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Abstract: We study the optimal reinsurance policy and dividends distribution of an insurance company under excess of loss reinsurance. The insurer gives part of its premium stream to another company in exchange of an obligation to support the difference between the amount of the claim and some retention level. The objective of the insurer is to maximise the expected discounted dividends. We suppose that in the absence of dividend distribution, the reserve process of the insurance company follows a compound Poisson process. We first prove existence and uniqueness results for this optimisation problem by using singular stochastic control methods and the theory of viscosity solutions. We then compute the optimal strategy of reinsurance, the optimal strategy of dividends pay-out and the value function by solving the associated integro-differential Hamilton-Jacobi-Bellman Variational Inequality numerically in the case of a Poisson process with constant intensity.

Key-words: Singular stochastic control, jump diffusion, dynamic programming principle, viscosity solution, Howard algorithm, insurance, reinsurance.

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Contrôle de risque et distribution de dividendes dans le cas d'un contrat de réassurance avec limitation d'excès de pertes

Résumé : On étudie la politique optimale de réassurance et de distribution de dividendes d'une compagnie d'assurance qui reverse une partie des primes qu'elle reçoit à un réassureur qui s'engage à payer la différence entre la taille de chaque sinistre qui survient et un certain niveau de rétention donné. Ce contrat est connu sous le nom de réassurance d'excès de pertes. L'objectif de l'assureur est de maximiser l'espérance du montant des dividendes actualisés. On modélise les sinistres par un processus de Poisson composé. On prouve l'existence et l'unicité de la solution de viscosité de l'inéquation variationnelle associée en utilisant des méthodes de contrôle stochastique singulier. Dans une seconde partie, on résoud numériquement le problème en utilisant un algorithme basé sur l'algorithme d'Howard dans le cas particulier d'un processus de Poisson.

Mots-clés : Contrôle stochastique singulier, diffusion avec sauts, principe de programmation dynamique, solution de viscosité, algorithme de Howard, assurance, réassurance.

1 Introduction

A basic problem in insurance is the problem of optimal risk control and dividends distribution. Indeed the surplus process of the insurer consists of a premium stream which commits him to pay the amount of the claims at their arrivals which is known to be a risky activity. To reduce the risk, the insurer gives part of the premium stream to another company in exchange of an obligation to cover a part of the claim. The insurer has also to pay a flow of dividends to shareholders. To make an optimal choice of the amount to insure and the dividends to pay out, stochastic control methods arise naturally.

In the literature, various criteria are used to formulate the problem of optimal risk control and/or dividends distribution such that (i) maximising expected utility of terminal surplus-process, (ii) minimising the ruin probability of the insurer or (iii) maximising the cumulative expected discounted dividend pay-outs.

Touzi (2000) studied the problem of maximising the expected utility of the terminal reserve in the case of a proportional reinsurance contract. He modelled the reserve process by a Doléans-Dade exponential of jump process and characterised the optimal strategy of reinsurance via a dual formulation. The criterion of maximising the expected utility of the terminal reserve is not relevant here since the insurer who is invited to cover a large risk wants to be risk neutral (see Aase (2002)). The second criterion is useful for consumers and supervisors and extremely conservative especially for rich companies. Schmidli (2001) studied the optimal proportional reinsurance policy which minimises the ruin probability in infinite horizon. He derived the associated Hamilton-Jacobi-Bellman equation, proved the existence of a solution and a verification theorem in the diffusion case. He proved that the ruin probability decreases exponentially and the optimal proportion to insure is constant. Moreover, he gave some conjecture in the Cramér-Lundberg case. The third criterion is preferable for shareholders. Jeanblanc and Shirayev (1995) studied the problem of optimal dividend distribution policy without optimal risk control. They modelled the evolution of the capital $X = (X_t)_{t\geq 0}$ of a company by $dX_t = \mu dt + \sigma dW_t - dZ_t$ where μ and σ are constants, $W = (W_t)_{t\geq 0}$ is a standard Brownian motion and $Z = (Z_t)_{t\geq 0}$is a nonnegative, nondecreasing right-continuous and adapted process. The process Z represents the strategy of payment of dividends by the company. They showed that there exists a threshold u_1 such that every excess of the reserve above u_1 is distributed as dividend instantaneously. Højgaard and Taksar (1999) studied the problem of risk control and dividends pay-out. They modelled the evolution of the process X of the company by $dX_t = a_t(\mu dt + \sigma dW_t) - dZ_t$, where $a = (a_t)_{t\geq 0}$ represents the risk exposure with $0 \leq a_t \leq 1$ for all $t \geq 0$. They found the optimal strategy which maximises the expected total discounted dividends when there is no restriction on the rate of dividend pay-out. They showed that there exists u_0 and u_1 with $u_0 \leq u_1$ such that every excess of the reserve above u_1 is distributed as dividend and the optimal risk exposure is given by $a(x) = \frac{u_0}{x} \wedge 1$ where x is the current reserve. Asmussen, Højgaard and Taksar (2000) considered the issue of optimal risk control and dividend distribution policies under excess of loss reinsurance which is the most common in the reinsurance industry. The insurer gives part of the premium stream to another company, in exchange of an obligation to support the difference between the amount of each claim and some fixed level called retention level. The authors used a diffusion approximation for the reserve process and reparametrized the problem by considering the drift term as the basic control parameter, which leads to a mixed regular/singular stochastic control problem. They derived an Hamilton Jacobi Bellman Variational Inequality (HJBVI in short) in the case of unbounded rate of dividends and proved that the value function is a classical solution of the associated HJBVI. They constructed the solution in the case of unbounded and bounded support of the distribution of the claims. In this paper, we study the same problem but we model the reserve process of the insurer by using a compound Poisson process. Due to the Markovian context, our problem may be studied by a direct dynamic programming approach leading to an integro-differential Hamilton Jacobi Bellman Variational Inequality. In general, the value function of control problems is not smooth enough to be a strong solution of the associated HJBVI. The notion of viscosity solution, first introduced by Crandall and Lions (1983), is known to be a powerful tool for this type of problems. We prove here an existence and uniqueness result for the associated HJBVI and then solve it by using an efficient numerical method, the convergence of which is ensured by the uniqueness result. The paper is organised as follows. The problem is formulated in Section 2. In Section 3, we prove that the value function is a viscosity solution of the associated HJBVI. In Section 4 we prove the uniqueness of the viscosity solution. Section 5 is devoted to the numerical analysis of the HJBVI in the case of a Poisson process with constant intensity: we perform a finite difference approximation of the HJBVI and then solve the problem by using an algorithm based on the "Howard algorithm". Numerical results are presented. They provide the optimal policy of reinsurance and the optimal strategy of dividends pay-out.

2 Formulation of the problem

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. We assume that the claims are generated by a compound Poisson process. More precisely, we consider an integer-valued random measure $\mu(dt, dy)$ with compensator $\pi(dy)dt$. We assume that $\pi(dy) = \beta G(dy)$ where $G(dy)$ is a probability distribution on $B \subseteq \mathbb{R}_+$ and β is a positive constant. In this case, the integral with respect to the random measure $\mu(dt, dy)$ is simply a compound Poisson process: we have $\int_0^t \int_B \mu(du, dy) = \sum_{i=1}^{N_t} Y_i$, where $N = \{N_t, t \geq 0\}$ is a Poisson process with intensity β and $\{Y_i, i \in \mathbb{N}\}$ is a sequence of random variables with common distribution G which represent the sizes of the claim. We suppose that N and $\{Y_i, i \in \mathbb{N}\}$ are independent.
We denote by $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ the filtration generated by the random measure $\mu(dt, dy)$.
A retention level process is an $\mathcal{F}_t$-adapted process $\alpha = (\alpha_t, t \geq 0)$ representing an excess of

loss treaty specifying that, of any claim of size y at time t, the direct insurer is to cover $y \wedge \alpha_t$ and the reinsurer is to cover the excess amount $(y-\alpha_t)_+$.
Given a retention level α_t at time t, we denote by $p(\alpha_t)$ the difference between the premium rate per unit of time received by the company and the premium rate per unit of time paid by the company to the reinsurer at time t.
From now on, we consider a premium rate of the same form as in Asmussen, Højgaard and Taksar (2000):

$$p(\alpha_t) \quad = \quad (1+k_1)\beta\nu - (1+k_2)\beta E\left[(Y_i-\alpha_t)_+\right] \quad \text{for all } t \geq 0, \tag{2.1}$$

where k_1 and k_2 are proportional factors satisfying $0 \leq k_1 \leq k_2$. In Equation (2.1), the term $\beta\nu$ represents the expectation of the amount of the claims during a unit of time. The second term of the r.h.s of Equation (2.1) is the premium paid to the reinsurance company to support the difference between the amount of the claims and the retention level α, during a unit of time. Regulations lay down that premiums must be sufficient to cover expenditures, which means that

$$p(\alpha_t) - \beta E[Y_i \wedge \alpha_t] \geq 0 \ \text{ for all } t \geq 0. \tag{2.2}$$

