Fluid Limit of Generalized Jackson Queueing Networks with Stationary and Ergodic Arrivals and Service Times

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Abstract: We use a sample-path technique to derive asymptotics of generalized Jackson queueing networks in the fluid scale, namely when space and time are scaled by the same factor n. The analysis only presupposes the existence of long-run averages and is based on some monotonicity and concavity arguments for the fluid processes. The results provide a functional strong law of large numbers for stochastic Jackson queueing networks since they apply to their sample paths with probability one. The fluid processes are shown to be piece-wise linear and an explicit formulation of the different drifts is computed. A few applications of this fluid limit are given. In particular, a new computation of the constant $\gamma(0)$ that appears in [2] in the stability condition for such networks is given. In the context of rare event as described in [3], the fluid limit of the network is also derived explicitly.

Key-words: generalized Jackson queueing networks, sample-path technique, fixed point, fluid limit.

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Limite fluide de réseaux de Jackson généralisés ayant des temps de service et des arrivées stationnaires ergodiques.

Résumé : Nous utilisons des techniques trajectorielles pour calculer des asymptotiques de réseaux de Jackson généralisés à une échelle fluide, c'est à dire quand le temps et l'espace sont renormalisés par le même facteur n. Le calcul présuppose uniquement l'existence de moyennes à long terme et est basé sur des arguments de monotonie et de concavité des processus fluides. Les résultats donnent des lois fortes des grands nombres fonctionnelles pour des réseaux de Jackson stochastiques puisqu'ils s'appliquent à leurs trajectoires avec probabilité un. Les processus fluides sont linéaires par morceaux et une formulation explicite des différentes pentes est calculée. Quelques applications de cette limite fluide sont données. En particulier, un nouveau calcul de la constante $\gamma(0)$ qui apparaît dans [2] pour la condition de stabilité de telles réseaux est donnée. Dans le contexte des événements rares décrits dans [3], la limite fluide du résau est aussi calculée explicitement.

Mots-clés : Réseaux de Jackson généralisés, techniques trajectorielles, point fixe, limite fluide.

1 Introduction

In this paper, we consider a (single class) generalized Jackson network and its fluid limit. Such networks have been considered among others by Jackson [10] or Gordon and Newell [8]. In [5], Chen and Mandelbaum derive the fluid approximation for generalized Jackson networks. The queue-length, busy-time and workload processes are obtained from the input processes through the oblique reflexion mapping due to Skorokhod [15] in a one-dimensional setting and to Harrison and Reiman [9] in the context of open networks. Using this fluid approach and assuming that service times and inter-arrival times are independent and identically distributed (i.i.d.), Dai shows in [6] that generalized Jackson networks are stable (i.e. positive Harris recurrent) when the nominal load is less than one at each station. The first stability result for generalized Jackson networks under ergodic assumptions can be found in the paper of Foss [7]. In [12], Majewski derives an unified formalism which allows discrete and fluid customers. The input for the model are the cumulative service times, the cumulative external arrivals and the cumulative routing decisions of the queues. A path space fixed point equation characterizes the corresponding behavior of the network.

The framework that we use here is that of Baccelli and Foss in [2], where only stationarity and ergodicity on the data are assumed. Denote by X_0^n the time to empty the system when n customers arrive at the same time from the outside world in the network. Thanks to a subadditive argument, the following limit is shown to hold in [2]

$$\lim_{n\to\infty} \frac{X_0^n}{n} = \gamma(0) \quad \text{a.s.} \tag{1}$$

The constant $\gamma(0)$ corresponds to the maximal throughout capacity of the network. In fact the saturation rule [1] makes this intuition rigorous and ensures that if $\rho = \lambda\gamma(0) < 1$ then the network is stable. In this paper, we provide a new proof of (1) using fluid approximations which gives an explicit formula for the constant $\gamma(0)$. One contribution of this paper is to provide a connection between the fluid approximation of a generalized Jackson network and the stability condition for this network under stationary and ergodic assumptions on the data. In particular, no i.i.d. assumptions are needed (on inter-arrival times or service times) and we can consider more general routing mechanism than Bernouilli routing.

The other application of this paper will be linked to the calculation of tails in generalized Jackson networks with subexponential service distributions in a companion paper [3]. We are able to give here the behavior (in the fluid scale) of the network on a "rare" event. We refer to [3] for an exact notion of what we mean by rare event.

Results of [5] or [12] will be of minor help for us since a lot of work would be required to obtain our explicit result form theirs. For these reasons, we took a different approach. For each time t, we are able to give an explicit formulation of the fluid limit. The simplicity of the result is due to the concavity of the processes in the fluid scale; a property which had not been proved yet to the best of our knowledge. In other words, given some drifts for the input processes, when a queue becomes empty, it remains empty forever. It seems that this basic fact has not been exploited yet. It allows us to reduce the computation of the fluid limits (which are solution of a fixed-point network equation in a functional space as described in

[12]) to the computation of some traffic intensity for a simplified network that evolves in time. Hence for a fixed time, we only have to compute a fixed point solution of some traffic equations (see Section 3). Property 4 gives the fluid approximation of generalized Jackson network. To obtain the time to empty the system, we just notice that if the network is processing fluid, then one of the queues has work since the initial time. This gives us a very compact way of obtaining the constant $\gamma(0)$ (Theorem 1 of Section 4.1). Property 5 is a slight extension of the main Theorem 1 and will be needed in the computation of the fluid picture of a generalized Jackson network in the specific case of one big jump see [3].

The paper is structured as follows. In Section 2, we introduce notation for single server queues and generalized Jackson networks. The fluid limits are established in Section 3. Then the computation of constant $\gamma(0)$ is given in Section 4 with connections with the stability condition of such networks. In Section 5, we give the fluid picture of the network in the one big jump framework.

2 General Setting and Notation

We will take the following notation

1. $\mathbb{A}_1$ (resp. $\mathbb{A}_1^*$) is the set of non-negative sequences: $u = \{u_i\}_{1\leq i\leq n}$, such that $n \leq +\infty$, and $u_i \geq 0$, (resp. $u_i > 0$) for all $i \leq n$;
2. $\mathbb{A}_2$ (resp. $\mathbb{A}_2^*$) is the set of non-decreasing sequences: $U = \{U_i\}_{1\leq i\leq n}$, such that $n \leq +\infty$, and $0 \leq U_i \leq U_{i+1}$ (resp. $0 < U_i < U_{i+1}$) for all $i \leq n-1$;

We will denote by $\mathbb{A}$ (resp. $\mathbb{A}^*$) the set of discrete measure on $\mathbb{R}_+$ such that there exists $U \in \mathbb{A}_2$ (resp. $U \in \mathbb{A}_2^*$) with $d\mathscr{U} = \sum_{1\leq i\leq n} \delta_{U_i}$. To such a measure we can associate a sequence $u \in \mathbb{A}_1$ (resp. $u \in \mathbb{A}_1^*$) in the following manner $u_i = U_i - U_{i-1}$, for $i \geq 1$ and with the convention $U_0 = 0$. $\mathbb{A}_3$ (resp. $\mathbb{A}_3^*$) will denote the set of counting functions: $\mathcal{U} : \mathbb{R}_+ \to \mathbb{N}$ such that $\mathcal{U}(t) = \sum_{1\leq i\leq n} \mathbb{1}_{\{U_i\leq t\}} = \int_0^t d\mathscr{U}$ with $d\mathscr{U} \in \mathbb{A}$ (resp. $d\mathscr{U} \in \mathbb{A}^*$). Clearly the spaces $\mathbb{A}, \mathbb{A}_1, \mathbb{A}_2$ and $\mathbb{A}_3$ are isomorphic and the same holds with $\mathbb{A}^*, \mathbb{A}_1^*, \mathbb{A}_2^*$ and $\mathbb{A}_3^*$.

2.1 Single Server Queue

A single server queue will be defined by $\mathbf{Q} = (\tau^A, \sigma)$, where $\tau^A = \{\tau_i^A\}_{1\leq i\leq n}$ and $\sigma = \{\sigma_i\}_{1\leq i\leq n}$ belong to $\mathbb{A}_2$ and $\mathbb{A}_1$ respectively. The interpretations are the following: customer i arrives in the queue at time τ_i^A and its service time is σ_i.
Associated to a queue $\mathbf{Q}$, we define the output process $\{\tau_i^D\}_{1\leq i\leq n} \in \mathbb{A}_2$ by

$$\begin{cases} \tau_1^D = \tau_1^A + \sigma_1, \\ \tau_i^D = \max[\tau_i^A, \tau_{i-1}^D] + \sigma_i, \quad 2 \leq i \leq n. \end{cases} \tag{2}$$

τ_i^D is the departure time of customer i. Expanding this recursion yields

$$\tau_i^D = \max_{j=1\ldots i} (\tau_j^A + \sigma(j,i)), \quad \text{for} \quad 1 \leq i \leq n, \tag{3}$$

with the notation $\sigma(j,i) = \sigma_j + \cdots + \sigma_i$. Hence we defined a mapping $\Phi : \mathbb{A} \times \mathbb{A} \mapsto \mathbb{A}$ such that:

$$\tau^D = \{\tau_i^D\}_{1 \le i \le n} = \Phi(\mathbf{Q}). \tag{4}$$

We will use the following notation for the different counting functions:

- $A(t) = \sum_{i=1}^{\infty} \mathbb{1}_{\{\tau_i^A \le t\}}$;
- $\Sigma(t) = \sum_{n=1}^{\infty} \mathbb{1}_{\{\sigma(1,n) \le t\}}$;
- $D(t) = \sum_{i=1}^{\infty} \mathbb{1}_{\{\tau_i^D \le t\}}$.

For any non-decreasing function F, we denote by $F^{\leftarrow}(x) = \inf\{t,\ F(t) \ge x\}$ the pseudo-inverse of F (which is left-continuous). We have $F^{\leftarrow}(x) \le u \Leftrightarrow x \le F(u)$. Moreover, we use the notation $\wedge$ for min and $\vee$ for max. The following lemma gives a new description of the output process in term of counting functions.

Lemma 1. *Given a queue $\mathbf{Q} \in \mathbb{A}^* \times \mathbb{A}$, let $D = \Phi(\mathbf{Q})$ where Φ is the mapping defined by Equations (3) and (4). In term of counting functions, we have:*

$$D(t) = A(t) \wedge \inf_{0 \le s \le t} \Sigma[t - s + \Sigma^{\leftarrow}(A(s))]. \tag{5}$$

Proof is postponed to appendix.

Remark 1. Equations (3) and (5) give two equivalent definitions of the mapping $\Phi : \mathbb{A}^* \times \mathbb{A} \to \mathbb{A}$. But for $\tau^A \in \mathbb{A}$, only Equation (3) gives the right definition of Φ. In particular notice that we always have $\tau_i^D \ge \sigma(1,i) \vee \tau_i^A$, from which we derive $D(t) \le \Sigma(t) \wedge A(t)$.

