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Design of Stabilizing Receding Horizon Controls for Constrained Linear Time-Varying Systems

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Abstract: In this paper, we propose a generalized stabilizing receding horizon control (RHC) scheme for input/state constrained linear discrete time-varying systems that improves feasibility and on-line computation on the constrained finite-horizon optimization problem, compared with existing schemes. The control scheme is based on a time-varying horizon cost function with time-varying terminal weighting matrices, which can easily be implemented via linear matrix inequality (LMI) optimization. We discuss modifications of the proposed scheme that improve feasibility or on-line computation time. Through various simulation examples, we illustrate the results of these schemes.

Key-words: RHC, constrained, time-varying, stability, feasibility, computation time.

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Conception de contrôles stabilisateurs à horizon fuyant pour des systèmes linéaires contraints et variables dans le temps

Résumé : Nous proposons dans cet article un nouveau cadre pour le contrôle stabilisateur à horizon fuyant pour des systèmes linéaires variables dans le temps, discrets et contraints en entrée/sortie. Par rapport aux cadres existants, cette nouvelle approche améliore la faisabilité et le calcul en ligne de l'optimisation sous contraintes à horizon fini. Ce cadre est basé sur une fonction de coût à horizon variable dans le temps avec des matrices de poids associées à l'état terminal qui peuvent facilement être obtenues par des optimisations basées sur des inégalités matricielles. Nous discutons des modifications du cadre proposé qui améliorent la faisabilité et le calcul en ligne. La simulation de divers exemples illustre les résultats obtenus dans ce cadre.

Mots-clés : RHC, contraintes, temps variable, stabilité, faisabilité, temps du calcul.

1 Introduction

Receding horizon control (RHC) uses the current control law obtained by solving un optimization problem at every sampling instant. Since the RHC can consider a finite horizon cost function for closed-loop stability, it takes advantage of handling input/state constraints, time-varying systems, etc. Finite horizon formulations with finite terminal weighting matrices have been widely investigated [1, 2, 3, 4, 5, 6, 7, 8], since they include the infinite horizon formulation in [9].

A constrained time-varying system must be designed to enhance feasibility and prevent heavy computational burden, so the terminal weighting matrix and cost horizon in the stabilizing RHC are designed with these issues in mind. In the literature, terminal weighting matrices have often been represented by inequality conditions that guarantee closed-loop stability [1, 3, 4, 5, 6].

For constrained-time varying systems, some results have conservative feasible initial-state sets since they assume a linear feedback RHC [9] or one-horizon cost function [4]. Although other results [1, 3] consider a nonlinear feedback RHC and finite horizon, the terminal inequality condition in [3] is not practical as shown in [7] and the fixed terminal weighting matrix in [1] still has a small feasible initial-state set.

Time-varying systems should solve many terminal inequality conditions for closed-loop stability as shown in [7]. Too many conditions may make the optimization problem infeasible numerically, and also cause heavy computational burden, so if time-varying systems are constrained, the optimization problem will become less feasible and have more computational burden. Thus, it will be interesting to investigate how to implement a stabilizing RHC to improve the feasibility and computational burden of constrained time-varying systems.

The horizon size is always fixed in the literature to our knowledge. For this reason, a large horizon size causes a lot of computational burden and may make the optimization problem infeasible numerically for constrained and/or time-varying systems. A small horizon size, however, has a small feasible initial-state set for constrained systems. Thus, it will also be interesting to investigate how to determine the horizon size given its affects on feasibility and computational burden.

In this paper, we propose a new RHC scheme[1] for input/state constrained discrete-time systems, which enhances feasibility and on-line computation on the constrained optimization problem compared with existing ones [1, 3, 4, 7, 9]. The control scheme is based on a time-varying finite horizon cost function with time-varying finite terminal weighting matrices, which can easily be implemented via linear matrix inequality (LMI) optimization. We discuss modifications of the proposed scheme that improve either feasibility or on-line computation. Through simulation examples, we illustrate the proposed results.

In Section 2, we propose a new RHC scheme and how to implement the proposed scheme. In Section 3, we illustrate our results through simulation examples. Finally, we present conclusions in Section 4.

[1]This work extends our conference paper [10].

2 A Generalized Stabilizing Receding Horizon Control Scheme

Consider the linear discrete time-varying system

$$x(i+1) \;=\; A(i)x(i) + B(i)u(i), \quad x(0) = x_0, \tag{1}$$

subject to the input and state constraints

$$u_{\min}(i) \;\le\; G_u(i)u(i) \;\le\; u_{\max}(i), \quad i = 0,\ 1,\ \cdots,\ \infty \tag{2}$$
$$x_{\min}(i) \;\le\; G_x(i)x(i) \;\le\; x_{\max}(i), \quad i = 0,\ 1,\ \cdots,\ \infty, \tag{3}$$

where $x(i) \in R^n$ is the state, $u(i) \in R^m$ the control, $G_u(i) \in R^{l\times m}$, $G_x(i) \in R^{p\times n}$, $u_{\min}(\cdot) < 0$, $u_{\max}(\cdot) > 0$, $x_{\min}(\cdot) < 0$, $x_{\max}(\cdot) > 0$, and the pair $(A(i), B(i))$ is uniformly stabilizable. In the literature, most existing RHC schemes assume that $u_{\min}(\cdot) = -u_{\lim}$, $u_{\max}(\cdot) = u_{\lim}$, $x_{\min}(\cdot) = -x_{\lim}$, and $x_{\max}(\cdot) = x_{\lim}$ for some positive constant vector $u_{\lim}$, making them more restrictive than assumptions (2) and (3).

