Meshing implicit surfaces with certified topology title

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Abstract: We describe a new algorithm for building piecewise linear approximations of an implicit surface. This algorithm is the first one guaranteeing that the implicit surface and its approximation are isotopic.

Key-words: implicit surfaces, meshing, Morse theory

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Maillage de surfaces implicites avec topologie certifiée

Résumé : Nous décrivons un nouvel algorithme pour construire une approximation polyédrique d'une surface implicite donnée. C'est le premier algorithme qui permette de garantir que la surface implicite et son approximation sont isotopes.

Mots-clés : surfaces implicites, maillage, théorie de Morse

1 Introduction

Implicit equations are a popular way to encode geometric objects [19]. Typical examples are CSG models, where objects are defined as results of boolean operations on simple geometric primitives. Given an implicit surface, associated geometric objects of interest, such as contour generators, are also defined by implicit equations. Another advantage of implicit representations is that they allow for efficient blending of surfaces, with obvious applications in CAD or metamorphosis. Finally, this type of representation is also relevant to other scientific fields, such as level sets methods or density estimation [5].

However, most graphical algorithms, and especially those implemented in hardware, cannot process implicit surfaces directly, and require that a piecewise linear approximation of the considered surface has been computed beforehand. As a consequence, polygonalization of implicit surfaces has been widely studied in the literature. Among the general classes of methods devoted to this problem, the most common one is the so-called extrinsic polygonalization method [19]. It consists in two steps : first build a tesselation of space, and then analyze the intersection of the considered surface with each cell of the tesselation to produce the approximation. The celebrated marching cube algorithm [13] belongs to this category. The goal of an implicit surface polygonizer is twofold : its output should be geometrically close to the original surface, and have the same topology. While the former is achieved by several polygonalization schemes [20], the latter has been barely addressed up to now.

Some algorithms achieve topological consistency, that is ensure that the result is indeed a manifold, by taking more or less arbitrary decisions when a topologically ambiguous configuration is encountered. This implies that their output might have a different topology from the original surface, except in very specific cases [12]. To the best of our knowledge, there is only one paper devoted to the more difficult problem of homeomorphic polygonalization [15]. The main theoretical tool used in this paper is Morse theory. The authors first find a level set of the considered function that can be easily polygonalized. This initial polygonalization is then progressively transformed into the desired one, by computing intermediate level sets. This requires in particular to perform topological changes when critical points are encountered. Unfortunately, this work is mostly heuristic, and the authors do not give any proof of the correctness of their algorithm.

In this chapter, we give the first certified algorithm for isotopic implicit surface polygonalization. Assuming the critical points of the function defining the surface are known, the whole algorithm can be implemented in the setting of interval analysis. We only assume that the considered isosurface is smooth, that is does not contain any critical point, which is generic by Sard's theorem [17]. Our polygonalization is the zero-set of the linear interpolation of the implicit function on a mesh of $\mathbb{R}^3$. We first exhibit a set of conditions on the mesh used for interpolation that ensure the topological correctness (section 2). Then, we describe an algorithm for building a mesh satisfying these conditions, thereby leading to a provably correct polygonalization algorithm (section 3).

2 A condition for isotopic meshing

Let f be a C^2 function from $\mathbb{R}^3$ to $\mathbb{R}$. We assume that $M = f^{-1}(0)$, the surface we want to polygonalise, is compact. In what follows, T denotes a triangulation of a domain $\Omega \subset \mathbb{R}^3$ containing M and $\hat{f}$ the function obtained by interpolating f linearly on T. A vertex v will be said *larger* (resp. *smaller*) than a vertex u if $f(v)$ is *larger* (resp. *smaller*) than $f(u)$; the sign of f at a vertex will be referred to as the sign of that vertex. We set $\hat{M} = \hat{f}^{-1}(0)$.

2.1 A glimpse at stratifi ed Morse theory

2.1.1 Classical Morse theory

The topology of implicit surfaces is usually investigated through Morse theory [14]. Given a real function f defined on a manifold, Morse theory studies the topological changes in the sets $f^{-1}(]-\infty, a])$ (lower level-sets) when a varies. In our case, as f is defined on $\mathbb{R}^3$, this amounts to study how the topology of the part of the graph of f lying below a horizontal hyperplane changes as this hyperplane sweeps $\mathbb{R}^4$. Classical Morse theory assumes that f is of class C^2. In this case, as is well known, these topological changes are related to the *critical points* of f, that is the points where the gradient ∇f of f vanishes. More precisely, the only topological changes occur when $f^{-1}(a)$ passes through a critical point p -a is then called a *critical value*. In the 2-dimensional case, the topology of $f^{-1}(]-\infty,a])$ can change in three possible ways, according to the type of critical point p (see figure 1).

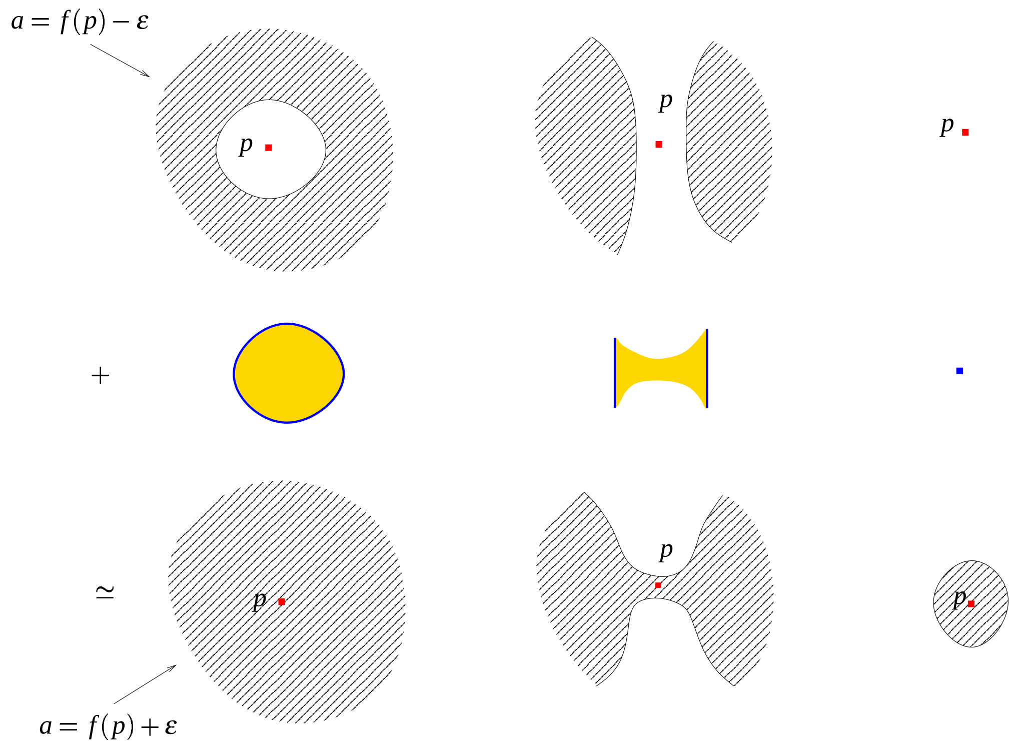

Figure 1: Smooth Morse theory in 2D.