2.2 Generalized Jackson Networks

We recall here the notation introduced in [2], to describe a generalized Jackson network with K nodes.

The networks we consider are characterized by the fact that service times and routing decisions are associated with stations and not with customers. This means that the j-th service on station k takes $\sigma_j^{(k)}$ units of time, where $\{\sigma_j^{(k)}\}_{j\ge1}$ is a predefined sequence. In the same way, when this service is completed, the leaving customer is sent to station $\nu_j^{(k)}$ (or leaves the network if $\nu_j^{(k)} = K+1$) and is put at the end of the queue on this station, where $\{\nu_j^{(k)}\}_{j\ge1}$ is also a predefined sequence, called the routing sequence. The sequences $\{\sigma_j^{(k)}\}_{j\ge1}$ and $\{\nu_j^{(k)}\}_{j\ge1}$, where k ranges over the set of stations, are called the driving sequences of the net. A generalized Jackson network will be defined by

$$\mathbf{JN} = \left\{ \{\sigma_j^{(k)}\}_{j\ge1},\ \{\nu_j^{(k)}\}_{j\ge1},\ n^{(k)},\ 0 \le k \le K \right\}.$$

where $(n^{(0)}, n^{(1)}, \dots, n^{(K)})$ describes the initial condition. The interpretation is as follows: for $k \neq 0$, at time $t = 0$, in node k, there are $n^{(k)}$ customers with service times $\sigma_1^{(k)}, \dots, \sigma_{n^{(k)}}^{(k)}$ (if appropriate, $\sigma_1^{(k)}$ may be interpreted as a residual service time).
Node 0 models the external arrival of customers in the network. Hence,

- if $n^{(0)} = 0$, there is no external arrival.
- if $\infty > n^{(0)} \geq 1$, then for all $1 \leq j \leq n^{(0)}$, the arrival time of the j-th customer in the network takes place at $\sigma_1^{(0)} + \cdots + \sigma_j^{(0)}$ and it joins the end of the queue of station $\nu_j^{(0)}$. Hence $\sigma_j^{(0)}$ is the j-th inter-arrival time. Note that in this case, there may be a finite number of customers passing through a given station so that the network is actually well defined once a finite sequence of routing decisions and service times are given on this station.
- if $n^{(0)} = \infty$, then when taking for instance the sequence $\{\sigma_j^{(0)}\}_{j\geq 1}$ i.i.d., the arrival process is a renewal process etc.

To each node of a generalized Jackson network, we can associate the following counting functions in $\mathbb{A}$:

1. $K + 1$ functions associated to the service times $\sigma^{(k)}$ (as in the single server queue);
2. $K(K+1)$ functions that counts the number of customers routed from a node $\{0, \dots, K\}$ to a node $\{1, \dots, K\}$;
3. $K + 1$ functions associated to $n^{(k)}$.

Hence a generalized Jackson network with K nodes is an object in $\mathbb{A}^{(K+1)(K+2)} = \mathbb{A}^{\mathbf{JN}}$.
We will use the following notation for each of these counting functions:

- $N = (n^{(0)}, \dots, n^{(K)})$, with $n^{(i)} \geq 0$;
- $\sigma^{(k)} = \{\sigma_j^{(k)}\}_{j\geq 1}$ and $\sigma^{(k)}(1, n) = \sum_{j=1}^{n} \sigma_j^{(k)}$, for $0 \leq k \leq K$;
- $\Sigma^{(i)}(t) = \sum_n \mathbb{1}_{\{\sigma^{(i)}(1,n)\leq t\}}$, for $0 \leq i \leq K$;
- $P_{i,j}(n) = \sum_{l\leq n} \mathbb{1}_{\{\nu_l^{(i)}=j\}}$, for $0 \leq i \leq K, \quad 1 \leq j \leq K+1$.

We denote the input and output processes of each queue k of the networks by $A^{(k)}$ and $D^{(k)}$ respectively, with the following notation $\mathbf{A} = (A^{(1)}, \dots, A^{(K)})$ and $\mathbf{D} = (D^{(1)}, \dots, D^{(K)})$. A procedure that constructs the processes $\mathbf{A}$ and $\mathbf{D}$ is given in Appendix 6.2.
We take the following notation: given a departure process for queue 0: $\Sigma^{(0)}$, and departure processes for the queues $i \in [1, K]$: $\mathbf{X} = \{X^{(i)}\}_{1\leq i\leq K}$, and an initial number of customers in each queue $n^{(i)}$, we construct the following input processes $\mathbf{Y} = \{Y^{(i)}\}_{1\leq i\leq K}$:

$$Y^{(i)}(t) = n^{(i)} + P_{0,i}(\Sigma^{(0)}(t) \wedge n^{(0)}) + \sum_{j=1}^{K} P_{j,i}(X^{(j)}(t)). \tag{6}$$

We denote this by $\mathbf{Y} = \Gamma(\mathbf{X}, \mathbf{JN})$.
Given an arrival process for each queue: $\mathbf{Y}$, we define the corresponding output process $\mathbf{X}$ and denote it by $\mathbf{X} = \Phi(\mathbf{Y}, \mathbf{JN})$. Hence, we have $X^{(i)} = \Phi(Y^{(i)}, \Sigma^{(i)})$, where Φ was defined for the single server queue in (3).

Property 1. *$\mathbf{A}$ and $\mathbf{D}$, the input and output processes of the generalized Jackson network are the unique solution of the fixed point equation*

$$\begin{cases} \mathbf{A} = \Gamma(\mathbf{D}, \mathbf{JN}), \\ \mathbf{D} = \Phi(\mathbf{A}, \mathbf{JN}). \end{cases} \tag{7}$$

We will denote by Ψ the mapping from $\mathbb{A}^{\mathbf{JN}}$ to $\mathbb{A}^2$ that to any Jackson network $\mathbf{JN}$ associates the corresponding couple $(\mathbf{A}, \mathbf{D})$.

Proof is postponed to appendix.

Remark 2. This property gives the connection between two possible descriptions of a generalized Jackson network. One of these descriptions has been given in words at the beginning of this section and is depicted with more rigor in Appendix 6.2. The other description is in term of fixed point Equation (7) which has already been introduced by Majewski in [12]. These two descriptions are equivalent in the special case of discrete inputs and an empty network at time $t = 0-$. For a more general result on this subject, we refer to [11].

3 Fluid Limit and Bottleneck Analysis

3.1 Fluid Limit for Single Server Queue

For any sequence of functions $\{f^n\}$, we define the corresponding scaled sequence $\{\hat{f}^n\}$ as follows: $\hat{f}^n(t) = \frac{f^n(nt)}{n}$. We say that $f^n \to f$ uniformly on compact sets, or simply $f^n \to f$ u.o.c. if for each $t > 0$,

$$\sup_{0 \le u \le t} |f^n(u) - f(u)| \to 0 \quad \text{as } n \to \infty.$$

We first recall the following lemma known as Dini's Theorem:

Lemma 2. *Let $\{f^n\}$ be a sequence of nondecreasing functions on $\mathbb{R}_+$ and let f be a continuous function on $\mathbb{R}$. Assume that $f^n(t) \to f(t)$ for all t (weak convergence is denoted by $f_n \to f$). Then $f^n \to f$ u.o.c.*

The following Lemma can be found in Billingsley [4] page 287:

Lemma 3. *If f_n are nondecreasing functions and $f_n \to f$, then $f_n^{\leftarrow} \to f^{\leftarrow}$.*

Property 2. *Consider a sequence of single server queues $\{\mathbf{Q}^n\} = \{\tau^{A,n}, \sigma^n\} \in (\mathbb{A} \times \mathbb{A})^{\mathbb{N}}$ with associated arrival process $\tau^{A,n}$ such that $\hat{A}^n(t) \to \hat{A}(t)$ for all $t > 0$, with $\hat{A}$ concave on $\mathbb{R}_+$, and associated service time process σ^n such that $\hat{\Sigma}^n(t) \to \mu t$ for all $t \ge 0$, with $\mu \ge 0$, then $\hat{D}^n \to \hat{D}$ u.o.c, with $\hat{D}(t) = \mu t \wedge \hat{A}(t)$.*

Proof:
First observe that thanks to Remark 1, we have $D^n(t) \leq \Sigma^n(t) \wedge A^n(t)$, hence making the fluid scaling and taking the limit in n, we have $\hat{D}(t) \leq \mu t \wedge \hat{A}(t)$. Property 2 follows in the case $\mu = 0$ by Lemma 2. We consider now the case $\mu > 0$ and first assume that: $\mathbf{Q}^n \in \mathbb{A}^* \times \mathbb{A}$ for all n and $\hat{A}(0) = 0$.
Since $\hat{A}(0) = 0$, $\hat{A}$ is continuous on $\mathbb{R}_+$ and Lemma 2 gives $\hat{A}^n \to \hat{A}$ u.o.c. Moreover thanks to Lemma 3, the sequences $\hat{\Sigma}^n$ and $\hat{\Sigma}^{n\leftarrow}$ converge u.o.c. to the respective functions $t \mapsto \mu t$ and $t \mapsto \frac{t}{\mu}$.
For fixed $t \geq 0$, we have thanks to uniformity on compact sets,

$$\begin{aligned}
\lim_{n\to\infty} \frac{D^n(nt)}{n} &= \lim_{n\to\infty} \inf_{0\leq u\leq t} \frac{1}{n}\Sigma^n[n(t-u) + (\Sigma^n)^{\leftarrow}(A^n(nu))] \wedge \frac{A^n(nt)}{n} \\
&= \inf_{0\leq u\leq t} \lim_{n\to\infty} \frac{1}{n}\Sigma^n[n(t-u) + (\Sigma^n)^{\leftarrow}(A^n(nu))] \wedge \lim_{n\to\infty} \frac{A^n(nt)}{n} \\
&= \inf_{0\leq u\leq t} [\mu(t-u) + \hat{A}(u)] \wedge \hat{A}(t) \\
&= \mu t \wedge \hat{A}(t),
\end{aligned}$$

where the last equality follows from concavity of $\hat{A}$. Now using Lemma 2, the result follows in this case.
To extend the result to the case: $\mathbf{Q}^n \in \mathbb{A} \times \mathbb{A}$, we consider the sequence $\tau_i^{B,n} = \tau_i^{A,n} + 1/i$ which belongs to $\mathbb{A}^*$. For any $\epsilon > 0$, we have for $n \geq 1/\epsilon$, $A^n(n(t-\epsilon)) \leq B^n(nt) \leq A^n(nt)$. Hence $\hat{A}(t-\epsilon) \leq \hat{B}(t) \leq \hat{A}(t)$ and since $\hat{A}$ is continuous, we have $\hat{B} = \hat{A}$. Moreover, since $\tau_i^{B,n} \geq \tau_i^{A,n}$, we have $D_B^n = \Phi(B^n, \Sigma^n) \leq \Phi(A^n, \Sigma^n)$, and we can apply the first part of the proof to $\hat{B}$, hence $D_B^n(t) \to \hat{A}(t) \wedge \mu t$ and the result follows in this case.
The case $\hat{A}(0) \neq 0$ can be dealt with the same monotonicity argument. For any $\epsilon > 0$, consider the sequence $\tau_i^{C,n} = \tau_i^{B,n} \vee i\epsilon$. We have $\hat{C}(t) = \frac{t}{\epsilon} \wedge \hat{A}(t)$ and $\tau_i^{C,n} \geq \tau_i^{A,n}$. We can apply the first part of the proof to $\hat{C}$, hence $D_C^n(t) \to \hat{C}(t) \wedge \mu t$. For $\epsilon \leq \mu^{-1}$, we get $\hat{D}(t) \geq \mu t \wedge \hat{A}(t)$. $\square$