Condition (2.2) is equivalent to

$$\alpha_t \geq \underline{\alpha} \quad \text{for all } (t,w) \text{ a.e} \tag{2.3}$$

where $\underline{\alpha}$, the lowest admissible retention, is the unique solution of

$$p(\underline{\alpha}) - \beta E[Y_i \wedge \underline{\alpha}] = 0.$$

Remark 2.1 *When the mark space is reduced to $B=\{\delta\}$ with $\delta>0$, then we have an explicit expression of $\underline{\alpha}$ which is $\underline{\alpha} = \frac{\delta(k_2-k_1)}{k_2}$.*

We denote by $L=(L_t, t\geq 0)$ the $\mathcal{F}_t$-adapted process of the cumulative amount of dividends paid out by the insurer. Given an initial reserve x and a policy (α, L), the reserve of the insurance company at time t under this excess of loss contract is then given by :

$$X_t^{x,\alpha,L} \quad = \quad x + \int_0^t p(\alpha_u)du - \int_0^t \int_B (y \wedge \alpha_u)\, \mu(du,dy) - \int_0^t dL_u. \tag{2.4}$$

A strategy (α, L) is said to be admissible if $\alpha = (\alpha_t, t \geq 0)$ satisfies (2.3) and

$$L \text{ is right-continuous, nondecreasing, } L_{0^-}=0 \text{ and } L_{t^+} - L_t \leq X_t^{x,\alpha,L} \text{ for } (t,w) \text{ a.e .} \tag{2.5}$$

The last hypothesis means that the insurer is not allowed to pay out dividends at time t which exceed the level of his reserve at this time.

Given an initial reserve x, we denote by $\Pi(x)$ the set of all admissible policies. For $(\alpha, L) \in \Pi(x)$, we define the return function as

$$J(x, \alpha, L) = E_x \int_0^{\bar{\tau}} e^{-rt} dL_t,$$

where r is a discount factor and $\bar{\tau}$ is the ruin time defined by

$$\bar{\tau} = \inf\{t \geq 0, X_t^{x,\alpha,L} \leq 0\}.$$

The value function is defined as

$$v(x) = \sup_{(\alpha,L)\in\Pi(x)} J(x, \alpha, L). \tag{2.6}$$

3 Characterisation of the value function as a viscosity solution of a HJBVI

In this section, we prove that the value function defined in (2.6) is a viscosity solution of the integro-differential Hamilton Jacobi Bellman Variational Inequality

$$\max\left\{H(x, v, v'), 1 - v'(x)\right\} = 0 \text{ in } I\!R_+^*, \tag{3.1}$$

with Dirichlet boundary conditions

$$v(0) = 0, \tag{3.2}$$

where

$$H(x, v, v') \equiv \sup_{\alpha \geq \underline{\alpha}} \left\{-rv(x) + p(\alpha)v'(x) + \int_B (v(x - y \wedge \alpha) - v(x))\pi(dy)\right\}.$$

We begin by stating some useful properties for the value function.

Theorem 3.1 *The value function v is nondecreasing in $I\!R_+$ and satisfies*

$$v(x) \leq x + K,$$

where K is a positive constant.

Proof. The first statement is obvious. Let $K = \frac{(1+k_1)\beta\nu}{r}$ and define ϕ on $I\!R_+^*$ by $\phi(x) = x + K$. Since ϕ is non decreasing and $p(\alpha)$ is bounded by $(1 + k_1)\beta\nu$, we have

$$H(x, \phi, \phi') \leq \sup_{\alpha \geq \underline{\alpha}} \{-r(x + K) + p(\alpha)\} \leq 0 \text{ for all } x \in I\!R_+^*. \tag{3.3}$$

Moreover

$$1-\phi^{'}(x)=0 \text{ for all } x\in I\!R^*_+. \tag{3.4}$$

Let $(\alpha, L)\in\Pi$ be given and $x\in I\!R^*_+$. We denote by τ the first exit time of $I\!R^*_+$ of the process $X^{x,\alpha,L}$ and define

$$\tau_n=\left\{s\geq 0, \left|\int_B e^{-rs}\left(\phi(X^{x,\alpha,L}_{s^-}-y\wedge\alpha_s)-\phi(X^{x,\alpha,L}_{s^-})\right)\pi(dy)\right|\geq n\right\}.$$

Applying Itô's formula to $e^{-ct}\phi(X^{x,\alpha,L}_t)$, we get

$$\begin{aligned}
& e^{-r(t\wedge\tau\wedge\tau_n)}\phi(X^{x,\alpha,L}_{t\wedge\tau\wedge\tau_n}) \\
= \; & \phi(x)+\int_0^{t\wedge\tau\wedge\tau_n} -re^{-rs}\phi(X^{x,\alpha,L}_s)+e^{-rs}p(\alpha_s)\phi^{'}(X^{x,\alpha,L}_s)ds \\
+ \; & \int_0^{t\wedge\tau\wedge\tau_n}\int_B e^{-rs}\left(\phi(X^{x,\alpha,L}_{s^-}-y\wedge\alpha_s)-\phi(X^{x,\alpha,L}_{s^-})\right)\mu(ds,dy) \\
- \; & \int_0^{t\wedge\tau\wedge\tau_n} e^{-rs}\phi^{'}(X^{x,\alpha,L}_s)dL^c_s \\
+ \; & \sum_{s=0}^{t\wedge\tau\wedge\tau_n} e^{-rs}\left(\phi(X^{x,\alpha,L}_{s^-}-\triangle L_s)-\phi(X^{x,\alpha,L}_{s^-})\right),
\end{aligned} \tag{3.5}$$

where L^c_s is the continuous part of L_s and $\triangle L_s=L_s-L_{s^-}$. Taking the expectation in (3.5), using (3.3), (3.4) and the martingale property of

$$\int_0^{t\wedge\tau\wedge\tau_n}\int_B e^{-rs}\left(\phi(X^{x,\alpha,L}_{s^-}-y\wedge\alpha_s)-\phi(X^{x,\alpha,L}_{s^-})\right)\tilde{\mu}(ds,dy),$$

we obtain

$$\phi(x)\geq E\left[\int_0^{t\wedge\tau\wedge\tau_n} e^{-rs}dL_s\right]+E\left[e^{-r(t\wedge\tau\wedge\tau_n)}\phi(X^{x,\alpha,L}_{t\wedge\tau\wedge\tau_n})\right].$$

Sending n to infinity and using $\phi(X^{x,\alpha,L}_{t\wedge\tau\wedge\tau_n})\geq 0$, we get

$$\phi(x)\geq E\left[\int_0^{t\wedge\tau} e^{-rs}dL_s\right].$$

Taking the supremum over all policies $(\alpha, L)\in\Pi(x)$, we obtain $v(x)\leq x+K$. □

We define now the upper and the lower semicontinuous envelope of the function v.

Definition 3.1 *(i) The upper semicontinuous envelope of the function v is*

$$v^*(x) = \limsup_{x' \to x} v(x'), \quad \textit{for all } x \in I\!R_+. \tag{3.6}$$

(ii) The lower semi-continuous envelope of the function v is

$$v_*(x) = \liminf_{x' \to x} v(x'), \quad \textit{for all } x \in I\!R_+. \tag{3.7}$$

Since the continuity of the Hamiltonian H in his arguments is not obvious, we define the upper and the lower semi-continuous envelope of H by $H^*(x, v, v^{'}) = \limsup_{x' \to x} H(x', v, v^{'})$ and $H_*(x, v, v^{'}) = \liminf_{x' \to x} H(x', v, v^{'})$.
Extending the definition of viscosity solutions introduced by Crandall and Lions (1983) and then by Soner (1986) and Sayah (1991) to first integro-differential operators, we define the viscosity solution as follows:

Definition 3.2 *(i)A function v is a viscosity super-solution of* (3.1) *in $I\!R_+^*$ if*

$$\max\left\{H_*(x, \psi, \psi^{'}), 1 - \psi^{'}(x)\right\} \leq 0 \tag{3.8}$$

whenever $\psi \in C^1(N_x)$, N_x is a neighbourhood of x and $v_* - \psi$ has a global strict minimum at $x \in I\!R_+^*$.
(ii)A function v is a viscosity sub-solution of (3.1) *in $I\!R_+^*$ if*

$$\max\left\{H^*(x, \psi, \psi^{'}), 1 - \psi^{'}(x)\right\} \geq 0 \tag{3.9}$$

whenever $\psi \in C^1(N_x)$, N_x is a neighbourhood of x and $v^* - \psi$ has a global strict maximum at $x \in I\!R_+^*$.
(iii) A function v is a viscosity solution of (3.1) *in $I\!R_+^*$ if it is both a super and a sub-solution in $I\!R_+^*$.*

We define

$$C_1(I\!R_+) = \{f : I\!R_+ \longrightarrow I\!R, f \text{ is nondecreasing and } \sup_{x \in R_+} \frac{v(x)}{1+x} < \infty\}.$$

Remark 3.1 *It is easy to check that v^* and v_* are in $C_1(I\!R_+)$.*

We assume that the dynamic programming principle holds, see e.g. Fleming-Soner (1993). For any stopping time τ and $t \geq 0$,

$$v(x) = \sup_{(\alpha, L) \in \Pi} E\left[e^{-r(t \wedge \tau)} v\left(X_{t \wedge \tau}^{x, \alpha, L}\right) + \int_0^{t \wedge \tau} e^{-rs} dLu\right], \tag{3.10}$$

where $a \wedge b = \min(a, b)$.

