Existence of Polynomial Solutions to Robust Convex Programming Problems

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Thème 4 — Simulation et optimisation
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Projet Sosso

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Abstract: We show in this note that, under general conditions, any convex programming problem depending continuously upon scalar parameters, and solvable for any value of the latter in a fixed compact set, admits a branch of solutions *polynomial* with respect to these parameters. This result may be useful to generate tractable approximations of robust convex programming problems with vanishing conservativeness.

Key-words: Robust Convex Programming, Polynomial Solutions

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Existence de solutions pôlynomiales pour les problèmes de programmation convexe robustes

Résumé : Nous montrons dans cette note que, sous des conditions générales, tout problème de programmation convexe dépendant continûment de paramètres scalaires et soluble pour toute valeur de ceux-ci dans un ensemble compact fixé, admet une branche de solutions *polynômiales* par rapport à ces paramètres. Ce résultat peut être utile pour générer des approximations calculables de problèmes de programmation convexe robustes, ayant un conservatisme tendant asymptotiquement vers zéro.

Mots-clés : Programmation convexe robuste, Solutions polynômiales

1 Introduction

In many practical applications of convex optimization, the data of the problem are subject to uncertainties, measurement errors, modelling approximations. To find tractable methods permitting to take into account these imperfections, is the goal of *robust convex programming* [3, 5, 8].

Previous recent works have extensively studied the basic robust convex programming problems, see [8] for a recent survey. It has been established that the robust counterpart of linear programming is equivalent to a standard convex programming problems, under usual constraints on the perturbations [3, 4, 8]. But in general, this good news does not hold any more for quadratic programming and conic quadratic programming problems [3, 7, 8], and for semidefinite programming problems [3, 8]. Indeed, except for special uncertainty structures, these robust convex programming problems are NP-hard.

In these conditions, efforts have been made to exhibit tractable approximations of the latter. For quadratic and conic quadratic programming problems [12, 7, 8] and for semidefinite programming problems [13, 5, 6, 8], such an operation is possible, and in certain cases, astute computations even permit to estimate (from above) some appropriately defined *levels of conservativeness*.

Yet, as satisfying as these progress may be, they do not offer, up to our knowledge, the possibility to decrease, and asymptotically remove, the approximation error. In an attempt to progress in this direction, we provide here a result on existence of polynomial solutions to general robust convex programming problems depending on parameters. As a matter of fact, optimal solutions of the considered problem may be seen, generally speaking, as functions of the parameters, with unprescribed regularity. Basically, the results to be exposed below permit to "replace" this untractable unknown function by new unknowns: the degree and coefficients of the polynomial. Moreover, they ensure that the conservativeness of this procedure vanishes when the degree increases. Thus, a natural next step to complete this procedure is to consider the theoretically simpler problem, obtained when assuming polynomial dependence with respect to the parameters, of the solution of the studied problem.

This idea has been applied to robust semidefinite programming. Based on a result on existence of polynomial solutions for this type of problems [9], such an approach has permitted to construct explicitly a family of standard semidefinite programming problems approximating with increasing, asymptotically perfect, precision, a given robust semidefinite programming problem [10]. The previous family is indexed by the degree of the underlying polynomial solution, and the coefficients of the latter may be deduced from the solution of the corresponding linear matrix inequality. The results given in this note are indeed extension of the work in [9] to general convex problems.

The central result presented here, Theorem 1, considers robust feasibility problem. It is afterwards applied in Corollary 2 to the estimation of the worst-case optimal value of convex programming problems depending upon parameters.

2 Main results

In all the paper, K is a compact set of $\mathbb{R}^m$, and $\mathcal{C}$ a proper cone in $\mathbb{R}^n$, in other words a closed convex solid and pointed cone. To $\mathcal{C}$ is associated as usual a partial ordering in $\mathbb{R}^n$, denoted $\leq_{\mathcal{C}}$: by definition

$$\forall \alpha, \alpha' \in \mathbb{R}^n,\ \alpha \leq_{\mathcal{C}} \alpha' \Leftrightarrow \alpha' - \alpha \in \mathcal{C}\ .$$

We also consider the strict partial ordering associated to $\mathcal{C}$:

$$\forall \alpha, \alpha' \in \mathbb{R}^n,\ \alpha <_{\mathcal{C}} \alpha' \Leftrightarrow \alpha' - \alpha \in \operatorname{int} \mathcal{C}\ .$$

Such generalized inequalities satisfy nice properties, among which the following will be especially important in the sequel:

$$\text{For any sequence } \alpha_k \leq_{\mathcal{C}} 0_n,\ \alpha_k \to \alpha_\infty \Rightarrow \alpha_\infty \leq_{\mathcal{C}} 0_n\ ,$$

and:

$$\text{For any } \alpha <_{\mathcal{C}} 0_n, \quad \text{there exists } \varepsilon > 0,\ \|\alpha'\|_n < \varepsilon \Rightarrow \alpha + \alpha' <_{\mathcal{C}} 0_n\ .$$

The first result of the present contribution is the following.