3.2 Bottleneck Analysis

We first define the **N**on **C**apture condition as follows:
Condition (NC): we say that the $K \times K$ matrix $P = (p_{i,j})_{1\leq i,j\leq K}$ satisfies **(NC)**, if P is a substochastic matrix such that the following stochastic matrix

$$R = \begin{pmatrix} p_{1,1} & \dots & p_{1,K} & 1 - \sum_i p_{1,i} \\ & p_{i,j} & & \vdots \\ p_{K,1} & \dots & p_{K,K} & 1 - \sum_i p_{K,i} \\ 0 & \dots & 0 & 1 \end{pmatrix}$$

has only $K+1$ as absorbing state, i.e if (X_n) is a Markov chain with transition matrix R, almost surely (X_n) is equal to $K+1$ eventually.
The following lemma is proved in appendix:

Lemma 4. *Let P be a $K \times K$ substochastic matrix. The following properties are equivalent:*

1. *P satisfies* **(NC)***;*
2. *the Perron Frobenius eigenvalue of P is $r < 1$;*
3. *$(I - P')$ is invertible.*

For $\mathbf{x}$ and $\mathbf{y}$ two vectors of $\mathbb{R}^K$, we will write $\mathbf{x} \geq \mathbf{y}$ if $x_i \geq y_i$, for all i.
For any matrix P, any vector $\alpha \in \mathbb{R}_+^K$ and any $\mathbf{y} \in \mathbb{R}_+^K$, we define $\mathbf{F}_\alpha : \mathbb{R}_+^K \to \mathbb{R}_+^K$ and $\mathbf{G}_\mathbf{y} : \mathbb{R}_+^K \to \mathbb{R}_+^K$ with

$$\begin{aligned}
(\mathbf{F}_\alpha)_i(x_1, \ldots, x_K) &= \alpha_i + \sum_{j=1}^{K} p_{j,i} x_j, \\
(\mathbf{G}_\mathbf{y})_i(x_1, \ldots, x_K) &= x_i \wedge y_i.
\end{aligned}$$

Property 3. *If the matrix P satisfies* **(NC)***, the fixed point equation $\mathbf{F}_\alpha \circ \mathbf{G}_\mathbf{y}(\mathbf{x}) = \mathbf{x}$ has an unique solution $\mathbf{x}(\alpha, \mathbf{y})$. Moreover, $(\alpha, \mathbf{y}) \mapsto \mathbf{x}(\alpha, \mathbf{y})$ is a continuous, non-decreasing function.*

Remark 3. These relations already appeared in Massey [13] and Chen and Mandelbaum [5] see section 3.1. In fact as pointed out in [5], we can use Tarski's fixed point theorem (Tarski [16]) to get the existence of this fixed point (called inflow in [5]). But we give here a self-contained proof that shows continuity and monotonicity properties of the solution.

Proof:
Existence of a solution to the fixed point equation is an easy consequence of monotonicity. Since $\mathbf{F}_\alpha$ and $\mathbf{G}_\mathbf{y}$ are non-decreasing functions and $\mathbf{F}_\alpha \circ \mathbf{G}_\mathbf{y}(0) \geq 0$, we see that $(\mathbf{F}_\alpha \circ \mathbf{G}_\mathbf{y})^n(0) \nearrow \mathbf{b}$. We have $\mathbf{b} \leq \mathbf{F}_\alpha(\mathbf{y})$ and $\mathbf{F}_\alpha \circ \mathbf{G}_\mathbf{y}(\mathbf{b}) = \mathbf{b}$.
For a given subset Δ of $[1, K]$ and $\mathbf{y} \in \mathbb{R}_+^K$ we define $\mathbf{F}_{\alpha,\mathbf{y}}^\Delta : \mathbb{R}_+^K \to \mathbb{R}_+^K$ by

$$(\mathbf{F}_{\alpha,\mathbf{y}}^\Delta)_i(x_1, \ldots, x_K) = \alpha_i + \sum_{j \in \Delta} p_{j,i} y_j + \sum_{j \in \Delta^c} p_{j,i} x_j.$$

$F_\bullet^\Delta(\bullet)$ depend only on $\{x_i,\ i \in \Delta^c\}$ and $\mathbf{F}_\alpha = \mathbf{F}_{\alpha,\mathbf{y}}^\emptyset$.
We fix $\mathbf{y} \in \mathbb{R}_+^K$ and first study the case $\mathbf{F}_{\alpha,\mathbf{y}}^\Delta(\mathbf{x}) = \mathbf{x}$.
This equation is

$$\left\{\begin{aligned}
x_1 &= \alpha_1 + \textstyle\sum_{j \in \Delta} p_{j,1} y_j + \sum_{j \in \Delta^c} p_{j,1} x_j, \\
&\ \vdots \\
x_K &= \alpha_K + \textstyle\sum_{j \in \Delta} p_{j,K} y_j + \sum_{j \in \Delta^c} p_{j,K} x_j.
\end{aligned}\right.$$

In fact, we only have to calculate $\{x_i, i \in \Delta^c\}$ and then, we obtain $\{x_i, i \in \Delta\}$. Renumbering the indexes of $\mathbf{x}$, and taking into account only those in Δ^c, we have

$$\left\{\begin{array}{lcl} x_1 & = & \lambda_1(\alpha,\mathbf{y}) + \sum_{j=1}^n p^\Delta_{j,1} x_j, \\ & \vdots & \\ x_n & = & \lambda_n(\alpha,\mathbf{y}) + \sum_{j=1}^n p^\Delta_{j,n} x_j. \end{array}\right. \tag{8}$$

$P^\Delta = (p^\Delta_{i,j}; i,j = 1,\dots,n)$ is a substochastic matrix and $I - P^\Delta$ is invertible (even for $\Delta = \emptyset$ see Lemma 4). Hence, if $\lambda(\alpha,\mathbf{y}) = (\lambda_1(\alpha,\mathbf{y}),\dots,\lambda_n(\alpha,\mathbf{y}))$, Equation (8) has only one solution given by:

$$\tilde{\mathbf{x}}^\Delta = \lambda(\alpha,\mathbf{y}) + \tilde{\mathbf{x}}^\Delta P^\Delta \quad \Leftrightarrow \quad \tilde{\mathbf{x}}^\Delta = \lambda(\alpha,\mathbf{y})(I - P^\Delta)^{-1}.$$

We now return to our fix point problem $\mathbf{x} = \mathbf{F}_\alpha \circ \mathbf{G}_\mathbf{y}(\mathbf{x})$. To show uniqueness of the solution, take any solution $\mathbf{z} = \mathbf{F}_\alpha \circ \mathbf{G}_\mathbf{y}(\mathbf{z})$. We have $\mathbf{z} \geq 0$ hence $\mathbf{F}_\alpha \circ \mathbf{G}_\mathbf{y}(\mathbf{z}) \geq \mathbf{F}_\alpha \circ \mathbf{G}_\mathbf{y}(0)$ and then $\mathbf{z} \geq \mathbf{b}$. Let $A = \{i,\ z_i > y_i\}$ and $B = \{i,\ b_i > y_i\}$. Of course, we have $B \subset A$ and $\mathbf{b} = \tilde{\mathbf{x}}^B$ since $\mathbf{F}^B_{\alpha,\mathbf{y}}(\mathbf{b}) = \mathbf{F}_\alpha \circ \mathbf{G}_\mathbf{y}(\mathbf{b}) = \mathbf{b}$. Moreover, we have

$$\begin{array}{rcl} z_i & = & \alpha_i + \sum_{j\in B} p_{j,i} y_j + \sum_{j\in A\setminus B} p_{j,i} y_j + \sum_{j\notin A} p_{j,i} z_j, \\ (\mathbf{F}^B_{\alpha,\mathbf{y}})_i(\mathbf{z}) & = & \alpha_i + \sum_{j\in B} r_{j,i} y_j + \sum_{j\in A\setminus B} p_{j,i} z_j + \sum_{j\notin A} p_{j,i} z_j, \end{array}$$

hence, we have $\mathbf{F}^B_{\alpha,\mathbf{y}}(\mathbf{z}) \geq \mathbf{z}$. But since $(\mathbf{F}^B_{\alpha,\mathbf{y}})^n(\mathbf{z}) \nearrow \tilde{\mathbf{x}}^B = \mathbf{b}$, we have $\mathbf{b} \geq \mathbf{z}$. Finally $\mathbf{z} = \mathbf{b}$. For any Δ, $(\alpha,\mathbf{y}) \mapsto \tilde{\mathbf{x}}^\Delta(\alpha,\mathbf{y}) = \lambda(\alpha,\mathbf{y})(I - P^\Delta)^{-1}$ is a continuous non-decreasing function. Fix any $(\alpha,\mathbf{y})$, and define $A = \{i,\ x_i(\alpha,\mathbf{y}) \geq y_i\}$, $B = \{i,\ x_i(\alpha,\mathbf{y}) > y_i\}$. We have of course $\mathbf{x}(\alpha,\mathbf{y}) = \tilde{\mathbf{x}}^A(\alpha,\mathbf{y}) = \tilde{\mathbf{x}}^B(\alpha,\mathbf{y})$ and for $(\beta,\mathbf{z})$ in a neighborhood of $(\alpha,\mathbf{y})$, we have $x(\beta,\mathbf{z}) \in \{\tilde{\mathbf{x}}^A(\beta,\mathbf{z}),\ \tilde{\mathbf{x}}^B(\beta,\mathbf{z})\}$, and the continuity of $(\alpha,\mathbf{y}) \mapsto \mathbf{x}(\alpha,\mathbf{y})$ follows from that of $(\alpha,\mathbf{y}) \mapsto \tilde{\mathbf{x}}^\Delta(\alpha,\mathbf{y})$.
Now to see that this function is non decreasing, take $(\beta,\mathbf{z}) \geq (\alpha,\mathbf{y})$, we have

$$\mathbf{F}_\beta \circ \mathbf{G}_\mathbf{z}(\mathbf{x}(\alpha,\mathbf{y})) \geq \mathbf{F}_\alpha \circ \mathbf{G}_\mathbf{y}(\mathbf{x}(\alpha,\mathbf{y})) = \mathbf{x}(\alpha,\mathbf{y})$$

and the sequence $\{(\mathbf{F}_\beta \circ \mathbf{G}_\mathbf{z})^n(\mathbf{x}(\alpha,\mathbf{y}))\}_{n\geq 0}$ increases to $\mathbf{x}(\beta,\mathbf{z})$. $\Box$