Theorem 3.2 *The value function v is a viscosity solution of (3.1) in $\mathbb{R}_+^*$.*

Proof. We first prove that v is a viscosity super-solution of (3.1) in $\mathbb{R}_+^*$. Let $v \in \mathbb{R}_+^*$ and $\psi \in C^1(\mathbb{R}_+^*)$ such that without loss of generality

$$0 = (v_* - \psi)(x) = \min_{\mathbb{R}_+^*}(v_* - \psi).$$

From the definition of v_*, there exists a sequence $(x_n)_n \in \mathbb{R}_+^*$ such that $x_n \longrightarrow x$ and $v(x_n) \longrightarrow v_*(x)$ when $n \longrightarrow \infty$.
For $\alpha \geq \underline{\alpha}$ and $\delta > 0$, we set $L_s = \delta$ and $\alpha_s = \alpha$ for all $s \geq 0$. Then $X_{0^+}^{x_n,\alpha,L} = x_n - \delta$. The dynamic programming principle (3.10) yields

$$\psi(x_n) + \gamma_n \geq E\left[e^{-r(t\wedge\tau)}\psi\left(X_{t\wedge\tau}^{x,\alpha,L}\right) + \int_0^{t\wedge\tau} e^{-rs}dLu\right], \tag{3.11}$$

where the sequence $\gamma_n := v_*(x_n) - \psi(x_n)$ is deterministic and converges to zero when n tends to infinity. Sending $n \longrightarrow \infty$ and $t \longrightarrow 0^+$ in (3.11), we get

$$\psi(x) \geq \psi(x - \delta) + \delta.$$

Sending now $\delta \longrightarrow 0^+$, we obtain

$$1 - \psi'(x) \leq 0. \tag{3.12}$$

It remains to prove

$$H_*(x, \psi, \psi') \leq 0. \tag{3.13}$$

We choose $L_s = 0$ and $\alpha_s = \alpha$ for all $s \geq 0$. We set

$$\theta_n = \inf\{t \geq 0, X_t^{x_n,\alpha,L} \notin B(x_n, \eta)\},$$

where η is a positive constant and $B(x_n, \eta) = \{x, |x - x_n| \leq \eta\}$. Applying Itô's formula to $e^{-r(t\wedge\theta_n)}\psi(X_{t\wedge\theta_n\wedge}^{x_n,\alpha,L})$ and using (3.11), we get

$$\begin{aligned} & E\left[\frac{1}{t}\int_0^{t\wedge\theta_n} -re^{-rs}\psi(X_s^{x_n,\alpha,L}) + e^{-rs}p(\alpha_s)\psi'(X_s^{x_n,\alpha,L})ds\right] \\ + \; & E\left[\frac{1}{t}\int_0^{t\wedge\theta_n}\int_B e^{-rs}\left(\psi(X_{s^-}^{x_n,\alpha,L} - y\wedge\alpha_s) - \psi(X_{s^-}^{x_n,\alpha,L})\right)\mu(ds, dy)\right] \leq \frac{\gamma_n}{t} \end{aligned} \tag{3.14}$$

From the definition of γ_n, two cases are possible:
Case 1: $\gamma_n = 0$. Using the martingale property of

$$\int_0^{t\wedge\theta_n} e^{-rs}\left(\psi(X_{s^-}^{x_n,\alpha,L} - y\wedge\alpha_s) - \psi(X_{s^-}^{x_n,\alpha,L})\right)\tilde{\mu}(ds, dy)$$

and sending n to infinity and t to zero, by dominated convergence theorem and mean value theorem, (3.14) implies

$$-r\psi(x) + p(\alpha)\psi^{'}(x) + \int_B (\psi(x - y \wedge \alpha) - \psi(x))\, \pi(dy) \leq 0$$

and so (3.13) is proved .
Case 2: $\gamma_n > 0$. We take $t = \sqrt{\gamma_n}$. By sending n to infinity we obtain also inequality (3.13). Combining (3.13) and (3.12), we conclude that v is a viscosity super-solution.
Let $\psi \in C^1(N_{x_0})$, $x_0 \in I\!R_+^*$ such that $(v^* - \psi)(x_0) = \max\limits_{I\!R_+^*}(v^* - \psi)$. For sub-solution inequality (3.9), we have to show

$$\max\left\{H^*\left(x_0, \psi, \psi^{'}\right), 1 - \psi^{'}(x_0)\right\} \geq 0. \tag{3.15}$$

Suppose that (3.15) does not hold. Hence the left-hand side of (3.15) is negative. By smoothness of ψ and since H^* is upper semi-continuous, there exists δ satisfying:

$$\max\left\{H^*\left(x, \psi, \psi^{'}\right), 1 - \psi^{'}(x)\right\} < 0 \tag{3.16}$$

for all $x \in B(x_0, \delta)$. By changing δ, we may assume that $B(x_0, \delta) \subset I\!R_+^*$.
From the definition of v^*, there exists a sequence $(x_n)_n \in I\!R_+^*$ such that $x_n \longrightarrow x_0$ and $v(x_n) \longrightarrow v^*(x_0)$ when $n \longrightarrow \infty$. We suppose that $x_n \in B(x_0, \delta)$ for all $n \in I\!N$. Let $(\alpha, L) \in \Pi(x)$ be given and define the stopping time τ_n as

$$\tau_n = \inf\{t \geq 0, X_t^{x_n,\alpha,L} \notin B(x_0, \delta)\}.$$

We truncate this stopping time by a constant T in order to make the stopping time τ_n bounded. We set $\tau^* = \tau_n \wedge T$. Applying Itô's formula to $e^{-r(t\wedge\tau^*)}\psi(X_{t\wedge\tau^*}^{x_n,\alpha,L})$ and using (3.11), we get (with L_t^c denoting the continuous part of L_t)

$$\begin{aligned}
&E\left[e^{-r(t\wedge\tau^*)}\psi(X_{t\wedge\tau^*}^{x_n,\alpha,L})\right] \qquad (3.17)\\
= \ & \psi(x_n) + E\left[\int_0^{t\wedge\tau^*} (-re^{-rs}\psi(X_s^{x_n,\alpha,L}) + e^{-rs}p(\alpha_s)\psi^{'}(X_s^{x_n,\alpha,L}))ds\right]\\
+ \ & E\left[\int_0^{t\wedge\tau^*}\int_B e^{-rs}\left(\psi(X_{s^-}^{x_n,\alpha,L} - y\wedge\alpha_s) - \psi(X_{s^-}^{x_n,\alpha,L})\right)\mu(ds, dy)\right]\\
- \ & E\left[\int_0^{t\wedge\tau^*} e^{-rs}\psi^{'}(X_s^{x_n,\alpha,L})dL_s^c\right]\\
+ \ & E\left[\sum_{s=0}^{t\wedge\tau^*} e^{-rs}\left(\psi(X_{s^-}^{x_n,\alpha,L} - \triangle L_s) - \psi(X_{s^-}^{x_n,\alpha,L})\right)\right],
\end{aligned}$$

where $\triangle L_s = L_s - L_{s^-}$. For $0 \leq s \leq t \wedge \tau^*$, (3.16) implies

$$-r\psi(X_s^{x_n,\alpha,L}) + p(\alpha_s)\psi^{'}(X_s^{x_n,\alpha,L}) + \int_B (\psi(X_s^{x_n,\alpha,L} - y \wedge \alpha) - \psi(X_s^{x_n,\alpha,L}))\pi(dy) < 0 \quad (3.18)$$

and

$$1 - \psi^{'}(X_s^{x_n,\alpha,L}) < 0. \quad (3.19)$$

Using (3.19), we have

$$E\left[\sum_{s=0}^{t\wedge\tau^*} e^{-rs}\left(\psi(X_{s^-}^{x_n,\alpha,L} - \triangle L_s) - \psi(X_{s^-}^{x_n,\alpha,L})\right)\right] \leq -E\left[\int_0^{t\wedge\tau^*} e^{-rs}\triangle L_s\right]. \quad (3.20)$$

