Stable set meeting every longest paths

Frédéric Havet

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

Stable set meeting every longest paths

Frédéric Havet

N° 5009

Novembre 2003

THÈME 1

Rapport de recherche

ISSN 0249-6399 ISRN INRIA/RR--5009--FR+ENG

Stable set meeting every longest paths

Frédéric Havet

Thème 1 — Réseaux et systèmes
Projet Mascotte

Rapport de recherche n° 5009 — Novembre 2003 — 7 pages

Abstract: Laborde, Payan and Xuong conjectured that every digraph has a stable set meeting every longest path. We prove that this conjecture holds for digraphs with stability at most 2.

Key-words: stable set, independant set, chromatic number, longest path

Unité de recherche INRIA Sophia Antipolis
2004, route des Lucioles, BP 93, 06902 Sophia Antipolis Cedex (France)
Téléphone : +33 4 92 38 77 77 — Télécopie : +33 4 92 38 77 65

Stable intersectant tous les plus longs chemins

Résumé : Laborde, Payan and Xuong ont conjecturé que tout graphe orienté possède un stable intersectant tous les plus longs chemins. Nous prouvons cette conjecture pour les digraphes de stabilité au plus 2.

Mots-clés : stable, ensemble indépendant, nombre chromatique, plus long chemin

1 Introduction

1.1 Prelimiary definitions

A *directed graph* D is a pair $(V(D), E(D))$ of disjoints sets (of *vertices* and *arcs*) together with two maps *tail* : $A(D) \rightarrow V(D)$ and *head* : $A(D) \rightarrow V(D)$ assigning to every arc e a *tail* $tail(e)$ and an *head* $head(e)$. The tail and the head of an arc are its *ends*. An arc with tail u and head v is denoted by uv; we say that u *dominates* v and write $u \rightarrow v$. We also say that u and v are adjacent. The *order* of a digraph is its number of vertices.

The *union* and *intersection* of the digraphs D_1 and D_2 are digraphs $D_1 \cup D_2 = (V(D_1) \cup V(D_2), A(D_1) \cup A(D_2))$ and $D_1 \cap D_2 = (V(D_1) \cap V(D_2), A(D_1) \cap A(D_2)$ respectively.

A *path* is a non-empty digraph P of the form

$$V(D) = \{v_0, v_1, \dots, v_k\} \qquad E = \{v_0v_1, v_1v_2, \dots, v_{k-1}v_k\},$$

where the v_i are all distinct. The vertices v_0 and v_k are respectively called the *origin* and *terminus* of P.

We often refer to a path by the natural sequence of its vertices, writing $P = v_0v_1 \dots v_k$.

If $P = v_0v_1 \dots v_k$ is a path then $C = (V(P), A(P) \cup \{v_kv_0\}\})$ is a *circuit*. It is often denoted by $v_0v_1 \dots v_kv_0$. The *predecessor* (resp. *successor*) of a vertex x in a circuit C is the vertex y such that $yx \in A(C)$, (resp. $xy \in A(C)$).

The *length* of a path or a circuit is its number of arcs. The length of a longest path in a digraph D is denoted by $l(D)$.

A path or a cycle in D is *hamiltonian* in D if it contains all the vertices of D.

Let $P = v_0v_1 \dots v_k$. For $0 \leq i \leq j \leq k$, we write

$$\begin{aligned} Pv_i &:= v_0 \dots v_i \\ v_iP &:= v_i \dots v_k \\ v_iPv_j &:= v_i \dots v_j \end{aligned}$$

for the appropriate subpaths of P. We use similar intuitive notation for subpaths of circuits and also for the concatenation of paths; for example the union $Pv \cup vQw \cup wR$ is denoted $PvQwR$.

A digraph is *strongly connected* or *strong* if for every two vertices u and v there is a path with origin u and terminus v. A maximal strong subdigraph of a digraph D is called a *component* of D. A component I of D is *initial* if there is no arc with tail in $V(D) \setminus V(I)$ and head in I.

Let D be a digraph. A *stable set* in D is a set S of vertices pairwise non adjacent. The *stability* of D, denoted $\alpha(D)$, is the maximum size of a stable set in D. A *colouring* of D is a partition of its vertex-set into stable sets. The *chromatic number* of D is the minimum number of stable sets in a colouring.

1.2 Problematic

Gallai-Roy Theorem [4, 6] relates the chromatic number to the order of a longest path. It states that *the chromatic number is at most as big as the order of a longest path.* A natural extension of this theorem is the following conjecture :

Conjecture 1 (Laborde, Payan and Xuong [7], 1982) *Every digraph has a stable set meeting every longest path.*

In order to prove Conjecture 1 Laborde, Payan and Xuong proposed the following conjecture which add an extra condition on the desired stable set.

Conjecture 2 (Laborde, Payan and Xuong [7]) *Every digraph has a stable set S such that S meets every longest path, and every vertex of S is an origin of a longest path.*

Laborde, Payan and Xuong [7] proved this conjecture for symmetric digraph. They also formulated the following conjecture implying it :

Conjecture 3 (Laborde, Payan and Xuong [7]) *For every digraph D, there exists a vertex x such that x is an origin of a longest path, and every longest path with origin in $N^-(x)$ contains x.*

Such a vertex is called *suitable*.

