Tails in Generalized Jackson Networks with Subexponential Service Distributions

François Baccelli, Serguei Foss, Marc Lelarge

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

Tails in Generalized Jackson Networks with Subexponential Service Distributions

François Baccelli — Serguei Foss — Marc Lelarge

N° 5081

January 2004

THÈME 1

Rapport de recherche

ISSN 0249-6399 ISRN INRIA/RR--5081--FR+ENG

Tails in Generalized Jackson Networks with Subexponential Service Distributions

François Baccelli * , Serguei Foss † , Marc Lelarge ‡

Thème 1 — Réseaux et systèmes
Projet TREC

Rapport de recherche n° 5081 — January 2004 — 21 pages

Abstract: We give the exact asymptotic of the tail of the stationary maximal dater in generalized Jackson networks with subexponential service times. This maximal dater, which is an analogue of the workload in an isolated queue, gives the time to clear all customers present at some time t when stopping all arrivals taking place later than t. We use the property that a large deviation of the maximal dater is caused by a single large service time in a single station at some distant time in the past of t and fluid limits of generalized Jackson networks to derive the asymptotic in question in closed form.

Key-words: generalized Jackson networks, subexponential random variable, heavy tail, integrated tail, Veraverbeke's Theorem, fluid limits.

* INRIA-ENS, ENS, 45 rue d'Ulm, 75005 Paris, France, Francois.Baccelli@ens.fr
† Institute of Mathematics, 630090 Novosibirsk, Russia & Department of Actuarial Mathematics and Statistics, Heriot-Watt University, Edinburgh, UK, S.Foss@ma.hw.ac.uk, the work of this author was supported by INTAS grant no 265.
‡ INRIA-ENS, ENS, 45 rue d'Ulm, 75005 Paris, France, Marc.Lelarge@ens.fr

Unité de recherche INRIA Rocquencourt
Domaine de Voluceau, Rocquencourt, BP 105, 78153 Le Chesnay Cedex (France)
Téléphone : +33 1 39 63 55 11 —Télécopie : +33 1 39 63 53 30

Queues de distributions dans des réseaux de Jackson généralisés avec des temps de service sous-exponentiels

Résumé : Nous donnons l'asymptotique exacte de la queue de distribution du dateur maximal stationnaire dans des réseaux de Jackson généralisés avec des temps de service sous-exponentiels. Ce dateur maximal, qui correspond à la charge dans le cas d'une file en isolation, donne le temps de traitement de tous les clients présents à l'instant t quand on arrête le processus des arrivées à partir de ce temps t. Pour obtenir l'asymptotique en question, nous utilisons la propriété qu'une grande déviation du dateur maximal est due à un unique grand service dans une unique station à un certain temps dans le passé t et des propriétés de limites fluides des réseaux de Jackson généralisés.

Mots-clés : réseaux de Jackson généralisés, variable aléatoire sous-exponentielle, distribution à queue lourde, distribution intégrée, Théorème de Veraverbeke, limites fluides.

1 Introduction

To the best of our knowledge, the literature on generalized Jackson networks with heavy tailed service times is limited to tandem queues. Bounds on the tail asymptotics of waiting and response times were considered in [2] and [8]. Exact asymptotics for these quantities were obtained in [5]. The present paper addresses the case of generalized Jackson networks with arbitrary topology. It focuses on a key state variable, already used in the past for determining the stability region of such networks [3], which is the time to empty the network when stopping the arrival process (this variable boils down to the virtual workload in an isolated queue or to the sojourn time for queues in tandem). The aim of the paper is to derive an exact asymptotic for the tail of this state variable in the stationary regime. The main ingredients for the derivation of this result are

- a generalization of the so called "single big event theorem", well known for isolated queues, to such generalized Jackson networks which was established in [5]; In the $GI/GI/1$ queue, this theorem states that in the case of subexponential service times, large workloads occur on a typical event where a single large service time has taken place in a distant past, and all other service time are close to their mean. Similarly, in generalized Jackson networks with subexponential service times, large maximal daters occur when a single large service time has taken place in one of the stations, and all other service times are close to their mean.
- a fluid limit for this class of networks which was proposed in [10];
- the computation of the multiplicative constants and of the argument of the second tail of the service times which is based on Markov chain analysis and on the fluid limit which significantly simplifies the step allowing one to get closed form formulas for the asymptotics.

Although this result sheds light on the way such a network experiences a deviation from its normal behavior, it is in no way final as the tail behavior of other state variables such as stationary queue size are still unknown. The derivation of the (more complex) asymptotic behavior of these other state variables was already obtained using a similar methodology in the particular case of tandem queues [5]. The extension of these queue size asymptotics to generalized Jackson networks with arbitrary topology seems to require much more effort and will not be pursued in the present paper. The proposed method should however extend to other characteristics of stationary workload like for instance the sum of the residual service times of all customers present in the network at some given time.

The paper is structured as follows. In Section 2, we introduce generalized Jackson networks and show that the 4 assumptions needed for applying the single big event theorem (called **(IA)**, **(AA)**, **(SE)** and **(H)** in [4]) hold. The main result is then established in Section 3.

Notation
Here and later in the paper, for positive functions f and g, the equivalence $f(x) \sim dg(x)$

with $d > 0$ means $f(x)/g(x) \to d$ as $x \to \infty$. By convention, the equivalence $f(x) \sim dg(x)$ with $d = 0$ means $f(x)/g(x) \to 0$ as $x \to \infty$, this will be written $f(x) = o(g(x))$. We will also use the notation $f(x) = \mathcal{O}(g(x))$ to mean $\limsup f(x)/g(x) < \infty$ and $\liminf f(x)/g(x) > 0$. In this paper, $\epsilon(x)$ denotes a function such that $\epsilon(x) \xrightarrow{x\to\infty} 0$. The function ϵ may vary from place to place; for example, $\epsilon(x) + \epsilon(x) = \epsilon(x)$, $\epsilon(x)(1 + \epsilon(x)) = \epsilon(x)$, etc. Similarly, we will write $\epsilon(x, y)$ for $\epsilon(x) + \epsilon(y)$, or $\epsilon(x)\epsilon(y)$, etc.

2 Generalized Jackson Networks

2.1 General Framework and Stochastic Assumptions

Service time and routing sequences
We recall here the notation introduced in [4], to describe a generalized Jackson network with K nodes.
The networks we consider are characterized by the fact that service times and routing decisions are associated with stations and not with customers. This means that the j-th service on station k takes $\sigma_j^{(k)}$ units of time, where $\{\sigma_j^{(k)}\}_{j\geq 1}$ is a predefined sequence. In the same way, when this service is completed, the leaving customer is sent to station $\nu_j^{(k)}$ (or leaves the network if $\nu_j^{(k)} = K + 1$) and is put at the end of the queue on this station, where $\{\nu_j^{(k)}\}_{j\geq 1}$ is also a predefined sequence, called the routing sequence. The sequences $\{\sigma_j^{(k)}\}_{j\geq 1}$ and $\{\nu_j^{(k)}\}_{j\geq 1}$, where k ranges over the set of stations, are called the driving sequences of the net. Node 0 models the external arrival of customers in the network, then the arrival time of the j-th customer in the network takes place at $\sigma_1^{(0)} + \cdots + \sigma_j^{(0)}$ and it joins the end of the queue of station $\nu_j^{(0)}$. Hence $\sigma_j^{(0)}$ is the j-th inter-arrival time.