In figure 1, the sets $f^{-1}(]-\infty,a])$ are displayed as striped regions. The leftmost column depicts the situation where p is a local maximum, that is when the Hessian of f at p is positive. In this case, $f^{-1}(]-\infty,a+\varepsilon])$ is obtained from $f^{-1}(]-\infty,a-\varepsilon])$ by gluing a topological disk along its boundary. In the case of a saddle point (i.e. the Hessian has signature $(1,1)$), passing a critical value amounts to glue a thickened topological line segment (in gold) along its "thickened" boundary (in blue). Finally, passing through a local minimum (negative Hessian) just amounts to add a disk disconnected from $f^{-1}(]-\infty,a-\varepsilon])$. If p does not fall in any of these categories, that is if the Hessian at p is degenerate, then classical Morse theory cannot be applied. C^2 functions whose critical points all have non-degenerate Hessian are called *Morse functions*. From now on, we will assume that f is a Morse function. Also, we require that 0 is not a critical value of f, which implies that M is a manifold.

2.1.2 Stratified Morse theory

As mentioned in the introduction, we chose to approximate the zero-set M of the smooth function f by the zero-set $\hat{M}$ of $\hat{f}$, which is piecewise linear. We thus need to be able to compare the topology of the level sets of $\hat{f}$ with the topology of those of f. Unfortunately, $\hat{f}$, being piecewise linear, falls out of the realm of classical Morse theory. Also, in the proof of lemma 14, we will need to apply Morse theory to a piecewise C^2 function. As a consequence, we have to resort to an extension of Morse theory developed by Goresky and MacPherson [9], called stratified Morse theory. This extension can handle a certain type of singular spaces, called *Whitney-stratified spaces*. Whitney-stratified spaces are unions of (open) smooth submanifolds of varying dimension, the strata, such that the boundary of each stratum is a union of lower dimensional strata[1]. These spaces can be rather complicated. For our purpose, we can restrict ourselves

[1]These spaces should also satisfy additional properties. For a precise definition, see [9].

to the case of a graph of a piecewise C^2 function g from $\mathbb{R}^3$ to $\mathbb{R}$. In this case, the 3-dimensional strata are the interior of the patches where the function is C^2, and lower dimensional strata are lower dimensional faces of these patches. g should also satisfy some conditions [2] for the theory to apply. In particular, the restriction of g to any stratum should be a Morse function. We will call such functions stratified Morse functions.

In stratified Morse theory, the critical points of a function are defined to be the critical points of the restriction of the function to a stratum. Note that points of 0-dimensional strata are by convention critical points. Just as in the classical case, the topology of the set $g^{-1}(]-\infty,a])$ changes only when a passes through a critical value, that is when $g^{-1}(a)$ passes through some critical point p. The difference is that the change in its topology can be much more involved than in the classical case. Still, like in the smooth case, it can be shown that the set $g^{-1}(]-\infty,a+\varepsilon])$ can always be obtained from $g^{-1}(]-\infty,a-\varepsilon])$ by gluing some set A along some subset $B \subset A$. The pair (A,B) is called the *local Morse data* of g at p. To put it more formally, if $B(p,\delta)$ denotes the ball centered on p and with radius δ, then one has :

$$A \simeq B(p,\delta) \cap g^{-1}([a-\varepsilon, a+\varepsilon])$$

and

$$B \simeq B(p,\delta) \cap g^{-1}(a-\varepsilon)$$

These definitions actually make sense, as one can show that the topology of each of the above spaces does not depend on ε and δ for $0 < \varepsilon \ll \delta \ll 1$. In the classical case, if critical point p has index λ, that it the Hessian of g at p has signature $(3-\lambda,\lambda)$, then A is homeomorphic to the product of a λ-dimensional disk with a $(3-\lambda)$-dimensional one, and B is homeomorphic to the product of $(\lambda-1)$-dimensional sphere with a $(n-\lambda)$-dimensional disk (see figure 1).

Together with each critical point p of a Morse function g defined on a stratified space is associated an integer, called the index of g at p, and denoted by $ind(p,g)$ or simply by $ind(p)$ when no confusion is possible. The index is defined to be the increase in the Euler characteristic of $g^{-1}(]-\infty,a])$ when a goes from $g(p)-\varepsilon$ to $g(p)+\varepsilon$. If p is not a critical point, then its index is set to 0. Note that this index is different from the one classically used in the smooth setting, that is the number λ considered in the previous paragraph. When p is a critical point of a smooth function, one actually has $ind(p) = (-1)^{\lambda}$. From now on, by index we will mean the number $ind(p)$. Almost by definition, we get the following counterpart of Hopf's theorem in the stratified setting :

Theorem 1 *Let Y be a compact subset of $\mathbb{R}^3$ and $g : Y \to \mathbb{R}$ be a stratified Morse function. Then, χ denoting the Euler characteristic :*

$$\chi(Y) = \sum_{p \in Y} ind(p)$$

In the sequel, we will use the following consequence of this theorem :

Lemma 2 *Let f,g be two stratified Morse functions defined on $\mathbb{R}^3$ and Y be a compact subset of $\mathbb{R}^3$ such that $f_{|Y}$ and $g_{|Y}$ are stratified Morse functions. If f and g coincide in a neighborhood of ∂Y, then :*

$$\sum_{p \in Y} ind(p,f) = \sum_{p \in Y} ind(p,g)$$

Proof. We have $\sum_{p \in Y} ind(p, f_{|Y}) = \chi(Y) = \sum_{p \in Y} ind(p, g_{|Y})$. Now the difference between $\sum_{p \in Y} ind(p,f)$ and $\sum_{p \in Y} ind(p, f_{|Y})$ is the sum of $ind(p,f) - ind(p, f_{|Y})$, where the sum runs over critical points of f lying on ∂Y, since both indices coincide for critical points lying in the interior of Y. As f and g coincide in a neighborhood of ∂Y, we have for each $p \in \partial Y$:

$$ind(p,f) - ind(p, f_{|Y}) = ind(p,g) - ind(p, g_{|Y})$$

and the result follows. □

In the following, we will call the quantity $\sum_{p \in Y} ind(p,f)$ the index of f on Y. We recall that if $f : \mathbb{R}^3 \to \mathbb{R}$ is a C^2 Morse function and $Y \subset \mathbb{R}^3$ is a 3-manifold with boundary, then ([10])

Lemma 3 *The index of f on Y is the degree of the map from ∂Y to the sphere S^2 that associates with each point $p \in \partial Y$ the normalized gradient of f at p.*

[2] Basically, the height function restricted to the graph of g should be a Morse function in the sense of [9].

Obviously, there is no such result in the stratified setting, as the normalized gradient is not continuous any more, so its degree is not defined. However, there is a simple situation in which a result in the same spirit holds. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a piecewise C^2 Morse function and p be a critical point of f.

Lemma 4 *Consider the set*[3] *:*

$$C_\varepsilon = convex\ hull\{\nabla f(x) |\ x \in B(p, \varepsilon)\ , \nabla f(x)\ is\ defined\}$$

If for sufficiently small ε, $0 \notin C_\varepsilon$, then the lower-level set $f^{-1}(]-\infty, f(p) - \eta])$ is a strong deformation retract of $f^{-1}(]-\infty, f(p) + \eta])$ for sufficiently small η. In particular, the index of f at p is 0.