For constrained system (1) subject to (2) and (3), consider the optimization problem

$$J^*(i, i+N_i) \;=\; \underset{u(i),u(i+1),\cdots,u(i+N_i-1),\ Q_f(i),\ H(\cdot),\ \beta_1(i),\ \beta_2(i)}{\text{Minimize}} \;\beta_1(i) + \beta_2(i) \tag{4}$$

subject to

$$\beta_1(i) \;\ge\; \sum_{\tau=i}^{i+N_i-1} x^T(\tau)Q(\tau)x(\tau) + u^T(\tau)R(\tau)u(\tau) \tag{5}$$
$$\beta_2(i) \;\ge\; x^T(i+N_i)Q_f(i)x(i+N_i) \tag{6}$$
$$u_{\min}(\tau) \;\le\; G_u(\tau)u(\tau) \;\le\; u_{\max}(\tau), \;\; \tau \in [i, i+N_i-1] \tag{7}$$
$$x_{\min}(\tau) \;\le\; G_x(\tau)x(\tau) \;\le\; x_{\max}(\tau), \;\; \tau \in [i, i+1, \cdots, i+N_i] \tag{8}$$
$$-u_{\lim}(\sigma) \;\le\; G_u(\sigma)u(\sigma) \;\le\; u_{\lim}(\sigma), \;\; \sigma \in [i+N_i, \infty) \tag{9}$$
$$-x_{\lim}(\sigma) \;\le\; G_x(\sigma+1)x(\sigma+1) \;\le\; x_{\lim}(\sigma), \;\; \sigma \in [i+N_i, \infty), \tag{10}$$

where $N_i \ge 1$, $u(\sigma) = -H(\sigma)x(\sigma)$ and $x(\sigma+1) = (A(\sigma) - B(\sigma)H(\sigma))x(\sigma)$ for $\sigma \in [i+N_i, \infty)$, the weighting matrices $Q(\tau)$, $R(\tau) = R^T(\tau)$ and $Q_f(i) = Q_f^T(i)$ are positive definite, $u_{\lim}(\sigma) = \min\{|u_{\min}(\sigma)|,\ u_{\max}(\sigma)\}$, and $x_{\lim}(\sigma) = \min\{|x_{\min}(\sigma)|,\ x_{\max}(\sigma)\}$. In Remarks 1 and 2 of this paper, we discuss the advantages of the above scheme in terms of feasibility and computational burden.

The following two lemmas, which are simple extensions of the previous results in [9] and in [11] respectively, investigate the feasibility of the optimization problem (4). To this end, we need the following terminal inequality condition which was first proposed in [7] as a condition for the closed-loop stability of unconstrained time-varying systems with a fixed horizon $N_i = N$:

$$\begin{aligned} Q_f(i) \;\;\ge\;\; & Q(\sigma) + H^T(\sigma)R(\sigma)H(\sigma) + (A(\sigma) - B(\sigma)H(\sigma))^T Q_f(i) \\ & (A(\sigma) - B(\sigma)H(\sigma)) \;\text{ for all }\; \sigma \ge i + N_i \;\text{ and some }\; H(\sigma). \end{aligned} \tag{11}$$

Lemma 1 *If problem (4) subject to (5)-(11) is feasible at some time i and $N_{j+1} \geq N_j - 1$ for all $j \geq i$, then it is feasible for all time $j \geq i$.*

Proof: Lemma 1 is clear from (9) and (10). Thus, we have feasible solutions for all times at time i, if the optimization problem is feasible at time i. ■

Throughout the rest of this paper, assume that $N_{i+1} \geq N_i - 1$ for all i.

At the initial time $i = 0$, we select the smallest N_0, with which problem (4) subject to (5)-(11) is feasible. Note that a larger N_0 increases the computational burden, while a very small N_0 has a small feasible region and may not have a good performance for constrained systems. In this paper, we do not discuss which horizon size, state and control weighting matrices give a better performance, but focus instead on the feasibility and on-line computational burden of the constrained optimization problem.

Next, we discuss how to satisfy (9) and (10). For this purpose, we introduce a simple extension of the ellipsoid constraints [11] for continuous time-invariant systems as follows:

$$Q_1(i) \;\geq\; (A(\sigma) - B(\sigma)H(\sigma))^T Q_1(i)(A(\sigma) - B(\sigma)H(\sigma)) \;\text{ for all } \sigma \geq i + N_i \tag{12}$$

$$\beta_3(i) \;\geq\; x^T(i+N_i)Q_1(i)x(i+N_i) \tag{13}$$

$$\begin{bmatrix} Z(\sigma) & G_u(\sigma)Y(\sigma) \\ Y^T(\sigma)G_u^T(\sigma) & S(i) \end{bmatrix} \;\geq\; 0 \;\text{ and }\; Z_{j,j}(\sigma) \;\leq\; u^2_{\lim,j}(\sigma) \;\text{ for all } \sigma \geq i + N_i \tag{14}$$

$$G_x(\sigma)S(i)G_x^T(\sigma) \;\leq\; E(\sigma) \;\text{ and }\; E_{j,j}(\sigma) \;\leq\; x^2_{\lim,j}(\sigma) \;\text{ for all } \sigma \geq i + N_i, \tag{15}$$

where $Q_1(i)$ is positive definite, $S(i) = \beta_3(i)Q_1^{-1}(i)$, $Y(\sigma) = H(\sigma)S(i)$, $u_{\lim,j}(\sigma)$ and $x_{\lim,j}(\sigma)$ are jth elements of $u_{\lim}(\sigma)$ and $x_{\lim}(\sigma)$, respectively, and $Z_{j,j}(\sigma)$ and $E_{j,j}(\sigma)$ are the (j,j) elements of the matrices $Z(\sigma)$ and $E(\sigma)$, respectively.

Lemma 2 *If there exist solutions $Q_1(i)$ and $\beta_3(i)$ satisfying (12)-(15), then (9) and (10) are satisfied.*

Proof: If (13) is satisfied, we have $\beta_3(i) \geq x^T(\sigma)Q_1(i)x(\sigma)$ for all $\sigma \geq i + N_i$ by (12). Thus,

$$\begin{aligned}
|[G_u(\sigma)u(\sigma)]_{j-th} \text{ row}|^2 &= |[G_u(\sigma)H(\sigma)x(\sigma)]_{j-th} \text{ row}|^2 \\
&= |[G_u(\sigma)H(\sigma)\beta_3^{\frac{1}{2}}(i)Q_1^{-\frac{1}{2}}(i)\beta_3^{-\frac{1}{2}}(i)Q_1^{\frac{1}{2}}(i)x(\sigma)]_{j-th} \text{ row}|^2 \\
&\leq |(G_u(\sigma)H(\sigma)\beta_3^{\frac{1}{2}}(i)Q_1^{-\frac{1}{2}}(i))_{j-th} \text{ row}|^2 \; \beta_3^{-1}(i)x^T(\sigma)Q_1(i)x(\sigma) \\
&\leq |(G_u(\sigma)H(\sigma)\beta_3^{\frac{1}{2}}(i)Q_1^{-\frac{1}{2}})_{j-th} \text{ row}(i)|^2 \;\text{ by }\; (13) \\
&\leq u^2_{\lim,j}(\sigma) \;\text{ by (14)}.
\end{aligned}$$

Similarly, (10) is satisfied by (12), (13), and (15). ■

It is easy to see that (12) and (13) are satisfied under the conditions (11) and (6) with $Q_1(i)$ and $\beta_3(i)$ replaced by $Q_f(i)$ and $\beta_2(i)$, respectively. Thus, we do not need to introduce new constraints

(12) and (13) in order to satisfy (9) and (10). From now on, we set $Q_1(i) = Q_f(i)$ and $\beta_3(i) = \beta_2(i)$ in (14) and (15), and do not use (12) and (13). Then, we can immediately get the following result from Lemmas (1) and (2).