Assumption 1, on the independence of routing and service times
All the sequences $\{\nu^{(k)}\}$ and $\{\sigma^{(k')}\}$ are mutually independent for k, k' ranging over the set of stations.

Assumption 2, on the independence of service times
We will assume the service times are independent for different stations and i.i.d. in each station with finite mean: $\mathbb{E}(\sigma^{(j)}) = \frac{1}{\mu^{(j)}} > 0$ for all $1 \leq j \leq K$.

Assumption 3, on routing
We assume that each of the successful routes used to build ν is obtained by a Markov chain

on the state space $\{0, 1, \ldots, K, K+1\}$ with transition matrix

$$R = \begin{pmatrix} 0 & p_{0,1} & \cdots & \cdots & p_{0,K} & 0 \\ \vdots & p_{1,1} & p_{1,2} & \cdots & p_{1,K} & p_{1,K+1} \\ \vdots & p_{2,1} & p_{2,2} & \cdots & p_{2,K} & p_{2,K+1} \\ \vdots & & & & & \vdots \\ 0 & \ldots & \ldots & \ldots & 0 & 1 \end{pmatrix}$$

This is equivalent to assuming that the routing decisions $\{\nu_j^{(k)}\}$ in station k are i.i.d. in j, independent of everything else, and such that the routing decision selects station i with probability $\mathbb{P}[\nu^{(k)} = i] = p_{k,i}$.
The fact that the routes built with this Markovian procedure are successful implies that state $K+1$ is the only absorbing state of this chain and all other states are transient; we then have the very same Markovian routing assumptions as in (exponential) Jackson Networks. More generally, when denoting by $\mathbb{E}_k$ the law of the chain with initial condition k, and V_j the number of visits of this absorbing chain in state j, we define:

$$\mathbb{E}_0[V_k] = \pi_k, \quad \mathbb{P}_0[V_k \geq 1] = p_k, \quad \mathbb{E}_k[V_j] = \pi_{k,j}, \quad \mathbb{P}_k[V_j \geq 1] = x_{k,j}. \tag{1}$$

We will use the following notation:

$$b_j = \frac{\pi_j}{\mu^{(j)}}, \quad b_{j,i} = \frac{\pi_{j,i}}{\mu^{(i)}}, \quad B_j = \max_i b_{j,i}.$$

With this notation, we denote by $b = \max_i \pi_i/\mu^{(i)} = \max_i b_i$. Let $\lambda^{-1} = \mathbb{E}[\sigma_0] = a$. Throughout this paper we will assume that:

$$\lambda b < 1. \tag{2}$$

We recall here some definitions

Definition 1. *A distribution function F on $\mathbb{R}_+$ is long tailed if for any $y > 0$,*

$$\overline{F}(x+y) \sim \overline{F}(x) \quad \textit{as} \quad x \to \infty.$$

We introduce a proper subset of the class of long tailed distributions, the class of subexponential distributions denoted by $\mathcal{S}$:

Definition 2. *A distribution function F on $\mathbb{R}_+$ is called subexponential if $\overline{F^{*2}}(x) \sim 2\overline{F}(x)$.*

Definition 3. *A positive measurable function f on $[0, +\infty)$ is called regularly varying with index $\alpha \in \mathbb{R}$ ($f \in \mathcal{R}(\alpha)$) if*

$$\lim_{x \to \infty} \frac{f(tx)}{f(x)} = t^\alpha \quad \textit{for all } t > 0.$$

Definition 4. *A positive measurable function h on $[0,+\infty)$ is called rapidly varying ($h \in \mathcal{R}(-\infty)$) if*

$$\lim_{x\to\infty} \frac{h(tx)}{h(x)} = 0 \quad \textit{for all } t > 1.$$

For example, Weibull or lognormal random variables have tail distributions that are rapidly varying.

Assumption 4, on the subexponentiality of service times

We will denote by $\mathcal{S}$ the class of subexponential distribution functions on the positive real line. For a distribution function F on the positive real line with finite first moment $M = \int_0^\infty \overline{F}(u)du$, $\overline{F}(u) = 1 - F(u)$ denotes the tail of F and F^s the integrated tail distribution:

$$F^s(x) = 1 - \min\left\{1, \int_x^\infty \overline{F}(u)du\right\} \stackrel{\text{def}}{=} 1 - \overline{F}^s(x).$$

The assumptions concerning service times are the following: there exists a distribution function F on $\mathbb{R}_+$ such that:

1. F is subexponential, with finite first moment M.
2. The integrated distribution F^s is subexponential.
3. The following equivalence holds when x tends to ∞:

$$\mathbb{P}(\sigma_1^{(k)} > x) \sim c^{(k)}\overline{F}(x),$$

for all $k = 1, \dots, K$ with $\sum_{k=1}^K c^{(k)} = c > 0$.

2.2 Main Result

Let $f^j(\sigma, n)$ be the following piece-wise linear function of (σ, n), where σ and n are non-negative real numbers:

$$f^j(\sigma,n) = \mathbb{1}_{\{\sigma > na\}}\{\sigma - na + np_j B_j\} + \mathbb{1}_{\{\sigma \le na\}} \max_k \left\{p_j b_{j,k}\frac{\sigma}{a} + \left(\frac{b_k}{a} - 1\right)(na - \sigma)\right\}^+ \quad (3)$$

and for all positive real numbers x, and all $j = 1, \dots, K$, let $\Delta^j(x)$ be the following domain:

$$\Delta^j(x) \quad = \quad \{(\sigma, t) \in \mathbb{R}_+^2,\ f^j(\sigma, t) > x\}. \quad (4)$$

Theorem 1. *Consider a stable generalized Jackson network with subexponential service time distributions satisfying assumptions 1-4. Let Z denote its stationary maximal dater at customer arrivals (see section 2.3 for a formal definition of Z). When $x \to \infty$,*

$$\mathbb{P}[Z > x] \sim \sum_{j=1}^K \pi_j \int\int_{\{(\sigma,t)\in\Delta^j(x)\}} \mathbb{P}\left[\sigma^{(j)} \in d\sigma\right] dt. \quad (5)$$

This equation may be rewritten with the constants $\{\alpha_i^j, \beta_i^j, \gamma_i^j\}_{0\le i\le l}$ *that will be calculated in Lemma 2 below as follows:*

$$\mathbb{P}[Z > x] \quad \sim \quad \sum_{j=1}^{K} \pi_j \left\{ \sum_{i=0}^{l} \sum_{\{\alpha_i^j x \le n < \alpha_{i+1}^j x\}} \mathbb{P}\left[\sigma^{(j)} > \frac{x}{\beta_i^j} + n\gamma_i^j \right] \right\}, \tag{6}$$

or with $\delta_i^j = 1/\beta_i^j + \alpha_i^j \gamma_i^j$ *and* $d^{(j)} = \pi_j c^{(j)}$,

$$\mathbb{P}[Z > x] \sim \sum_{j=1}^{K} d^{(j)} \left\{ \sum_{i=0}^{l} \frac{1}{\gamma_i^j} \left[\overline{F}^s(\delta_i^j x) - \overline{F}^s(\delta_{i+1}^j x) \right] \right\}. \tag{7}$$