We recall that loosely speaking, a space B is a strong deformation retract [4] of $A \supset B$ if A can be continuously collapsed to B without being torn (see figure 2). In particular, one has $\chi(A) = \chi(B)$. For a precise definition see any topology textbook, such as [11] or [6]. Lemma 4 is proved in [1] (proposition 1.2).

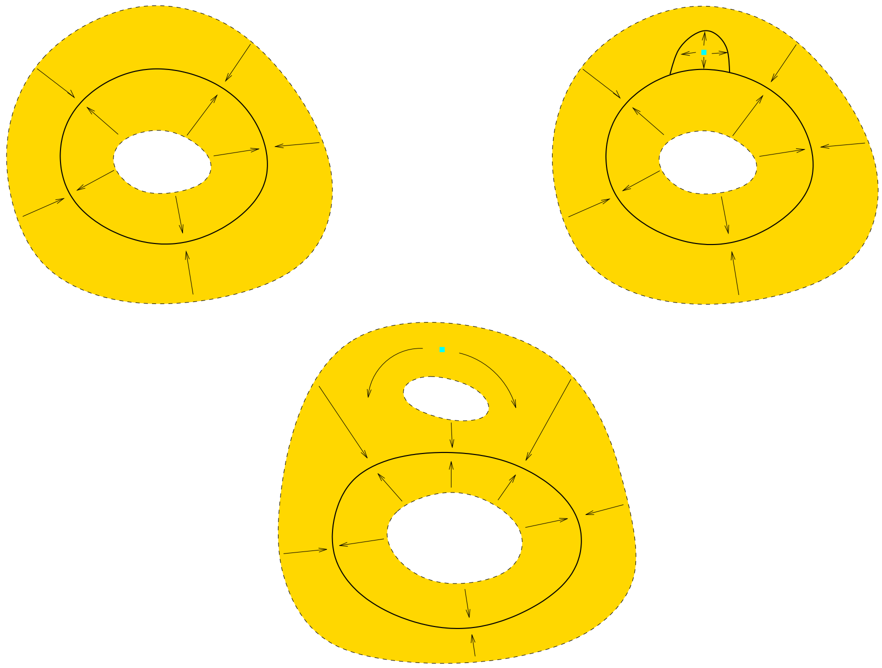

Figure 2: The gold region on the upper left corner deformation retracts on the bold curve. This is not true for the two other cases, since the gold region has to be torn to be collapsed on the bold curve (light blue points).

2.1.3 PL case

We now apply stratified Morse theory to the simple case of the piecewise linear function $\hat{f}$. For piecewise linear functions, being a stratified Morse function means that no two neighboring vertices map to the same value by f (hyp. a), which we will assume from now on. If this is not the case, f can be perturbed so that this property holds.

Let us now recall some well-known definitions [7, 9] :

Definition 1 *The* star *of a vertex is the union of all simplices*[5] *containing this vertex. The* link *of a vertex is the boundary of its star.*

[3] The limit of the set C_ε as ε goes to 0 is known as the Clarke's subdifferential of f at p.
[4] In what follows, we write "deformation retract" for short.
[5] By simplex we mean a closed cell of T of any dimension.

Definition 2 *The* lower star $St^-(v)$ *of* $\hat{f}$ *at a vertex* v *is the union of all simplices incident on* v *all vertices of which but* v *are smaller than* v*. The* lower link $Lk^-(v)$ *of* $\hat{f}$ *at a vertex* v *is the union of all simplices of the link of* v *all vertices of which are smaller than* v*.*

Because $\hat{f}$ is linear on each simplex of T, its only critical points are the vertices of T. To guarantee that $\hat{M}$ is a manifold, we assume that no vertex of T maps to 0 by f (hyp. **b**). Again, this can be ensured by perturbing f slightly if necessary. We refer to hypothesis **a** and **b** as the *genericity assumptions*.

Proposition 5 *The local Morse data at a vertex* v *of* T *is homotopy equivalent to* $(St^-(v), Lk^-(v))$*.*

We recall that homotopy equivalence is a coarser relation than homeomorphy, allowing for instance for changes in the dimensions of the spaces involved. For precise definitions of homotopy equivalence of topological spaces and of pair of spaces, see [11] or [6].

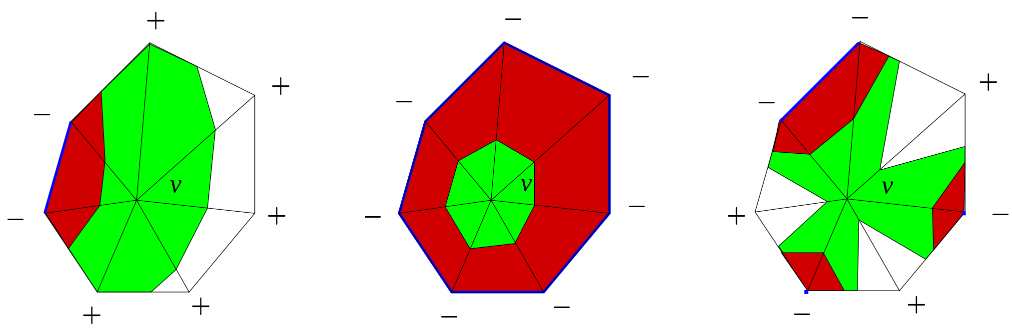

Figure 3: Morse theory for PL functions in 2D. Plus and minus signs indicate whether neighbors of v are larger or smaller than v. Lower links are displayed in blue, sets $\hat{f}^{-1}(]-\infty, f(v)-\varepsilon])$ in red, and sets $\hat{f}^{-1}(]-\infty, f(v)+\varepsilon])$ in green.

Figure 3 shows the local Morse data in 2D in the case of a vertex with connected lower link (left), of a maximum (ie lower link equal to the link, middle), and of a "3-fold saddle" (lower link with 3 components, right). In the sense of stratified Morse theory, the vertex v in the left of figure 3 is a critical point, as any vertex. Still, no topological change in the lower level-sets occurs at such a point. This is what incited us to modify the definition of critical points in the PL case :

Definition 3 *A* critical point *of* $\hat{f}$ *is a vertex whose lower link is not contractible*[6]*. A vertex that is not a critical point of* $\hat{f}$ *will be called* regular.

With this definition, any critical point induces a change in the homotopy type of lower level-sets. The index of a critical point v is 1 minus the Euler characteristic of $Lk^-(v)$ [2]. In figure 3 v respectively has index 0, 1, and -2. In 2D the critical points are exactly the vertices with non-zero index. This is not true any more in 3D. For instance, vertices whose lower link has the topology of the disjoint union of an annulus and a disk are critical but have index 0. Still, regular points all have index 0. In 3D, a point is regular if and only if its lower link and its upper link (similarly defined) are connected, which yields an easy way detect critical points. Finally, remark that if a vertex meets the assumptions of proposition 4, then by proposition 5 its lower stars retracts by deformation on its lower link, so that its lower star is contractible, i.e. the vertex is regular.

2.2 Main result

0. We assume that f does not vanish on any tetrahedron of T containing a critical point of f.

[6] A topological space is contractible if it is homotopy equivalent to a point.

