_1} - \epsilon\delta_{i,i_0}\delta_{j,j_2}$ (δ is the Kronecker symbol),
- $(A_2)_{ij} = A_{ij} - \epsilon\delta_{i,i_0}\delta_{j,j_1} + \epsilon\delta_{i,i_0}\delta_{j,j_2}$.

For ϵ small enough A_1 and A_2 are in V, hence $H = \mathcal{T}(A_1) - T$ and $-H = \mathcal{T}(A_2) - T$ are both in $\mathcal{H}^T$. We can apply Lemma 2 and it follows :

$$\begin{aligned} 0 &= \langle Hv, w \rangle \\ &= (\sum_k l_{i_0 k} v_k)(\frac{w_{j_1}}{l_{i_0 j_1}} - \frac{w_{j_2}}{l_{i_0 j_2}}) \end{aligned}$$

The last equality implies, since $l_{i_0 k} > 0$ and $v_k > 0$:

$$\frac{w_{j_1}}{l_{i_0 j_1}} = \frac{w_{j_2}}{l_{i_0 j_2}} \tag{15}$$

The end of the proof relies on a combinatorial argument on graphs. Without loss of generality, we can suppose $I = \{1, ..., |I|\}$. Let $K_i = \{j, (i,j) \in K\}$.

We now define the embedded non-oriented graphs $\mathcal{G}_i$ on the set {1,..,N} of base stations. We put an edge in $\mathcal{G}_i$ between j_1, j_2 if there exists an integer $i_0 \leq i$ such that j_1 and j_2 are in $K(i_0)$. From what precedes, this implies (15).

Similarly we define the graph $\mathcal{K}_i$ by putting an edge between j_1 and j_2 if j_1 and j_2 are in $K(i)$. By construction, we have $\cup_{i=1}^{l} \mathcal{K}_i = \mathcal{G}_l$.

We now remark that Hypothesis (10) together with Equation (15) implies that if there is a path leading from j_1 to j_2 in $\mathcal{G}_i$, there cannot be any edge between j_1 and j_2 in $\mathcal{K}_{i+1}$. More precisely, a set of connected nodes in $\mathcal{G}_i$ and a set of connected nodes in $\mathcal{K}_{i+1}$ cannot have more than two common nodes.

Let N_i be the number of non-isolated nodes in $\mathcal{G}_i$ and $n_c(i)$ be the number of connected components in $\mathcal{G}_i$ not reduced to an isolated node. We obtain :

$$N_1 = |K_1|.$$

The constraint on our embedded graphs implies that adding the edges of $\mathcal{K}_{i+1}$ to $\mathcal{G}_i$ can either merge two distinct connected components of $\mathcal{G}_i$, increase a connected component or add a new connected component. In these three possible cases, the following formula is satisfied :

$$N_{i+1} = N_i + |K_{i+1}| + n_c(i+1) - n_c(i) - 1,$$

at last, by summing this last equation from 1 to $|I| - 1$, we obtain

$$|K| - |I| \leq N - n_c(|I|),$$

which in turn implies (11).

Since $|K_i| \geq 2$, $|K| \geq 2|I|$ and the corollary follows. □

5.2 Sufficient Condition (7)

We use the notations introduced in proposition 3. In this part we study more precisely, the sufficient conditions given by :

$$J_1 = \min_{A \in V} \max_j \sum_i a_{ij} f_{ij} = \min_{T \in W} \max_j \rho_j(T) < 1.$$

For $T \in \mathbb{R}^{N\times N}$, $\|T\|_1 = \max_j \sum_k \|T_{jk}\|$. We define $W_1^* = \{T \in W : \|T\|_1 = \min_{T\in W} \|T\|_1\}$ and V_1^* similarly.

We suppose for all i, j, $f_{ij} > 0$. In fact, if $f_{ij} = 0$ then, either : $h_i = 0$ (the user has no bit rate requirement) or $\forall k, \quad l_{ik} = 0$ (the user's signal on interference ratio is zero). Thus, $f_{ij} = 0$ is of no interest.

Proposition 5. *If $T \in W_1^*$ then $\rho_j(T) = \sum_k T_{jk}$ does not depend on j.*

proof Suppose, $\sum_i a_{ij_1} f_{ij_1} < \sum_i a_{ij_2} f_{ij_2}$. There exists i_0 such that $a_{i_0 j_2} > 0$. Now, we define a new matrix $A^\epsilon = (a^\epsilon_{ij})$ by $a^\epsilon_{ij} = a_{ij} + \epsilon\delta_{i,i_0}\delta_{j,j_1} - \epsilon\delta_{i,i_0}\delta_{j,j_2}$. For ϵ small enough, $A^\epsilon \in V$. We immediately check, using $f_{ij} > 0$, for ϵ small enough, $\max(\rho_{j_1}(\mathcal{T}(A^\epsilon)), \rho_{j_2}(\mathcal{T}(A^\epsilon))) < \max(\rho_{j_1}(\mathcal{T}(A)), \rho_{j_2}(\mathcal{T}(A)))$. □

This proposition has an intuitive interpretation. For an optimal power allocation, all the traffic loads in each base station : $\rho_j = \sum_i a_{ij} f_{ij}$ are equal. However, they are many different ways of equalizing loads in the network and we will now prove a condition on V_1^* which is similar to Theorem 2. To this end, we define for $A \in V$, $I(A) = \{i \in \{1, .., N\}, \exists a_{i,j} \in (0,1)$ for some $j\}$ and $K(A) = \{(i,j), a_{i,j} \in (0,1)\}$.