1. *If* $\overline{F}^s \in \mathcal{R}(-\alpha)$, *with* $\alpha > 0$, *we can rewrite Equation (7) as:*

$$\frac{\mathbb{P}[Z > x]}{\overline{F}^s(x)} \to \sum_{j=1}^{K} d^{(j)} \left\{ \sum_{i=0}^{l} \frac{1}{\gamma_i^j} \left[(\delta_i^j)^{-\alpha} - (\delta_{i+1}^j)^{-\alpha} \right] \right\}.$$

2. *If* $\overline{F}^s \in \mathcal{R}(-\infty)$, *then, we have*

$$\frac{\mathbb{P}[Z > x]}{\overline{F}^s(x)} \to \sum_{j=1}^{K} \frac{d^{(j)}}{a - p_j B_j}.$$

2.3 Sample Path Construction of the Maximal Dater

The sample path construction we introduce here is that of [4]. The main interest of such a construction is that some monotinicity properties are preserved as explained in [4]. These monotonicity properties as shown in [5] are crucial for our asymptotic calculation.
A generalized Jackson network will be defined by

$$\mathbf{JN} = \left\{ \{\sigma_j^{(k)}\}_{j\ge 1},\ \{\nu_j^{(k)}\}_{j\ge 0},\ n^{(k)},\ 0 \le k \le K \right\},$$

where $N = (n^{(0)}, n^{(1)}, \ldots, n^{(K)})$ describes the initial condition. The interpretation is as follows: for $i \neq 0$, at time $t = 0$, in node i, there are $n^{(i)}$ customers with service times $\sigma_1^{(i)}, \ldots, \sigma_{n^{(i)}}^{(i)}$ (if appropriate, $\sigma_1^{(i)}$ may be interpreted as a residual service time).
The interpretation of $n^{(0)}$ is as follows:

- if $n^{(0)} = 0$, there is no external arrival.
- if $\infty > n^{(0)} \ge 1$, then for all $1 \le j \le n^{(0)}$, the arrival time of the j-th customer in the network takes place at $\sigma_1^{(0)} + \cdots + \sigma_j^{(0)}$. Note that in this case, there may be a finite number of customers passing through a given station so that the network is actually well defined once a finite sequence of routing decisions and service times is given on this station.

- if $n^{(0)} = \infty$, then when taking for instance the sequence $\{\sigma_j^{(0)}\}_{j\geq 1}$ i.i.d., the arrival process is a renewal process etc.

Euler route, Euler network
Consider a route $p = (p_1, \ldots, p_\phi)$ with $1 \leq p_i \leq K$ for $i = 2 \ldots \phi - 1$. Such a route is *successful* if $p_1 = 0$ and $p_\phi = K + 1$. To such a route, we associate a routing sequence $\nu = (\nu^{(0)}, \ldots, \nu^{(K)})$ as follows ($\oplus$ means here concatenation and $\emptyset$ the empty sequence):

Procedure(p) :

1 **for** $k = 0 \ldots K$ **do**
  $\nu^{(k)} := \emptyset;$
  $\phi^{(k)} := 0;$
od

2 **for** $i = 1 \ldots \phi - 1$ **do**
  $\nu^{(p_i)} := \nu^{(p_i)} \oplus p_{i+1};$
  $\phi^{(p_i)} := \phi^{(p_i)} + 1;$
od

Note that $\phi^{(j)}$ is the number of visits to node j in such a route. In particular in our stochastic framework, we have $\mathbb{E}[\phi^{(j)}] = \pi_j$.
A simple Euler network is a generalized Jackson network

$$E = \{\sigma, \nu, N\},$$

with $N = (1, 0, \ldots, 0) = 1$, such that the routing sequence $\nu = \{\nu_i^{(k)}\}_{i=1}^{\phi^{(k)}}$ is generated by a successful route and such that $\sigma = \{\sigma_i^{(k)}\}_{i=1}^{\phi^{(k)}}$ is a sequence of real-valued non-negative numbers, representing service times.

Consider a sequence of simple Euler networks, say $\{E(n)\}_{n=-\infty}^{0}$, where $E(n) = \{\sigma(n), \nu(n), 1\}$. For $m \leq n \leq 0$, we define $\sigma_{[m,n]}$ and $\nu_{[m,n]}$ to be the concatenation of $\{\sigma(k)\}_{m\leq k\leq n}$ and $\{\nu(k)\}_{m\leq k\leq n}$ and then define the *composed* generalized Jackson network:

$$\mathbf{JN}_{[m,n]} = \{\sigma_{[m,n]}, \nu_{[m,n]}, N_{[m,n]}\}, \quad \text{with} \quad N_{[m,n]} = (m - n + 1, 0, \ldots, 0).$$

Maximal dater
As proved in [4], for all possible values of $\nu(p)$ and $\sigma(p)$ in the simple Euler networks, for all integers $m \leq n$, the composed network $\mathbf{JN}_{[m,n]}$ stays empty forever after some finite time. We denote by $X_{[m,n]}$ the time to empty $\mathbf{JN}_{[m,n]}$ forever and by $Z_{[m,n]} =$

$X_{[m,n]} - \sum_{i=1}^{m-n+1} \sigma^{(0)}_{[m,n],i}$ the associated maximal dater. The sequence $Z_{[-n,0]}$ is an increasing sequence. We define the maximal dater of the generalized Jackson network $\mathbf{JN} = \{\sigma, \nu, N\}$ where σ and ν are the infinite concatenation of the $\{\sigma(n)\}_n$ and $\{\nu(n)\}_n$ and $N = (+\infty, 0, \ldots, 0)$, by

$$Z = \lim_{n\to\infty} Z_{[-n,0]}. \tag{8}$$

Theorem 13 of [4] applies so that if $\lambda b < 1$ then $Z < \infty$ a.s.; conversely, if $\lambda b > 1$, $Z = \infty$ a.s.
To all generalized Jackson network $\mathbf{JN}_{[m,n]}$, we also associate the generalized Jackson network $\mathbf{JN}_{[m,n]}(Q)$ in which driving sequences are the same as in the original network except for the sequence $\{\sigma_j^{(0)}\}$ that is now $\sigma_j^{(0)} = 0$ for all j. Similarly we define $Z_{[m,n]}(Q)$ the time to empty the generalized Jackson network $\mathbf{JN}_{[m,n]}(Q)$.
Let

$$Y_i^{(k)} = \sum_{j=1}^{\phi^{(k)}(i)} \sigma_j^{(k)}(i) \tag{9}$$

be the total load brought by (external) customer i to station k. Note that

$$\begin{aligned} Z_i &= Z_{[i,i]} = Y_i^{(1)} + \cdots + Y_i^{(K)}, \quad \forall i \\ Z_{[n,0]}(Q) &\geq \max_{j=1,\ldots,K} \sum_{i=n}^{0} Y_i^{(j)}, \quad \forall n \leq 0. \end{aligned}$$

Lemma 4 of [3] also implies that

$$\lim_{n\to\infty} \frac{Z_{[-n,0]}(Q)}{n} = b \quad \text{a.s.}. \tag{10}$$

2.4 Technical Conditions

Under Assumption 1-3, the properties **(IA)** and **(AA)** of [4], which read

- **(IA)** the sequence of simple Euler networks $\{E(n)\}_{n=0}^{-\infty}$ consists of i.i.d. random variables.
- **(AA)** the random variables $\{Y_i^{(k)}\}$ are independent of the inter-arrival times, and such that the sequence of random vectors $(Y_i^{(1)}, \ldots, Y_i^{(K)})$ is i.i.d. (general dependences between the components of the vector $(Y_i^{(1)}, \ldots, Y_i^{(K)})$ are allowed),

are both satisfied.