Substituting (3.18), (3.19) and (3.20) into (3.17) and using the martingale property of

$$\int_0^{t\wedge\tau^*} e^{-rs}\left(\psi(X_{s^-}^{x_n,\alpha,L} - y\wedge\alpha_s) - \psi(X_{s^-}^{x_n,\alpha,L})\right)\tilde{\mu}(ds,dy),$$

we obtain

$$\psi(x_n) \geq E\left[e^{-r(t\wedge\tau^*)}\psi(X_{t\wedge\tau^*\wedge\tau_\pi}^{x_n,\alpha,L}) + \int_0^{t\wedge\theta_n} e^{-rs}\psi^{'}(X_s^{x_n,\alpha,L})dL_s\right]. \quad (3.21)$$

Since x_0 is a strict global maximiser of $v^* - \psi$ (we can assume without any loss of generality $(v^* - \psi)(x_0) = 0$), there exists $\xi > 0$ such that

$$\max_{x\notin \overset{\circ}{B}(x_0,\delta)} (v^* - \psi)(x) = -\xi,$$

which implies $v^*(x) \leq -\xi + \psi(x)$ for all $x \notin \overset{\circ}{B}(x_0,\delta)$ and so inequality (3.21) implies

$$v(x_n) + \delta_n \geq E\left[e^{-r(t\wedge\tau^*)}v(X_{t\wedge\tau^*}^{x_n,\alpha,L}) + \int_0^{t\wedge\tau^*} e^{-rs}\psi^{'}(X_s^{x_n,\alpha,L})dL_s\right] + \xi P(\tau_n < T), \quad (3.22)$$

where $\delta_n := \psi(x_n) - v(x_n)$. Since $\delta_n = \psi(x_n) - \psi(x_0) + v^*(x_0) - v(x_n)$, there exists $n_0 \in I\!N$ such that for all $n \geq n_0$, $\delta_n \leq \frac{\xi}{2}P(\tau_n < T)$ and so inequality (3.22) implies

$$v(x_n) \geq \sup_{(\alpha,L)\in\Pi} E\left[e^{-r(t\wedge\tau^*)}v(X_{t\wedge\tau^*}^{x_n,\alpha,L}) + \int_0^{t\wedge\tau^*} e^{-rs}\psi^{'}(X_s^{x_n,\alpha,L})dL_s\right] + \frac{\xi}{2}P(\tau_n < T),$$

which is a contradiction with the dynamic programming principle. □

We need now to specify the boundary conditions for the *usc* and *lsc* envelopes of v. Since the continuity of v is not obvious, we need to characterise $v^*(0)$ and $v_*(0)$.

Theorem 3.3 *The upper and the lower semi-continuous envelope of v satisfy the following equalities*

$$v^*(0) = v_*(0) = 0. \tag{3.23}$$

Proof. Since $v(x) \geq 0$ for all $x \in I\!R_+^*$, we have from the definition of the lower semi-continuous envelope of the function v, $v_*(x) \geq 0$ and so $v_*(0) \geq 0$. The opposite inequality is true since $v_*(0) \leq v(0) = 0$.
It remains to prove $v^*(0) = 0$. The first inequality is obvious since we have $v^*(0) \geq v(0) = 0$. We need to show $v^*(0) \leq 0$. Suppose it is not true: there exists $\eta > 0$ such that $v^*(0) \geq 2\eta$. From the definition of v^*, there exists a sequence $(x_n)_n \in I\!R_+^*$ such that $x_n \longrightarrow 0$ and $v(x_n) \longrightarrow v^*(0)$ when $n \longrightarrow \infty$, which implies that there exists $n_0 \in I\!N$ such that for all $n \geq n_0$, we have $v(x_n) \geq \eta$. Let $(\alpha, L) \in \Pi$ be given and define the stopping time τ_n as

$$\tau_n = \inf\{t \geq 0, X_t^{x,\alpha,L} \leq 0\}.$$

Since $x_n \longrightarrow 0$ when $n \longrightarrow \infty$, we have $\tau_n \longrightarrow 0^+$. From hypothesis (2.5), we have $L_{0^+} - L_0 \leq X_0^{x,\alpha,L} = x_n - L_0$ and so $L_{0^+} \longrightarrow 0$ when $n \longrightarrow \infty$. Let $\epsilon > 0$, there exists $n_1 \in I\!N$ such that for all $n \geq n_1$, we have $\int_0^{\tau_n} e^{-rs} dL_s \leq \epsilon$. Taking the expectation and then the supremum over all admissible strategies we obtain $v(x_n) \leq \epsilon$. Sending ϵ to 0^+, we obtain a contradiction. $\square$

4 Uniqueness of the viscosity solution

Some uniqueness proofs for viscosity solutions of first-order integro-differential operators are given in Soner (1986) for bounded viscosity solutions and in Sayah (1991) and in Pham (1998) for unbounded viscosity solutions. As in Soner (1986) Lemma 2.1 or in Sayah (1991) Proposition 2.1, we give an equivalent formulation for viscosity solutions which is needed to prove a comparison theorem.

Proposition 4.1 *Let v be a function defined on $I\!R_+$, then*
i) v is a viscosity super-solution of (3.1) *in $I\!R_+^*$ if and only if*

$$\max\left\{H_*(x_0, v_*, \psi'), 1 - \psi'(x_0)\right\} \leq 0 \tag{4.1}$$

whenever $\psi \in C^1(N_{x_0})$, $v_* - \psi$ has a global strict minimum at $x_0 \in I\!R_+^*$, N_{x_0} is a neighbourhood of x_0 and

$$H_*(x_0, v_*, \psi') = \liminf_{x \longrightarrow x_0} \sup_{\alpha \geq \underline{\alpha}} \left\{-rv_*(x) + p(\alpha)\psi'(x) + \int_B (v_*(x - y \wedge \alpha) - v_*(x))\pi(dy)\right\}.$$

ii) v is a viscosity sub-solution of (3.1) *in $I\!R_+^*$ if and only if*

$$\max\left\{H^*(x_0, v^*, \psi'), 1 - \psi'(x_0)\right\} \geq 0 \tag{4.2}$$

whenever $\psi \in C^1(N_{x_0})$, $v^* - \psi$ *has a global strict maximum at* $x_0 \in I\!R_+^*$, N_{x_0} *is a neighbourhood of* x_0 *and*

$$H^*(x_0, v^*, \psi') = \limsup_{x \longrightarrow x_0} \sup_{\alpha \geq \underline{\alpha}} \left\{ -rv^*(x) + p(\alpha)\psi'(x) + \int_B (v^*(x - y \wedge \alpha) - v^*(x))\pi(dy) \right\}.$$

Proof. We prove the statement for sub-solutions only, the other statement is proved similarly. Let v such that

$$\max \left\{ H^*(x_0, v^*, \psi'), 1 - \psi'(x_0) \right\} \geq 0,$$

whenever ψ and x_0 are as above. Since $v^*(x) - v^*(x_0) \leq \psi(x) - \psi(x_0)$ for all $x \in I\!R_+^*$, then

$$H^*(x_0, v^*, \psi') \leq H^*(x_0, \psi, \psi').$$

Hence v is a viscosity sub-solution of (3.1) in $I\!R_+^*$.
Conversely, let $\psi \in C^1(N_{x_0})$ and $x_0 \in I\!R_+^*$ such that

$$(v^* - \psi)(x_0) = \max_{I\!R_+^*} (v^* - \psi)(x) = 0.$$

For each ϵ, $\delta > 0$, we define

$$\Phi_{\epsilon,\delta}(x) \quad = \quad \begin{cases} \psi(x) & \text{if} \quad x \in B(x_0, \epsilon) \\ v^*(x) + \delta & \text{if} \quad x \notin B(x_0, \epsilon). \end{cases}$$

We have $v^*(x_0) = \Phi_{\epsilon,\delta}(x_0)$ and $v^*(x) - \Phi_{\epsilon,\delta}(x) < 0$ for all $x \in I\!R_+^* - \{x_0\}$. Hence

$$(v^* - \Phi_{\epsilon,\delta})(x_0) = \max_{x \in I\!R_+^*} (v^* - \Phi_{\epsilon,\delta})(x).$$