Theorem 3. *We suppose : for all integer n, for all sequences $i_1, ..., i_n$ of $\{1, ..., M\}$ and for all sequences of distinct integers $j_1, ..., j_n$ of $\{1, ..., N\}$, we have (with $j_{n+1} = j_1$) :*

$$\prod_{k=1}^{n} \frac{f_{i_k,j_k}}{f_{i_k,j_{k+1}}} \neq 1. \tag{16}$$

Then :

$$\exists A \in V_1^* \quad : \quad |K(A)| - |I(A)| \leq N - 1. \tag{17}$$

This theorem and Theorem 2 have the same kind of conclusion. However, we cannot expect to have an equation (17) true for all $A \in V_1^*$. Nevertheless, an optimal solution satisfying (17) appears, in a sense, to be the simplest in V_1^* : this solution minimizes the number of users in macrodiversity.

Remarks done on Theorem 2 are still available. This theorem allows to expect that I and J_1 do not differ strongly : the sufficient condition (7) could not be far from a necessary solution under mild assumptions.

proof First we consider a slighty different optimization problem :

$$\min_{A\in V} \sum_j \rho_j^n = \min_{A\in V} F_n(A). \tag{18}$$

(18) is an optimization of a strictly convex function on a compact convex set. There is a unique solution, say $A_n = (a_{ij})$ to simplify notations.

The Lagrangian associated to (18) is :

$$\mathcal{L}(A) = \sum_j (\sum_i f_{ij} a_{ij})^n + \sum_i \lambda_i (1 - \sum_j a_{ij}) - \sum_{i,j} \mu_{ij} a_{ij}.$$

When the minimum is reached : $\mu_{ij} a_{ij} = 0$, $\lambda_j (1 - \sum_i a_{ij} f_{ij}) = 0$ and

$$\frac{\partial \mathcal{L}}{\partial a_{ij}}(A_n) = 0.$$

By differentiating, we thus obtain the following property :

$$a_{ij} \neq 0 \Rightarrow n f_{ij} \rho_j^{n-1} = \lambda_i. \tag{19}$$

Now if $i_0 \in I$, exists j_1 and j_2, such that $a_{i_0 j_1}$ and $a_{i_0 j_2}$. Then, from (19), we deduce :

$$f_{i_0 j_1} \rho_{j_1}^{n-1} = f_{i_0 j_2} \rho_{j_2}^{n-1}.$$

This last equation is of type (15). We conclude in the same manner : the unique solution A_n of (18) verifies (17). Since $F_n(A)^{1/n}$ converges monotonically toward $\max_j \rho_j(A)$, we have : $\lim_n A_n \in V_1^*$ and this solution satisfies (17). □

6 Macrodiversity Model in Infinite Networks : Uplink Case

From now, we do not suppose anymore, the number of users, M, or the number of base stations, N, to be finite. In this section, we give an extend the work done by Hanly in [11] in the finite networks to infinite networks.

6.1 General Setting

In infinite CDMA networks, our macrodiversity model is still given by the set of inequalities (1). A rate $(h_i)_{i \geq 0} > 0$ is feasible, if there exists a power allocation $(S_i)_{i \geq 0}$ such that :

$$\forall i \in \mathbb{N}, \quad \sum_{j=0}^{\infty} \frac{S_i L(X_i, Y_j)}{W_j + \sum_{m=0}^{\infty} S_m L(X_m, Y_j)} \geq h_i. \tag{20}$$

Consider :

$$G : \begin{cases} {\mathbb{R}^+}^{\mathbb{N}} & \rightarrow & {\mathbb{R}^+}^{\mathbb{N}} \\ (S_i)_i & \mapsto & h_i (\sum_j \frac{L(X_i, Y_j)}{W_j + \sum_{m=0}^{\infty} S_m L(X_m, Y_j)})^{-1} \end{cases}$$

The power allocation problem is equivalent to finding $S \in {\mathbb{R}_*^+}^{\mathbb{N}}$ such that, componentwise : $G(S) \leq S$.

6.2 Ergodic Macrodiversity Network

In this part, we follow the probabilistic setting of [2].