Under Assumption 1, the variable Z associated to $\mathbf{JN} = \{\sigma, \nu, N\}$ represents the stationary maximal dater of the generalized Jackson network, namely the time that it would take in steady state to clear the worlkoad of all customers present in the system when stopping future arrivals.

Under Assumption 4, the assumptions **(SE)** and **(H)** of [4] are satisfied:

- **(SE)** For all $k = 1, \dots, K$

$$\mathbb{P}(Y_1^{(k)} > x) \sim \pi^k \mathbb{P}(\sigma^{(k)} > x) \sim d^{(k)} \overline{F}(x),$$

  with $d^{(k)} = c^{(k)} \pi_k$ and then $d \stackrel{\text{def}}{=} \sum_k d^{(k)} > 0$.

- **(H)**

$$\mathbb{P}(\sum_{k=1}^{K} Y_1^{(k)} > x) \sim \mathbb{P}(\max_{1 \le k \le K} Y_1^{(k)} > x) \sim \sum_{k=1}^{K} \mathbb{P}(Y_1^{(k)} > x) \sim d\overline{F}(x).$$

See Sections 4.4.2 and 7.2 of [5].
Under Assumption 4, there exists a non-decreasing integer-valued function $N_x \to \infty$ and such that, for all finite real numbers b,

$$\sum_{n=0}^{N_x} \overline{F}(x + nb) = o\left(\overline{F}^s(x)\right), \quad x \to \infty \tag{11}$$

(see Section 4.1.2 of [5]).

3 Exact Tail Asymptotics

3.1 Single Big Event Theorem

As already mentioned, one of the tools we will use within this setting is the "single big event theorem" for generalized Jackson networks. More precisely, Theorems 7 and 8 of [5] give the following result:

Property 1. *Let Z be the stationary maximal dater of the generalized Jackson network defined in (8). For any x and for $j = 1, \dots, r$, let $\{K_{n,x}^j\}$ be a sequence of events such that*

1. *for any n and j, the event $K_{n,x}^j$ and the random variable $Y_{-n}^{(j)} = \sum_{k=1}^{\phi^{(j)}(-n)} \sigma_k^{(j)}(-n)$ are independent;*
2. *for any j, $\mathbb{P}(K_{n,x}^j) \to 1$ uniformly in $n \ge N_x$ as $x \to \infty$.*

For all sequences $\epsilon_n \to 0$, we denote $x_n = x + n(a - b + \epsilon_n)$. Then, as $x \to \infty$,

$$\mathbb{P}[Z > x] \sim \sum_{j=1}^{K} \sum_{n \geq N_x} \mathbb{P}[Z > x, Y_{-n}^{(j)} > x_n, K_{n,x}^j],$$

and

$$\mathbb{P}[Z > x] = \mathcal{O}(\overline{F}^s(x)).$$

This property leads to the following and more handy result:

Corollary 1. *Take any sequence of events $\{K_n^j\}$ such that for any j, $\mathbb{P}(K_n^j) \to 1$ uniformly in $n \geq N_x$ as $x \to \infty$. Take $z_x \to \infty$, $z_x = o(x)$, such that $\overline{F}^s(x \pm z_x) \sim \overline{F}^s(x)$, and denote:*

$$G(x) = \sum_{j=1}^{K} \sum_{n \geq N_x} \mathbb{P}\left[Z_{[-n,0]} > x, K_n^j, Y_{-n}^{(j)} > x_n, \phi^{(j)}(-n) \leq L\right].$$

Then, we have:

$$(1 + \epsilon(x))G(x) \leq \mathbb{P}[Z > x] \leq (1 + \epsilon(x))G(x - z_x) + \epsilon(L, x)\overline{F}^s(x). \qquad (12)$$

If G is long tailed, we have as $x \to \infty$

$$\mathbb{P}[Z > x] \sim G(x).$$

The proof is forwarded to the appendix.

3.2 Fluid Limit

We have to find sequences of events $\{K_n^j\}$ allowing one to calculate the sum

$$\sum_{n \geq N_x} \mathbb{P}\left[Z_{[-n,0]} > x, K_n^j, Y_{-n}^{(j)} > x_n, \phi^{(j)}(-n) \leq L\right] \qquad (13)$$

where as above, $x_n = x + n(a - b + \epsilon_n)$.
The events in question will be based on the piece-wise linear functions $f^j(\sigma, n)$ defined in (3). Let us describe the intuitive reason for introducing this function. Assume the big service time is equal to σ and takes place on station j and within the set of service times of the simple Euler network $E(-n)$. Let us look at the maximal dater $Z_{[-n,0]}$ in the fluid scale suggested by the a.s. limit of (10):

- if $\sigma > na$, then the number of customers blocked in station j at time σ is of the order of np_j, whereas the number of customers in the other stations is small. So, according to (10), the time to empty the network from time σ on should be of the order np_jB_j; hence, in this case, the maximal dater in question should be of the order of $f^j(\sigma, n)$ indeed;

- if $\sigma < na$, then at time σ, the number of customers blocked in station j is of the order of $p_j \frac{\sigma}{a}$, and the other stations have few customers; from time σ to the time of the last arrival (which is of the order of na), station k has to serve approximately the load $p_j \frac{\sigma}{a} b_{j,k}$ generated by these blocked customers plus the load $(na - \sigma)\frac{b_k}{a}$ generated by the external arrivals on the time interval from σ to the last arrival. On this time interval, the service capacity is of the order of $(na - \sigma)$. Hence the maximal dater should again be of the order of $f^j(\sigma, n)$.

Particularly for the last case, a more thourough understanding of the fluid limit is clearly necessary. This will be provided by the results in [10]. We now return to rigor.
For all simple Euler networks $E = (\sigma, \nu, 1)$, let $Y^{(j)}(E) = \sum_{u=1}^{\phi^{(j)}} \sigma_u^{(j)}$.

Consider a generalized Jackson network built from the i.i.d. sequence of simple Euler networks $\{E(k)\}$. To all simple Euler networks E and all positive integers n, we associate the network $\mathbf{JN}^n(E)$ with input $\{\tilde{E}(k)\}_{k=-n}^{\infty}$, where $\tilde{E}(k) = E(k)$ for all $k > -n$ and $\tilde{E}(-n) = E$. That is, if we denote by $\sigma^{(k),n}$ and $\nu^{(k),n}$ the concatenations $(\{\sigma^{(k)}(E)\}, \{\sigma^{(k)}(-n+1)\}, \dots, \{\sigma^{(k)}(0)\}, \dots)$ and $(\{\nu^{(k)}(E)\}, \{\nu^{(k)}(-n+1)\}, \dots, \{\nu^{(k)}(0)\}, \dots)$ respectively, then

$$\mathbf{JN}^n(E) \quad = \quad \{\sigma^n(E), \nu^n(E), 0, N^n\}, \quad \text{with} \quad N^n = (n, 0, \dots, 0).$$

The maximal dater of order $[-n, 0]$ in this nework will be denoted by $\tilde{Z}^n(E)$. Of course $\tilde{Z}^n(E(n)) = Z_{[-n,0]}$. More generally, we will add the superscript n to any other function associated to a network to mean that of network $\mathbf{JN}^n(E)$.
For all simple Euler networks $E = (\sigma, \nu, 1)$, let $Y^{(j)}(E) = \sum_{u=1}^{\phi^{(j)}} \sigma_u^{(j)}$.