3.3 Fluid Limit for Generalized Jackson Networks

We consider the following sequence of Jackson networks:

$$\begin{array}{rcl} \mathbf{JN}^n & = & \{\sigma^n, \nu^n, N^n\}, \quad \text{with}, \\ \lim_{n\to\infty} \frac{N^n}{n} & = & (n^{(0)}, n^{(1)}, \dots, n^{(K)}), \quad n^{(0)} \leq +\infty, \quad n^{(i)} < \infty,\ i \neq 0. \end{array}$$

Thanks to **Procedure 1** given in appendix, we can construct the corresponding input and output processes $\mathbf{A}^n$ and $\mathbf{D}^n$. We assume that the driving sequences satisfy

$$\begin{array}{rrcl} & \hat{\Sigma}^{(0),n}(t) & \to & \Sigma^{(0)}(t), \quad \text{where } t \mapsto \Sigma^{(0)}(t) \wedge n^{(0)} \text{ is a concave function,} \\ \forall k \geq 1, & \hat{\Sigma}^{(k),n}(t) & \to & \mu^{(k)}t, \quad \forall t \geq 0 \quad (\mu^{(k)} \geq 0), \\ & \hat{P}^n_{i,j}(t) & \to & p_{i,j}t \quad \forall t \geq 0. \end{array}$$

We suppose that the routing matrix $P = (p_{i,j})_{1 \leq i,j \leq K}$ satisfies **(NC)**.

Property 4. *The processes $\mathbf{A}^n$ and $\mathbf{D}^n$ converge uniformly on compact sets to a fluid limit defined by*

$$\hat{A}^{(i)}(t) = n^{(i)} + p_{0,i}(\Sigma^{(0)}(t) \wedge n^{(0)}) + \sum_{j=1}^{K} p_{j,i}\hat{D}^{(j)}(t), \tag{9}$$

$$\hat{D}^{(i)}(t) = \hat{A}^{(i)}(t) \wedge \mu^{(i)}t. \tag{10}$$

Remark 4.

1. Existence and uniqueness of solutions to Equations (9) and (10) follow directly from Property 3 as shown in the proof. Moreover, it easily follows from the proof that each component of $\mathbf{A}$ and $\mathbf{D}$ is concave and if $\Sigma^{(0)}$ is piece-wise linear then so are the processes $\mathbf{A}$ and $\mathbf{D}$.
2. Theorem 7.1 of [5] gives the fluid approximation of a generalized Jackson network, if we take a linear function for $\Sigma^{(0)}$, then from $(\hat{\mathbf{A}}, \hat{\mathbf{D}})$, we can calculate explicitly the solution of the equations of this Theorem.

Proof:
For any fixed $n \geq 1$, we define the sequences of processes $\{\mathbf{A}^n_t(k), \mathbf{D}^n_t(k)\}_{k \geq 0}$ and $\{\mathbf{A}^n_b(k), \mathbf{D}^n_b(k)\}_{k \geq 0}$ with the same recurrence equation:

$$\left\{ \begin{array}{l} \mathbf{A}^n(k+1) = \Gamma(\mathbf{D}^n(k), \mathbf{JN}^n), \\ \mathbf{D}^n(k+1) = \Phi(\mathbf{A}^n(k+1), \mathbf{JN}^n), \end{array} \right.$$

but with different initial conditions $\mathbf{D}^n_t(0) = (\Sigma^{(1),n}, \ldots, \Sigma^{(K),n})$ and $\mathbf{D}^n_b(0) = (0, \ldots, 0)$. We recall the notation:

$$\begin{aligned} \Gamma_i(\mathbf{X}, \mathbf{JN}^n)(t) &= n^{(i),n} + P^n_{0,i}(\Sigma^{(0),n}(t) \wedge n^{(0),n}) + \sum_{j=1}^{K} P^n_{j,i}(X_j(t)), \\ \Phi_i(\mathbf{X}, \mathbf{JN}^n)(t) &= \Phi(X_i, \sigma^{(i),n})(t), \end{aligned}$$

and we will use the scaled sequences $\hat{\mathbf{A}}^n(k)(t) = \frac{\mathbf{A}^n(k)(nt)}{n}$ and $\hat{\mathbf{D}}^n(k)(t) = \frac{\mathbf{D}^n(k)(nt)}{n}$. We introduce the mappings $\Gamma : \mathbb{C}^K \to \mathbb{C}^K$ and $\Phi : \mathbb{C}^K \to \mathbb{C}^K$ that appear in Equations (9) and (10) (where $\mathbb{C}$ is the set of continuous functions on $\mathbb{R}_+$):

$$\begin{aligned} \Gamma^s_i(x_1, \ldots, x_K)(t) &= n^{(i)} + p_{0,i}(\Sigma^{(0)}(t) \wedge n^{(0)}) + \sum_{j=1}^{K} p_{j,i}x_j(t), \\ \Phi^s_i(x_1, \ldots, x_K)(t) &= x_i(t) \wedge \mu^{(i)}t. \end{aligned}$$

The following lemma holds for both top and bottom sequences, hence we omit the $._t$ or $._b$.

Lemma 5. *Assume that for a fixed k, $\hat{\mathbf{D}}^n(k) \to \hat{\mathbf{D}}(k)$ u.o.c. and that each component of $\hat{\mathbf{D}}(k)$ is a concave function. Then we have*

$$\hat{\mathbf{A}}^n(k+1) \xrightarrow{n\to\infty} \Gamma^s(\hat{\mathbf{D}}(k)) = \hat{\mathbf{A}}(k+1) u.o.c.$$

and

$$\hat{\mathbf{D}}^n(k+1) \xrightarrow{n\to\infty} \Phi^s(\hat{\mathbf{A}}(k+1)) = \hat{\mathbf{D}}(k+1) \quad u.o.c.$$

and components of $\hat{\mathbf{A}}(k+1)$ and $\hat{\mathbf{D}}(k+1)$ are concave functions.

Proof of the lemma:
For any fixed t, we have

$$\frac{A^{(i),n}(k+1)(nt)}{n} = \frac{n^{(i),n}}{n} + \frac{P^n_{0,i}(\Sigma^{(0),n}(nt) \wedge n^{(0),n})}{n} + \sum_{j=1}^{K} \frac{P^n_{i,j}(D^{(j),n}(k)(nt))}{n}.$$

Hence thanks to Lemma 2, we have $\hat{\mathbf{A}}^n(k+1) \xrightarrow{n\to\infty} \Gamma^s(\hat{\mathbf{D}}(k))$ u.o.c. and each component of $\hat{\mathbf{A}}(k+1) = \Gamma^s(\hat{\mathbf{D}}(k))$ is clearly a concave function. Now thanks to Property 2 the result follows. □

We now return to the proof of Property 4.

We have $\hat{\mathbf{A}}(k+1) = \Gamma^s \circ \Phi^s(\hat{\mathbf{A}}(k))$. This equation gives the relation between 2 functions of a real parameter t. But we can fix this parameter and then we obtain for any fixed t an equation between real numbers that we write $\hat{\mathbf{A}}(k+1)(t) = \Gamma^s \circ \Phi^s(\hat{\mathbf{A}}(k)(t))$ (even if $\Gamma^s \circ \Phi^s$ is supposed to act on functions). Moreover as a consequence of Property 3, we know that the fixed point equation $\Gamma^s \circ \Phi^s(\zeta(t)) = \zeta(t)$ has an unique solution, namely $\zeta(t) = \mathbf{x}(\alpha, \mu^{(1)}t, \dots, \mu^{(K)}t)$, with $\alpha = (n^{(1)} + p_{0,1}(\Sigma^{(0)}(t) \wedge n^{(0)}), \dots, n^{(i)} + p_{0,i}(\Sigma^{(0)}(t) \wedge n^{(0)}), \dots, n^{(K)} + p_{0,K}(\Sigma^{(0)}(t) \wedge n^{(0)}))$. For any t, the sequence $\{\hat{\mathbf{A}}_b(k)(t)\}_{k\geq 1}$ (resp. $\{\hat{\mathbf{A}}_t(k)(t)\}_{k\geq 1}$) is non decreasing (resp. non increasing). We have $\hat{\mathbf{A}}_b(k)(t) \xrightarrow{k\to\infty} \zeta(t)$ and $\hat{\mathbf{A}}_t(k)(t) \xrightarrow{k\to\infty} \zeta(t)$ and $\hat{\mathbf{D}}_b(k)(t) \xrightarrow{k\to\infty} \Phi^s(\zeta(t))$ and $\hat{\mathbf{D}}_t(k)(t) \xrightarrow{k\to\infty} \Phi^s(\zeta(t))$.

Moreover, fix any $n \geq 1$, the mappings $. \mapsto \Gamma(., \mathbf{JN}^n)$ and $. \mapsto \Phi(., \mathbf{JN}^n)$ are non decreasing and :

$$\begin{cases} \mathbf{A}^n = \Gamma(\mathbf{D}^n, \mathbf{JN}^n), \\ \mathbf{D}^n = \Phi(\mathbf{A}^n, \mathbf{JN}^n). \end{cases}$$

Hence, for all $k \geq 0$, we have :

$$\begin{array}{lll} \mathbf{A}^n_b(k) \leq & \mathbf{A}^n & \leq \mathbf{A}^n_t(k), \\ \mathbf{D}^n_b(k) \leq & \mathbf{D}^n & \leq \mathbf{D}^n_t(k). \end{array}$$

We have :

$$\begin{array}{rcccl} \frac{\mathbf{A}_b^n(k)(nt)}{n} & \leq & \frac{\mathbf{A}^n(nt)}{n} & \leq & \frac{\mathbf{A}_t^n(k)(nt)}{n}, \\ \hat{\mathbf{A}}_b(k)(t) & \leq & \liminf_n \frac{\mathbf{A}^n(nt)}{n} \leq \limsup_n \frac{\mathbf{A}^n(nt)}{n} & \leq & \hat{\mathbf{A}}_t(k)(t), \end{array}$$

hence, we have

$$\forall t, \quad \lim_n \frac{\mathbf{A}^n(nt)}{n} = \zeta(t).$$

The result follows from Lemma 2. □

4 Maximal Dater Asymptotic

4.1 Motivation

We first recall the definition of simple Euler network from Section 4.1 of [2]. Consider a route $p = (p_1, \ldots, p_L)$ with $1 \leq p_i \leq K$ for $i = 2, \ldots, L-1$. Such a route is successful if $p_1 = 0$ and $p_L = K+1$. We can associate to such a route a routing sequence ν and a vector ϕ as follows ($\oplus$ means concatenation):