We model base stations and users by considering two point processes on $\mathbb{R}^2$: $\Pi_{bs} = \{(Y_j, W_j)\}_j$ and $\Pi_m = \{(X_i, h_i)\}_i$. W_j and h_i are marks of point processes. We can suppose $W_j > 0$ and $h_i > 0$ for all i, j. Moreover, $\Pi_m \times \Pi_{bs}$ is supposed to be a stationary and ergodic marked point process. We note λ_m (resp. λ_{bs}) the intensity of Π_m (resp. Π_{bs}). The Palm probability of the process Π_m (resp. Π_{bs}) is denoted by P_m^0 (resp. P_{bs}^0). At last, we consider a radial positive attenuation function, that is : $L(x, y) = l(|x - y|)$.

This proposition explains why our model is interesting :

Proposition 6. *Let Π_m and Π_{bs} as above. There exists a power allocation satisfying (20) with probability 0 or 1.*

proof With the notations introduced in the previous subsection, the event $\{(20)$ has a solution$\} = \{$exists $S \in {\mathbb{R}^+}^{\mathbb{N}}$, such that $G(S) \leq S\}$ is invariant under a translation on $\mathbb{R}^2$ since the value T(S) does not change if we translate simultaneously all users and all base stations. Thus, from ergodicity, this event has probability 0 or 1. □

We define the traffic intensity by :

$$\rho = \frac{\lambda_m}{\lambda_{bs}} E_m^0(h_0). \tag{21}$$

The following result is a natural extension of Theorem 1 :

Theorem 4. *Let Π_m and Π_{bs} as above.*

We suppose that one of the two following conditions holds :

- *$x \mapsto xl(x)$ is in $L^1(\mathbb{R})$ and $x \mapsto xl(x)$ is non-increasing on a neighborhood of $+\infty$,*
- *or, exists $\epsilon > 0$ such that $x \mapsto x^{1+\epsilon} l(x)$ is in $L^1(\mathbb{R})$.*

Moreover we suppose :

$$E_{bs}^0(W_0^{-1}) < +\infty, \tag{22}$$

then :

- *If $\rho > 1$, then (20) has almost surely no solution.*
- *If $\rho < 1$, then (20) admits almost surely a solution.*

A parallel can be made between this theorem and the existence of a stationary regime in G/G/s queues. The intensity of user arrival is λ_m and $\frac{E^0_m(h_0)}{\lambda_{bs}}$ plays the role of the mean amount of required service time on the number of service booths. As for $G/G/s$ queues, the limit case $\frac{\lambda_m}{\lambda_{bs}} E^0(h_0) = 1$ is much harder and the power allocation problem is not solved for these networks.

Once again, as for finite CDMA networks on the uplink, the geometry does not appear in the feasibility condition $\rho < 1$. Only the bit rates requirement and the density of the network is involved.

The technical hypothesis on $l(x)$ is used to ensure a rapid decay of the tail of the shot-noise process $\sum_i l(|X_i|)$. It covers a usual model for the attenuation function : $l(x) \sim x^{-\alpha}$, $\alpha > 2$. Hypothesis (22) simplifies the proof of the sufficient condition. The result should hold for weaker assumptions.

The proof of Theorem 4 is not easy and can be found in Appendix. The main idea is to follow the lines of the original proof of Theorem 1 and use ergodicity to ensure convergence and some uniform bounds on shot noise processes.

7 Macrodiversity Model in Infinite Networks : Downlink Case

7.1 General Setting

In infinite CDMA networks, our macrodiversity model is still given by the set of inequalities (3) and Proposition 1 remains obviously true :

Proposition 7. *An allocation $(S_{ij})_{i,j\in\mathbb{N}}$ is a solution of (3) if and only if there exists a non-negative matrix $A = (a_{ij})_{i,j\in\mathbb{N}}$ verifying $\forall i$, $\sum_j a_{ij} = 1$ and :*

$$\forall i, j \quad a_{ij} C_i \leq \frac{L(X_i, Y_j) S_{ij}}{W_i + \sum_j L(X_i, Y_j) \sum_{m \neq i} S_{mj}}. \tag{23}$$

Thus, we can still follow the line of [2] and most results of the previous sections will remain true. We can still define V, W and the linear mapping $\mathcal{T}$. Proposition 2 has an infinite dimensional analogue.

First, we recall some results on infinite recurrent matrices. Let us denote by $T^n = (T^n_{jk})$, the n^{th} power of T. The power series $T_{jk}(z) = \sum_n T^n_{jk} z^n$ have a common convergence radius $R = \frac{1}{\rho}$; ρ is by definition the spectral radius of T. $T_{jj}(R)$ is finite or infinite at the same time for all j, making T respectively *transient* or *recurrent*. For more refer to [15].