We are now in a position to state the main result pertaining to the fluid limit. Let ϵ_n, z_n be some sequences of positive real numbers; we define:

$$\begin{aligned}
\mathbf{U}^j(n) &= \{E \text{ is a simple Euler network such that } Y^{(k)}(E) \le z_n \; \forall k \neq j\}, \\
\mathbf{V}^j(n) &= \{E \in \mathbf{U}^j(n),\ Y^{(j)}(E) \ge n(a-b),\ \phi^{(j)} \le L\}, \\
K_n^j &= \left\{ \sup_{\{E \in \mathbf{V}^j(n)\}} \left| \frac{\tilde{Z}^n(E) - f^j(Y^{(j)}(E), n)}{n} \right| \le \epsilon_n \right\} \cap \left\{ E(-n) \in \mathbf{U}^j(n) \right\}. \quad (14)
\end{aligned}$$

We first recall a result that derives directly from Property 6 and the remark following this property in [10].

Property 2. *Under the previous assumptions, there exists a sequence $z_n \to \infty$ with $\frac{z_n}{n} \to 0$, such that we have*

$$\sup_{\{E \in \mathbf{V}^j(n)\}} \left| \frac{\tilde{Z}^n(E) - f^j(Y^{(j)}(E), n)}{n} \right| \xrightarrow{n \to \infty} 0 \quad a.s.$$

Lemma 1. *Let $\{K_n^j\}$ be the sequence of events defined above. There exist sequences $\epsilon_n \to 0$ and $z_n \to \infty$ with $\frac{z_n}{n} \to 0$, such that we have $\mathbb{P}[K_n^j] \to 1$ uniformly in $n \ge N_x$ as $x \to \infty$.*

Proof:
The left-hand part of the definition of K_n^j depends of $\{E(k)\}_{k=-n+1}^{0}$ and the right-hand part depends only on $E(-n)$, hence we have

$$\mathbb{P}[K_n^j] = \mathbb{P}\left[\sup_{\{E\in\mathbf{V}^j(n)\}}\left|\frac{\tilde{Z}^n(E) - f^j(Y^{(j)}(E),n)}{n}\right| \le \epsilon_n\right]\mathbb{P}\left[E(-n)\in\mathbf{U}^j(n)\right].$$

The distribution of $E(-n)$ does not depend on n, hence $Y^{(i)}(E(-n))/n \to 0$ a.s. since its mean is finite. Therefore, there exists a sequence $z_n \to \infty$, $z_n/n \to 0$ such that

$$\mathbb{P}(Y^{(i)}(E(-n)) \le nz_n, \forall i \ne j) = \mathbb{P}\left[E(-n)\in\mathbf{U}^j(n)\right] \to 1$$

uniformly in $n \ge N_x$ as $x \to \infty$.
The first term derives directly from Property 2. Therefore, there exist sequences $\epsilon_n \to 0$ and $z_n \to \infty$ and $\frac{z_n}{n} \to 0$, such that we have $\mathbb{P}[K_n^j] \to 1$ uniformly in $n \ge N_x$ as $x \to \infty$. □

3.3 Computation of the Exact Asymptotics

Thanks to Lemma 1, it is easy to see that the sequence of events $\{K_n^j\}$ defined in (14) satisfies assumptions of Corollary 1. Moreover, we will see that we are now able to calculate the sum in Equation (13) which will give the exact assymptotic for $\mathbb{P}[Z > x]$ in Theorem 1. Before stating this result, we need to introduce some notation.
On the event $K_n^j \cap \left\{Y_{-n}^{(j)} > x_n, \phi^{(j)}(-n) \le L\right\}$, we have

$$Z_{[-n,0]} = f^j(Y_{-n}^{(j)}, n) + n\eta_n, \quad \text{with } \eta_n \text{ r.v. such that } |\eta_n| \le \epsilon_n.$$

Then $\{Z_{[-n,0]} > x\} = \{f^j(Y_{-n}^{(j)}, n) > x - n\eta_n\}$. In order to prove equivalence (5), we will first give an explicit form for the domains Δ^j defined in (4).

Lemma 2. *There exist constants $\{\alpha_i^j, \beta_i^j, \gamma_i^j\}_{0\le i\le l}$ (given in closed form in the proof of the lemma in function of the quantities p_j and $b_{j,k}$ defined in Section 2.1) with $0 = \alpha_0^j \le \alpha_1^j \ldots \le \alpha_l^j$, such that:*

$$\Delta^j(z) \quad = \quad \bigcup_{i=0}^{l}\left\{\alpha_i^j z \le t < \alpha_{i+1}^j z,\ \sigma > \frac{z}{\beta_i^j} + t\gamma_i^j\right\}, \tag{15}$$

with the convention $\alpha_{l+1}^j = +\infty$. Moreover, we have

$$\alpha_0^j = 0, \quad \alpha_1^j = 1/p_jB_j, \quad \beta_0^j = 1, \quad \gamma_0^j = a - p_jB_j,$$

for all j. In addition, $\beta_i^j \le 1$ for all i,j and the following inclusion holds:

$$\{\sigma \ge z + t(a - p_jB_j)\} \subset \Delta^j(z). \tag{16}$$

Proof:
The domain Δ^j may be divided in two parts:

$$\begin{aligned}\Delta^j(z) &= \{(\sigma,t),\ f^j(\sigma,t) > z\}.\\ &= \{\sigma > ta,\ \sigma > z + t(a - p_j B_j)\} \cup \left\{\sigma \le ta,\ \sigma > a \min_k \frac{z + t(a-b_k)}{a - b_k + p_j b_{j,k}}\right\}.\end{aligned}$$

For the first part, we have (see Figure 1):

$$\begin{aligned}\{\sigma > ta,\ \sigma > z + t(a - p_j B_j)\} &= \left\{0 \le t < \frac{z}{p_j B_j},\ \sigma > z + t(a - p_j B_j)\right\}\\ &\quad \cup \left\{\frac{z}{p_j B_j} \le t,\ \sigma > ta\right\}.\end{aligned}$$