Lemma 3 *If problem (4) subject to (5)-(8), (11), (14) and (15) is feasible at the initial time, then it is always feasible.*

Proof: >From Lemma 2, (9) and (10) are satisfied if there exist solutions $u(\cdot)$, $Q_f(i)$, $H(\cdot)$, $\beta_1(i)$, and $\beta_2(i)$ for the problem (4) subject to (5)-(8), (11), (14) and (15) at the time i. Thus, if the problem is feasible at some time i, it is also feasible at the next time $i+1$ with $\beta_2(i+1)$ and $Q_f(i+1)$ replaced by $\beta_2(i)$ and $Q_f(i)$, respectively. ■

>From Lemma 3, we can define the feasible initial-state set as

$$\chi_0 = \{x(0) \quad | \quad \text{there exist feasible } u(0),\ u(1),\ \cdots,\ u(N_0-1),\ Q_f(0),\ H(\cdot),\ \beta_1(0), \text{ and } \beta_2(0) \text{ for problem } (4) \text{ subject to } (5)-(8), (11), (14) \text{ and } (15) \text{ with } x(0)\}. \tag{16}$$

For computational simplicity, we can set $H(\sigma) = H(i)$ for all $\sigma \geq i + N_i$. In this case, however, we have a smaller feasible initial-state set.

The first control $u^*(i)$ at each time i is called a receding horizon control (RHC), which is obtained by solving problem (4) subject to (5)-(8), (11), (14) and (15). Accordingly, the proposed scheme is called a RHC scheme.

Next, we investigate closed-loop stability of the proposed RHC. To this end, we introduce the following lemma:

Lemma 4 *If problem (4) subject to (5)-(8), (11), (14) and (15) is feasible, the resulting optimal cost is monotonically nonincreasing, i.e.,* $J^*(\tau_1, \tau_2) \geq J^*(\tau_1, \tau_2+1)$ *for* $\tau_2 \geq \tau_1 + 1$. *Thus,* $J^*(\tau_1, \tau_2) \geq J^*(\tau_1, \tau_3)$ *for all* $\tau_3 \geq \tau_2$.

Proof: Let $u_1(\cdot)$ and $u_2(\cdot)$ be optimal controls of $J^*(\tau_1, \tau_2+1)$ and $J^*(\tau_1, \tau_2)$, respectively. Define $\Delta J^*(\tau_1)$ =$J^*(\tau_1, \tau_2+1)$ - $J^*(\tau_1, \tau_2)$. Then, if we replace $u_1(\tau)$ by $u_2(\tau)$ for all $\tau \in [\tau_1, \tau_2 - 1]$, then $\Delta J^*(\tau_1) \leq x_2^T(\tau_2)Q(\tau_2)x_2(\tau_2) + u^T(\tau_2)R(\tau_2)u(\tau_2) + x^T(\tau_2+1)Q_f(\tau_1)x(\tau_2+1) - x_2^T(\tau_2)Q_f(\tau_1)x_2(\tau_2)$, where $x_2(\cdot)$ is the state trajectory due to $u_2(\cdot)$ and $x(\tau_2+1) = A(\tau_2)x_2(\tau_2) + B(\tau_2)u(\tau_2)$.

Since this inequality holds for any $u(\tau_2)$ satisfying the input and state constraints (2) and (3), with $u(\tau_2) = -H(\tau_2)x_2(\tau_2)$, we have $\Delta J^*(\tau_1) \leq x_2^T(\tau_2)\{Q(\tau_2) + H^T(\tau_2)R(\tau_2)H(\tau_2) + (A(\tau_2) - B(\tau_2)H(\tau_2))^T Q_f(\tau_1)(A(\tau_2) - B(\tau_2)H(\tau_2)) - Q_f(\tau_1)\}x_2(\tau_2) \leq 0$ by (11). ■

Theorem 1 *If problem (4) subject to (5)-(8), (11), (14) and (15) is feasible at the initial time, then the closed-loop system with the resulting RHC* $u^*(i)$ *is uniformly asymptotically stable.*

Proof: We know from Lemma 3 that the optimization problem is always feasible. By using the proof of Lemma 4, we can easily show that $J^*(i, i+N_i) \geq x^T(i)Q(i)x(i) + u^{*T}(i)R(i)u^*(i) + J^*(i+1, i+N_i+1)|_{Q_f(i),\beta_2(i)}$, where $J^*(i+1, i+N_i+1)|_{Q_f(i),\beta_2(i)}$ is the optimal cost at $i+1$ when

$Q_f(i+1)$ and $\beta_2(i+1)$ are replaced by $Q_f(i)$ and $\beta_2(i)$, respectively. Here, $Q_f(i)$ and $\beta_2(i)$ are solutions for $J^*(i, i+N_i)$. When $N_{i+1} \geq N_i$, Lemma 4 shows that $J^*(i+1, i+N_i+1)|_{Q_f(i),\beta_2(i)}$ $\geq J^*(i+1, i+N_{i+1}+1)|_{Q_f(i),\beta_2(i)}$ and optimality shows that $J^*(i+1, i+N_{i+1}+1)|_{Q_f(i),\beta_2(i)}$ $\geq J^*(i+1, i+N_{i+1}+1)$. Thus, $J^*(i, i+N_i) \geq x^T(i)Q(i)x(i) + u^{*T}(i)R(i)u^*(i) + J^*(i+1, i+1+N_{i+1})$. This nonincreasing cost monotonicity also holds when $N_{i+1} = N_i - 1$. Since $Q(\cdot)$ is positive definite, $J^*(i, i+N_i)$ is a Lyapunov function. When the pair $(A(i), C(i))$ is uniformly detectable and $C(i)$ is not positive definite, we can guarantee that the resulting closed-loop system is uniformly attractive and bounded. If the system has no input/state constraints, the closed-loop system is uniformly asymptotically stable under the uniform detectability. ■