Theorem 6 *Let W be a union of open simplices and vertices of T.*
If W satisfies the following conditions :
1. *f does not vanish on ∂W.*
2. *W contains no tetrahedron of T containing a critical point of f.*
2'. *W contains no critical point of $\hat{f}$.*
3. *$\hat{M}$ is a deformation retract of W*[7].
4. *f and $\hat{f}$ have the same index on each bounded component of $\Omega \setminus W$.*

Then M and $\hat{M}$ are isotopic in W. Moreover, the Hausdorff distance between M and $\hat{M}$ is smaller than the "width" of W, that is the maximum over the components V of W of the Hausdorff distance between the subset of ∂V where f is positive and the one where f is negative.

In the conclusion of the theorem, isotopic in W means that M can be continuously deformed into $\hat{M}$ while remaining a manifold embedded in W, so that M could not be a knotted torus if $\hat{M}$ is an unknotted one, for instance. We first prove that under the conditions of the theorem, M and $\hat{M}$ are homeomorphic. The fact that they actually are isotopic will be shown in the next section. Before proving the theorem, we first show by some examples that none of its assumptions can be removed. In the three following pictures, minima of f are represented by *min*, maxima by *max*, and saddle points by s. Critical points of $\hat{f}$ are represented similarly but with a hat. The sign preceding a critical point symbol indicates the sign of the considered function (f or $\hat{f}$) at the critical point.

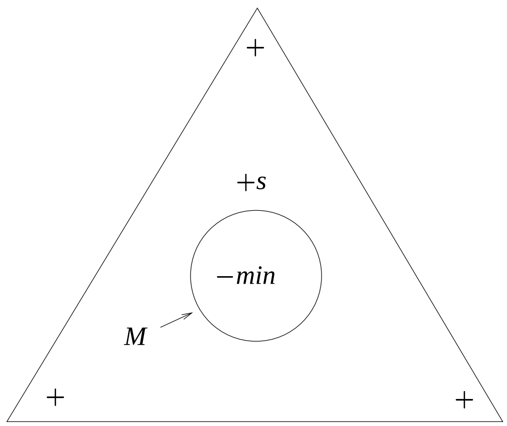

Figure 4: Condition **0.** is needed.

Figure 4 shows that condition **0.** cannot be removed even in the 2D case. By allowing for critical points of f inside a triangle of T with positive vertices, one can build an example where M has an extra component w.r.t. $\hat{M}$ without violating conditions involving critical points and their indices. Indeed, in figure 4, f has index 0 on the triangle, since minima have index 1 and saddle points have index -1.

Figure 5 is a 2D example of two zero-sets M (boundary of the gold region) and M' which are not homeomorphic, though their defining functions have the same critical points, with the same indices. Dashed curves represent two other level-sets of the function defining M' (one in green and one in blue). Such an example can also be built such that $M' = \hat{M}$ for some mesh T. This shows the importance of the set W in the theorem. In particular, conditions **1.** and **3.** cannot be removed. Indeed, if one drops **1.**, taking for W any set satisfying **2.** and **3.** makes the theorem fail. On the other hand, if one drops **3.**, any W satisfying **2.** and **1.** also makes the theorem fail.

Figure 6 is a 3D example where M is a torus whereas $\hat{M}$ is a sphere. This is because $\hat{f}$ has an extra negative minimum inside $\hat{f}^{-1}(]-\infty,0])$ whereas f has an index 1 saddle point rejected outside the bounding box Ω. Depending on whether this extra minimum lies in W or not (see the circular double arrow in figure 6), one obtains counterexamples to the theorem if assumptions **2'.** or **4.** are dropped. One can build similar examples showing that condition **2.** is also needed.

We now return to the proof of theorem 6.

[7] See figure 2.

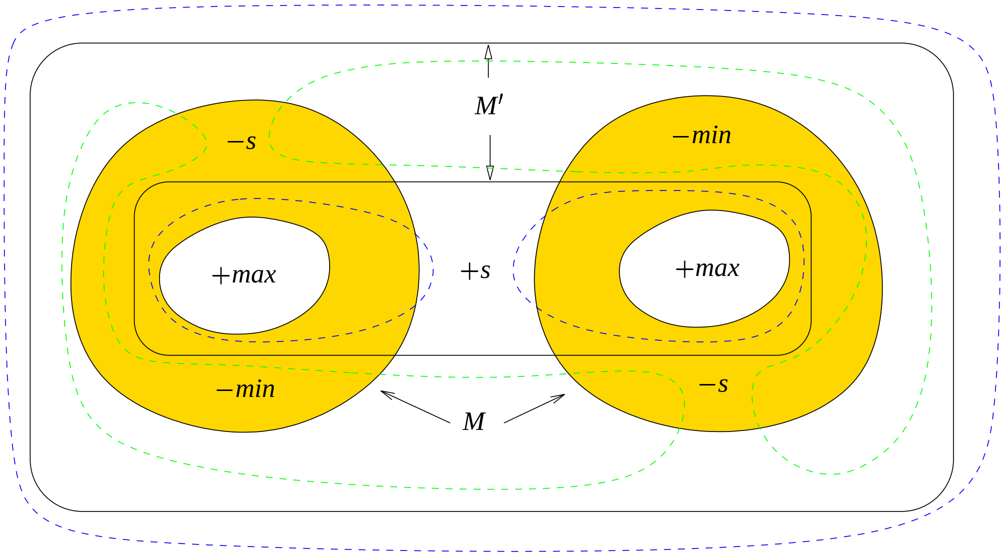

Figure 5: Critical points do not determine the topology of level-sets.

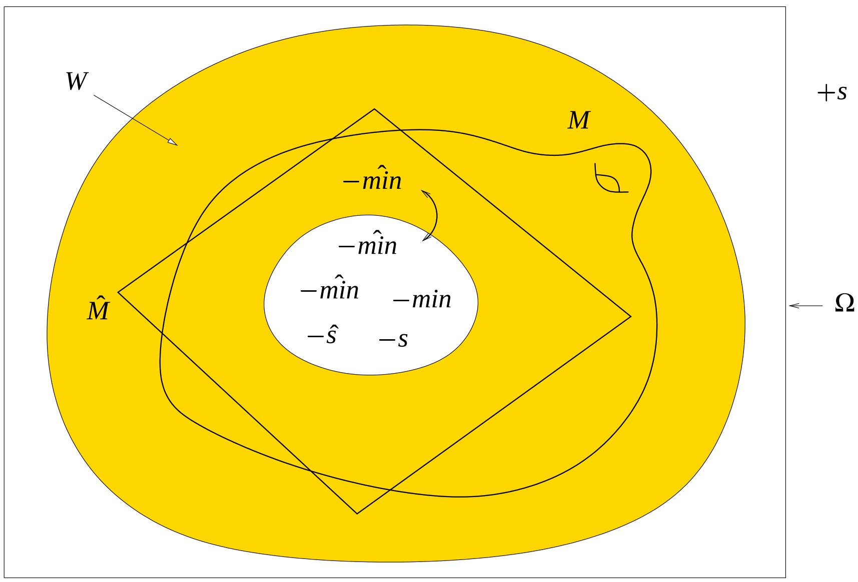

Figure 6: Condition **2'.** and **4.** are needed.