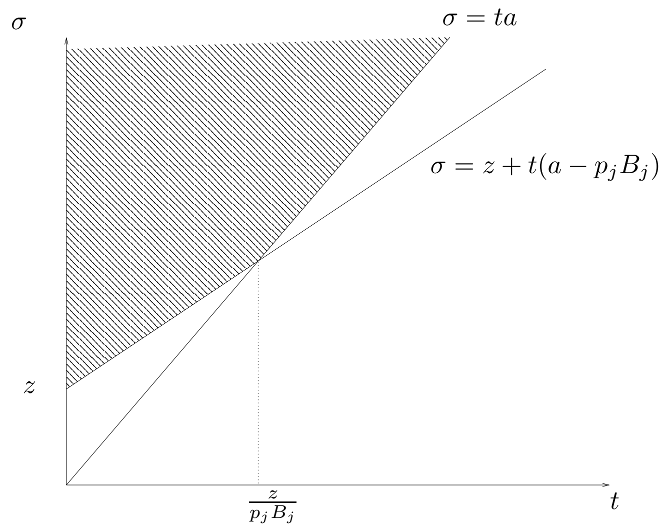

Figure 1: First part of $\Delta^j(z)$

For the second part, we have (see Figure 2):

$$\left\{\sigma \le ta,\ \sigma > a \min_k \frac{z + t(a-b_k)}{a - b_k + p_j b_{j,k}}\right\} = \bigcup_k \left\{\frac{z}{p_j b_{j,k}} \le t,\ a\frac{z + t(a-b_k)}{a - (b_k - p_j b_{j,k})} < \sigma \le ta\right\}.$$

Now, it is easy to see that the lemma holds (see Figure 3).

The inequality on the β's follows directly from the fact that $p_j b_{j,k} \le b_k$ from which, we have

$$\frac{a}{a + p_j b_{j,k} - b_k} \ge 1.$$

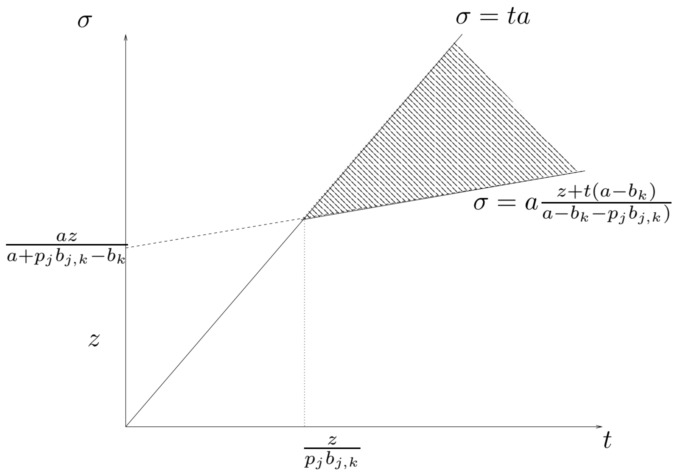

Figure 2: Construction of the second part of $\Delta^j(z)$

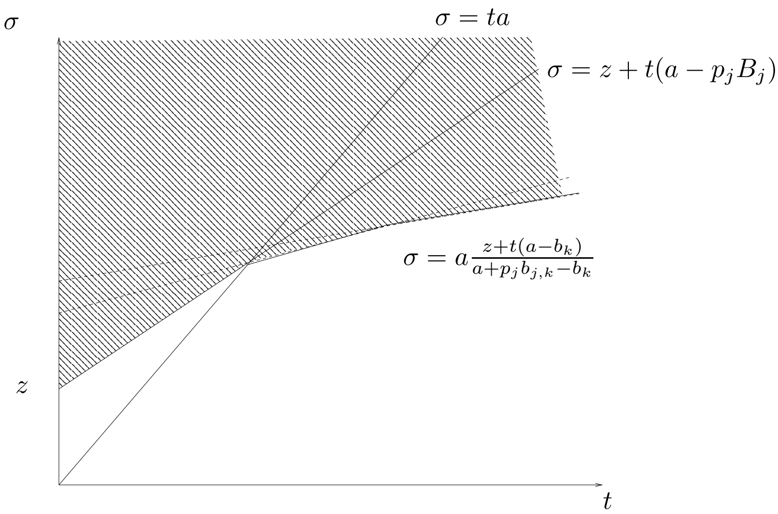

Figure 3: Domain $\Delta^j(z)$

□

Lemma 3. *Let X be a random variable such that $\overline{F}^s \in \mathcal{S}$, $\epsilon_n \to 0$ as $n \to \infty$ and*

$$\frac{a(x)}{x} \to a, \quad \frac{b(x)}{x} \to b, \quad \textit{with } 0 < a < b \textit{ as } x \to \infty.$$

If $\overline{F}(x) = \mathbb{P}[X > x]$, *for* $\alpha \geq 1$, $\beta > 0$, *we have as* $x \to \infty$

$$\sum_{a(x)\leq n<b(x)} \mathbb{P}[X > \alpha x + n(\beta+\epsilon_n)] - \sum_{ax\leq n<bx} \mathbb{P}[X > \alpha x + n\beta] = o(\overline{F}^s(x)).$$

Proof:
For the simplicity of notation, we assume that $a(x) \leq ax$ for all x. We have

$$\begin{aligned}\sum_{a(x)\leq n\leq ax} \mathbb{P}[X > \alpha x + n(\beta+\epsilon_n)] &= \frac{1+\epsilon(x)}{\beta}\int_{\alpha x+a(x)\beta}^{\alpha x+ax\beta}\overline{F}(u)du \\ &\leq \frac{1+\epsilon(x)}{\beta}\frac{ax-a(x)}{ax}\overline{F}^s(\alpha x)\end{aligned}$$

since $\overline{F}(x)$ is non-increasing. Hence, we have only to prove the lemma for $a(x) = ax$ and $b(x) = bx$. We have the following bound with $\delta_x = \sup_{n\geq ax}\epsilon_n$.

$$\begin{aligned}\sum_{ax\leq n<bx} \mathbb{P}[X > \alpha x + n(\beta+\epsilon_n)] - \mathbb{P}[X > \alpha x + n\beta] &\leq \sum_n \mathbb{P}[X \in (\alpha x + n\beta, \alpha x + n(\beta+\delta_x)] \\ &= (1+\epsilon(x))\overline{F}^s(\alpha x)\left(\frac{1}{\beta} - \frac{1}{\beta+\delta_x}\right) \\ &= o(\overline{F}^s(\alpha x)) = o(\overline{F}^s(x)).\end{aligned}$$

□

Proof of Theorem 1:
Thanks to Corollary 1, we know that the tail asymptotic of the maximal dater is linked to the quantity $S(j)$ defined by

$$S(j) = \sum_{n\geq N_x} \mathbb{P}\left[Z_{[-n,0]} > x, K_n^j, Y_{-n}^{(j)} > x_n, \phi^{(j)}(-n) \leq L\right].$$

On the event $A_{n,x}^j = K_n^j \cap \left\{Y_{-n}^{(j)} > x_n, \phi^{(j)}(-n) \leq L\right\}$, we have

$$\begin{aligned}\{Z_{[-n,0]} > x\} &= \{f^j(Y_{-n}^{(j)}, n) > x - n\eta_n\} \\ &= \{(Y_{-n}^{(j)}, n) \in \Delta^j(x - n\eta_n)\}.\end{aligned}$$

Clearly $\Delta^j(z)$ is a non-increasing function of z and we define

$$D_-^j = \Delta^j(x - n\epsilon_n) \supset \Delta^j(x - n\eta_n) \supset \Delta^j(x + n\epsilon_n) = D_+^j.$$