Proposition 8. *Let,*

$$I = \min_{A \in V} \rho(\mathcal{T}(A)) = \min_{T \in W} \rho(T),$$

- *if $I < 1$ then (23) has a solution ,*
- *if $I > 1$ then (23) does not admit any solution,*
- *if $I = 1$, let $I = \rho(T^*)$, $T^* \in W$, (23) has a solution if T^* is transient.*

proof We have $I = \mathcal{T}(A^*)$, $A^* \in V$, with a classical diagonal extraction argument. Then, this proposition appears to be a consequence of propositions 3.1 to 3.3 of [2]. □

Similarly, we can find an analogous set of sufficient conditions to proposition 3.

7.2 Poisson/General Macrodiversity Network

We model base stations and users by considering two point processes on $\mathbb{R}^2$: $\Pi_{bs} = \{Y_j\}_j$ and $\Pi_m = \{(X_i, h_i, W_i)\}_i$, h_i and W_i are the marks of the point process. The marks are supposed identically distributed, independent and independent of the rest of the model. Without loss of generality, we can suppose $h_i > 0$ almost surely. We suppose that the point process of users Π_m is a stationary Poisson process of intensity λ_m. At last, we consider a radial attenuation function, that is : $L(x, y) = l(|x - y|)$.

We have the following negative result :

Theorem 5. *If $r \mapsto l(r)$ is differentiable and satisfies, for some real C :*

$$|l'(r)| \leq Cl(r), \tag{24}$$

then :

$$\min_{A \in V} \rho(\mathcal{T}(A)) = \min_{T \in W} \rho(T) = \infty, \quad \textit{almost surely.}$$

Hypothesis (24) covers the classical model used for attenuation functions : $l(r) \sim r^{-\alpha}$, $\alpha > 0$. However, the theorem should be true for a larger class of attenuation functions.

This result is disappointing : whatever the intensity of base stations is, there is no chance of finding a solution to the power allocation problem. It implies that some admission congestion protocol must be enforced in a CDMA network on the downlink. Otherwise, there will always be a local concentration of users which saturates the whole network. If we compare to Theorem 4, this result is in complete opposition with what happens on uplink.

The proof of Theorem 5 relies on classical results on spectral radius (see [15] for details).

Lemma 3. *Let T and S be non-negative matrices (possibly infinite), then :*
- *if $\forall j, k$ $T_{jk} \geq S_{jk}$, then $\rho(T) \geq \rho(S)$,*
- *for all square sub-matrix $\tilde{T}$ of T, $\rho(T) \geq \rho(\tilde{T})$.*

proof of theorem 5 Let R, h be some positive real numbers and M an integer. The event $A_i = \{\Pi_m(B(X_i,R)) \geq M\} \cap \{\forall X_k \in B(X_i,R), \quad h_k > h\}$ has a positive probability, provided h small enough. Hence by independent marking and independency property of Poisson processes, $\sum_i \mathbb{1}_{A_i} = \infty$ almost surely. We consider one of these configurations.

Without loss of generality, we suppose $i = 1$, $\forall k \in \{1 \ldots M\}$, $X_k \in B(X_1, R)$ and $h_k > h$.

Fix $1 > \epsilon > 0$ from hypothesis (24), for ϵ small enough, exists R such that :

$$\forall x \in B(0,R), \forall y \in \mathbb{R}^2, \quad |l(|x-y|) - l(|y|)| \leq l(|y|)\epsilon.$$

Hence, for all $X_i \in B(0,R)$ we easily check :

$$|\frac{L(0,Y_k)}{L(0,Y_j)} - \frac{L(X_i,Y_k)}{L(X_i,Y_j)}| \leq \frac{2\epsilon}{1-\epsilon}\frac{L(0,Y_k)}{L(0,Y_j)}. \tag{25}$$

Let $T = \mathcal{T}(A) \in W$, we have

$$T_{jk} \geq \tilde{T}_{jk} = h\sum_{i=1}^{M} a_{ij}\frac{L(X_i,Y_k)}{L(X_i,Y_j)},$$

and, by lemma 3, $\rho(T) \geq \rho(\tilde{T})$.

Now, if $\tilde{T}^{(N)}$ denotes the sub-matrix of $\tilde{T}$ extracted from the first N rows and N columns, from (25), we deduce :

$$\tilde{T}^{(N)}_{jk} \geq h(1 - \frac{2\epsilon}{1-\epsilon})\frac{L(0,Y_k)}{L(0,Y_j)}\sum_{i=1}^{M} a_{ij}. \tag{26}$$

Moreover, exists N such that $\sum_{j=1}^{N}\sum_{i=1}^{M} a_{ij} \geq M(1-\epsilon)$. For such N, define, the $N \times N$ matrix, $\hat{T}^{(N)}$, with $\hat{T}^{(N)}_{jk}$ is equal to the right hand side of (26). From lemma 3, $\rho(\tilde{T}) \geq \rho(\tilde{T}^{(N)}) \geq \rho(\hat{T}^{(N)})$. In the 4^{th} paragraph of the proof of proposition 4, we have already computed the spectral radius of such matrices. We find :

$$\rho(T) \geq \rho(\hat{T}^{(N)}) \geq hM(1-3\epsilon).$$

We thus have proved that $\rho(T)$ cannot be upper bounded. □

8 Conclusion and Future Works

In this paper, we have in a general setting, studied what macrodiversity can bring in a CDMA network. We have shown that uplink and downlink have drastically different behaviors toward macrodiversity.