Procedure 2(p) :

−1− **for** $k = 0 \ldots K$ **do**
  $\nu^{(k)} := \emptyset;$
  $\phi^{(k)} := 0;$
od

−2− **for** $i = 1 \ldots L-1$ **do**
  $\nu^{(p_i)} := \nu^{(p_i)} \oplus p_{i+1};$
  $\phi^{(p_i)} := \phi^{(p_i)} + 1;$
od

Note that $\phi^{(j)}$ is the number of visits to node j in such a route.
A simple Euler network is a generalized Jackson network $E = \{\sigma, \nu, N\}$, with $N = (1, 0, \ldots, 0)$. The routing sequence $\nu = \{\nu_i^{(k)}\}_{i=1}^{\phi^{(k)}}$ is generated by a successful route and $\sigma = \{\sigma_i^{(k)}\}_{i=1}^{\phi^{(k)}}$ is a sequence of real-valued non-negative numbers, representing service times.
Consider now a sequence of simple Euler networks, say $\{E(l)\}_{l=1}^{+\infty}$ where $E(l) = \{\sigma(l), \nu(l), 1\}$. We define σ and ν to be the infinite concatenation of the $\{\sigma(l)\}_{l=1}^{+\infty}$ and $\{\nu(l)\}_{l=1}^{+\infty}$. Denote by σ_c the sequence obtained from σ in the following manner

$$\sigma_c = (c\sigma^{(0)}, \sigma^{(1)}, \ldots, \sigma^{(K)}).$$

We consider the corresponding sequence of Jackson networks $\mathbf{JN}_c^n = \{\sigma_c, \nu, N^n\}$, with $N^n = (n, 0, \ldots, 0)$. The Jackson network $\mathbf{JN}_c^n$ corresponds to an empty network with n

customers in node 0 at time $t = 0$. We will denote by X_c^n the time to empty the system $\mathbf{JN}_c^n$, called maximal dater of the network. Thanks to the Euler property of $\{E(i)\}_{i\geq 1}$, we know that for all n, $X_c^n < +\infty$ (see [2]). We suppose that

$$\lim_{n\to\infty} \frac{\sigma_c^{(0)}(1,n)}{n} = \frac{c}{\lambda}, \tag{11}$$

$$\lim_{n\to\infty} \frac{\sigma^{(k)}(1,n)}{n} = \frac{1}{\mu^{(k)}}, \quad 1 \leq k \leq K, \quad (\mu^{(k)} > 0) \tag{12}$$

$$\lim_{n\to\infty} \frac{P_{i,j}(n)}{n} = p_{i,j}, \quad 0 \leq i \leq K, \quad 1 \leq j \leq K+1. \tag{13}$$

We assume that $P = (p_{i,j})_{1\leq i,j\leq K}$ satisfies **(NC)**. We denote by π_i the solution of the following system

$$\forall i \in [1,K], \quad \pi_i = p_{0,i} + \sum_{j=1}^{K} p_{j,i}\pi_j. \tag{14}$$

The constant π_i is the expected number of visits to site i for the Markov chain with transition matrix P and with initial distribution $p_{0,i}$ (see proof of Lemma 4). We will prove the following theorem:

Theorem 1. *Under the previous conditions, we have for all $c \geq 0$*

$$\lim_{n\to\infty} \frac{X_c^n}{n} = \max_{1\leq i\leq K} \frac{\pi_i}{\mu^{(i)}} \vee \frac{c}{\lambda}.$$

4.2 Proof of Theorem 1

Given a routing matrix $P = (p_{i,j}; i,j = 0,\ldots,K+1)$ that satisfies **(NC)** and a vector $\alpha = (\alpha_1,\ldots,\alpha_K) \in \mathbb{R}_+^K$, we denote by π_i^α the solution of the following system (see Lemma 4)

$$\forall i \in [1,K], \quad \pi_i^\alpha = \alpha_i + \sum_{j=1}^{K} p_{j,i}\pi_j^\alpha.$$

Property 5. *Consider a sequence of Jackson networks as in Property 4 such that $\mu^{(k)} > 0$ for all k, $\Sigma^{(0)}(t) = \lambda t/c$ with $\lambda > 0$ and $c \geq 0$ (with the convention: $1/0 = +\infty$) and $X^n < +\infty$ for all n, we denote $\alpha = (n^{(1)} + n^{(0)}p_{0,1},\ldots,n^{(K)} + n^{(0)}p_{0,K})$, we have*

$$\lim_{n\to\infty} \frac{X_c^n}{n} = \max_{1\leq i\leq K} \frac{\pi_i^\alpha}{\mu^{(i)}} \vee \frac{cn^{(0)}}{\lambda}.$$

Proof:
Lower bound:
Consider the auxiliary Jackson network $\tilde{\mathbf{JN}}^n = \{0, \nu^n, N^n\}$, and the associated vector $\mathbf{Y}(n)$, where $Y^{(i)}(n)$ is the total number of customers that go through node i in this network. We have

$$Y^{(i)}(n) = n^{(i),n} + P^n_{0,i}(n^{(0),n}) + \sum_{j=1}^{K} P^n_{j,i}(Y^{(j)}(n)).$$

Hence $\lim_n \frac{Y^{(i)}(n)}{n} = \pi_i^\alpha$ thanks to assumption **(NC)** on P.
Now consider the original network $\mathbf{JN}^n_c$. The number of customers that go through node i is still $Y^{(i)}(n)$. Hence we have the following inequality for the maximal dater of node $i \geq 1$, $X^{(i),n} \geq \sigma^{(i),n}(1, Y^{(i)}(n))$. And for node 0, $X_c^{(0),n} \geq \sigma_c^{(0),n}(1, n^{(0),n})$. Hence, we have

$$\liminf_{n\to\infty} \frac{X^{(i),n}}{n} \geq \lim_{n\to\infty} \frac{\sigma^{(i),n}(1, Y^{(i)}(n))}{n} = \frac{\pi_i^\alpha}{\mu^{(i)}},$$
$$\liminf_{n\to\infty} \frac{X_c^{(0),n}}{n} \geq \lim_{n\to\infty} \frac{\sigma_c^{(0),n}(1, n^{(0),n})}{n} = \frac{cn^{(0)}}{\lambda}.$$

Since $X_c^n = \max_{1\leq i\leq K} X^{(i),n} \vee X_c^{(0),n}$, the lower bound follows.
Upper bound:
We consider the original Jackson network. Thanks to Property 4, we know that the corresponding input and output processes $\mathbf{A}^n$ and $\mathbf{D}^n$ converge to a fluid limit $\hat{A}$ and $\hat{D}$ respectively. Let $T^{(i)} = \inf\{t > 0,\ \hat{A}^{(i)}(t) = \hat{D}^{(i)}(t)\}$, $T = \max_{i\in[1,K]} T^{(i)}$ and $M = T \vee cn^{(0)}/\lambda$. We have

$$\forall t \geq M, \quad \hat{A}^{(i)}(t) = n^{(i)} + p_{0,i}n^{(0)} + \sum_{j=1}^{K} p_{j,i}\hat{A}^{(j)}(t),$$

hence, we have

$$\forall t \geq M, \quad \hat{A}^{(i)}(t) = \hat{D}^{(i)}(t) = \pi_i^\alpha. \tag{15}$$

We denote $i_0 = \arg\max\{T^{(i)}\}$ and we have $\hat{A}^{(i_0)}(T) = \hat{D}^{(i_0)}(T) = \mu^{(i_0)}T$ by concavity of $\hat{A}^{(i_0)}$, hence $T = \frac{\pi_{i_0}^\alpha}{\mu^{(i_0)}}$. Moreover, Equation (15) implies:

$$\forall t \geq M, \quad \frac{Y^{(i)}(n) - D^{(i),n}(nt)}{n} \xrightarrow{n\to\infty} 0,$$

where $Y^{(i)}(n)$ is the total number of customers that go through node i. Since $X_c^n < +\infty$, we know that for any t,

$$X_c^n \quad \leq \quad nt + \sum_{i=1}^{K} \sigma^{(i),n}(D^{(i),n}(nt), Y^{(i)}(n)) + \sigma_c^{(0),n}(\Sigma^{(0),n}(nt), n^{(0),n}),$$

taking $t = M$, we have $\limsup_{n\to\infty} \frac{X_c^n}{n} \le M = T \vee \frac{cn^{(0)}}{\lambda} = \frac{\pi_{i_0}^{\alpha}}{\mu^{(i_0)}} \vee \frac{cn^{(0)}}{\lambda}$, and the result follows. □

proof of Theorem 1:

It is easy to see that assumptions of Property 4 hold for the Jackson networks $\mathbf{JN}_c^n = \{\sigma_c, \nu, N^n\}$, with $n^{(i)} = 0$, except the $n^{(0)} = 1$.□

4.3 Stability of Generalized Jackson Networks

We now give the connection between this fluid limit and the stability region of generalized Jackson networks under stationary ergodic assumptions following [2].

Assume that we have a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, endowed with an ergodic measure-preserving shift θ. Consider a sequence of simple Euler networks, say $\{E(n)\}_{n=-\infty}^{\infty}$ where $E(n) = \{\sigma(n), \nu(n), 1\}$. Let $\xi(n) = \{\{\sigma(n)\}, \{\nu(n)\}\}$. The stochastic assumptions of Section 4.1 of [2] are as follows:

- the variables $\{\sigma(n)\}$, $\{\nu(n)\}$ are random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$;
- the random variable $\xi(n)$ satisfy the relation $\xi(n) = \xi(0) \circ \theta^n$ for all n, which implies that $\{\xi(n)\}_n$ is stationary and ergodic;
- all the expectations $\mathbb{E}\phi^{(k)}(0)$ and $\mathbb{E}\sum_{i=1}^{\phi^{(k)}(0)} \sigma_i^{(k)}(0)$ are finite ($\phi^{(j)}(n)$ is obtained by **Procedure 2** on $E(n)$).