For simplicity of notation, we write $Y^{(j)} = Y^{(j)}_{-n}$ and $\phi^{(j)} = \phi^{(j)}(-n)$. We assume w.l.o.g. that ϵ_n is a decreasing sequence, hence for $n \geq N_x$, $\epsilon_n \leq \epsilon_{N_x} = \epsilon_x$ and we have for $n \geq N_x$

$$\begin{aligned} A_+(n) &= \mathbb{P}\left[(Y^{(j)}, n) \in D^j_+\right] \\ &= \sum_{i=0}^{l} \mathbb{1}_{\{\alpha^j_i(x+n\epsilon_n) \leq n < \alpha^j_{i+1}(x+n\epsilon_n)\}} \mathbb{P}\left[Y^{(j)} > \frac{x}{\beta^j_i} + n\gamma^j_i + \frac{n\epsilon_n}{\beta^j_i}\right] \\ &\leq \sum_{i=0}^{l} \mathbb{1}_{\{\alpha^j_i x \leq n < \alpha^j_{i+1}(x+n\epsilon_x)\}} \mathbb{P}\left[Y^{(j)} > \frac{x}{\beta^j_i} + n\gamma^j_i + \frac{n\epsilon_n}{\beta^j_i}\right]. \end{aligned}$$

Then we have

$$\begin{aligned} \sum_{n \geq N_x} A_+(n) \leq & \sum_{\{N_x \leq n < \alpha^j_1 x(1+\epsilon(x))\}} \mathbb{P}\left[Y^{(j)} > x + n(a - p_j B_j) + n\epsilon_n\right] \\ & + \sum_{i=1}^{l} \sum_{\{\alpha^j_i x \leq n < \alpha^j_{i+1} x(1+\epsilon(x))\}} \mathbb{P}\left[Y^{(j)} > \frac{x}{\beta^j_i} + n\gamma^j_i + \frac{n\epsilon_n}{\beta^j_i}\right]. \end{aligned}$$

Thanks to Assumption 2, we know that $Y^{(j)}$ satisfies assumption of Lemma 3 and we have

$$\begin{aligned} \sum_{n \geq N_x} A_+(n) &= \sum_{\{0 \leq n < \alpha^j_1 x\}} \mathbb{P}\left[Y^{(j)} > x + n(a - p_j B_j)\right] \\ &\quad + \sum_{i=1}^{l} \sum_{\{\alpha^j_i x \leq n < \alpha^j_{i+1} x\}} \mathbb{P}\left[Y^{(j)} > \frac{x}{\beta^j_i} + n\gamma^j_i\right] + \epsilon(x)\overline{F}^s(x) \\ &= (1+\epsilon(x)) \sum_{i=0}^{l} \sum_{\{\alpha^j_i x \leq n < \alpha^j_{i+1} x\}} \mathbb{E}[\phi^{(j)}]\mathbb{P}\left[\sigma^{(j)} > \frac{x}{\beta^j_i} + n\gamma^j_i\right] + \epsilon(x)\overline{F}^s(x), \end{aligned}$$

where the last equality follows from assumption **(SE)**. But we have

$$S(j) \leq \sum_{n \geq N_x} A_+(n).$$

We now look at the lower bound. With the same arguments as above, we easily get with

$$A_-(n) = \mathbb{P}\left[(Y^{(j)}, n) \in D^j_-\right],$$

that,

$$\sum_{n \geq N_x} A_-(n) = \sum_{n \geq N_x} A_+(n) + \epsilon(x)\overline{F}^s(x).$$

We now show that

$$\sum_{n\geq N_x} A_-(n) = \sum_{n\geq N_x} \mathbb{P}\left[(Y^{(j)}, n) \in D_-^j, A_{n,x}^j\right] + \epsilon(x,L)\overline{F}^s(x).$$

Consider the difference

$$\begin{aligned} A_-(n) - \mathbb{P}\left[(Y^{(j)}, n) \in D_-^j, A_{n,x}^j\right] &\leq \mathbb{P}\left[(Y^{(j)}, n) \in D_-^j, \phi^{(j)}(-n) > L\right] \\ &\leq \mathbb{P}\left[Y^{(j)} \geq x + n(a - p_j B_j - \epsilon_n), \phi^{(j)}(-n) > L\right] \end{aligned}$$

where the last inequality follows from inclusion (16) of Lemma 2. With the same kind of argument as in Corollary 1, we have

$$\sum_{n\geq N_x} A_-(n) - \mathbb{P}\left[(Y^{(j)}, n) \in D_-^j, A_{n,x}^j\right] \quad \leq \quad \epsilon(x,L)\overline{F}^s(x).$$

Hence, we proved that when $x \to \infty$, we have

$$S(j) \sim \sum_{i=0}^{l} \sum_{\{\alpha_i^j x \leq n < \alpha_{i+1}^j x\}} \mathbb{E}[\phi^{(j)}]\mathbb{P}\left[\sigma^{(j)} > \frac{x}{\beta_i^j} + n\gamma_i^j\right].$$

Now since this quantity is long tailed, we use Corollary 1 to derive the asymptotic for $\mathbb{P}[Z > x]$. □

Appendix: Proof of Corollary 1

The proof is based on Property 1, which shows that we have

$$\mathbb{P}[Z > x] = (1 + \epsilon(x))\left(\sum_{j=1}^{K}\sum_{n\geq N_x} \mathbb{P}[Z > x, Y_{-n}^{(j)} > x_n, K_n^j]\right).$$

Since $Z \geq Z_{[-n,0]}$, we have

$$\begin{aligned} \mathbb{P}[Z > x, Y_{-n}^{(j)} > x_n, K_{n,x}^j] &\geq \mathbb{P}[Z_{[-n,0]} > x, Y_{-n}^{(j)} > x_n, K_n^j] \\ &\geq \mathbb{P}[Z_{[-n,0]} > x, Y_{-n}^{(j)} > x_n, K_n^j, \phi^{(j)}(-n) \leq L]. \end{aligned}$$

Hence we have

$$\mathbb{P}[Z > x] \quad \geq \quad (1 + \epsilon(x))\left(\sum_{j=1}^{K}\sum_{n\geq N_x} \mathbb{P}[Z_{[-n,0]} > x, Y_{-n}^{(j)} > x_n, K_n^j, \phi^{(j)}(-n) \leq L]\right).$$

We now derive the upper bound. Take $z_x \to \infty$ such that $\overline{F}^s(x+z_x) \sim \overline{F}^s(x)$, then when $x \to \infty$, we have

$$\mathbb{P}[Z_{[-\infty,-n-1]} < z_x] = \mathbb{P}[Z < z_x] \to 1.$$

We define $\tilde{x} = x + z_x$, and $\tilde{K}^j_{n,x} = K^j_n \cap \{Z_{[-\infty,-n-1]} \le z_x\}$. Observe that $\tilde{K}^j_{n,x}$ satisfies also assumptions of Property 1. By sub-additivity, we have $Z \le Z_{[-\infty,-n-1]} + Z_{[-n,0]}$ (see [4]), hence

$$\begin{aligned}
\mathbb{P}(Z > \tilde{x}, \tilde{K}^j_{n,x}, Y^{(j)}_{-n} > \tilde{x}_n) &\le \mathbb{P}(Z_{[-\infty,-n-1]} + Z_{[-n,0]} > \tilde{x}, \tilde{K}^j_{n,x}, Y^{(j)}_{-n} > \tilde{x}_n) \\
&\le \mathbb{P}(Z_{[-n,0]} > x, \tilde{K}^j_{n,x}, Y^{(j)}_{-n} > x_n) \\
&\le \mathbb{P}(Z_{[-n,0]} > x, K^j_n, Y^{(j)}_{-n} > x_n).
\end{aligned}$$