On the downlink, the results tend to limit the impact of macrodiversity. From Theorem 2, we know that the main improvement between a fixed cell network and a macrodiversity network seems to be in the flexibility into affecting each user to a specific base station and not on the possibility to share a user between several base stations. Thus a starting point of a decentralized power allocation protocol would be to consider an adaptative allocation for each user to a single base station. This would use the geometry of the network better that it has been done so far. Future work will concentrate on this issue.

On the contrary, as expected, for the uplink, in view of Theorem 4, macrodiversity has a much larger impact and appears as a major improvement compared to traditional cellular network structure.

9 Appendix : Proof of Theorem 4

The following lemma on shot noise processes is needed in the proof. In what follows, $|\bullet|$ is the Euclidean norm and $B(x,R)$ is the closed ball of center x and radius R.

Lemma 4. *Let $\Pi = \{(X_i, Z_i)\}_i$ be a stationary marked point process on $\mathbb{R}^d \times \mathbb{R}^+$. We suppose Π has a finite intensity λ and $E^0(Z_0) < \infty$. Let $\alpha < 1$ and $x \mapsto l(x)$ a non-negative function on $\mathbb{R}$. If $x \mapsto x^{d-1}l(x)$ is in $L^1(\mathbb{R})$ and $x \mapsto x^{d-1}l(x)$ is non-increasing on a neighborhood of $+\infty$, or if exists $\epsilon > 0$ such that $x \mapsto x^{d-1+\epsilon}l(x)$ is in $L^1(\mathbb{R})$. Then, almost surely :*

$$\liminf_{R\to+\infty} \sup_{x\in B(0,\alpha R)} \sum_{X_i\notin B(0,R)} Z_i l(|x-X_i|) = 0.$$

proof Suppose for exemple, $x \mapsto x^{d-1}l(x)$ is non-increasing on a neighborhood of $+\infty$ For n integer, let $C_n(R) = \{x \in \mathbb{R}^d : x \in B(0,(n+1)R)\backslash B(0,nR)\}$. We can write for all $x \in B(0,\alpha R)$:

$$\sum_{X_i\notin B(0,R)} Z_i l(|x-X_i|) \leq \sum_{n=1}^{\infty} l((n-\alpha)R) \sum_{X_i} Z_i \mathbb{1}_{X_i\in C_n(R)}.$$

From Campbell formula, we deduce :

$$\begin{aligned}
E \sup_{x\in B(0,\alpha R)} \sum_{X_i\notin B(0,R)} Z_i l(|x-X_i|) &\leq \lambda \sum_{n=1}^{\infty} l((n-\alpha)R) \int_{\mathbb{R}^d}\int_0^{+\infty} z\mathbb{1}_{x\in C_n(R)} P^0(dz)dx \\
&\leq \lambda \sum_{n=1}^{\infty} l((n-\alpha)R)\nu_d E^0(Z_0)\pi R^d((n+1)^d - n^d) \\
&\leq \lambda C_d R E^0(Z_0) \sum_{n=1}^{\infty} l((n-\alpha)R)R^{d-1}n^{d-1}.
\end{aligned}$$

From the hypothesis on $x \mapsto x^{d-1}l(x)$, we can apply the dominated convergence theorem to conclude :

$$\lim_{R\to+\infty} E \sup_{x\in B(0,\alpha R)} \sum_{X_i\notin B(0,R)} Z_i l(|x-X_i|) = 0.$$

In order to get the result in almost sure convergence, it suffices to recall that from any sequence converging in L^1, we can extract a sequence converging almost surely. We thus obtain the stated result. The case $x \mapsto x^{d-1+\epsilon}l(x)$ in $L^1(\mathbb{R})$ is similar. □

This lemma will be used to build a stationary solution.

Lemma 5. *With the hypothesis of theorem 4, the mapping G as it is defined in paragraph 6.1 is continuous on $G^{-1}({\mathbb{R}_*^+}^{\mathbb{N}})$.*

proof We can prove continuity for L^∞-norm on ${\mathbb{R}^+}^{\mathbb{N}}$: $\|S\|= \sup_{i\in\mathbb{N}}|S_i|$. The proof is a classical dominated convergence argument. □

proof of theorem 4 The idea is to follow the proof of Hanly in the finite case and use ergodicity and the uniform bound given by Lemma 4 to extend to infinite case.