[3] proposed a terminal inequality condition that is different from (11) for the closed-loop stability of time-varying systems:

$$\begin{aligned} Q_f(i) \quad &\geq \quad Q(i+N) + H^T(i)R(i+N)H(i) + (A(i+N) - B(i+N)H(i))^T Q_f(i+1) \\ &\quad\quad (A(i+N) - B(i+N)H(i)) \ \text{ for some } \ H(\sigma). \end{aligned} \tag{17}$$

However, as pointed out in [7], it is impossible to solve (17) for all i since $Q_f(i+1)$ is a design parameter of the optimization problem at the next time $i+1$. Other results [4, 9] consider uncertain time-varying systems that are included in the convex hull

$$\begin{aligned} \Omega \quad &= \quad \{(A(i), B(i)) |\ (A(i), B(i)) = \lambda_1(i)(A_1, B_1) + \lambda_2(i)(A_2, B_2) + \cdots + \lambda_L(i)(A_L, B_L)\} \\ &\quad\quad \text{with } \sum_{k=1}^{L} \lambda_k(i) = 1, \quad \lambda_k(i) \geq 0 \ \text{ for all } \ i. \end{aligned} \tag{18}$$

It is easy to see that the closed-loop stability of the uncertain system (18) is guaranteed if

$$Q_f(i) \quad \geq \quad Q + H^T(i)RH(i) + (A_k - B_kH(i))^T Q_f(i)(A_k - B_kH(i)) \ \text{ for all } k \in [1, L] \tag{19}$$

(see [7] for a detailed proof). Note that for implementation, the cost monotonicity with time-varying terminal weighting matrices should be handled carefully as shown in [7].

Next, we show that the proposed RHC scheme includes previous schemes [1, 3, 4, 7, 9].

Remark 1 *For convenience of comparison, we introduce*

$$\underset{u(\cdot), Q_f(i), H(i)}{\text{Minimize}} \sum_{\tau=i}^{i+N-1} x^T(\tau)Q(\tau)x(\tau) + u^T(\tau)R(\tau)u(\tau) + x^T(i+N)Q_f(i)x(i+N). \tag{20}$$

[3], [9], and [4] consider the following problems: (20) subject to (7), (8), (14), (15), and (17) with $\beta_2(i) = 1$ and an additional parameter $Q_f(i+1)$; (20) subject to (9) and (10) for $\tau \in [i, \infty)$, (14) and (15) with $\sigma \geq i$ and $\beta_2(i) = 1$, and (19) (i.e., without $Q_f(i)$); and (4) subject to (5)-(8), (14), (15), and (19) with $N_i = 1$, respectively. The scheme in [1] can be seen as an extension of that in [3] for nonlinear systems, and the scheme in [7] can be handled as a special case of the proposed one in this paper. Hence, the control scheme proposed in this paper can handle all existing ones [9, 1, 3, 4, 7] as a special case.

Based on Remark 1, we compare the feasibility and computation of the proposed RHC scheme with existing schemes in [3, 4, 9] for constrained linear time-varying systems. Note that N_i is set to be N for all i in the literature.

Remark 2 *Existing results [1, 3, 4, 9] assume that $H(\sigma) = H(i)$ for all $\sigma \geq i + N$. So, they have less computational burden, but smaller feasible initial-state sets than the proposed scheme. For comparison with existing results, let us assume that $H(\sigma) = H(i)$ for all $\sigma \geq i + N_i$ in the proposed RHC scheme. As N_0 increases, the feasible initial-state set in this paper gets larger than that in [4] with $N_i = 1$ for all i, although it holds after some finite horizon depending on the system, constraints, and weighting matrices. When N_0 is small, the feasible set in this paper is also larger than that in [3] with $N_i = N_0$ and $\beta_2(i) = 1$ for all i. As N_0 increases, the feasible set in this paper becomes similar to that with $\beta_2(i)$ fixed as $\beta_2(i) = 1$. However, a large N_0 causes a lot of computational burden for constrained and/or time-varying systems. The linear feedback RHC scheme with the smallest feasible state-set [9] has less computational burden than other schemes [3, 4] and the proposed RHC scheme with the fixed horizon size N. Since N_i can be reduced as $N_i = N_{i-1} - 1$ in this paper, the proposed scheme with the reduced N_i has a much smaller computation time than that with the fixed horizon size. Thus, the proposed scheme in this paper is much more flexible and profitable than the existing ones in terms of feasibility and computation.*

As another method to reduce the computational burden for time-varying systems, we present the following result based on Theorem 1.

Corollary 1 *Let $Q_f(0)$ be the solution for (11), (14) and (15). If problem (4) with $Q_f(i)$ replaced by $Q_f(0)$ subject to (5)-(8) is feasible at the initial time, then the closed-loop system with the resulting RHC $u^*(i)$ is uniformly asymptotically stable.*

The proof for Theorem 1 applies to Corollary 1. The RHC from Corollary 1 has less on-line computational burden than that from Theorem 1 since (11), (14) and (15) is solved once before optimization; however, it does not perform as well as the RHC from Theorem 1.

Next, we suggest how to implement the proposed stabilizing RHCs.

2.1 Implementation of the RHC via LMI optimization

Here, we convert the proposed RHC scheme to the equivalent linear matrix inequality (LMI) problem for implementation of the RHCs.