2.3 Proof of the theorem

Lemma 7 *Let S and T be two subsets of $\mathbb{R}^3$ that meet.*
Assume the boundary of S is connected, as well as T and its complement.
If the complements of S and T meet but not their boundaries, then S is contained in the interior of T or the other way around.

Proof. Let S and T be two such sets. ∂S is the disjoint union of $\partial S \cap \text{int}(T)$ and $\partial S \cap \text{int}(\text{compl}(T))$ since $\partial S \cap \partial T$ is empty. So we have a partition of ∂S in two relatively open sets. As it is connected, one has to be empty.
If $\partial S \cap \text{int}(T)$ is empty then $\partial S \subset \text{int}(\text{compl}(T))$ that is $T \cap \partial S$ is empty. As a consequence, T is included in $\text{int}(S)$ or

in $\text{int}(\text{compl}(S))$ by connectedness. Since S and T meet, we have that $T \subset \text{int}(S)$.
Now if $\partial S \cap \text{int}(\text{compl}(T))$ is empty then $\text{compl}(T)$ is contained in $\text{int}(S)$ or in $\text{int}(\text{compl}(S))$ by connectedness again. Similarly as above it has to be contained in $\text{int}(\text{compl}(S))$, which means that $S \subset T$. Thus $\text{int}(S) \subset \text{int}(T)$ so $\partial S \supset S \setminus \text{int}(T) = S \cap \partial T$. If S would meet ∂T, then ∂S and ∂T would meet, which is impossible : S is included in the interior of T. □

Note that this lemma could have been stated in an arbitrary topological space.

Lemma 8 *Let V be a component of W.*
$M \cap V$ is a connected smooth compact manifold without boundary.

Proof. Hypothesis **3** implies easily that $\hat{M} \cap V$ is a deformation retract of V. Thus V contains the interior of a simplex having positive and negative vertices. As a consequence, f vanishes on $\overline{V}$. Since f does not vanish on ∂W (**1**), M intersects V. Also, M does not meet the boundary of V (**1**), so $M \cap V$ is a smooth compact manifold without boundary.

Because $\hat{M} \cap V$ is a deformation retract of V which is connected, it is a connected closed surface. Therefore, the complement of $\hat{M} \cap V$ has exactly two components, one of which is bounded. Because V retracts by deformation on $\hat{M}$ (**3**), $\mathbb{R}^3 \setminus V$ also has exactly one bounded component which we denote by A and one unbounded component we denote by B. The complement of A, which is $B \cup V$, is connected, because B and V are connected. For the same reason, $A \cup V$ is also connected. Moreover, since the complement of $A \cup V$ is B, it is also connected. In summary, A is connected as well as its complement, and the same is true for $A \cup V$.

Call now M_i, $i = 1..n$ the connected components of $M \cap V$. For each i, let N_i be the bounded component of $\mathbb{R}^3 \setminus M_i$. $M_i = \partial N_i$ does not meet $\partial(A \cup V) \subset \partial W$ (**1**), and $A \cup V$ is connected as is its complement. So N_i is included in $A \cup V$ thanks to lemma 7. Now N_i contains at least one critical point of f. But as $N_i \subset A \cup V$, such a point has to lie in A, by **2**. So N_i meets A, but since $\partial N_i = M_i$ does not meet $\partial A \subset \overline{W}$, N_i contains A by lemma 7 again. Suppose $M \cap V$ is not connected. Then N_1 and N_2 both contain A so they intersect. Because M is smooth, their boundaries do not intersect. So one has w.l.o.g. $N_1 \subset N_2$. Now f vanishes on $\partial(N_2 \setminus N_1) = \partial N_1 \cup \partial N_2$, and therefore has an extremum in $N_1 \setminus N_2$, which is impossible because $N_1 \setminus N_2 \subset V$. □

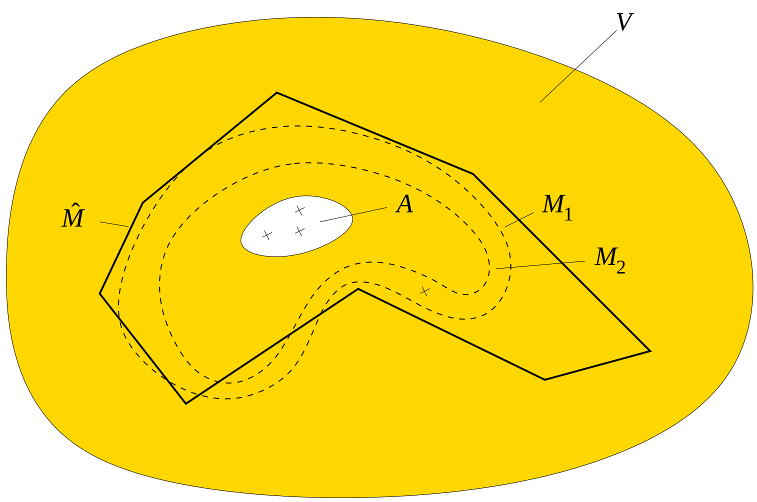

Figure 7: Proof of lemma 8.

So $M \cap V$ and $\hat{M} \cap V$ are connected compact surfaces without boundary. As seen in the preceding proof, A contains all critical points of f enclosed by $M \cap V$, with the same notations. Also, A contains all critical points of $\hat{f}$ enclosed by $\hat{M} \cap V$ by **2'**. From condition **4.**, we deduce that the volumes enclosed by $M \cap V$ and by $\hat{M} \cap V$ have the same Euler characteristic, since the Euler characteristic of a lower level set is the index of the considered function on that lower level set (theorem 1). So $M \cap V$ and $\hat{M} \cap V$ have the same genus and are thus homeomorphic. To complete the proof that M and $\hat{M}$ are homeomorphic, it remains to check that :

Lemma 9 *M is included in W.*

Proof. Let D be some component of $\Omega \setminus W$. We claim that $M \cap D$ is empty. First $\hat{M} \cap D$ is empty by **3** so w.l.o.g vertices lying in the closure of D are all positive. If $M \cap D$ is not empty then some component E of $f^{-1}(]-\infty, 0])$ meets D. Moreover, by condition **1**, ∂D does not meet E. Indeed, f is positive at vertices of ∂D and does not vanish on $\partial D \subset \partial W \cup \partial\Omega$. So E, being connected, is included in the interior of D. But then E is compact and thus f reaches its minimum on E : E contains a (negative) critical point of f. This is impossible since the tetrahedron containing this critical point would have negative vertices by condition **0**, though being included in D. □

Now that we know that M and $\hat{M}$ are homeomorphic, the fact that they are isotopic is a consequence of proposition 10, which is proved in [4].

Proposition 10 *Let $\hat{S}$ be a orientable compact connected surface without boundary and let S be a surface such that*

- *$\hat{S}$ is homeomorphic to S,*
- *S is embedded in $V = \hat{S} \times [0,1]$,*
- *$S \cap (\hat{S} \times \{0\}) = \emptyset$ and $S \cap (\hat{S} \times \{1\}) = \emptyset$,*
- *$V \setminus S$ has two connected components, one containing $\hat{S} \times \{0\}$ and the other one containing $\hat{S} \times \{1\}$.*

Then S is isotopic to $\hat{S}$ in V.