Case $\rho > 1$.

Suppose exists a solution of (20) with a positive probability. Let $\underline{0} = (0)_{i\in\mathbb{N}}$, from Hypothesis (22), for all i, $G(\underline{0})_i > 0$. From Proposition 6, there is almost surely a solution $S = (S_i)$. We have component-wise $G(S) \le S$. The function G is monotonous component-wise : if $S \le S'$ then $G(S) \le G(S')$. We deduce that $G(\underline{0}) \le G(S) \le S$ and for all i, $G^n(\underline{0})_i$ is an increasing sequence and is upper bounded by S_i. This sequence converges toward S_i^*, which by continuity (Lemma 5) satisfies $S^* = G(S^*)$. Since G is invariant under a translation, we can define a solution (S_i^*) as a mark on Π_m. For the sake of simplicity, we drop the $''*''$ exponent in S^* and suppose directly $G(S) = S$, $S_i > 0$.

We consider the thinned point process : $\Pi_m^\sigma = \sum_i \mathbb{1}_{\sigma^{-1}<S_i<\sigma}\delta_{\{X_i,h_i,S_i\}}$, this marked point process is still stationary and ergodic. Let λ_m^σ be its intensity. The Palm probability of Π_m^σ is $P_{m,\sigma}^0(\bullet) = P_m^0(\bullet|S_0 \in (\sigma^{-1},\sigma))$, (see [1]). Since, S_i is finite and positive for all i, for σ large enough, we still have :

$$\frac{\lambda_m^\sigma}{\lambda_{bs}} E_{m,\sigma}^0(h_0) > 1,$$

thus we can suppose directly that $\sigma^{-1} < S_i < \sigma$ for all i.

Exists $\eta > 0$ such that : $\rho > (1-\eta)^{-1}$. Also, exists $\alpha < 1$, such that : $\alpha^2\rho > (1-\eta)^{-1}$. To simplify notations, let $N_R = \sum_j \mathbb{1}_{Y_j\in B(0,R)}$ and, similarly, $M_R = \sum_i \mathbb{1}_{X_i\in B(0,R)}$. Now, from ergodicity of our model, exist R_1 such that, for all $R > R_1$:

$$\frac{1}{N_R}\sum_{i=1}^{M_{\alpha R}} h_i > (1-\eta)^{-1},$$

and for $C > 1$:

$$\frac{M_{\alpha R}}{N_R} < C\frac{\lambda_m}{\lambda_{bs}}\alpha^2.$$

Let $Z_j = W_j^{-1}$, now, from Lemma 4, almost surely :

$$\liminf_{R\to+\infty} \sup_{X_i\in B(0,\alpha R)} \sum_{Y_j\notin B(0,R)} Z_j l(|X_i - Y_j|) = 0.$$

Let $\epsilon > 0$, we can find $R > R_1$ such that :

$$\sup_{X_i\in B(0,\alpha R)} \sum_{Y_j\notin B(0,R)} Z_j l(|X_i - Y_j|) < \epsilon.$$

Now, for such R,

$$\begin{aligned}
(1-\eta)^{-1} &< \frac{1}{N_R}\sum_{i=1}^{M_{\alpha R}} h_i \\
&< \frac{1}{N_R}\sum_{i=1}^{M_{\alpha R}}\sum_{j=1}^{\infty} \frac{S_i L(X_i,Y_j)}{W_j + \sum_{m=1}^{\infty} S_m L(X_m,Y_j)} \\
&< \frac{1}{N_R}\sum_{i=1}^{M_{\alpha R}}\sum_{j=1}^{N_R} \frac{S_i L(X_i,Y_j)}{W_j + \sum_{m=1}^{M_{\alpha R}} S_m L(X_m,Y_j)} + \frac{1}{N_R}\sum_{i=1}^{M_{\alpha R}} \sum_{Y_j\notin B(0,R)} \sigma Z_j l(|X_i - Y_j|) \\
&< 1 + \epsilon\sigma C\frac{\lambda_m}{\lambda_{bs}}\alpha^2.
\end{aligned}$$

For ϵ small enough we get a contradiction.

Case $\rho < 1$ and $h_i < h$ for all i.