For simplicity, throughout the rest of this paper, define $\bar{F}(i)$ and $\hat{F}(i)$ as

$$\bar{F}(i) = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ F(i) & \ddots & \ddots & \ddots & \vdots \\ 0 & F(i+1) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & F(i+N_i-2) & 0 \end{bmatrix}, \quad \hat{F}(i) = \begin{bmatrix} F(i) & 0 & \cdots & 0 \\ 0 & F(i+1) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & F(i+N_i-1) \end{bmatrix},$$

respectively. $\bar{F}(i)$ and $\hat{F}(i)$ are used to define $\bar{A}(i)$, $\bar{B}(i)$, $\hat{Q}(i)$, $\hat{R}(i)$, $\hat{G}_u(i)$, and $\hat{G}_x(i)$ which consist of $A(i)$, $B(i)$, $Q(i)$, $R(i)$, $G_u(i)$, and $G_x(i)$, respectively. Also define $\Phi_A(i+N_i,i)$ as $\Phi_A(i+N_i,i) = A(i+N_i-1)A(i+N_i-2)\cdots A(i)$. Then, problem (4) subject to (5)-(8), (11), (14) and (15) can be converted to the equivalent problem formulation

$$\underset{U(i),\, S(i),\, Y(\sigma),\, Z(\sigma),\, E(\sigma),\, \beta_1(i),\, \beta_2(i)}{\text{Minimize}} \quad \beta_1(i) + \beta_2(i) \tag{21}$$

subject to (14), (15),

$$\begin{bmatrix} \beta_1(i) - W_1(i)U(i) - W_0(i) & (W_2^{\frac{1}{2}}(i)U(i))^T \\ W_2^{\frac{1}{2}}(i)U(i) & I \end{bmatrix} \geq 0 \tag{22}$$

$$\begin{bmatrix} 1 & (\Phi_A(i+N_i,i)x(i) + \bar{B}_2(i)U(i))^T \\ \Phi_A(i+N_i,i)x(i) + \bar{B}_2(i)U(i) & S(i) \end{bmatrix} \geq 0 \tag{23}$$

$$\begin{aligned} \left[u_{\text{min}}^T(i),\ u_{\text{min}}^T(i+1),\ \cdots,\ u_{\text{min}}^T(i+N_i-1)\right]^T &\leq \hat{G}_u(i)U(i) \\ &\leq \left[u_{\text{max}}^T(i),\ u_{\text{max}}^T(i+1),\ \cdots,\ u_{\text{max}}^T(i+N_i-1)\right]^T \end{aligned} \tag{24}$$

$$\begin{aligned} \left[x_{\text{min}}^T(i),\ x_{\text{min}}^T(i+1),\ \cdots,\ x_{\text{min}}^T(i+N_i-1)\right]^T &\leq \hat{G}_x(i)(I-\bar{A}(i))^{-1} \\ (\bar{B}(i)U(i) + X_0(i)) &\leq \left[x_{\text{max}}^T(i),\ x_{\text{max}}^T(i+1),\ \cdots,\ x_{\text{max}}^T(i+N_i-1)\right]^T \end{aligned} \tag{25}$$

$$\begin{bmatrix} S(i) & (A(\sigma)S(i) - B(\sigma)Y(\sigma))^T & (C(\sigma)S(i))^T & (R^{\frac{1}{2}}(\sigma)Y(\sigma))^T \\ A(\sigma)S(i) - B(\sigma)Y(\sigma) & S(i) & 0 & 0 \\ C(\sigma)S(i) & 0 & \beta_2(i)I & 0 \\ R^{\frac{1}{2}}(\sigma)Y(\sigma) & 0 & 0 & \beta_2(i)I \end{bmatrix} \geq 0 \tag{26}$$

for all $\sigma \geq i + N_i$, where $W_1(i) = 2X_0^T(i)\hat{Q}_{\bar{A}}(i)\bar{B}_2$, $U(i) = [u^T(i),\ u^T(i+1),\ \cdots,\ u^T(i+N_i-1)]^T$, $W_0(i) = X_0^T(i)\hat{Q}_{\bar{A}}(i)X_0(i)$, $W_2(i) = \hat{R}(i) + \bar{B}^T(i)\hat{Q}_{\bar{A}}(i)\bar{B}(i)$, $\hat{Q}_{\bar{A}}(i) = (I-\bar{A}(i))^{-T}\hat{Q}(i)(I-\bar{A}(i))^{-1}$, $X_0(i) = [x^T(i),\ 0,\ \cdots,\ 0]^T$. When $N_i = 1$, (22)-(25) are changed into

$$\begin{bmatrix} \beta_1(i) - x^T(i)Q(i)x(i) & (R^{\frac{1}{2}}(i)u(i))^T \\ R^{\frac{1}{2}}(i)u(i) & I \end{bmatrix} \geq 0, \quad \begin{bmatrix} 1 & (A(i)x(i) + B(i)u(i))^T \\ A(i)x(i) + B(i)u(i) & S(i) \end{bmatrix} \geq 0$$

$$u_{\text{min}}(i) \leq G_u(i)u(i) \leq u_{\text{max}}(i), \quad x_{\text{min}}(i) \leq G_x(i)x(i) \leq x_{\text{max}}(i).$$

The RHC is obtained by solving the above LMI optimization problem at each time i. Note that $x_{\text{min}}(i+N_i) \leq G_x(i+N_i)x(i+N_i) \leq x_{\text{max}}(i+N_i)$ is satisfied by (15) and (23).

Another stabilizing RHC in Corollary 1 that has less on-line computational burden than the RHC from the above LMI problem can be obtained by solving the following LMI problem:

$$\underset{U(i),\, \beta(i)}{\text{Minimize}} \quad \beta(i) \tag{27}$$

subject to (24), (25),

$$\begin{bmatrix} \beta(i) - W_1(i)U(i) - W_0(i) & (W_2^{\frac{1}{2}}(i)U(i))^T & (\Phi_A(i+N_i,i)x(i) + \bar{B}_2(i)U(i))^T \\ W_2^{\frac{1}{2}}(i)U(i) & I & 0 \\ \Phi_A(i+N_i,i)x(i) + \bar{B}_2(i)U(i) & 0 & Q_f^{-1}(0) \end{bmatrix} \geq 0 \quad (28)$$

$$\begin{bmatrix} 1 & (\Phi_A(i+N_i,i)x(i) + \bar{B}(i)U(i))^T \\ \Phi_A(i+N_i,i)x(i) + \bar{B}(i)U(i) & Q_f^{-1}(0) \end{bmatrix} \geq 0, \quad (29)$$

where $Q_f(0)$ is obtained at the initial time by solving (11), (14), and (15) with $\beta_2(0) = 1$. When $N_i = 1$, (28) and (29) are changed into $\begin{bmatrix} \beta(i) - x^T(i)Q(i)x(i) & (R^{\frac{1}{2}}(i)u(i))^T & (A(i)x(i) + B(i)u(i))^T \\ R^{\frac{1}{2}}(i)u(i) & I & 0 \\ A(i)x(i) + B(i)u(i) & 0 & Q_f^{-1}(0) \end{bmatrix} \geq 0$, $\begin{bmatrix} 1 & (A(i)x(i) + B(i)u(i))^T \\ A(i)x(i) + B(i)u(i) & Q_f^{-1}(0) \end{bmatrix} \geq 0$.