In such a setting, we can find Ω_0, such that on Ω_0 conditions (12), (13) and **(NC)** hold and $\mathbb{P}(\Omega_0) = 1$. Thanks to the strong law of large numbers, we have almost surely:

$$\frac{\phi^{(j)}(1) + \cdots + \phi^{(j)}(n)}{n} \to \mathbb{E}\phi^{(j)}(0) < +\infty,$$

$$\frac{\sum_{i=1}^{\phi^{(j)}(1)} \sigma_i^{(j)}(1) + \cdots + \sum_{i=1}^{\phi^{(j)}(n)} \sigma_i^{(j)}(n)}{n} \to \mathbb{E} \sum_{i=1}^{\phi^{(j)}(0)} \sigma_i^{(j)}(0) < +\infty.$$

From these equations, we derive condition (12):

$$\lim_{n\to\infty} \frac{\sigma^{(j)}(1,n)}{n} = \frac{\mathbb{E}\sum_{i=1}^{\phi^{(j)}(0)} \sigma_i^{(j)}(0)}{\mathbb{E}\phi^{(j)}(0)} \overset{\text{def}}{=} \frac{1}{\mu^{(j)}} \quad \text{a.s.}$$

With the same kind of arguments, we show that limit (13) holds almost surely. To show that P satisfies **(NC)**, we denote $V^{(j)} = \mathbb{E}\phi^{(j)}(0)$ and $V^{(j)}(n) = \phi^{(j)}(1) + \cdots + \phi^{(j)}(n)$ and thanks to the Euler's property of the graphs, we have $V^{(i)}(n) = P_{0,i}(n) + \sum_{j=1}^{K} P_{j,i}(V^{(j)}(n))$, hence $V^{(i)} = p_{0,i} + \sum_{j=1}^{K} p_{j,i} V^{(j)}$. Equation (14) has a finite solution, hence P satisfies **(NC)** and $V^{(i)} = \pi_i$ (see Lemma 4). Now we can define Ω_0 as follows:

$$\Omega_0 = \left\{ \frac{\sigma^{(k)}(1,n)}{n} \to \frac{1}{\mu^{(k)}}, \ \frac{P_{i,j}(n)}{n} \to p_{i,j}, \ \frac{V^{(j)}(n)}{n} \to \pi_j \right\}.$$

We will take the conventional notation: $\mu^{(0)} = \lambda$ for the intensity of the external arrival. The limit calculated in Theorem 1 is exactly the constant $\delta(c)$ defined in Equation (85) of [2]. On the event Ω_0, Theorem 1 applies and gives a new proof of Theorem 15 of [2] which says that $\delta(0) = \gamma(0) = \max_i \pi_i/\mu^{(i)}$. Moreover, the lower bound of Lemma 6 (in [2]) is shown to be in fact the exact value of $\delta(c)$. Theorems 13 and 14 of [2] give the stability condition of a Jackson-type queueing networks in an ergodic setting. To be more precise: for $m \leq n \leq 0$, we define $\sigma_{[m,n]}$ and $\nu_{[m,n]}$ to be the concatenation of the $\{\sigma(k)\}_{m\leq k\leq n}$ and $\{\nu(k)\}_{m\leq k\leq n}$ and then define the corresponding generalized Jackson networks:

$$\mathbf{JN}_{[m,n]} = \{\sigma_{[m,n]}, \nu_{[m,n]}, N_{[m,n]}\}, \quad \text{with} \quad N_{[m,n]} = (m-n+1, 0, \ldots, 0).$$

We define $X_{[m,n]}$ to be the time to empty the generalized Jackson network $\mathbf{JN}_{[m,n]}$ and $Z_{[m,n]} = X_{[m,n]} - \sum_{i=1}^{n} \sigma^{(0)}_{[m,n],i}$ the associated maximal dater. Note that notation is consistent with [2]. The sequence $Z_{[-n,0]}$ is an increasing sequence. So there exists a limit $Z = \lim_{n\to\infty} Z_{[-n,0]}$ (which may be either finite or infinite). We call this limit Z the maximal dater of the generalized Jackson network $\mathbf{JN} = \{\sigma, \nu, N\}$ where σ and ν are the infinite concatenation of the $\{\sigma(k)\}_{k\leq 0}$ and $\{\nu(k)\}_{k\leq 0}$ and $N = (+\infty, 0, \ldots, 0)$. Let A be the event

$$A = \{Z = \lim_{n\to\infty} Z_{[-n,0]} = \infty\}. \tag{16}$$

This event is of crucial interest since a finite stationary construction of the state of the network can only be made on the complementary part of A. In other word, $Z < \infty$ iff the network is stable. The following Theorem follows from Theorem 13 and 14 of [2]

Theorem 2. *Let $\rho = \lambda \max_{1\leq i\leq K} \frac{\pi_i}{\mu^{(i)}}$. If $\rho < 1$, then $\mathbb{P}(A) = 0$. If $\rho > 1$, then $\mathbb{P}(A) = 1$.*

5 Rare Events in Generalized Jackson Networks

The aim of this section is to give a picture of one kind of rare event when the maximal dater of a generalized Jackson network becomes very big. Under some stochastic assumptions, one can prove that large maximal daters occur when a single large service time has taken place in one of the stations, and all other service times are close to their mean see [3]. We now give the corresponding fluid picture.

5.1 The One Big Jump Framework

We consider a sequence of simple Euler networks, say $\{E(n)\}_{n=-\infty}^{+\infty}$ where $E(n) = \{\sigma(n), \nu(n), 1\}$. Considering the corresponding $\mathbf{JN}_{[-n,+\infty]}$ network, we assume that

$$\hat{\Sigma}^{(0),n}(t) \to t/a, \quad \forall t, \tag{17}$$

$$\forall k \geq 1, \quad \hat{\Sigma}^{(k),n}(t) \to \mu^{(k)} t, \quad \forall t, \tag{18}$$

$$\forall i,j, \quad \hat{P}^n_{i,j}(t) \to p_{i,j} t, \quad \forall t. \tag{19}$$

We assume that $P = (p_{i,j})_{1\leq i,j\leq K}$ satisfies **(NC)** and we take the following notation:

$$\forall i \in [1,K], \quad \pi_i = p_{0,i} + \sum_{k=1}^{K} p_{k,i}\pi_k, \tag{20}$$

$$\forall i \in [1,K], \quad x_i = p_{0,i} + \sum_{k\neq j} p_{k,i}x_k \Rightarrow x_j = p_j, \tag{21}$$

$$\forall i \in [1,K], \quad \pi_{j,i} = \delta_{j,i} + \sum_{k=1}^{K} p_{k,i}\pi_{j,k}. \tag{22}$$

Equation (20) is the traditional traffic equation of the network in term of number of customers. In Equation (21), p_j corresponds to the amount of traffic coming in queue j if this one is blocked (its departure process is null). Note that in this case $x_i \leq \pi_i$. Equation (22) corresponds to the traffic equation in the network when there is no input from the outside world and only queue j is active. We introduce the corresponding loads:

$$b_j = \frac{\pi_j}{\mu^{(j)}}, \quad b = \max_j b_j \quad \text{and,} \quad b_{j,i} = \frac{\pi_{j,i}}{\mu^{(i)}}, \quad B_j = \max_i b_{j,i}.$$

We assume that the stability condition $b < a$ holds. We suppose that the big jump occurs in the simple Euler network $-n$, hence we replace $E(-n)$ by an extra E which is not "typical" in the following sense: a big service time σ takes place on station j and within the set of service times of the simple Euler network E. Let us look at the corresponding maximal dater $Z_{[-n,0]}(E)$ in the fluid scale suggested by the limit of Property 5:

- if $\sigma > na$, then the number of customers blocked in station j at time σ is of the order of np_j, whereas the number of customers in the other stations is small. So, according to Property 5, the time to empty the network from time σ on should be of the order np_jB_j; hence, in this case, the maximal dater in question should be of the order of $\sigma - na + np_jB_j$;
- if $\sigma < na$, then at time σ, the number of customers blocked in station j is of the order of $p_j\frac{\sigma}{a}$, and the other stations have few customers; from time σ to the time of the last arrival (which is of the order of na), station k has to serve approximately the load $p_j\frac{\sigma}{a}b_{j,k}$ generated by these blocked customers plus the load $(na-\sigma)\frac{b_k}{a}$ generated by the external arrivals on the time interval from σ to the last arrival. On this time interval, the service capacity is of the order of $(na-\sigma)$. Hence the maximal dater should be of the order of $\max_k \left\{p_jb_{j,k}\frac{\sigma}{a} + (na-\sigma)\frac{b_k}{a} - (na-\sigma)\right\}^+$.

It is now natural to introduce the following function:

$$f^j(\sigma,n) = \mathbb{1}_{\{\sigma>na\}}\left\{\sigma - na + np_jB_j\right\} + \mathbb{1}_{\{\sigma\leq na\}} \max_k \left\{p_jb_{j,k}\frac{\sigma}{a} + \left(\frac{b_k}{a} - 1\right)(na-\sigma)\right\}^+. \tag{23}$$

We now return to rigor and consider the network $\mathbf{JN}^n(E)$ with input $\{\tilde{E}(k)\}_{k=-n}^{\infty}$, where $\tilde{E}(k) = E(k)$ for all $k > -n$ and $\tilde{E}(-n) = E$. That is, if we denote by $\sigma^{(k),n}$ and $\nu^{(k),n}$ the concatenations $(\{\sigma^{(k)}(E)\}, \{\sigma^{(k)}(-n+1)\}, \dots, \{\sigma^{(k)}(0)\}, \dots)$ and $(\{\nu^{(k)}(E)\}, \{\nu^{(k)}(-n+1)\}, \dots, \{\nu^{(k)}(0)\}, \dots)$ respectively, then

$$\mathbf{JN}^n(E) \quad = \quad \{\sigma^n(E), \nu^n(E), 0, N^n\}, \quad \text{with} \quad N^n = (n, 0, \dots, 0).$$

The maximal dater of order $[-n, 0]$ in this network will be denoted by $\tilde{Z}^n(E)$. Of course $\tilde{Z}^n(E(n)) = Z_{[-n,0]}$. For all simple Euler networks $E = (\sigma, \nu, 1)$, let $Y^{(j)}(E) = \sum_{u=1}^{\phi^{(j)}} \sigma_u^{(j)}$.