Theorem 1. *Let $G : \mathbb{R}^p \times K \to \mathbb{R}^n$ be a continuous function, $\mathcal{C}$-convex with respect to the first variable, that is:*

$$\forall x, x' \in \mathbb{R},\ \forall \delta \in K,\ \forall \lambda \in [0,1],\ G(\lambda x + (1-\lambda)x', \delta) \leq_{\mathcal{C}} \lambda G(x,\delta) + (1-\lambda) G(x', \delta)\ . \tag{1}$$

Assume that:

$$\forall \delta \in K,\ \exists x \in \mathbb{R}^p,\ G(x,\delta) <_{\mathcal{C}} 0_n\ . \tag{2}$$

Then, there exists a polynomial *function $x^* : K \to \mathbb{R}^p$ such that*

$$\forall \delta \in K, G(x^*(\delta), \delta) <_{\mathcal{C}} 0_n\ .$$

For fixed value of the parameter δ, to find $x \in \mathbb{R}^p$ such that $G(x,\delta) <_{\mathcal{C}} 0_n$, is a convex programming problem. Thus, problem (2) is a robust convex program. Theorem 1 states that, under very general assumptions, solvability of the latter for any value of the perturbation vector δ in K, is *equivalent* to existence of a solution polynomial with respect to the components of δ.

Remark that no convexity or connectedness assumption is made on the compact set K.

Proof. We first show the existence of a certain $\alpha \in \text{int } \mathcal{C}$ such that

$$\forall \delta \in K,\ \{x \in \mathbb{R}^p\ :\ G(x,\delta) \leq_{\mathcal{C}} -4\alpha\} \neq \emptyset\ . \tag{3}$$

Otherwise, for any $\alpha >_{\mathcal{C}} 0$, there exists $\delta^\alpha \in K$ such that the previous set is empty. In this case, consider δ^0 an accumulation point of the sequence δ^α, $\alpha \to 0$, and $x^0 \in \mathbb{R}^p$ such that $G(x^0, \delta^0) <_{\mathcal{C}} 0_n$. By continuity, there exist points δ^α, $\alpha >_{\mathcal{C}} 0_n$, arbitrarily close from δ^0, and $\alpha^0 >_{\mathcal{C}} 0_n$ such that, say, $G(x^0, \delta^\alpha) \leq_{\mathcal{C}} -4\alpha^0$. Thus, for such an α with $0 <_{\mathcal{C}} \alpha <_{\mathcal{C}} \alpha^0$, we have $x^0 \in \{x \in \mathbb{R}^p\ :\ G(x,\delta^\alpha) \leq_{\mathcal{C}} -4\alpha^0\} \subset \{x \in \mathbb{R}^p\ :\ G(x,\delta^\alpha) \leq_{\mathcal{C}} -4\alpha\} \neq \emptyset$, so we are led to a contradiction. This establishes the validity of (3) for a certain positive $\alpha \in \text{int } \mathcal{C}$.

Now, for the vector α previously exhibited, define

$$F\ :\ K \to 2^{\mathbb{R}^p},\ \delta \mapsto F(\delta) = \{x \in \mathbb{R}^p\ :\ G(x,\delta) \leq_{\mathcal{C}} -2\alpha\}\ . \tag{4}$$

Notice that the set-valued map F maps K into the non-void closed convex subsets of $\mathbb{R}^p$. As a matter of fact, for any $\delta \in K$, if $x_k \to x_\infty$ for a sequence $x_k \in F(\delta)$, then $G(x_k, \delta) \to G(x_\infty, \delta)$ by continuity, and $G(x_\infty, \delta) \leq_{\mathcal{C}} -2\alpha$, so $x_\infty \in F(\delta)$: the set $F(\delta)$ is thus closed. On the other hand, $\mathcal{C}$-convexity property (1) implies that, for any $\delta \in K$, any $x, x' \in F(\delta)$ and any $\lambda \in [0,1]$, $G(\lambda x + (1-\lambda)x', \delta) \leq_{\mathcal{C}} \lambda G(x,\delta) + (1-\lambda) G(x',\delta) \leq_{\mathcal{C}} -2\lambda\alpha - 2(1-\lambda)\alpha = -2\alpha$, and this establishes the convexity of the set $F(\delta)$.

At this point, let us establish that F fulfils the following property of *lower semicontinuity*, see e.g. [2].