Next, we discuss how to deal with (14), (15) and (26) (or (11)) for all $\sigma \geq i + N_i$, and address some computational issues on the proposed optimization schemes.

2.2 Computational Issues

In many cases, we can express the set of all pairs $(A(i), B(i))$ using a finite number of pairs and we have $G_u(\sigma) = G_u$ and $G_x(\sigma) = G_x$ for all σ. Thus, we have only to solve finite number of constraints (14), (15), and (26) even for infinite number of pairs $(A(i), B(i))$ in many cases. For instance, suppose that the pair $(A(i), B(i))$ belongs to a polytope like (18). Then, (14), (15), and (26) are guaranteed to hold with $Q(\sigma) = Q$ and $R(\sigma) = R$ if we have

$$\begin{bmatrix} Z(i) & G_u Y(i) \\ Y^T(i)G_u^T & S(i) \end{bmatrix} \geq 0,\ Z_{j,j}(i) \leq \bar{u}^2_{\lim,j}(i),\ G_x S(i) G_x^T \leq E(i),\ E_{j,j}(i) \leq \bar{x}^2_{\lim,j}(i)$$

$$\begin{bmatrix} S(i) & (A_jS(i) - B_jY(i))^T & (Q^{\frac{1}{2}}S(i))^T & (R^{\frac{1}{2}}Y(i))^T \\ A_jS(i) - B_jY(i) & S(i) & 0 & 0 \\ Q^{\frac{1}{2}}S(i) & 0 & I & 0 \\ R^{\frac{1}{2}}Y(i) & 0 & 0 & I \end{bmatrix} \geq 0 \quad (30)$$

for all $j \in [1, L]$, where $Y(i) = H(i)S(i)$, $\bar{u}_{\lim,j}(i) \equiv \min\{u_{\lim,j}(i+N_i),\ u_{\lim,j}(i+N_i+1),\ \cdots\}$, and $\bar{x}_{\lim,j}(i) \equiv \min\{x_{\lim,j}(i+N_i),\ x_{\lim,j}(i+N_i+1),\ \cdots\}$. This assertion can be easily proved from Lemma 3 in [7] and Lemma 2 in this paper.

However, even if the set of all pairs $(A(i), B(i))$ can be expressed using a finite number of pairs, the optimization problem (21) can be numerically infeasible when the maximum and minimum eigenvalues of $W_2(i)$ have a big difference and the computer recognizes the minimum eigenvalue as zero numerically, even though the problem formulation ensures that all eigenvalues of $W_2(i)$ are positive theoretically. The numerical problem may come, when there are a lot of different pairs $(A(i), B(i))$, and eigenvalues of $A(i)$s have big differences. In addition, too many inequality constraints (30) require a lot of computational burden.

In these cases, we can consider (14), (15), and (26) for only $i + N_i \leq \sigma \leq i + N_i + T_i$ with some finite T_i instead of all $\sigma \geq i + N_i$, where T_i is a user-designed nonnegative integer. Then, the optimization problem becomes easier to handle in terms of on-line computation and numerical feasibility; however, Lemma 4 shows that cost monotonicity is no longer guaranteed theoretically. Nevertheless, simulation examples with $T_i = 0$ in the next section show that the cost is monotonically nonincreasing in many cases.

>From this discussion, we see that the proposed schemes are much more flexible than existing schemes in terms of feasibility and on-line computation.

In the next section, our results are illustrated through the simulation examples.

3 Simulation Examples

To illustrate the RHCs that result from the proposed schemes, we consider two types of constrained time-varying systems.

For all examples, assume that $x_{\text{lim}}(i) = 10$, $G_u(i) = 1$, $G_x(i) = [1 \;\; 1]$, $Q(i)$ is an identity matrix, and $R(i) = 1$. In this section, the RHCs from (21) subject to (14), (15), (22)-(26) for all $\sigma \geq i+N_i$ with $N_i = N_{i-1}-1$, for all $\sigma \geq i+N_i$ with $N_i = N$, for $\sigma = i+N_i$ with $N_i = N_{i-1}-1$, and for $\sigma = i + N_i$ with $N_i = N$ are called RH1, RH2, RH3, and RH4, respectively. The RHCs from (27) subject to (24), (25), (28), (29) with $N_i = N_{i-1} - 1$ and with $N_i = N$ are called RH5 and RH6, respectively. If we have infinite number of pairs $(A(\sigma), B(\sigma))$, then we set $H(\sigma) = H(i)$ for all $\sigma \geq i + N_i$.

Even if we have finite number of pairs $(A(\sigma), B(\sigma))$, we also set $H(\sigma) = H(i)$ for all $\sigma \geq i + N_i$, since having different $H(\sigma)$ for all $\sigma \geq i + N_i$ has almost the same feasible sets and performances as setting $H(\sigma) = H(i)$ in these examples, while having $H(\sigma)$ for all $\sigma \geq i + N_i$ requires much more computation than setting $H(\sigma) = H(i)$.

For a polytopic time-varying system as in (18), RH2 with $N = 1$ corresponds to the RHC scheme in [4]. Since the papers [3, 4] show that their schemes have larger feasible sets and better performances than the scheme in [9], we compare our proposed schemes with the existing ones in [4] for time-varying systems. Note that there are no implementable algorithms in [3] for time-varying systems.

For all simulations, we use $\sqrt{\|x\|^2}$ as a performance criterion.