If the digraph has a hamiltonian path then Conjecture 3 holds. Indeed every origin of a longest path satisfies the conditions of Conjecture 3. Since every digraph with stability 1 has a hamiltonian path according to Redei's Theorem [5], it follows that Conjecture 3 holds and thus so do Conjectures 2 and 2

The aim of this paper is to prove Conjecture 3 for digraphs with stability 2.

Theorem 1 *Every digraph with stability 2 has a suitable vertex.*

If the digraph is strong, the result holds according to the following result :

Theorem 2 (Chen and Manalastas [3], 1983) *Every strong digraph with stability* 2 *has a hamiltonian path.*

So it remains to prove the result for non-strong digraphs.

2 The proof

In this section, we prove Theorem 1. Therefore, we need the following preliminary results. The first one is the well known Theorem of Camion.

Theorem 3 (Camion [2], 1959) *Every strong digraph with stability 1 has a hamiltonian circuit.*

The second one is a structural theorem, due to Chen and Manalastas [3], implying directly Theorem 2 (See also [1] for a short proof) :

Theorem 4 (Chen and Manalastas [3], 1983) *Let D be a strong digraph with stability 2. If D has no hamiltonian circuit then D contains circuits C_1, C_2 such that $C_1 \cup C_2$ includes all the vertices of D and $C_1 \cap C_2$ is either empty or a path (possibly of length 0).*

The following proposition follows immediatly from the definitions of component and initial component. The proof is left to the reader.

Proposition 1 *Let D be a digraph, F one of its component and P a path in D.*
(i) $F \cap P$ is a path;
(ii) if F is initial and $x \in V(F \cap P)$ then Px is in I. In particular, its origin is in I.

Lemma 1 *Let D be a digraph and I of its initial components. If I has a hamiltonian circuit then every longest path meeeting I contains all the vertices of I. In particular, every vertex in $V(I)$ which is an origin of a longest path of D is suitable.*

Proof. Let C be a hamiltonian circuit of I. Let P be a path meeting I that does not contain all the vertices of I. Let x be the last vertex on P which is in I. Then $V(Px) \subset V(I)$. Let x^+ be the successor of x in C. Then x^+CxP is a path longer than P. □

Lemma 2 *Let D be a digraph with a unique initial component I.*
If u is an origin of a path of length $l(D-I)+|I|$ then u is suitable.

Proof. Let P be a longest path. Let y be the first vertex on P that is not in I and x its predecessor. By Proposition 1, $yP \cap I$ is empty and Px is in I so has length at most $|I|-1$ so yP has length at least $l(D-I)$. Hence yP has length $l(D-I)$ so Px contains every vertex of I, in particular u. □

Proof of Theorem 1. We prove this result by induction on the number of vertices of D, the result being obviously true if D has two vertices.

If D is strong, then by Theorem 2, D has a hamiltonian path with origin s. Then s is suitable.

Hence, we now assume that D is not strong. Let I be an initial component of D.

Suppose first that I is hamiltonian. If there is a vertex v of I that is an origin of a longest path then Lemma 1 gives the result. If there is no origin of a longest path in I then by Proposition 1, no longest path intersect I. So the longest paths of D are the longest paths of $D-I$. By induction hypothesis, there is a suitable vertex v in $D-I$, which is also a suitable vertex in D.

Hence we may assume that D has a unique initial component I without hamiltonian circuit.

By Theorem 3, $\alpha(I)=2$. By Theorem 4, we are in one of the two following subcases :

a) I contains circuits C_1, C_2 such that $C_1 \cup C_2$ includes all the vertices of I and $C_1 \cap C_2$ is a path. We may also assume that C_1 and C_2 are such that the length of $C_1 \cap C_2$ is maximum. Let x be the origin of $C_1 \cap C_2$ and y its terminus. For $i = 1, 2$, let x_i be the predecessor of x in C_i and y_i the successor of y_i in C_i. Because the length of $C_1 \cap C_2$ is maximum, $\{x_1, x_2\}$ is a stable set. Otherwise, by symmetry, $x_1 \rightarrow x_2$ and $C_1' = xC_1x_1x_2x$ and C_2 yield a contradiction.

   Let s be the origin of a longest path Q in $D - I$. Then it is dominated by one vertex of $\{x_1, x_2\}$, say x_1. Thus $y_2C_2yC_1x_1sQ$ is a path of length $l(D-I)+|I|$. So by Lemma 2, y_2 is suitable.

b) I contains circuits C_1, C_2 such that $C_1 \cup C_2$ includes all the vertices of I and $C_1 \cap C_2$ is empty.