Let z_n be some sequence of positive real numbers, we define:

$$\begin{aligned}
\mathbf{U}^j(n) &= \{E \text{ is a simple Euler network such that } Y^{(k)}(E) \le z_n \ \forall k \ne j\}, \\
\mathbf{V}^j(n) &= \{E \in \mathbf{U}^j(n),\ Y^{(j)}(E) \ge n(a-b),\ \phi^{(j)} \le L\},
\end{aligned}$$

Property 6. *Under the previous assumptions, there exists a sequence $z_n \to \infty$ with $\frac{z_n}{n} \to 0$, such that we have*

$$\sup_{\{E \in \mathbf{V}^j(n)\}} \left| \frac{\tilde{Z}^n(E) - f^j(Y^{(j)}(E), n)}{n} \right| \xrightarrow{n \to \infty} 0. \tag{24}$$

5.2 Computation of the Fluid Limit

We take a sequence of simple Euler networks $F_n \in \mathbf{V}^j(n)$ and denote by $\mathbf{JN}^n = \mathbf{JN}^n(F_n)$. Since z_n/n tends to 0, we have

$$\begin{aligned}
&\hat{\Sigma}^{(0),n}(t) \to t/a, \quad \forall t, \quad \text{a.s.} \\
\forall k \ne j \ge 1, \quad &\hat{\Sigma}^{(k),n}(t) \to \mu^{(k)} t, \quad \forall t, \quad \text{a.s.} \\
\forall i, j, \quad &\hat{P}_{i,j}^n(t) \to p_{i,j} t, \quad \forall t, \quad \text{a.s.}
\end{aligned}$$

We denote by $\zeta_n = Y^{(j)}(F_n) \in [n(a-b), +\infty)$ and by T_n the time for station j to complete its $\phi^{(j)}(F_n)$ first services in the network $\mathbf{JN}^n$. From monotonicity, we get $\zeta_n \le T_n \le \zeta_n + \sum_{k \ne j} Y^{(k)}(F_n)$. Hence, we have $\lim_{n\to\infty} \frac{T_n}{\zeta_n} = 1$, since $z_n/\zeta_n \le z_n/n(a-b) \to 0$. We first suppose that $\zeta_n/n \to \zeta < +\infty$. Then $\mathbf{JN}^n$ is such that $\Sigma^{(j),n}(t) \le L$ for $t \le T_n$. Hence $\frac{\Sigma^{(j),n}(nt)}{n} \le \frac{L}{n}$ for $nt \le T_n$, so that $\hat{\Sigma}^{(j),n}(t) \to 0$, for all $t \le \zeta$. We see that this last fluid limit does not hold on the whole positive real line. Nevertheless consider the Jackson networks with the same driving sequences as $\mathbf{JN}^n$ except for station j where we take the concatenation of $(\{\sigma^{(j)}(F_n)\}, \infty, \dots)$. For this new network, the fluid limit for station j holds on $\mathbb{R}_+$ and we can directly apply Property 4. But it is easy to see that for $t \le T_n$, this network and the original Jackson network $\mathbf{JN}^n$ have exactly the same dynamic. Hence, Property 4 applies for $t \le \zeta$, so that for each k, the sequence $\{\hat{A}^{(k),n}\}$ converges u.o.c. to a limit $\hat{A}^{(k)}$ when n tends to ∞, with a similar result and notation for the departure process. We have with $\lambda = a^{-1}$,

$$\begin{aligned}
\hat{A}^{(i)}(t) &= p_{0,i}\lambda(t\wedge a) + \sum_{k=1}^{K} p_{k,i}\hat{D}^{(k)}(t),\\
\hat{D}^{(i)}(t) &= \hat{A}^{(i)}(t) \wedge \tilde{\mu}^{(i)} t \quad \text{with } \tilde{\mu}^{(i)} = \mu^{(i)} \text{ for } i \neq j \text{ and } \tilde{\mu}^{(j)} = 0.
\end{aligned}$$

We can rewrite the first expression:

$$\hat{A}^{(i)}(t) = \lambda p_{0,i}(t\wedge a) + \sum_{k\neq j} p_{k,i}\hat{D}^{(k)}(t).$$

Hence with the notation introduced in previous section, we have

$$\begin{aligned}
\hat{A}^{(i)}(t) &= \hat{D}^{(i)}(t) = \lambda x_i(t\wedge a) \leq \lambda \pi_i (t\wedge a) \quad \text{for} \quad t\leq \zeta \quad \text{and} \quad i\neq j,\\
\hat{A}^{(j)}(t) &= \lambda p_j (t\wedge a) \quad \text{for} \quad t \leq \zeta.
\end{aligned}$$

In what follows, we will consider the new Jackson network obtained by taking as initial condition the state of the initial network at time T_n and as routing and service sequences the routing decisions and (residual) service still unused at this time. This network will be denoted by $\bar{\mathbf{JN}}^n = \{\bar{\sigma}^n, \bar{\nu}^n, 0, \bar{N}^n\}$, with

$$\begin{aligned}
\bar{\sigma}^{(0),n} &= \left\{ \Sigma^{(0),n\leftarrow}(\Sigma^{(0),n}(T_n)+1) - T_n, \sigma^{(0),n}_{\Sigma^{(0),n}(T_n)+2}, \dots \right\},\\
\bar{\nu}^{(0),n} &= \left\{ \nu^{(0),n}_{\Sigma^{(0),n}(T_n)+1}, \nu^{(0),n}_{\Sigma^{(0),n}(T_n)+2}, \dots \right\},\\
\bar{N}^{(0),n} &= N^{(0),n} - \Sigma^{(0),n}(T_n),
\end{aligned}$$

and for $i \neq 0$,

$$\begin{aligned}
\bar{\sigma}^{(i),n} &= \left\{ \sigma^{(i),n}_{D^{(i),n}(T_n)+1} - r^{(i),n}, \sigma^{(i),n}_{D^{(i),n}(T_n)+2}, \dots \right\},\\
\bar{\nu}^{(i),n} &= \left\{ \nu^{(i),n}_{D^{(i),n}(T_n)+1}, \nu^{(i),n}_{D^{(i),n}(T_n)+2}, \dots \right\},\\
\bar{N}^{(i),n} &= A^{(i),n}(T_n) - D^{(i),n}(T_n),\\
r^{(i),n} &= \begin{cases} \sigma^{(i),n}_{D^{(i),n}(T_n)+1} & \text{if } A^{(i),n}(T_n) = D^{(i),n}(T_n),\\ \max(A^{(i),n\leftarrow}\left(D^{(i),n}(T_n)+1\right), D^{(i),n\leftarrow}\left(D^{(i),n}(T_n)\right)) - T_n & \text{else.} \end{cases}
\end{aligned}$$

We have:

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} A^{(i),n}(T_n) &= \lim_{n\to\infty}\frac{1}{n} D^{(i),n}(T_n) = \hat{A}^{(i)}(\zeta) = \hat{D}^{(i)}(\zeta) \quad \text{for } i\neq j,\\
\lim_{n\to\infty} \frac{1}{n} A^{(j),n}(T_n) &= \lambda p_j(\zeta\wedge a) \quad \text{and} \quad \lim_{n\to\infty}\frac{1}{n} D^{(j),n}(T_n) = 0.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\frac{\bar{N}^n}{n} &\rightarrow (\lambda(a-\zeta)^+, 0, \ldots, \lambda p_j(\zeta \wedge a), \ldots, 0) \\
\hat{\bar{\Sigma}}^{(0),n}(t) &\rightarrow \lambda t, \quad \forall t, \\
\forall i \geq 1, \quad \hat{\bar{\Sigma}}^{(i),n}(t) &\rightarrow \mu^{(i)} t, \quad \forall t, \\
\forall i,j, \quad \hat{\bar{P}}^n_{i,j}(t) &\rightarrow p_{i,j} t, \quad \forall t.
\end{aligned}$$

We can apply Property 5 with a parameter α that depends on the quantity $a - \zeta$:

- if $\zeta \geq a$, then we have $\alpha = p_j e_j$, with $e_j = (0, \ldots, 1, \ldots, 0)$ with the one is in j-th position and

$$\pi_i^\alpha = p_j \pi_{j,i}.$$

  Property 5 gives

$$\frac{\tilde{Z}_n(F_n) + na - T_n}{n} \quad \rightarrow \quad p_j \max_i \frac{\pi_{j,i}}{\mu^{(i)}}. \tag{25}$$

  Hence, we have

$$\tilde{Z}_n(F_n) \quad = \quad (T_n - na + np_j B_j)(1 + o(n)) = f^j(T_n, n)(1 + o(n)). \tag{26}$$

- if $\zeta < a$, we have $\alpha = \lambda(a-\zeta)P_0 + \lambda p_j \zeta e_j /$, where $P_0 = (p_{0,1}, \ldots, p_{0,K})$ and

$$\pi_i^\alpha = \lambda \left[(a-\zeta)\pi_i + p_j \pi_{j,i} \zeta\right].$$

  Property 5 gives

$$\frac{\tilde{Z}_n(F_n) + na - T_n}{n} \rightarrow (a-\zeta) \vee \lambda \max_i \left[\frac{(a-\zeta)\pi_i + p_j \pi_{j,i}\zeta}{\mu^{(i)}}\right]. \tag{27}$$

  Hence, we have

$$\tilde{Z}_n(F_n) \quad = \quad (1 + o(n)) \max_i \left[p_j b_{j,i} \frac{T_n}{a} + (na - T_n)(\frac{b_i}{a} - 1)\right]^+ = f^j(T_n, n)(1 + o(n)) \tag{28}$$

The case $\zeta_n / n \rightarrow \infty$ corresponds to $\zeta = \infty$. Results until Equation (25) hold true in this context, hence Equation (26) holds true.
Finally we proved that for any sequence $F_n \in \mathbf{V}^j(n)$ with $Y^{(j)}(F_n) = \zeta_n \in [n(a-b), +\infty)$ such that $\zeta_n / n \rightarrow \zeta \leq +\infty$,

$$\left|\frac{\tilde{Z}_n(F_n) - f^j(\zeta_n, n)}{n}\right| \xrightarrow{n \rightarrow \infty} 0. \tag{29}$$

But the result holds for any sequence $F_n \in \mathbf{V}^j(n)$. Consider any sequence $F_n \in \mathbf{V}^j(n)$ and suppose that

$$\limsup_{n\to\infty}\left|\frac{\tilde{Z}_n(F_n)-f^j(Y^{(j)}(F_n),n)}{n}\right| = l > 0.$$

By extracting a subsequence of $\{F_n\}$, we can replace lim sup by lim. Moreover by doing once more an extraction, we may suppose that $Y^{(j)}(F_n)/n \to \zeta \le +\infty$ and for this subsequence, limit (29) is violated. Hence for any sequence F_n, we have

$$\left|\frac{\tilde{Z}_n(F_n)-f^j(Y^{(j)}(F_n),n)}{n}\right| \xrightarrow{n\to\infty} 0. \tag{30}$$

We consider now a sequence $F_n \in \mathbf{V}^j(n)$ such that

$$\left|\frac{\tilde{Z}_n(F_n)-f^j(Y^{(j)}(F_n),n)}{n}\right| \ge \sup_{\{E\in\mathbf{V}^j(n)\}}\left|\frac{\tilde{Z}_n(E)-f^j(Y^{(j)}(E),n)}{n}\right| - \epsilon_n,$$

with $\epsilon_n \to 0$. Thanks to (30), we see that (24) holds. □

Remark 5. In the stochastic framework of section 4.3, we see that assumptions on the limits (17), (18) and (19) are fulfilled. In particular, if the sequence of simple Euler networks $\{E(n)\}_{n=-\infty}^{+\infty}$ is i.i.d, then we deduce from previous property that:

$$\sup_{\{E\in\mathbf{V}^j(n)\}}\left|\frac{\tilde{Z}^n(E)-f^j(Y^{(j)}(E),n)}{n}\right| \xrightarrow{n\to\infty} 0 \quad \text{a.s.}$$

6 Appendices

6.1 Proof of Lemma 1

For $1 \le j$, we define the point process Γ_j as follows:

$$\begin{aligned}\tau_n^{\Gamma_j} &= 0 \quad \text{for } 1 \le n \le j-1,\\ \tau_n^{\Gamma_j} &= \tau_j^A + \sigma(j,n) \quad \text{for } n \ge j.\end{aligned}$$