To prove theorem 6, one applies proposition 10 taking for S a component of M and for $\hat{S}$ a smooth isotopic approximation of the corresponding component of $\hat{M}$ (recall we work with smooth objects throughout this section). Both surfaces are embedded in a component of interior of W, which we identify with $\hat{S} \times]0,1[$[8]. Under this identification, $\hat{S}$ and S can be regarded as embedded in $\hat{S} \times [0,1] = V$.

The proof of the bound on the Hausdorff distance between M and $\hat{M}$ is not difficult. Pick any point p in $\hat{M}$ and let V be the component of W containing it. Assume w.l.o.g. that $f(p) > 0$ and let p' be the closest point of p on the component of ∂V where f is negative. The line segment pp' meets M at a point q. The distance between p and q is smaller than the distance between p and p' which is smaller than the Hausdorff distance between the two components of ∂V. This shows one half of the bound. The other half can be proved in a similar way.

3 Algorithm

In the algorithm, we take as V a set that is related to the notion of watershed. This set satisfies properties **2.** and **3.** by construction. In section 3.1, we give its definition, basic properties, and construction algorithms. Section 3.2 describes the meshing algorithm itself, which ensures that V fulfills also conditions **0.**, **1.**, **2'.**, and **4.**, and proves its correctness.

3.1 PL watersheds

We first assume that the mesh T conforms to $\hat{M}$, i.e. $\hat{M}$ is contained in a union of triangles of T. We will see later how to alleviate this assumption. Strictly speaking, this is in contradiction with the genericity assumptions. However, this does not cause any problem for our purpose. Define W^+ as the result of the following procedure :

Positive Watershed Algorithm

set $W^+ = \hat{M}$.
mark all vertices of $\hat{M}$.
while there is a positive unmarked vertex v s.t. the vertices of $Lk^-(v)$ are :
  -regular for $\hat{f}$
  -marked
do

[8] We omit the proof that this can be done.

set $W^+ = W^+ \cup St^-(v)$.
if v is critical for $\hat{f}$, set $W^+ = W^+ \setminus \{v\}$.
mark v.
end while
return W^+

W^- is defined as the result of the same algorithm applied to $-f$. We set $W = W^+ \cup W^-$. Note that W contains no critical point of $\hat{f}$. Also, positive marked vertices are exactly the vertices of $\overline{W^+}$. Critical positive marked vertices are exactly the vertices of $\overline{W^+} \setminus W^+$.

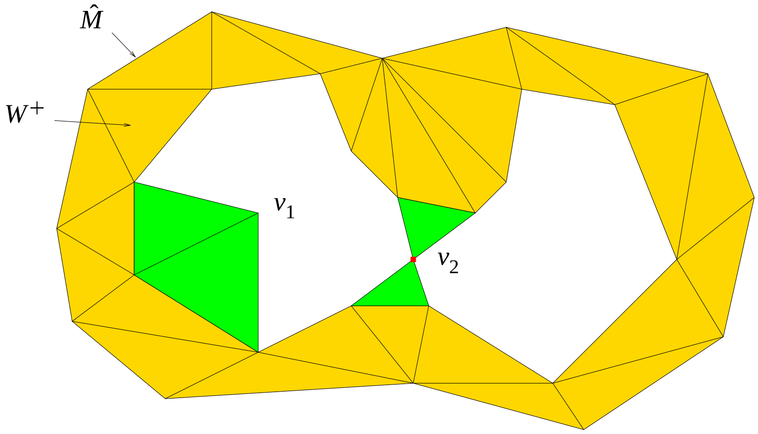

Figure 8: Construction of W^+ : lower stars are added one by one, their apex being removed if it is critical (v_2, in red).

Lemma 11 *$\hat{M}$ is a deformation retract of W.*

Proof. It is sufficient to show the result for W^+. Let W_i^+ be the state of W^+ after i steps of the algorithm, and let v_i be the i-th marked vertex. As $W_0^+ = \hat{M}$, the only thing we have to show is that W_i^+ is a deformation retract of W_{i+1}^+ for all i. Let us first show that $Lk^-(v_i)$ is included in W_i^+. If it is not the case, let u be the largest vertex of some simplex s in $Lk^-(v_i) \setminus W_i^+$. s is in $St^-(u)$ which is hence not included in W_i^+. So u is either critical or not marked yet, which is a contradiction since v_i is marked. So $Lk^-(v_i) \subset W_i^+$. Now if v_i is regular, $Lk^-(v_i)$ is contractible so $St^-(v_i)$ retracts by deformation on $Lk^-(v_i)$: W_{i+1}^+ retracts by deformation on W_i^+. If v_i is critical, W_{i+1}^+ also retracts by deformation on W_i^+. Indeed, $St^-(v_i) \setminus \{v_i\}$ retracts by deformation on $Lk^-(v_i)$ because cones minus their apex always retract by deformation on their base. This concludes the proof. □

One may prefer a more intrinsic definition of W^+. In the same spirit as in [8], one can define a partial order on the vertices of T by the closure of the acyclic relation $\prec$ defined by $u \prec v$ if $u \in St^-(v)$. We will note this order $\prec$ again and say that v *flows* into u whenever $u \prec v$. The next lemma shows that the vertices of W^+ do not depend on the order the vertices are considered in the construction.

Lemma 12 *The vertices of W^+ are exactly the positive vertices that do not flow in any positive critical point.*

Proof. The vertices of W^+ have this property by construction. Let $p \notin W^+$ be a positive vertex and assume p does not flow in any positive critical point. In particular, p is regular. Hence, as $p \notin W^+$, the lower link of p, which is not empty, has to contain either a critical vertex or an unmarked one. It cannot contain a critical point because as T conforms to $\hat{M}$, vertices in $Lk^-(p)$ are all non-negative, and so p would flow into a positive critical point. There is thus an unmarked vertex p_1 in $Lk^-(p)$. If p_1 can be chosen positive, then p_1 satisfies the same assumptions as p so one can define p_2 in a similar way. By going on, one obtains a strictly decreasing sequence of positive vertices, that thus has to end. Let p_k its last term. $Lk^-(p_k)$ contains no positive unmarked vertices. But as T conforms to $\hat{M}$, vertices in $Lk^-(p_k)$ are all non-negative. As vertices of $\hat{M}$ are marked, we get a contradiction. □

Note that W is the union of simplices with all their vertices in W, minus the critical points of $\hat{f}$. As a result, we get an intrinsic definition of W, and not only of its vertices. From an algorithmic point of view, it may be efficient to examine the vertices in increasing order in the construction of W^+. One can for instance maintain the ordered list of vertices neighboring W, always consider the first element of this list for marking, and discard it if it cannot be marked. Indeed, with this strategy, a vertex that cannot be marked at some point will never be marked.
Another consequence of lemma 12, which will be useful later, goes as follows. Call c the minimum of $|\hat{f}(v)| = |f(v)|$ over all critical points v of $\hat{f}$.