   Suppose that there are four distinct vertices $a_1, b_1 \in V(C_1)$ and $a_2, b_2 \in V(C_2)$ such that $a_1 \rightarrow b_2$ and $a_2 \rightarrow b_1$. Moreover, take four such vertices such that $l(a_1C_1b_1) + l(a_2C_2b_2)$ is minimum. For $i = 1, 2$, let c_i be the predecessor of b_i in C_i. If $c_1 = a_1$ then we are in subcase a) with $b_1C_1a_1b_2C_2a_2b_1$ and C_2. So we may assume that $c_1 \neq a_1$ and $c_2 \neq a_2$ (by symmetry). Since $l(a_1C_1b_1) + l(a_2C_2b_2)$ is minimum $\{c_1, c_2\}$ is a stable set.
   Let s be the origin of a longest path Q in $D - I$. Then s is dominated by one vertex of $\{c_1, c_2\}$ say c_1. Thus, setting d_2 the successor of a_2 in C_2, $d_2C_2a_2b_1C_1c_1sQ$ is a path of length $l(D - I) + |I|$. So by Lemma 2, d_2 is a suitable.

   Suppose now that there are no four distinct vertices $a_1, b_1 \in V(C_1)$ and $a_2, b_2 \in V(C_2)$ such that $a_1 \rightarrow b_2$ and $a_2 \rightarrow b_1$. Then since I is strong, there are three vertices $a_1, b_1 \in V(C_1)$ and $a_2 \in V(C_2)$ such that $a_1 \rightarrow a_2$ and $b_1 \rightarrow a_2$. Moreover assume that a_1 and b_1 are such that $l(a_1C_1b_1)$ is minimum. Let c_1 be the predecessor of b_1 in C_1. If $a_1 = c_1$ then we are in the subcase a) with $b_1C_1a_1a_2b_1$ and C_2. So we may assume that $a_1 \neq c_1$. Then $\{c_1, a_2\}$ is stable by minimality of $l(a_1C_1b_1)$. And for any vertex $b_2 \in C_2$, the set $\{c_1, b_2\}$ is stable otherwise we get four vertices giving a contradiction. Let s be the origin of a longest path Q in $D - I$.
   If s is dominated by c_1, then, setting d_2 the successor of a_2 in C_2, $d_2C_2a_2b_1C_1c_1sQ$ is a path of length $l(D - I) + |I|$. So by Lemma 2, d_2 is a suitable.
   If s is not dominated by c_1 then s is dominated by every vertex in C_2 in particular e_2 the predecessor of a_2 in C_2. Setting d_1 the successor of a_1 in C_1, $d_1C_1a_1b_2C_2e_2sQ$ is a path of length $l(D - I) + |I|$. So by Lemma 2, d_1 is suitable.

□

References

[1] J. A. Bondy, A short proof of the Chen-Manalastas theorem. *Discrete Math.* **146** (1995), no. 1-3, 289–292.

[2] P. Camion, Chemins et circuits hamiltoniens des graphes complets. *C. R. Acad. Sci. Paris* **249** (1959), 2151–2152.

[3] C. C. Chen and P. Manalastas Jr., Every finite strongly connected digraph of stability 2 has a Hamiltonian path. *Discrete Math.* **44** (1983), no. 3, 243–250.

[4] T. Gallai, On directed paths and circuits. In *Theory of Graphs (Proc. Colloq., Titany, 1966), pages 115–118*, Academic Press, New York, 1968.

[5] L. Rédei, Ein kombinatorischer Satz. *Acta Litt. Szeged* **7** (1934), 39–43.

[6] B. Roy, Nombre chromatique et plus longs chemins d'un graphe. *Rev. Française Informat. Recherche Opérationelle* **1** (1967), no. 5, 129–132.

[7] J.-M. Laborde, C. Payan and N. H. Xuong, Independent sets and longest directed paths in digraphs, In *Graphs and other combinatorial topics (Prague, 1982)*, 173–177, Teubner, Leipzig, 1983

[8] M. Las Vergnas, Sur les arborescences dans un graphe orienté. *Discrete Math.* **15** (1976), no. 1, 27–39.

[9] S. Thomassé, Covering a strong digraph by $\alpha-1$ disjoint paths: a proof of Las Vergnas' conjecture. *J. Combin. Theory Ser. B* **83** (2001), no. 2, 331–333.

Unité de recherche INRIA Sophia Antipolis
2004, route des Lucioles - BP 93 - 06902 Sophia Antipolis Cedex (France)

Unité de recherche INRIA Futurs : Parc Club Orsay Université - ZAC des Vignes
4, rue Jacques Monod - 91893 ORSAY Cedex (France)
Unité de recherche INRIA Lorraine : LORIA, Technopôle de Nancy-Brabois - Campus scientifique
615, rue du Jardin Botanique - BP 101 - 54602 Villers-lès-Nancy Cedex (France)
Unité de recherche INRIA Rennes : IRISA, Campus universitaire de Beaulieu - 35042 Rennes Cedex (France)
Unité de recherche INRIA Rhône-Alpes : 655, avenue de l'Europe - 38334 Montbonnot Saint-Ismier (France)
Unité de recherche INRIA Rocquencourt : Domaine de Voluceau - Rocquencourt - BP 105 - 78153 Le Chesnay Cedex (France)

Éditeur
INRIA - Domaine de Voluceau - Rocquencourt, BP 105 - 78153 Le Chesnay Cedex (France)
http://www.inria.fr
ISSN 0249-6399