3.1 Constrained Time-Varying System I

Consider the polytopic time-varying system:

$$\begin{aligned} A(i) &= \left|\frac{1}{3}sin(\frac{i}{\gamma_1}\pi)\right| \begin{bmatrix} 1.5 & 0.1 \\ 0 & 1.5 \end{bmatrix} + \left|\frac{1}{3}cos(\frac{i}{\gamma_2}\pi)\right| \begin{bmatrix} 1 & 0.1 \\ 0 & 1.2 \end{bmatrix} + \frac{1}{3}\alpha(i) \begin{bmatrix} 1 & 0.1 \\ 0 & 0 \end{bmatrix} \\ B(i) &= \left|\frac{1}{3}sin(\frac{i}{\gamma_1}\pi)\right| \begin{bmatrix} 0 \\ 0.08 \end{bmatrix} + \left|\frac{1}{3}cos(\frac{i}{\gamma_2}\pi)\right| \begin{bmatrix} 0 \\ 0.08 \end{bmatrix} + \frac{1}{3}\alpha(i) \begin{bmatrix} 0 \\ 0.08 \end{bmatrix}, \end{aligned} \tag{31}$$

where $u_{\text{lim}}(i) = 1$, $\gamma_1 = 5$, $\gamma_2 = 8$, and $\alpha(i) = 1 - \left|\frac{1}{3}sin(\frac{i}{\gamma_1}\pi)\right| - \left|\frac{1}{3}cos(\frac{i}{\gamma_2}\pi)\right|$. Note that the pair $(A(i), B(i))$ in (31) belongs to the polytope (18) with three pairs of vertices.

First, we compare the feasible initial-state set of each proposed scheme. When $N_0 = N = 1$, RH1(=RH2) is infeasible if $|x_1(0)| \geq 0.02982$ and $x_2(0) = 0$, or $x_1(0) = 0$ and $|x_2(0)| \geq 0.13695$. RH3(=RH4) is infeasible if $|x_1(0)| \geq 10.00000$ and $x_2(0) = 0$, or $x_1(0) = 0$ and $|x_2(0)| \geq 10.00000$. RH5(=RH6) is infeasible if $|x_1(0)| \geq 0.00553$ and $x_2(0) = 0$, or if $x_1(0) = 0$ and $|x_2(0)| \geq 0.04973$. Table 1 shows performances of each scheme when $i = 0, 1, \cdots, 14$ and the initial state is $x_0 = \begin{bmatrix} 0 \\ 0.04972 \end{bmatrix}$.

Table 1: $\sqrt{\|x\|^2}$ & Total Online Simulation Time (sec.)

	$RH2_{N=1}$ [4] (=RH1)	$RH4_{N=1}$ (=RH3)	$RH6_{N=1}$ (=RH5)
$\sqrt{\|x\|^2}$	0.0608	0.1021	0.0615
Time	7.1	3.2	1.3

If we increase the horizon size N_i, then we have a larger feasible set as mentioned in Remark 2, while RH3 and RH4 has almost the same feasible set for this example. When $N_0 = N = 5$, RH1 and RH2 are infeasible if $|x_1(0)| \geq 0.97002$ and $x_2(0) = 0$, or $x_1(0) = 0$ and $|x_2(0)| \geq 0.99280$. RH5 and RH6 are infeasible if $|x_1(0)| \geq 0.35115$ and $x_2(0) = 0$, or if $x_1(0) = 0$ and $|x_2(0)| \geq 0.44344$. Figures 1 and 2, and Table 2 show performances of each scheme when $i = 0, 1, \cdots, 14$ and the initial state is $x_0 = \begin{bmatrix} 0 \\ 0.44343 \end{bmatrix}$. The optimal costs from RH1, RH2, RH5, and RH6 decrease monotonically as Theorem 1 and Corollary 1 prove. Note that the optimal costs from RH3 and RH4 also seem to decrease monotonically for many different values of x_0, N_i, γ_1 and γ_2, although, as mentioned in Subsection 2.2, the cost monotonicity is not guaranteed theoretically.

These results illustrate Remark 2, Subsection 2.2, and Corollary 1: RH3 and RH4 have the widest feasible initial-state set and RH5 has the smallest on-line computation time.

Table 2: $\sqrt{\|x\|^2}$ & Total Online Simulation Time (sec.)

	$RH1_{N_0=4}$	$RH2_{N=4}$	$RH3_{N_0=2}$	$RH4_{N=2}$	$RH5_{N_0=5}$	$RH6_{N=5}$
$\sqrt{\|x\|^2}$	0.7221	0.7846	0.9110	0.9110	0.7233	0.8241
Time	6.9	9.2	3.1	3.5	1.7	3.2

3.2 Constrained Time-Varying System II

Consider the periodic time-varying system

$$A(i) = \begin{bmatrix} 1.2 + \alpha_1 \; sin(\frac{i}{\gamma_3}\pi) & 0.2 \\ 0.3 & 1 + \alpha_2 \; cos(\frac{i}{\gamma_4}\pi) \end{bmatrix}, \quad B(i) = \begin{bmatrix} \alpha_3 \; cos(\frac{i}{\gamma_5}\pi) \\ 0.08 + \alpha_4 \; sin(\frac{i}{\gamma_6}\pi) \end{bmatrix}, \tag{32}$$