The base argument of Hanly is a change of variables and an application of Brouwer's fixed point theorem (see [9]). Hanly defines :

$$g : \begin{cases} \bigotimes_{i\in\mathbb{N}}(h_i,+\infty] & \to & {\mathbb{R}^+}^{\mathbb{N}} \\ (t_i)_{i\in\mathbb{N}} & \mapsto & (\frac{h_i}{t_i - h_i})_i \end{cases},$$

and

$$f_i : \begin{cases} {\mathbb{R}^+}^{\mathbb{N}} & \to & \mathbb{R}^+ \\ (S_m)_{m\in\mathbb{N}} & \mapsto & \sum_{j=1}^{\infty} \frac{(S_i+1)L(X_i,Y_j)}{W_j + \sum_{m=1}^{\infty} S_m L(X_m,Y_j)} \end{cases}.$$

Let $\epsilon > 0$ and define :

$$\phi^\epsilon : \left\{ \begin{array}{lcl} \bigotimes_{i\in\mathbb{N}}[h_i(1+\epsilon),\frac{1}{\epsilon}] & \to & \bigotimes_{i\in\mathbb{N}}[h_i(1+\epsilon),\frac{1}{\epsilon}] \\ (t_i)_{i\in\mathbb{N}} & \mapsto & \left(\left\{ \begin{array}{lll} f_i\circ g_i(t_i) & if & f_i\circ g(t)\in[h_i(1+\epsilon),\frac{1}{\epsilon}] \\ h_i(1+\epsilon) & & f_i\circ g(t) < h_i(1+\epsilon) \\ \frac{1}{\epsilon} & & f_i\circ g(t) > \frac{1}{\epsilon} \end{array} \right. \right)_{i\in\mathbb{N}} \end{array} \right. .$$

From Hypothesis (22) and Lemma 4, it is easy to see that f_i is continuous on $\bigotimes_{i\in\mathbb{N}}[\frac{h_i}{\epsilon^{-1}-h_i},\frac{1}{\epsilon}]$ for the L^∞-norm. Thus, ϕ^ϵ is a continuous map. $\bigotimes_{i\in\mathbb{N}}[h_i(1+\epsilon),\frac{1}{\epsilon}]$ is a compact convex set and hence by Brouwer's fixed point theorem : exists t^ϵ such that $\phi^\epsilon(t^\epsilon)=t^\epsilon$. We will first show that we can extract a converging sequence from t^ϵ.

We consider the thinned point process : $\Pi_{bs}^{q,w}=\sum_j \mathbb{1}_{W_j>w}\mathbb{1}_{\sum_i l(|X_i-Y_j|)<q}\delta_{\{Y_j,W_j\}}$, this point process is still stationary and ergodic. Let $\lambda_{bs}^{q,w}$ be its intensity. Since, $\sum_j l(|X_i-Y_j|)$ is almost surely finite for all j and $W_j>w$, for q large and w small, we still have :

$$\frac{\lambda_m}{\lambda_{bs}^{q,w}}E_m^0(h_0)>1,$$

thus we can suppose directly that $\sum_i l(|X_i-Y_j|)<q$ and $W_j>w$ for all j.

Let $a>h$ large enough to guarantee : $\frac{h}{a-h}\frac{q}{w}<a$ and suppose $t_i^\epsilon\geq a$. Then $S_i^\epsilon=(g(t^\epsilon))_i\leq\frac{h_i}{a-h_i}$. Hence $a\leq t_i^\epsilon\leq f_i(S^\epsilon)\leq\frac{h_i}{a_i-h_i}\sum_j\frac{L(X_i,Y_j)}{W_j}<a$. Thus, we have proved : for all i, $t_i^\epsilon\in[h_i,a]$. We thus can extract a sequence t^ϵ converging toward $t\in\bigotimes_{i\in\mathbb{N}}[h_i,a]$. We now want to show that $\lim_{\epsilon\to 0}g(t^\epsilon)$ exists. To do so, we prove that for all i, exists ϵ_i such that for all $\epsilon<\epsilon_i$, t_i^ϵ satisfies : $t_i^\epsilon>h_i(1+\epsilon_i)$.

Suppose that for some i, for all $\eta>0$, exists $\epsilon<\eta$ such that : $t_i^\epsilon=h_i(1+\epsilon)$. We consider a sequence of such ϵ. Let $S_m^\epsilon=(g(t^\epsilon))_m$ and $I_j^\epsilon=\sum_m S_m^\epsilon L(X_m,Y_j)$, the interference at base station j. We have $I_j^\epsilon\geq\epsilon^{-1}L(X_i,Y_j)$, thus for all j : $\lim_{\epsilon\to 0}I_j^\epsilon=+\infty$. Since $t_k^\epsilon=\max(\sum_j\frac{(S_k^\epsilon+1)L(X_k,Y_j)}{W_j+I_j^\epsilon},h_k(1+\epsilon))$, by a dominated convergence argument we deduce that S_k^ϵ cannot be bounded, hence for all k :

$$\lim_{\epsilon\to 0}t_k^\epsilon=h_i.$$

Since $\rho<1$, exists $\alpha>1$ such that :

$$\frac{\lambda_m\alpha^2}{\lambda_{bs}}E_m^0(h_0)<1.$$

Thus, ergodicity implies :

$$\lim_{\epsilon\to 0}\lim_R\frac{1}{N_R}\sum_{i=1}^{M_{\alpha R}}t_i^\epsilon=\frac{\lambda_m\alpha^2}{\lambda_{bs}}E_m^0(h_0)<1. \tag{27}$$