Thus the hypothesis of the Proposition yields

$$\max \left\{ H^*(x_0, \Phi_{\epsilon,\delta}, \Phi'_{\epsilon,\delta}), 1 - \psi'(x_0) \right\} \geq 0.$$

From the definition of H, we have the following estimation

$$H^*(x_0, \Phi_{\epsilon,\delta}, \psi') - H^*(x_0, v^*, \psi') \leq G^*(x_0), \tag{4.3}$$

where $G^*(x_0) := \limsup_{x \longrightarrow x_0} G(x)$ and

$$\begin{aligned} G(x) \quad := \quad & \sup_{\alpha \geq \underline{\alpha}} \left\{ -r\left(\Phi_{\epsilon,\delta}(x) - v^*(x)\right) + p(\alpha) \left(\Phi'_{\epsilon,\delta}(x) - \psi'(x) \right) \right. \\ + \quad & \left. \int_B \left(\Phi_{\epsilon,\delta}(x - y \wedge \alpha) - v_*(x - y \wedge \alpha)\right) \pi(dy) - \int_B \left(\Phi_{\epsilon,\delta}(x) - v_*(x)\right) \pi(dy) \right\}. \end{aligned}$$

From the definition of G^*, there exists a sequence $(x_n)_n \in I\!R_+^*$ such that $x_n \longrightarrow x_0$ and $G(x_n) \longrightarrow G^*(x_0)$ when $n \longrightarrow \infty$. We suppose that $x_n \in B(x_0, \epsilon)$ for all $n \in I\!N$. From the definition of $\Phi_{\epsilon,\delta}$ we have $\Phi'_{\epsilon,\delta}(x)(x_n) = \psi'(x_n)$ and $v^*(x_n) - \Phi_{\epsilon,\delta}(x_n) \leq 0$ and so

$$G(x_n) \leq \sup_{\alpha \geq \underline{\alpha}} \left\{ \int_B (\Phi_{\epsilon,\delta}(x_n - y \wedge \alpha) - v_*(x_n - y \wedge \alpha))\, \pi(dy) - \int_B (\Phi_{\epsilon,\delta}(x_n) - v_*(x_n))\, \pi(dy) \right\}.$$

We choose $\alpha \geq \underline{\alpha}$, and consider the two cases: $\underline{\alpha} > 0$ and $\underline{\alpha} = 0$. If $\underline{\alpha} > 0$, then $\alpha > 0$. We set $B_1^\alpha = \{y \in B, x_n - y \wedge \alpha \in B(x_0, \epsilon)\}$ and $B_2^\alpha = \{y \in B, x_n - y \wedge \alpha \notin B(x_0, \epsilon)\}$. Observe that for $(x_n - y \wedge \alpha) \notin B(x_0, \epsilon)$, we have $\Phi_{\epsilon,\delta}(x_n - y \wedge \alpha) - v^*(x_n - y \wedge \alpha) = \delta$ and for $(x_n - y \wedge \alpha) \in B(x_n, \epsilon)$, we have $\Phi^{\epsilon,\delta}(x_0 - y \wedge \alpha) - v^*(x_0 - y \wedge \alpha) = \psi(x_n - y \wedge \alpha) - v^*(x_n - y \wedge \alpha)$. Since $v^*(x_n) - \Phi_{\epsilon,\delta}(x_n) \leq 0$ we get

$$\begin{aligned} & \int_B \left(\Phi^{\epsilon,\delta}(x_n - y \wedge \alpha) - v^*(x_n - y \wedge \alpha)\right) \pi(dy) - \int_B (\Phi_{\epsilon,\delta}(x_n) - v_*(x_n))\, \pi(dy) \\ \leq \; & \delta \int_{B_2^\alpha} \pi(dy) + K\pi(B_1^\alpha) \\ \leq \; & \delta \int_B \pi(dy) + K\pi([0, \epsilon]) \end{aligned} \tag{4.4}$$

where K is a constant independent of α and the last inequality is derived for ϵ sufficiently small ($\epsilon \leq \frac{\alpha}{2}$).
If $\underline{\alpha} = 0$, then $\alpha = 0$ or $\alpha > 0$. If $\alpha = 0$, we have

$$\int_B (\Phi_{\epsilon,\delta}(x_n - y \wedge \alpha) - v^*(x_n - y \wedge \alpha))\, \pi(dy) - \int_B (\Phi_{\epsilon,\delta}(x_n) - v_*(x_n))\, \pi(dy) = 0. \tag{4.5}$$

The case $\alpha > 0$ is similar to the first one. From (4.4) and (4.5) we deduce that

$$\sup_{\alpha \geq \underline{\alpha}} \left\{ \int_B (\Phi_{\epsilon,\delta}(x_n - y \wedge \alpha) - v^*(x_n - y \wedge \alpha))\pi(dy) \right\} \longrightarrow 0$$

when ϵ and δ tend to 0^+. Sending n to infinity, inequality (4.3) implies

$$H(x_0, v^*, \psi') \geq 0,$$

and so (4.2) is proved. □
Uniqueness of the solution of the HJBVI (3.1) with boundary conditions (3.2) is a consequence of the following theorem.

Theorem 4.1 (Comparison theorem) *Let v_1 and v_2 in $C_1(I\!R_+)$ be a viscosity sub-solution and a super-solution respectively of (3.1) in $I\!R_+^*$ such that $v_1^*(0) = v_{2*}(0) = 0$. Then*

$$v_1^*(x) \leq v_{2*}(x) \;\; \textit{for all } x \in I\!R_+^*. \tag{4.6}$$

Proof.

Due to the linear growth of the viscosity sub-solution v_1 (resp. super-solution v_2), the function u_1 (resp u_2) defined by $u_1(x) = v_1(x)e^{-\lambda x}$ (resp $u_2(x) = v_2(x)e^{-\lambda x}$) for $\lambda \in I\!\!R_+^*$ and $x \in I\!\!R_+$ is bounded. For $\epsilon > 0$, we define $\Phi : I\!\!R_+ \times I\!\!R_+ \longrightarrow I\!\!R \cup \{-\infty\}$ as

$$\Phi(x,z) \;:=\; u_1^*(x) - u_{2*}(z) - \frac{1}{\epsilon}(x-z)^2.$$

Since u_1 and u_2 are bounded and Φ is upper semi-continuous, Φ has a global maximum at point $(x^*, z^*) \in I\!\!R_+ \times I\!\!R_+$. Using $\Phi(0,0) \leq \Phi(x^*, z^*)$ and that u_1 and u_2 are bounded, it follows that

$$\begin{aligned} |x^* - z^*|^2 &\leq \epsilon(u_1^*(x^*) - u_{2*}(z^*)) \\ &\leq C_\lambda \epsilon, \end{aligned} \tag{4.7}$$

where C_λ is a constant depending only on λ.
If $x^* = 0$, then using $\Phi(x,x) \leq \Phi(0, z^*)$ for all $x \in I\!\!R_+$ and $u_1^*(0) = 0$, we get

$$u_1^*(x) - u_{2*}(x) \;\leq\; -u_{2*}(z^*). \tag{4.8}$$

From inequality (4.7) and since $x^* = 0$, we deduce that $z^* \longrightarrow 0$ when $\epsilon \longrightarrow 0$.
Since u_{2*} is lower semi-continuous, it follows that $\liminf_{\epsilon \longrightarrow 0} u_{2*}(z^*) \geq u_{2*}(0)$.
Taking the limit when $\epsilon \longrightarrow 0^+$ in (4.8), we obtain $u_1^*(x) \leq u_{2*}(x)$ and so

$$v_1^*(x) \leq v_{2*}(x).$$

If $z^* = 0$, then for all $x \in I\!\!R_+$, we have

$$u_1^*(x) - u_{2*}(x) \leq u_1^*(x^*). \tag{4.9}$$

From inequality (4.7) and since $z^* = 0$, we deduce that $x^* \longrightarrow 0$ when $\epsilon \longrightarrow 0$.
Since u_1^* is upper semi-continuous, it follows that $\limsup_{\epsilon \longrightarrow 0} u_1^*(x^*) \leq u_1^*(0)$.
Taking the limit when $\epsilon \longrightarrow 0^+$ in (4.9), we obtain $u_1^*(x) \leq u_{2*}(x)$ and so

$$v_1^*(x) \leq v_{2*}(x).$$

It remains to study the case when $z^* \neq 0$ and $x^* \neq 0$. Writing the variational inequality for the function u, we obtain

$$\max\left\{H'(x,u,u'), 1 - e^{\lambda x}(u'(x) + \lambda u(x))\right\} = 0 \text{ in } I\!\!R_+^*,$$

where
$H'(x,u,u') = \sup_{\alpha \geq \underline{\alpha}} \left\{-ru(x) + p(\alpha)\left(u'(x) + \lambda u(x)\right) + \int_B (u(x - y\wedge\alpha)e^{\lambda y \wedge \alpha} - u(x))\pi(dy)\right\}$. Since u_1^* is a subsolution and $(u_1^* - \psi_1)(x)$ reaches its maximum in x^* where

$$\psi_1(x) \equiv u_{2*}(z^*) + \frac{1}{\epsilon}(x - z^*)^2,$$

we have from Proposition (4.1)

$$\max\left\{H^{'*}(x^*, u_1^*, \psi_1'), 1 - e^{\lambda x^*}(\psi_1'(x^*) + u_1^*(x^*))\right\} \geq 0,$$

which implies

$$\max\left\{H^{'}(x^*, u_1^*, \psi_1'), 1 - e^{\lambda x^*}(\psi_1'(x^*) + u_1^*(x^*))\right\} \geq 0.$$