Lemma 13 *W contains all vertices whose image by $|f|$ is smaller than c.*

Proof. Let p be such that $|f(p)| < c$. Without loss of generality, assume that p is positive. Any critical point v in which p flows satisfies $f(v) < f(p)$. So it cannot be positive by definition of c : by lemma 12, p lies in W^+. □

Non conforming case. We now drop the assumption that T conforms to $\hat{M}$ and assume genericity again. From T and $\hat{M}$ one can build a mesh S that is finer than T, conforms to $\hat{M}$, and has all its extra vertices on $\hat{M}$. Indeed, it suffices to triangulate the overlay of $\hat{M}$ and T without adding extra vertices except those of $\hat{M} \cap T$. This can be done as the cells of the overlay are convex. The construction of W described above can then be applied to S. A positive vertex of T has its lower link in S containing only vertices of $\hat{M}$ if and only if its lower link in T contains only negative vertices. Thus, in order to find the say positive vertices of $W \cap T$, one can apply the positive watershed algorithm described above to T, if at the initialization step one marks all negative vertices having a positive neighbor instead of those of $\hat{M}$. Still, note that if a negative critical point has a positive neighbor, then this neighbor will not be marked by this modified algorithm, whereas it could have been marked by the standard algorithm applied to S. However, if we assume that vertices having a neighbor of opposite are regular (hyp. **c**), then this does not happen and the result W' of the modified is equal to W.

Updating W'. The intrinsic definition of W -or W'- given above yields an efficient way of updating W when T undergoes local transformations. It is sufficient to describe the algorithm for updating the vertices of W^+. Let T_1 be a mesh obtained from T by removing some set of tetrahedra E and remeshing E. Call A the set of positive critical points of the linear interpolation of f on T_1 that lie in E. Then the vertex set of the positive watershed W_1^+ associated with T_1 can be computed from the vertex set of W^+ by performing the following two operations. To begin with, the set of vertices of T_1 that flow in A must be removed from W^+ (lemma 12), which amounts to a graph traversal. Remaining vertices all belong to W_1^+. Then, mark these vertices and apply the positive watershed algorithm loop to get the vertex set of W_1^+.

Remark. The presented definition of a watershed seems quite well-behaved and leads to an easy construction algorithm, but it is not fully satisfactory. In particular, the watershed we compute is in general strictly included in the 'true watershed'. The 'true watershed' seems hard to compute, though, and can intersect a triangle in a very complicated way. There might be interesting intermediate definitions between ours and the true one, for instance based on the PL analog of the Morse complex introduced in [7].

3.2 Main algorithm

Assume the critical points of f are given. Theorem 6 enables us to build a mesh isotopic to M using only one simple predicate, *vanish*. *vanish* takes a triangle or a box and returns true if f vanishes on that triangle or that box. We actually not even need a predicate, but rather a filter. More precisely, *vanish* may return true even if f does not vanish on the considered element, but not the other way around. Still, we require that *vanish* returns the correct answer for sufficiently small elements.

Our algorithm also requires to build a refinable triangulation of space such that $\hat{f}$ (resp. $\nabla\hat{f}$) converges to f (resp. ∇f) when the size of elements tends to 0. As noticed by Shewchuk [18], this is guaranteed provided all tetrahedra have dihedral and planar angles bounded away from π. In [3], Bern, Eppstein and Gilbert described an octree-based algorithm yielding meshes whose angles are bounded away from 0. In our case, which is much easier, the desired triangulation can simply be obtained by adding a vertex at the center of each square and each cube of the octree, triangulating the squares radially from their center, and doing the same with the cubes. Indeed, resulting planar and dihedral angles are all bounded away from 180°. One can expect that this scheme does not produce too many elements upon refinement, because the size of elements is allowed to change rapidly as we do not require that these have a bounded aspect ratio (see figure 9). The main algorithm uses an octree O, the associated triangulation T, the watershed

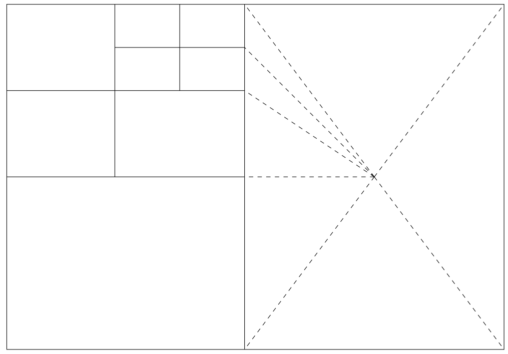

Figure 9: Octree and triangulation used in the algorithm. In this 2D example, only the edges of the triangulation of the box on the right are shown (dashed).

W'. We will say that two (closed) boxes of O are neighbors if they intersect. O is initialized to a bounding box Ω of M. Such a bounding box can be found by computing the critical points of the coordinate functions restricted to M. Besides, we maintain four sets of boxes ordered by decreasing size. *Critical* contains all boxes containing a critical point of $\hat{f}$ that is not in a box containing a critical point of f. *Index* contains all boxes neighboring a box b containing a critical point of f and such that f and $\hat{f}$ have different indices on b. *Boundary1* contains all boxes containing two neighbors -in T- of opposite signs one of which is critical for $\hat{f}$ (hyp. **c**, see paragraph **Non conforming case**). Finally, *Boundary2* contains all boxes that contain a triangle t of $\partial W'$ such that $vanish(t)$ is true and that are not included in W'.

Main Algorithm

Initialization Refine O until $vanish(b)$ is false for all boxes containing at least one critical point of f.
compute T and W', and the four sets.
while (1) **do**
  update T, W', and the four sets.
  if *Critical*$\neq \emptyset$ **then**
    split its first element.
  else if *Boundary1*$\neq \emptyset$ **then**
    split its first element.
  else if *Boundary2*$\neq \emptyset$ **then**
    split its first element.
  else if f and $\hat{f}$ have different indices on some bounded component of W' **then**
    split the first element of *Index*.
  else
    return $\hat{M}$
  end if
end while

Thanks to theorem 6 applied to W', the correctness of this algorithm almost amounts to its termination. The only problem is that W' might contain some critical point of f, thereby violating condition **2.**. It thus seems that the definition of W' needs to be slightly modified. The modification consists in taking as W'^{+} vertices -and the same for W'^{-}- the positive vertices that do not flow into positive critical points of $\hat{f}$ nor into vertices lying in a box containing a positive critical point of f. With this modification, lemma 11 still holds and lemma 13 holds if one replaces c by the minimum c' of c and the minimum of $|f|$ on the boxes containing a critical point of f. c' is positive as f does not vanish on these boxes.

We now show that the main algorithm terminates. First note that after the initialization step, no box containing a critical point of f is split. The magnitude of ∇f is thus larger than a certain constant g_{min} on the complement C of the union of these boxes. Let us show that the size of the boxes of *Critical* that are split at some point is bounded from below. As $\nabla \hat{f}$ converges to ∇f, there is a number s_1 such that for each tetrahedron with diameter smaller than s_1, $||\nabla f - \nabla \hat{f}||$ is smaller than $g_{min}/2$ on the interior of that tetrahedron. If the tetrahedron is included in C, this implies that $\nabla \hat{f}$ and ∇f make an angle smaller than $\pi/6$.

Lemma 14 *Let $A \subset \mathbb{R}^3$ be such that ∂A is a manifold included in C and containing no vertex of T. Suppose that all boxes meeting ∂A are smaller than s_1.*
Then f and $\hat{f}$ have the same index on A.