Since t^ϵ is a fixed point, we have for $\epsilon < a^{-1}$:

$$\frac{1}{N_R}\sum_{i=1}^{M_{\alpha R}} t_i^\epsilon = \frac{1}{N_R}\sum_{i=1}^{M_{\alpha R}} \phi_i^\epsilon(t^\epsilon) \geq \frac{1}{N_R}\sum_{i=1}^{M_{\alpha R}} f_i \circ g(t^\epsilon), \tag{28}$$

We write :

$$\begin{aligned}
\frac{1}{N_R}\sum_{i=1}^{M_{\alpha R}} f_i \circ g(t^\epsilon) &\geq \frac{1}{N_R}\sum_{i=1}^{M_{\alpha R}}\sum_{j=1}^{N_R} \frac{L(X_i,Y_j)(S_i^\epsilon+1)}{W_j+I_j^\epsilon} \\
&\geq \frac{1}{N_R}\sum_{j=1}^{N_R}\frac{I_j^\epsilon+\sum_i L(X_i,Y_j)}{W_j+I_j^\epsilon} - \frac{1}{N_R}\sum_{j=1}^{N_R}\sum_{X_i\notin B(0,\alpha R)}\frac{L(X_i,Y_j)(\epsilon^{-1}+1)}{w} \\
&\geq \frac{1}{N_R}\sum_{j=1}^{N_R}\frac{I_j^\epsilon+\sum_i L(X_i,Y_j)}{W_j+I_j^\epsilon} - \sup_{y\in B(0,R)}\sum_{X_i\notin B(0,\alpha R)}\frac{L(X_i,y)(\epsilon^{-1}+1)}{w}.
\end{aligned}$$

Now, by letting R tends toward infinity, using Lemma 4, we obtain :

$$\liminf_{R\to+\infty}\frac{1}{N_R}\sum_{i=1}^{M_{\alpha R}} f_i\circ g(t^\epsilon) \geq \lim_R \frac{1}{N_R}\sum_{j=1}^{N_R}\frac{I_j^\epsilon+\sum_i L(X_i,Y_j)}{W_j+I_j^\epsilon},$$

We can apply the ergodic theorem for point processes (see [7]) :

$$\liminf_{R\to+\infty}\frac{1}{N_R}\sum_{i=1}^{M_{\alpha R}} f_i\circ g(t^\epsilon) \geq E^0_{bs}(\frac{I_0^\epsilon}{W_0+I_0^\epsilon}) + E^0_{bs}(\frac{\sum_i L(X_i,0)}{W_0+I_0^\epsilon}),$$

Letting ϵ tends toward 0 and using the dominated convergence theorem, we conclude that :

$$\lim_{\epsilon\to 0}\liminf_{R\to+\infty}\frac{1}{N_R}\sum_{i=1}^{M_{\alpha R}} f_i\circ g(t^\epsilon) \geq 1.$$

This last inequality together with (28) contradicts (27). Thus we cannot have $t_i^\epsilon = h_i(1+\epsilon)$ an infinite number of times. We have proved that for $\epsilon < \epsilon_i$, $t_i^\epsilon > h_i(1+\epsilon_i)$. Since $g_i(t) = \frac{h_i}{t-h_i}$ is a continuous map on $[h_i(1+\epsilon_i), a]$, we can define : $S_i^* = g_i(t_i) = \lim_{\epsilon\to 0} g_i(t_i^\epsilon)$. From the continuity of f_i :

$$f_i(S^*) = h_i\frac{S_i^*+1}{S_i^*},$$

which is equivalent to :

$$h_i = \sum_{j=1}^{\infty}\frac{S_i^* L(X_i,Y_j)}{W_j+\sum_{m=1}^{\infty} S_m^* L(X_m,Y_j)}.$$

This concludes the proof of the theorem when $h_i < h$ for all i.

Case $\rho < 1$, general case.
Let $h > 0$, we consider a new user point process : $\Pi'_m = \sum_i \lceil \frac{h_i}{h} \rceil \delta_{\{X_i, h\lceil \frac{h_i}{h} \rceil^{-1}\}}$. Since, by hypothesis, the marked point process $\{(X_i, h_i)\}$ is ergodic, Π'_m is a stationary ergodic marked point process, its marks : $h\lceil \frac{h_i}{h} \rceil^{-1}$ are upper bounded by h. Moreover, if we find a power allocation satisfying (20) for Π'_m, by additivity of (20), we have found a solution of (20) for Π_m. A direct computation shows that $\lambda'_m \leq \lambda_m(\frac{E^0_m(h_0)}{h} + P^0_m(h_0 \geq h))$. Hence for h large enough, $\frac{\lambda'_m}{\lambda_{bs}} E^0_{m'}(h_0) < 1$. This conclude the proof in the general case. □

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