Similarly $(u_{2*} - \psi_2)(z)$ reaches its minimum in z^* where

$$\psi_2(z) \equiv u_1^*(x^*) - \frac{1}{\epsilon}(x^* - z)^2.$$

Since u_{2*} is a super-solution, we have

$$\max\left\{H_*^{'}(z^*, u_{2*}, \psi_2'), 1 - e^{\lambda z^*}(\psi_2'(z^*) + u_{2*}(z^*))\right\} \leq 0,$$

which implies

$$\max\left\{H^{'}(z^*, u_{2*}, \psi_2'), 1 - e^{\lambda z^*}(\psi_2'(z^*) + u_{2*}(z^*))\right\} \leq 0.$$

Observing that $\max\{a, b\} - \max\{d, e\} \leq 0$ implies either $a \leq d$ or $b \leq e$, we divide our consideration into two cases:

(i) the case

$$H^{'}(z^*, u_{2*}, \psi_2') - H^{'}(x^*, u_1^*, \psi_1') \leq 0,$$

which implies

$$\begin{aligned} 0 \;\leq\; & \sup_{\alpha \geq \underline{\alpha}} \{-r\,(u_1^*(x^*) - u_{2*}(z^*)) + p(\alpha)\,(\psi_1'(x^*) - \psi_2'(z^*) + \lambda\,(u_1^*(x^*) - u_{2*}(z^*))) \\ & + \int_B \left(u_1^*(x^* - y \wedge \alpha)e^{-\lambda(y\wedge\alpha)} - u_{2*}(z^* - y \wedge \alpha)e^{-\lambda(y\wedge\alpha)} - u_1^*(x^*) + u_{2*}(z^*)\right)\pi(dy)\right\}. \end{aligned} \tag{4.10}$$

Since (x^*, z^*) is a maximum point of Φ in $I\!R_+ \times I\!R_+$ and $\Phi(x^*, z^*) \geq \Phi(0, 0) = 0$, we have

$$\Phi(x^*, z^*) \geq \Phi(x^* - y \wedge \alpha, z^* - y \wedge \alpha)e^{-\lambda(y\wedge\alpha)} \text{ for all } y \in B,$$

which implies

$$(u_1^*(x^* - y \wedge \alpha) - u_{2*}(z^* - y \wedge \alpha))\,e^{-\lambda(y\wedge\alpha)} - u_1^*(x^*) + u_{2*}(z^*) \leq 0 \text{ for all } y \in B.$$

From inequality (4.10) and using the fact that $\psi_1'(x^*) = \psi_2'(z^*) = \frac{1}{2\epsilon}(x^* - z^*)$, we have

$$\sup_{\alpha \geq \underline{\alpha}} \{(-r + \lambda p(\alpha))\,(u_1^*(x^*) - u_{2*}(z^*))\} \geq 0.$$

Since $p(\alpha)$ is bounded, choosing λ sufficiently small, we obtain

$$u_1^*(x^*) - u_{2*}(z^*) \leq 0.$$

Using that $\Phi(x,x) \leq \Phi(x^*,z^*)$, we conclude that $u_1^*(x) \leq u_{2*}(x)$ and

$$v_1^*(x) \leq v_{2*}(x).$$

(ii) the second case occurs if

$$e^{\lambda x^*}(\frac{1}{2\epsilon}(x^*-z^*) + \lambda u_1^*(x^*)) - e^{\lambda z^*}(\frac{1}{2\epsilon}(x^*-z^*) + \lambda u_{2*}(z^*)) \leq 0,$$

which implies

$$e^{\lambda x^*} u_1^*(x^*) - e^{\lambda z^*} u_{2*}(z^*) \leq \frac{1}{2\epsilon}(e^{\lambda z^*} - e^{\lambda x^*})(x^*-z^*) \leq 0,$$

and so we obtain

$$v_1^*(x) \leq v_{2*}(x).$$

□

5 Numerical study

Here we restrict ourselves to the case when the compound Poisson process is a Poisson process with constant intensity π. We assume that all the claims have the same size denoted by δ. As seen in Remark (2.1), we have an explicit expression of $\underline{\alpha} = \frac{\delta(k_2-k_1)}{k_2}$. Given an initial reserve x and a policy (α, L), the reserve of the insurance company at time t is then given by :

$$X_t^{x,\alpha,L} = x + \int_0^t p(\alpha_u)du - \int_0^t \alpha_u \, dN_u - \int_0^t dL_u,$$

where $\alpha_t \in [\underline{\alpha}, \delta]$ for (t,w) a.e, L satisfies (2.5) and $p(\alpha_t) = (k_1 - k_2)\beta\nu + (1 + k_2\alpha_t)$.
Our purpose is to solve the following equation

$$\begin{cases} \max\left\{ \sup_{\alpha \in [\underline{\alpha},\delta]} \left\{A^\alpha(x,v,v^{'})\right\}, 1 - v^{'}(x)\right\} = 0 \quad \text{in} \quad I\!R_+^* \\ v(0) = 0, \end{cases} \tag{5.1}$$

where

$$A^\alpha(x,v,v^{'}) = -rv(x) + p(\alpha)v^{'}(x) + \pi(v(x-\alpha) - v(x)).$$

We proceed with a technical change of variable which brings $\mathbb{R}_+$ into $[0,1)$, namely

$$\begin{cases} z = \frac{x}{1+x} \\ \psi(t,z) = v(t,x). \end{cases}$$

The function ψ is defined in $[0,1)$ and satisfies

$$\begin{cases} \max\left\{\sup_{\alpha\in[\underline{\alpha},\delta]}\left\{\bar{A}^\alpha(z,\psi,\psi^{'})\right\}, 1-(1-z)^2\psi^{'}(z)\right\} = 0 \ \text{ in } (0,1) \\ \psi(0) = 0, \end{cases} \tag{5.2}$$

where

$$\bar{A}^\alpha(z,\psi,\psi^{'}) = -r\psi(z) + p(\alpha)(1-z)^2\psi^{'}(z) + \pi\left(\psi(\frac{z-(1-z)\alpha}{1-(1-z)\alpha}) - \psi(z)\right).$$

In Sections 3 and 4, we have proved that the value function (2.6), within a change of variables, is the unique viscosity solution of HJBVI (5.2). This solution is approximated by performing the following numerical method:
(i) approximate HJBVI (5.2) by using a consistent finite difference approximation which satisfies the discrete maximum principle (DMP) (see Lapeyre, Sulem and Talay),
(ii) solve the discrete equation by means of the Howard algorithm (policy iteration) (see Howard (1960)). Finally a reverse change of variables is performed in order to display the solution of Equation (5.1).

5.1 Finite difference approximation

Let $p = \frac{1}{M}, (M \in \mathbb{N}^*)$ denote the finite difference step in the state coordinate. Let $z_i = ip$ denote the points of the grid $\Omega_p = (0,1)\cap(p\mathbb{Z})$, $0 \le i \le M-1$. The variational inequality (5.2) is approximated by using the following approximations

$$(1-z_i)^2\psi^{'}(z_i) \ \simeq \ (1-ip)^2\frac{\psi\left((i+1)p\right)-\psi\left(ip\right)}{p} \text{ for } z_i \in \Omega_p$$

and

$$-(1-z_i)^2\psi^{'}(z_i) \ \simeq \ -(1-ip)^2\frac{\psi\left(ip\right)-\psi\left((i-1)p\right)}{p} \text{ for } z_i \in \Omega_p.$$

For the boundary conditions, we set $\psi(0) = 0$.
This finite difference approximation leads to a system of $(M-1)$ inequalities with $(M-1)$ unknowns $\{\psi(z_i)\ , z_i \in \Omega_{p,h}\}$:

$$\begin{cases} \max\left\{\sup_{\alpha\in[\underline{\alpha},\delta]}\left\{\bar{A}^\alpha_p\psi_p\right\}, \bar{L} - \bar{B}\psi_p\right\} = 0 \text{ in } \Omega_p, \\ \psi(0) = 0 \end{cases} \tag{5.3}$$

where ψ_p is the vector $(\psi(z_i))_{i=1\ldots M-1}$, $\bar{A}_p^\alpha$ is the $(M-1)\times(M-1)$ matrix associated to the approximation of the operator $\bar{A}^\alpha$, $\bar{B}$ is a $(M-1)\times(M-1)$ matrix associated to the second term of our variational inequality, which verifies

$$\begin{cases} \bar{B}(i,i) = -\frac{(1-z_i)^2}{p} \text{ for all } 1 \le i \le M-1 \\ \bar{B}(i,i-1) = \frac{(1-z_i)^2}{p} \text{ for all } 1 \le i \le M-1 \\ \bar{B}(i,j) = 0 \text{ if } j \notin \{i, i-1\}, \end{cases}$$

and $\bar{L}$ is the vector which satisfies $\bar{L}(i) = 1$ for all $1 \le i \le M-1$. HJBVI (5.3) can be solved by using the Howard algorithm described below.