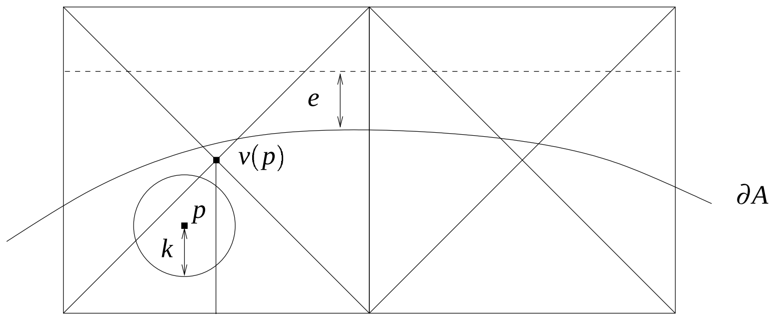

Figure 10: Proof of lemma 14.

Proof. Let $p \in \partial A$ and $d(p)$ denote the local feature size of p with respect to the 2-skeleton of T, as defined -in 2D- by Ruppert [16]. Simplices of T that meet the open ball centered at p of radius $d(p)$ all share a vertex $v(p)$ -by definition, $d(p)$ is the largest number such that this holds. We call d_{min} the minimum of d, which is known to be positive, and set k equal to the minimum of d_{min} and e, the half of the distance from ∂A to the closest box that does not meet ∂A. Let us now consider a smooth nonnegative function $\phi : \mathbb{R}^3 \to \mathbb{R}$ with support included in the open ball centered at 0 of radius k. The convolution of $\hat{f}$ and ϕ is a smooth function $\tilde{f}$. Let p be a point at distance less than e from ∂A. The gradient of $\tilde{f}$ at p is a weighted average of the gradients of $\hat{f}$ at points lying in the open ball centered at p and with radius k. All gradients involved in this average are gradients of $\hat{f}$ on tetrahedra incident on $v(p)$. Moreover, the size of these tetrahedra is smaller than s_1 because $k \leq e$. As a consequence, considered gradients all make an angle smaller than $\pi/6$ with the gradient of f at $v(p)$. As the weights in the average are nonnegative, we have that the angle between $\nabla \tilde{f}(p)$ and $\nabla f(v(p))$ is smaller than $\pi/6$. Also, the angle between $\nabla f(v(p))$ and $\nabla f(p)$ is less than $\pi/3$ since both vectors make an angle smaller than $\pi/6$ with the gradient of $\hat{f}$ on some tetrahedron containing p and $v(p)$. Finally, we get that $\nabla \tilde{f}(p)$ and $\nabla f(p)$ make a positive dot product.

Let now U_1 be a neighborhood of ∂A whose closure does not contain any vertex of T and U_2 be an open set such that $U_1 \cup U_2 = \mathbb{R}^3$. We also require that the Hausdorff distance between U_1 and ∂A is smaller than e and that $U_2 \cap \partial A = \emptyset$. Denote by $\{u_1, u_2\}$ a partition of unity subordinate to the covering $\{U_1, U_2\}$. This means that for $i = 1..2$, u_i is a non negative smooth function defined on $\mathbb{R}^3$, with support in U_i, and such that $u_1 + u_2$ is identically 1. In particular, u_2 equals 1 on the complement of U_1, and vice versa. So the function $g = u_2 \hat{f} + u_1 \tilde{f}$ coincide with $\hat{f}$ on $\mathbb{R}^3 \setminus U_1$ and with $\tilde{f}$ on $\mathbb{R}^3 \setminus U_2 \supset \partial A$. Now recall that $\nabla \tilde{f}$ and ∇f make a positive dot product on ∂A. Hence the linear homotopy between both vector fields does not vanish on ∂A : by normalization, one gets a homotopy between $\nabla \tilde{f}/||\nabla \tilde{f}||$ and $\nabla f/||\nabla f||$, considered as maps from ∂A to the unit sphere. Because the degree is invariant under homotopy, we deduce that these maps have the same degree, which shows that f and $\tilde{f}$ have the same index on A (lemma 3). Now as g and $\tilde{f}$ coincide in a neighborhood of ∂A, f and g have the same index on A by lemma 2. To complete the proof, it thus suffices to show that g and $\hat{f}$ also have the same index on A. Now the critical points of $\hat{f}$ are critical for g, with the same index, as U_1 contains no such point. Potential other critical points of g can only lie in U_1. But the gradient of g at any point p of U_1 where it is defined is a convex combination of $\nabla \tilde{f}(p)$ and $\nabla \hat{f}(p)$: it thus makes a positive dot product with $\nabla f(p)$. As a consequence, 0 is not in the convex hull of the image of a small neighborhood of p by ∇g, which implies that g has index 0 at p (lemma 4). We thus proved the announced claim. □

Suppose that some box b of *Critical* of size smaller than s_1 is split. Let v be a critical point of $\hat{f}$ included in b. All the boxes containing v are in *Critical* and their size is smaller than s_1 since we consider boxes in decreasing order.

Now the gradients of $\hat{f}$ on tetrahedra incident on v all make a positive dot product with ∇f which is a contradiction with lemma 4 which implies that v is not critical. So the conclusion is that *Critical* becomes -at least temporary- empty after a finite number of consecutive splitting of boxes in *Critical*.

Now if the algorithm splits a box b in *Boundary1*, then b contains a say positive critical point of $\hat{f}$, which belongs to a box containing a critical point of f as *Critical* is empty. So the maximum of $|f|$ on b is larger than the minimum of $|f|$ on the boxes containing a critical point of f (i.e. c'). On the other hand, f vanishes on b since b contains a negative vertex. This cannot happen if the size of b is below a certain value, so that boxes in *Boundary1* cannot be split eternally.

Suppose that the algorithm splits arbitrarily small boxes in *Boundary2*. If a small enough box b is split, then b contains a triangle t of W' on which f vanishes. So, if the size of b is small enough, the maximum of $|f|$ on b will be smaller than c'. By lemma 13, all vertices of b belong to W' so $b \subset W'$ which is a contradiction. Thus the size of split boxes in *Boundary2* is also bounded from below.

To complete the proof of termination, we need to prove that *Index* does not contain too small boxes. This is true by applying lemma 14 to small offsets of the boxes containing critical points of f. Finally :

Theorem 15 *The main algorithm returns an isotopic piecewise linear approximation of M.*

Furthermore, if one wishes to guarantee that the Hausdorff distance between M its approximation is less than say ε, it suffices to modify the positive watershed algorithm so as to control that the width of W is smaller than ε, thanks to theorem 6.

Conclusion

We have given an algorithm that approximates regular level sets of a given function with piecewise linear manifolds having the same topology. Though no implementation has been carried out yet, we believe that it should be rather efficient due to the simplicity of the involved predicates and the relative coarseness of the required space decomposition. The main drawback of our algorithm is that it requires a priori knowledge of the critical points of the considered function. A closer look shows that we almost only need to find a set of boxes containing all the critical points, and on which the function does not vanish. This task, corresponding to the initialization step in the main algorithm, can be done in a certified way using interval analysis. The only problem with this approach is that it does not give a way to compute the index of the function on these boxes, which we also need. Designing an efficient and certified method for this purpose would lead to a complete solution to the problem. Also, we plan to adapt the algorithm to the case of surfaces with boundaries, which is useful for instance when one wants to study the considered level set inside a user-specified bounding box.

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