The construction of Γ_j is depicted in Figure 1 and we have

$$\begin{aligned}\Gamma_j(t) &= j-1 \quad \text{for } t < \tau_j^A,\\ \Gamma_j(t) &= \Sigma(t-\tau_j^A+\sigma(1,j-1)) \quad \text{for } t \ge \tau_j^A.\end{aligned}$$

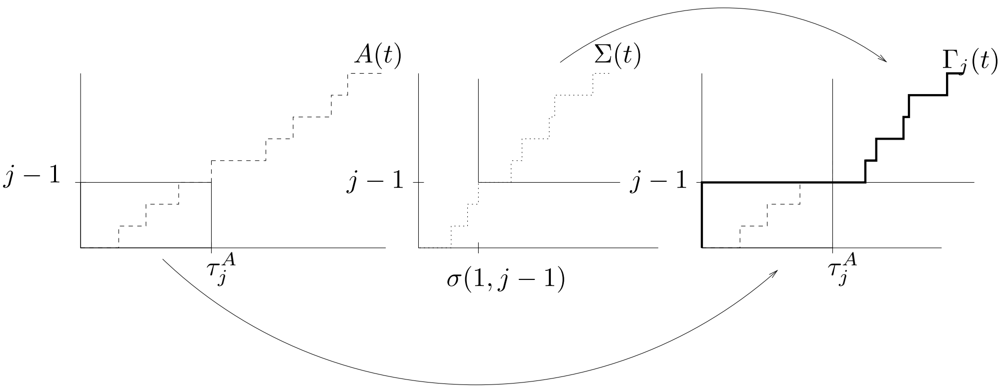

Figure 1: Construction of Γ_j

Thanks to (3), we have $t \geq \tau_n^D \Leftrightarrow \forall j \leq n,\ t \geq \tau_n^{\Gamma_j}$, hence we have

$$D(t) \geq n \quad \Leftrightarrow \quad \inf_{j \leq n} \Gamma_j(t) \geq n,$$

but we have for all $j \geq n+1$, $\Gamma_j(t) \geq n$, for all t, hence $D(t) = \inf_{j\geq 1} \Gamma_j(t)$. We have

$$\inf_{j\geq 1} \Gamma_j(t) = \inf_{\{j \geq 1,\ \tau_j^A \leq t\}} \Sigma[t - \tau_j^A + \sigma(1, j-1)] \wedge A(t).$$

We now show that $\inf_{j\geq 1} \Gamma_j(t) = A(t) \wedge \inf_{0\leq s \leq t} \Sigma[t - s + \Sigma^{\leftarrow}(A(s))]$. Since $\tau_j^A \in \mathbb{A}_2^*$, on each interval $[\tau_{j-1}^A, \tau_j^A)$ (we use the convention $\tau_0^A = 0$), we have $A(s) = j-1$ and the function $s \mapsto \Sigma[t - s + \Sigma^{\leftarrow}(j-1)]$ is non-increasing, hence we have

$$\inf_{s \in [\tau_{j-1}^A, \tau_j^A)} \Sigma[t - s + \Sigma^{\leftarrow}(A(s))] \quad = \quad \Sigma[t - \tau_j^A + \sigma(1, j-1)].$$

Moreover, we have for $\tau_k^A \leq t < \tau_{k+1}^A$,

$$\inf_{s \in [\tau_k^A, t)} \Sigma[t - s + \Sigma^{\leftarrow}(A(s))] = \Sigma(\Sigma^{\leftarrow}(k)) \geq k = A(t).$$

Finally we have

$$\begin{aligned} A(t) \wedge \inf_{0 \leq s \leq t} \Sigma[t - s + \Sigma^{\leftarrow}(A(s))] &= A(t) \wedge \inf_{\{j\geq 1,\ \tau_j^A \leq t\}} \Sigma[t - \tau_j^A + \sigma(1, j-1)] \\ &= \inf_{j \geq 1} \Gamma_j(t). \end{aligned}$$

□

6.2 Construction of Arrival and Departure Processes

We give a procedure that constructs the processes **A** and **D**:

Procedure 1(JN) :

−1− $t := 0$;
for $i \geq 0$ **do**
$R^{(i)}(t) := \sigma_1^{(i)}$; $A^{(i)}(t) := n^{(i)}$; $D^{(i)}(t) := 0$;
od
−2− $V := \min_{\{i,\, A^{(i)}(t)-D^{(i)}(t)\geq 1\}} R^{(i)}(t)$; $\gamma := \arg \min_{\{i,\, A^{(i)}(t)-D^{(i)}(t)\geq 1\}} R^{(i)}(t)$;
−3− **if** $V = \infty$ **then** END;
fi
−4− $D^{(\gamma)}(t+V) := D^{(\gamma)}(t) + 1$; $A^{(\gamma)}(t+V) := A^{(\gamma)}(t)$;
if $A^{(\gamma)}(t+V) - D^{(\gamma)}(t+V) \geq 1$ **then** $R^{(\gamma)}(t+V) := \sigma^{(\gamma)}_{D^{(\gamma)}(t+V)+1}$; **fi**
$j := \nu^{(\gamma)}_{D^{(\gamma)}(t+V)}$;
if $j \neq K+1$ **then** $A^{(j)}(t+V) := A^{(j)}(t) + 1$; $D^{(j)}(t+V) := D^{(j)}(t)$;
  if $A^{(j)}(t) - D^{(j)}(t) = 0$ **then** $R^{(j)}(t+V) := \sigma^{(j)}_{A^{(j)}(t+V)}$; **fi**
fi
for $i \notin \{\gamma, j\}$ **do**
$R^{(i)}(t+V) := R^{(i)}(t) - V$; $A^{(i)}(t+V) := A^{(i)}(t)$; $D^{(i)}(t+V) := D^{(i)}(t)$;
od
$t := t + V$;
−5− **goto** 2;

Remark 6. Since each sequence $\{\sigma_j^{(k)}\}_{j\geq 1}$ or $\{\nu_j^{(k)}\}_{j\geq 1}$ is infinite, the variables

$$\nu^{(\gamma)}_{D^{(\gamma)}(t+V)}, \sigma^{(\gamma)}_{D^{(\gamma)}(t+V)}, \sigma^{(j)}_{A^{(j)}(t+V)}$$

in step 4 are always available:

- if $\sum_{i=0}^K n^{(i)} < +\infty$ then the procedure ends in step 3;
- if $\sum_{i=0}^K n^{(i)} = +\infty$, the procedure never ends, this corresponds to a network with infinite number of customers. In this case there exists $T \leq \infty$ such that $\lim_{t\to T} A(t) = \lim_{t\to T} D(t) = \infty$.

Proof of Property 1:
If we define $J^{(k)} = \sup\{j, \sum_{i=1}^j \sigma_i^{(k)} = 0\}$, the generalized Jackson network is equivalent to

the following

$$\left\{ \{\sigma_j^{(k)}\}_{j \geq J^{(k)}+1}, \{\nu_j^{(k)}\}_{j \geq J^{(k)}+1}, n^{(k)} + \sum_{i=0}^{K} P_{i,k}(J^{(i)}) \right\}.$$

Hence, we can assume that $J^{(k)} = 0$, for all k and we have for time $t = 0$, $A^{(i)}(0) = n_i$, $D^{(i)}(0) = 0$. For $t \geq 0$ let

$$\begin{aligned}
\tilde{D}^{(i)}(t) &= A^{(i)}(0) \wedge \inf_{0 \leq s \leq t} \Sigma^{(i)}[t - s + \Sigma^{(i)\leftarrow}(A^{(i)}(0))] \\
\tilde{A}^{(i)}(t) &= n^{(i)} + P_{0,i}(\Sigma^{(0)}(t) \wedge n^{(0)}) + \sum_{j=1}^{K} P_{j,i}(\tilde{D}^{(j)}(t)).
\end{aligned}$$

Now consider $t_1 = \inf\{t \geq 0, \exists i, \tilde{D}^{(i)}(t-) \neq \tilde{D}^{(i)}(t), \text{or } \Sigma^{(0)}(t-) \neq \Sigma^{(0)}(t)\}$ the first time of jump for processes $\tilde{D}$ and $\tilde{A}$. Thus

$$\begin{aligned}
A^{(i)}(t) &= \tilde{A}^{(i)}(t) \quad \text{for } 0 \leq t \leq t_1, \\
D^{(i)}(t) &= \tilde{D}^{(i)}(t) \quad \text{for } 0 \leq t \leq t_1,
\end{aligned}$$

provide a solution pair to (7) over $t \in [0, t_1]$, moreover this solution is exactly the one constructed by the previous procedure. Now suppose a solution pair $(\mathbf{A}, \mathbf{D})$ has been constructed on $[0, t_n]$, where t_n is a jump point for one of the $A^{(i)}$, $D^{(i)}$. As above let $X(s) = \mathbf{A}(s)$ for $s \leq t_n$, $X(s) = \mathbf{A}(t_n)$ for $s > t_n$, and for $t \geq t_n$ define,

$$\begin{aligned}
\tilde{D}^{(i)}(t) &= X^{(i)}(t) \wedge \inf_{0 \leq s \leq t} \Sigma^{(i)}[t - s + \Sigma^{(i)\leftarrow}(X^{(i)}(s))], \\
\tilde{A}^{(i)}(t) &= n^{(i)} + P_{0,i}(\Sigma^{(0)}(t) \wedge n^{(0)}) + \sum_{j=1}^{K} P_{j,i}(\tilde{D}^{(j)}(t)).
\end{aligned}$$

Letting $t_{n+1} = \inf\{t \geq t_n, \exists i, \tilde{D}^{(i)}(t-) \neq \tilde{D}^{(i)}(t), \text{or } \Sigma^{(0)}(t-) \neq \Sigma^{(0)}(t)\}$, one concludes as above. The uniqueness of $(\mathbf{A}, \mathbf{D})$ and the fact that $(\mathbf{A}, \mathbf{D})$ are constructed by **Procedure 1** are a consequence of this construction procedure. □

Remark 7. This construction is very similar to the construction of the reflection mapping made in the proof of Theorem 2.1 of [5].

6.3 Proof of Lemma 4

Proof:
$1 \Rightarrow 2 \Rightarrow 3$ by Corollary 1 page 8 and Corollary 2 page 31 of Seneta [14]. To see that $3 \Rightarrow 1$, just write the equations for the expected number of visits for the Markov chain (X_n) with transition matrix R, to state $i \neq K+1$, $V_i = \mathbb{E}\left[\sum_n \mathbb{1}_{\{X_n = i\}}\right]$:

$$V_i = \mathbb{P}[X_0 = i] + \sum_{j=1}^{K} p_{j,i} V_j \quad \text{for all } i \in [1, K]. \tag{31}$$

Since $(I - P')$ is invertible, (31) has a finite solution. Hence the only absorbing state of (X_n) is $K + 1$. □

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