5.2 The Howard algorithm

To solve Equation (5.3), we use the Howard algorithm (see Lapeyre Sulem and Talay), also named policy iteration. It consists of computing two sequences $(\alpha^n)_{n\ge1}$ and $(\psi_p^n)_{n\ge1}$, (starting from ψ_p^1) defined by:

- Step $2n-1$. Given ψ_p^n, compute a strategy α^n defined as

$$\alpha^n \in \operatorname{argmax}_{\alpha\in[\underline{\alpha},\delta]} \left\{\bar{A}_p^\alpha \psi_p^n\right\}.$$

- Step $2n$. Compute a partition $(D_1^n \cup D_2^n)$ such that

$$\bar{A}^{\alpha^n}\psi_p^{n+1} \ge \bar{L} - \bar{B}\psi_p^{n+1} \text{ on } D_1^n,$$

$$\bar{A}^{\alpha^n}\psi_p^{n+1} < \bar{L} - \bar{B}\psi_p^{n+1} \text{ on } D_2^n.$$

  Define ψ_p^{n+1} as the solution of the linear systems:

$$\bar{A}^{\alpha^n}\psi_p^{n+1} = 0 \text{ on } D_1^n,$$

  and

$$\bar{L} - \bar{B}\psi_p^{n+1} = 0 \text{ on } D_2^n.$$

- If $|\psi_p^{n+1} - \psi_p^n| \le \epsilon$ stop, otherwise, go to step $2n+1$.

When the matrix of the linear system satisfies the discrete maximum principle, the sequence $(\psi_p^n)_{n\in I\!N}$ increases and is bounded and so converges to the viscosity solution of (5.3).

	k_1	k_2	r	π	δ
Test 1	0.2	0.25	0.07	10	1
Test 2	0.15	0.25	0.07	5	2

Table 1: The parameters of the numerical tests

5.3 Numerical results

Variational inequality (5.1) is solved by using the Howard algorithm. This algorithm is very efficient and converges in five iterations. Two tests are performed with parameter values given in Table 1.
The optimal policy has the following form: every excess of the reserve above some critical threshold u is distributed as dividend. When the reserve process is below $l \equiv \underline{\alpha} = \frac{(k_2-k_1)\delta}{k_2}$, it is optimal to distribute all the current reserve as dividends because of the constraint (2.3): See Table 2.

Test 1	l=0.2	u=9.52
Test 2	l=0.8	u=14.38

Table 2: Lower and upper critical thresholds

When the reserve process is in (l, u), then the insurer doesn't distribute any dividend. Figures 1 and 2 display the optimal retention level $\alpha(x)$ given in feedback form of the reserve level for two different sizes of the claims $\delta = 1$ and $\delta = 2$. The optimal retention level satisfies: $\alpha(x) = x$ for $l \leq x \leq \delta$ and $\alpha(x) = \delta$ for $\delta \leq x \leq u$. A similar result was obtained by Højgaard and Taksar (1999) in the case of a diffusion model and proportional reinsurance. Figures 3 and 5 display the value function v in terms of the reserve level x. The value function v is nondecreasing. It is linear when $x \in [0, l]$ and $x \in [u, \infty)$. Note that contrary to the paper by Asmussen, Højgaard and Taksar (2000) where the reserve is modelled by a diffusion, our value function is discontinuous in l. This justifies the use of generalised viscosity solutions. Figures 4 and 6 enlarge the region of discontinuity.

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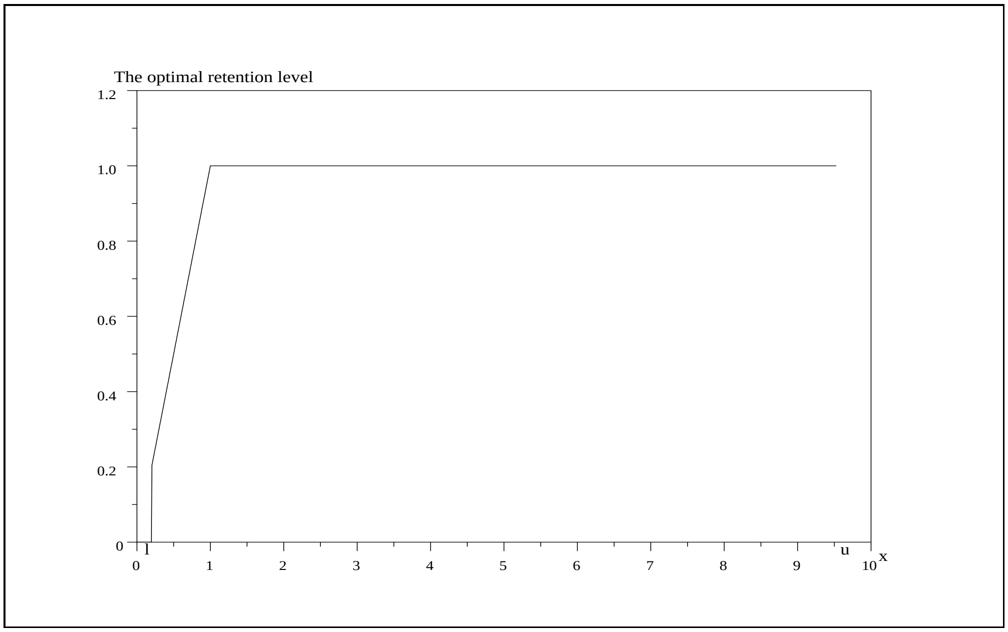