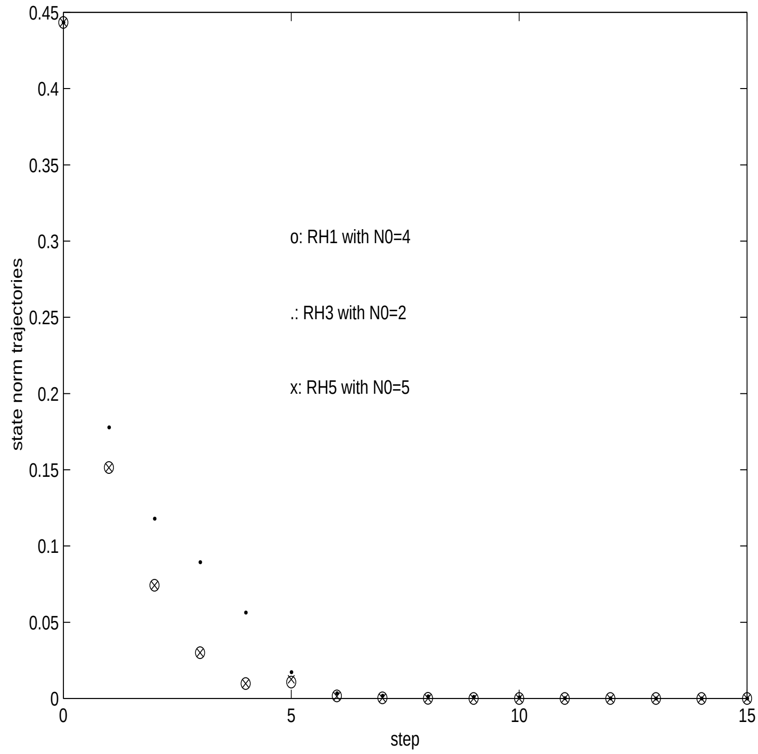

Figure 1: $\sqrt{||x||^2}$ trajectories of RH1, RH3, RH5

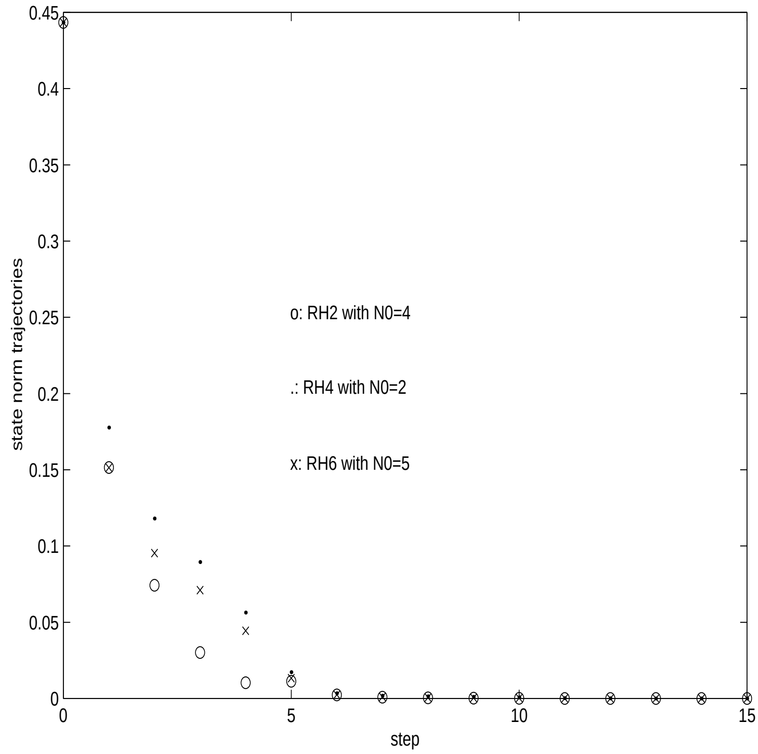

Figure 2: $\sqrt{||x||^2}$ trajectories of RH2, RH4, RH6

where $u_{\lim}(i) = 2$, $\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = \frac{1}{2}$, $\alpha_4 = 1$, $\gamma_3 = 2$, $\gamma_4 = 3$, $\gamma_5 = 5$, and $\gamma_6 = 7$.

As mentioned in Subsection 2.2, the optimization problems for RH1 and RH2 are numerically infeasible due to too many on-line constraints for many different values of α_1 through γ_6.

When $N_0 = N = 4$, RH3 and RH4 are infeasible if $|x_1(0)| \geq 2.54591$ and $x_2(0) = 0$, or $x_1(0) = 0$ and $|x_2(0)| \geq 3.73438$. RH5 and RH6 are infeasible if $|x_1(0)| \geq 1.60879$ and $x_2(0) = 0$, or if $x_1(0) = 0$ and $|x_2(0)| \geq 2.29653$. Figure 3 and Table 3 show their performances when $i = 0, 1, \cdots, 19$ and the initial state is $x_0 = \begin{bmatrix} 0 \\ 2.29652 \end{bmatrix}$. The optimal costs from RH5 and RH6 decrease monotonically. Note that the optimal costs from RH3 and RH4 seem to decrease monotonically for many different values of x_0, N_i, and $\alpha_1 - \gamma_6$.

Table 3: $\sqrt{||x||^2}$ & Total Online Simulation Time (sec.)

	$\text{RH3}_{N_0=4}$	$\text{RH4}_{N=4}$	$\text{RH5}_{N_0=4}$	$\text{RH6}_{N=4}$
$\sqrt{\|\|x\|\|^2}$	14.3099	14.7375	13.2038	13.9514
Time	4.6	6.1	2.2	4.7

4 Conclusion

In this paper, we proposed a generalized stabilizing receding horizon control (RHC) scheme for input/state constrained linear discrete-time systems, which can easily be implemented using linear matrix inequality (LMI) optimization. We discussed modifications to the proposed scheme; for constrained time-varying systems, these modifications make either the optimization problem more feasible numerically or the on-line computation time smaller than the original proposed scheme.

The proposed schemes are more flexible and profitable in terms of feasibility and on-line computation than existing ones in [1, 3, 4, 7, 9]. We expect that this work will be useful for other regulation problems, which consider constrained time-varying systems.

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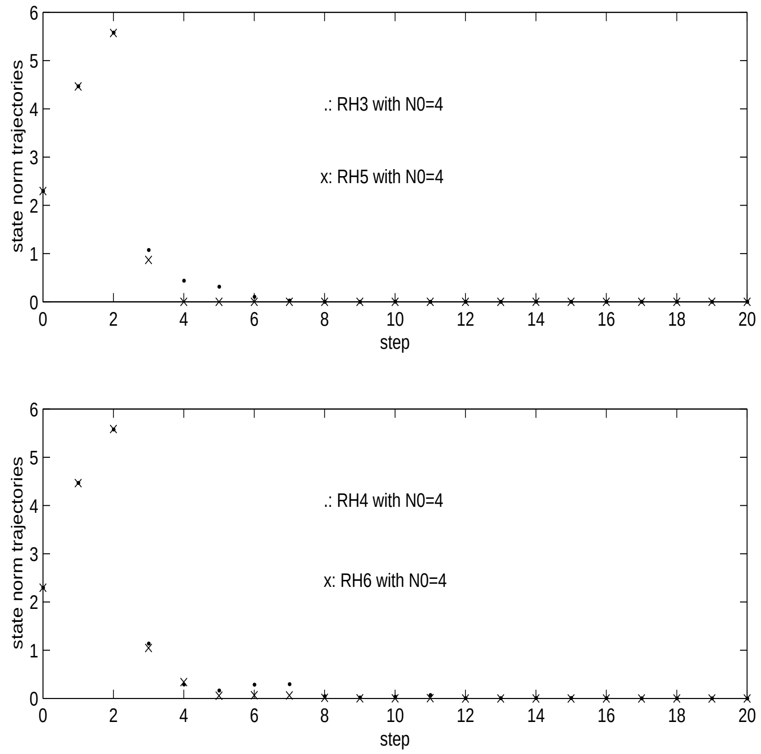

Figure 3: $\sqrt{||x||^2}$ trajectories of RH3, RH4, RH5, RH6

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