We now make the truncation of ϕ.

$$\begin{aligned}
A(n) &= \mathbb{P}\left[Z_{[-n,0]} > x, K^j_n, Y^{(j)}_{-n} > x_n\right] \\
&\le \mathbb{P}\left[Z_{[-n,0]} > x, K^j_n, Y^{(j)}_{-n} > x_n, \phi^{(j)}(-n) \le L\right] + \mathbb{P}\left[Y^{(j)}_{-n} > x_n, \phi^{(j)}(-n) > L\right] \\
&= \mathbb{P}\left[Z_{[-n,0]} > x, K^j_n, Y^{(j)}(E(-n)) > x_n, \phi^{(j)}(-n) \le L\right] + B(n).
\end{aligned}$$

We will use the following result due to Kesten (for a proof see Athreya and Ney [1]):

Lemma 4. *Let $X \in \mathcal{S}$ and let S_n be the sum of n independent copies of X. Then for every $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that*

$$\sup_{x \ge 0} \frac{\mathbb{P}[S_n > x]}{\mathbb{P}[X > x]} \le K(\epsilon)(1+\epsilon)^n, \quad n = 1, 2, \ldots$$

Recall that $\mathbb{P}(\phi^{(j)}(0) = l) = \delta^l(1-\delta)$ for some $0 < \delta < 1$, hence take ϵ such that $(1+\epsilon)\delta < 1$, and we have

$$\begin{aligned}
B(n) &= \sum_{l \ge L+1} \mathbb{P}[\phi^{(j)}(-n) = l] \mathbb{P}\left[\sum_{k=1}^{l} \sigma^{(j)}_k(-n) > x_n\right] \\
&\le \sum_{l \ge L+1} \delta^l (1-\delta) K(\epsilon)(1+\epsilon)^l \mathbb{P}[\sigma^{(j)} > x_n] \\
&\le (1-\delta) K(\epsilon) \mathbb{P}[\sigma^{(j)} > x_n] \frac{((1+\epsilon)\delta)^{L+1}}{1-(1+\epsilon)\delta}.
\end{aligned}$$

Then, we have

$$\sum_{n \ge N_x} A(n) \le \sum_{n \ge N_x} \mathbb{P}\left[Z > x, K^j_n, Y^{(j)}_{-n} > x_n, \phi^{(j)}(-n) \le L\right] + \epsilon(x, L)\overline{F}^s(x).$$

Since $\tilde{K}^j_{n,x}$ satisfies assumptions of Property 1, we have

$$\begin{aligned}
\mathbb{P}(Z > \tilde{x}) &= (1+\epsilon(\tilde{x}))\sum_{j=1}^{K}\sum_{n\geq N_{\tilde{x}}} \mathbb{P}(Z > \tilde{x}, \tilde{K}^j_{n,x}, Y^{(j)}_{-n} > \tilde{x}_n) \\
&\leq (1+\epsilon(x))\sum_{j=1}^{K}\sum_{n\geq N_x} \mathbb{P}(Z_{[-n,0]} > x, K^j_n, Y^{(j)}_{-n} > x_n) \\
&\leq (1+\epsilon(x))\sum_{j=1}^{K}\sum_{n\geq N_x} \mathbb{P}(Z_{[-n,0]} > x, K^j_n, Y^{(j)}_{-n} > x_n, \phi^{(j)}(-n) \leq L) + \epsilon(x,L)\overline{F}^s(x).
\end{aligned}$$

Hence, we have showed with the notation of the lemma

$$\begin{aligned}
(1+\epsilon(x))G(x) &\leq \mathbb{P}(Z > x) \\
\mathbb{P}(Z > x + z_x) &\leq (1+\epsilon(x))G(x) + \epsilon(x,L)\overline{F}^s(x).
\end{aligned}$$

From these inequalities, we directly derive inequality (12). If G is long tailed, we can choose $z_x \to \infty$ such that $G(x+z_x) \sim G(x)$ and $\overline{F}^s(x+z_x) \sim \overline{F}^s(x)$, and the last statement of the lemma follows. □

References

[1] K.B. Athreya, P.E. Ney (1972) *Branching Processes*. Springer, Berlin.

[2] F. Baccelli, S. Schlegel and V. Schmidt, *Asymptotics of Stochastic Networks with Subexponential Service Times*, QUESTA, 33, 205–23, 1999.

[3] F. Baccelli, S. Foss (1995) *On the Saturation Rule for the Stability of Queues* J. Appl. Prob., 32, pp. 494-507.

[4] F. Baccelli, S. Foss (1994) *Ergodicity of Jackson-type queueing networks* QUESTA, 17, pp. 5-72.

[5] F. Baccelli, S. Foss (2003) *Moments and Tails in Monotone-Separable Stochastic Networks*, INRIA Report 4197, 2001, to appear in the Annals of Applied Probability.

[6] H. Chen, A. Mandelbaum (1991) *Discrete flow networks: Bottelnecks analysis and fluid approximations*, Math. Op. Res. 16, pp. 408-446.

[7] P. Embrechts, C. Goldie, N. Veraverbeke (1979) *Subexponentiality and infinite divisibility*, Z. Wahrscheinlichkeitstheorie verw. Geb. 49, pp. 335-347.

[8] T. Huang, K. Sigman *Delay asymptotics for tandem, split & match, and other feed forward queues with heavy tailed service*. QUESTA, 33, 1999.

[9] C.M. Goldie, C. Klüppelberg, (1997) Subexponential distributions, in *A Practical Guide to Heavy Tails : Statistical Techniques and Applications*, eds R.J. Adler, R.E. Feldman. M.S. Taqqu, Birkhäuser, pp. 435-459.

[10] M. Lelarge *Fluid Limit of Generalized Jackson Queueing Networks with Stationary and Ergodic Arrivals and Service Times.*

Unité de recherche INRIA Rocquencourt
Domaine de Voluceau - Rocquencourt - BP 105 - 78153 Le Chesnay Cedex (France)

Unité de recherche INRIA Lorraine : LORIA, Technopôle de Nancy-Brabois - Campus scientifique
615, rue du Jardin Botanique - BP 101 - 54602 Villers-lès-Nancy Cedex (France)
Unité de recherche INRIA Rennes : IRISA, Campus universitaire de Beaulieu - 35042 Rennes Cedex (France)
Unité de recherche INRIA Rhône-Alpes : 655, avenue de l'Europe - 38330 Montbonnot-St-Martin (France)
Unité de recherche INRIA Sophia Antipolis : 2004, route des Lucioles - BP 93 - 06902 Sophia Antipolis Cedex (France)

Éditeur
INRIA - Domaine de Voluceau - Rocquencourt, BP 105 - 78153 Le Chesnay Cedex (France)
http://www.inria.fr
ISSN 0249-6399