Figure 1: The optimal retention level for k_1=0.2, k_2=0.25 and δ=1

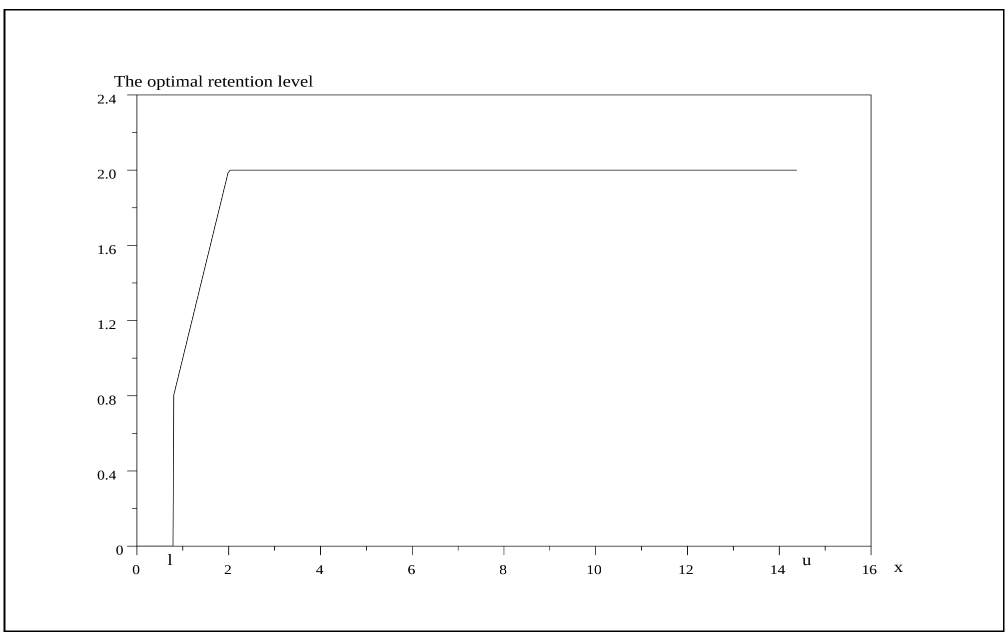

Figure 2: The optimal retention level for k_1=0.15, k_2=0.25 and δ=2

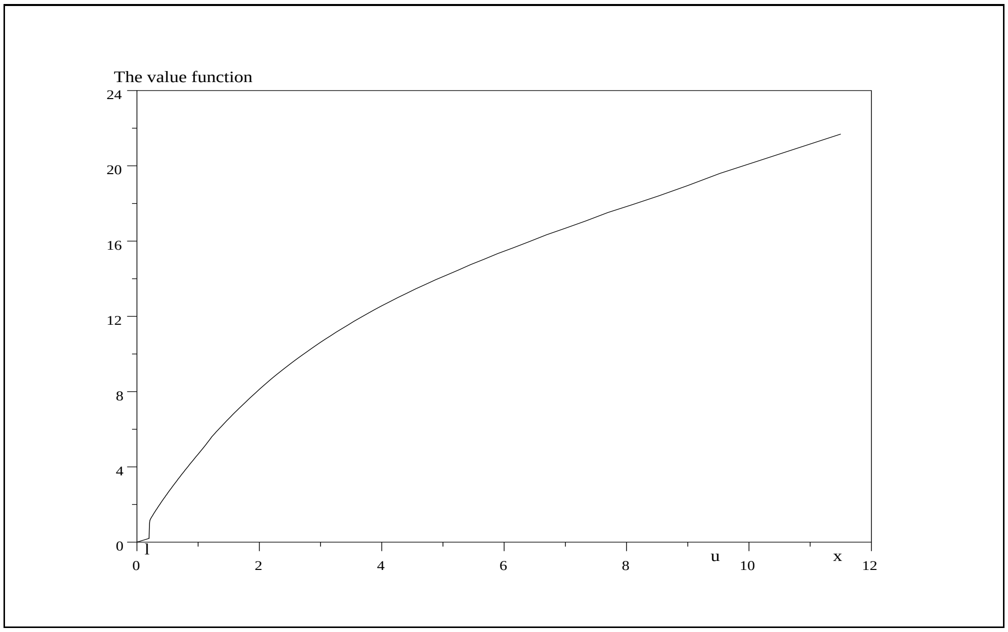

Figure 3: The value function for k_1=0.2, k_2=0.25 and δ=1

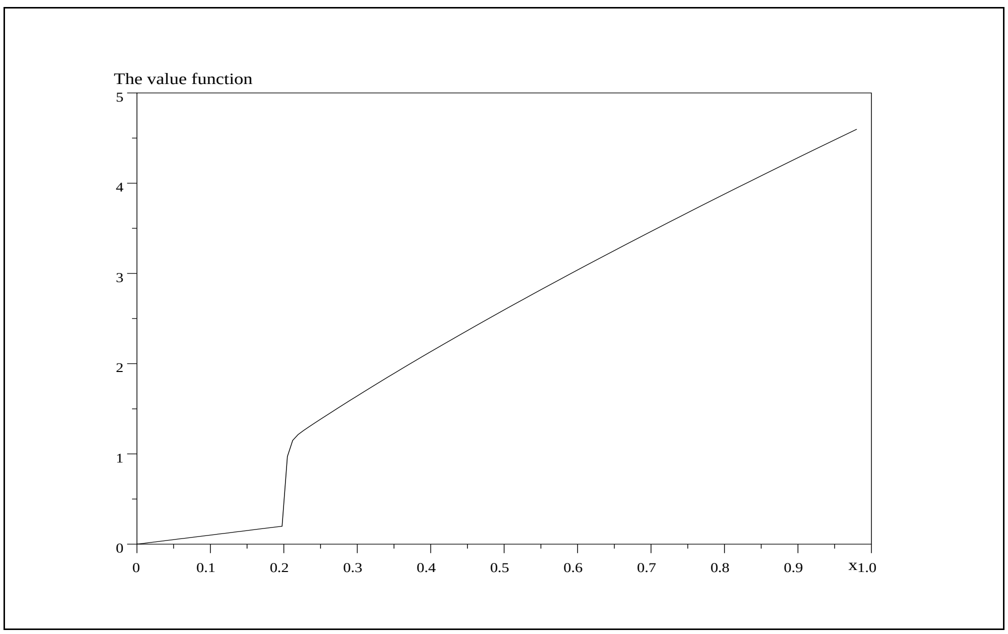

Figure 4: Enlargement of Figure 3 in the neighbourhood of l=0.2

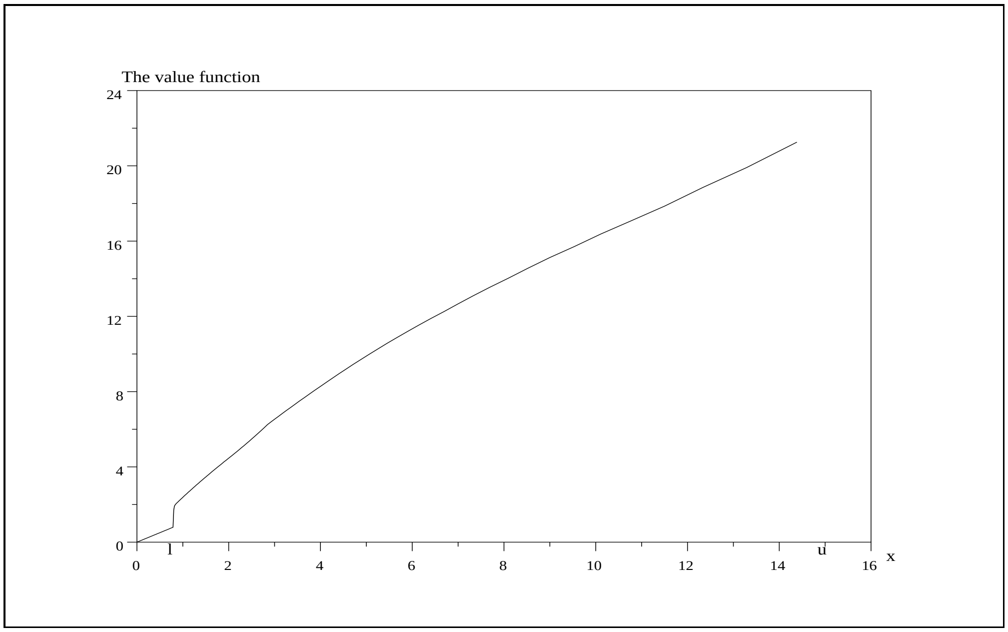

Figure 5: The value function for k_1=0.15, k_2=0.25 and δ=2

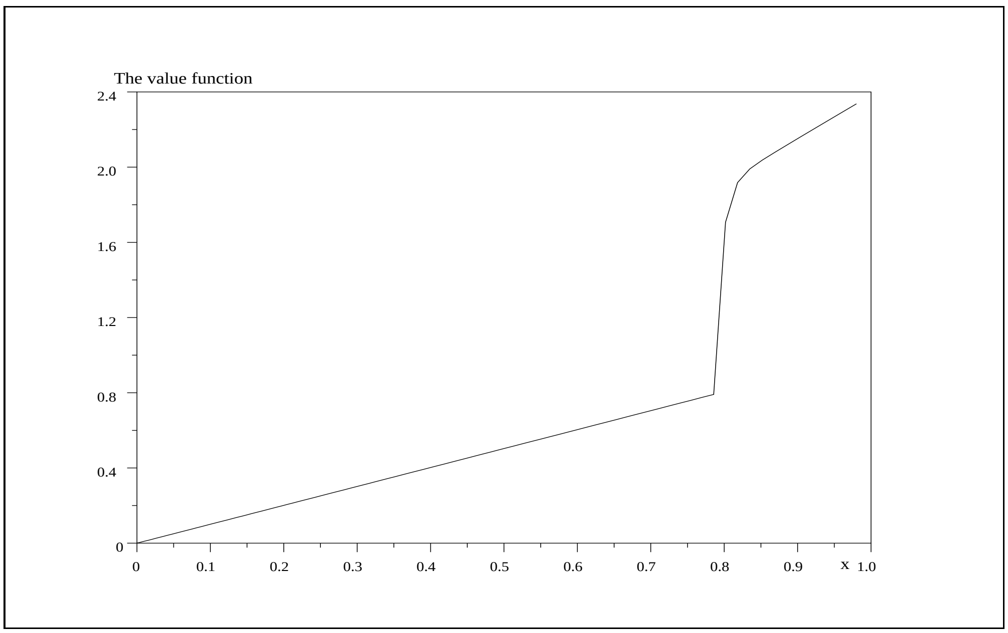

Figure 6: Enlargement of figure 5 in the neighbourhood of l=0.8

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