Definition. *Let X be a topological space, Y a metric space. A set-valued map F from X to Y is said* lower semicontinuous *at $x^0 \in X$ if for any $y^0 \in F(x^0)$ and any neighborhood $N(y^0)$ of y^0, there exists a neighborhood $N(x^0)$ such that*

$$\forall x \in N(x^0),\ F(x) \cap N(y^0) \neq \emptyset\ .$$

F is said lower semicontinuous *if it is lower semicontinuous at every point $x^0 \in X$.* ■

Let $\delta^0 \in K$, $x^0 \in F(\delta^0)$, $\varepsilon > 0$. To prove lower semicontinuity of F at δ^0, we exhibit $\eta > 0$ such that for every $\delta \in K$ with $\|\delta - \delta^0\|_m < \eta$, there exists $x \in F(\delta)$, $\|x - x^0\|_p < \varepsilon$.

Indeed, by assumption, there exists $x^{\delta^0} \in \mathbb{R}^p$ such that $G(x^{\delta^0}, \delta^0) \leq_{\mathcal{C}} -4\alpha$. For $\lambda \in (0,1]$ such that

$$\lambda \leq \frac{\varepsilon}{2\|x^{\delta^0} - x^0\|_p}\ , \tag{5}$$

let $x \stackrel{\text{def}}{=} (1-\lambda)x^0 + \lambda x^{\delta^0}$. In particular, this implies $\|x - x^0\|_p = \lambda \|x^{\delta^0} - x^0\|_p \leq \varepsilon/2 < \varepsilon$.

The $\mathcal{C}$-convexity property (1) implies that, for any $\eta > 0$ and every $\delta \in K$,

$$\begin{aligned} G(x,\delta) &\leq_{\mathcal{C}} (1-\lambda)G(x^0,\delta) + \lambda G(x^{\delta^0},\delta) \\ &= (1-\lambda)G(x^0,\delta^0) + \lambda G(x^{\delta^0},\delta^0) + (1-\lambda)\left(G(x^0,\delta) - G(x^0,\delta^0)\right) + \lambda\left(G(x^{\delta^0},\delta) - G(x^{\delta^0},\delta^0)\right) \\ &\leq_{\mathcal{C}} -2(1-\lambda)\alpha - 4\lambda\alpha + (1-\lambda)\left(G(x^0,\delta) - G(x^0,\delta^0)\right) + \lambda\left(G(x^{\delta^0},\delta) - G(x^{\delta^0},\delta^0)\right) . \end{aligned}$$

For λ fulfilling (5), one has $\lambda\alpha \in \text{int } \mathcal{C}$. Thus, by continuity of G, for any $\varepsilon > 0$ and any λ in $(0,1]$ fulfilling (5), there exists $\eta > 0$ such that

$$\|\delta - \delta^0\|_m < \eta \Rightarrow G(x^0,\delta) - G(x^0,\delta^0) \leq_{\mathcal{C}} 2\lambda\alpha,\ G(x^{\delta^0},\delta) - G(x^{\delta^0},\delta^0) \leq_{\mathcal{C}} 2\lambda\alpha .$$

With this choice for η, one has $G(x,\delta) \leq_{\mathcal{C}} -2(1+\lambda)\alpha + 2\lambda\alpha = -2\alpha$ when $\|\delta - \delta^0\|_m < \eta$. Thus $x \in F(\delta)$, provided that $\delta \in K$ and $\|\delta - \delta^0\|_m < \eta$. We conclude that F is lower continuous at δ^0. This achieves the proof of the lower semicontinuity of F defined in (4).

We now apply to F Michael's Selection Theorem [14], see also [2].

Theorem (Michael's Selection Theorem). *Let X be a metric space, Y a Banach space. Let F, a set-valued map from X into the closed convex subsets of Y, be lower semicontinuous. Then there exists $f : X \to Y$, a continuous selection from F.* ■

Recall that a selection from F is any single valued map f such that, for any $x \in X$, $f(x) \in F(x)$. Application of the previous result yields existence of a continuous selection $f : K \to \mathbb{R}^p$ from F defined in (4). This function is such that

$$\forall \delta \in K,\ G(f(\delta),\delta) \leq_{\mathcal{C}} -2\alpha .$$

It remains to apply to each of the p components of f the following result, see e.g. [11].

Theorem (Weierstrass Approximation Theorem). *Every continuous real-valued function defined on a compact subset K of $\mathbb{R}^m$, is the limit of a sequence of polynomials, which converges uniformly in K.* ■

Thus, the selection f previously exhibited is uniform limit in K of a sequence of (vector-valued) polynomials in δ. In particular, there exists a polynomial function $x^* : K \to \mathbb{R}^p$ such that

$$\forall \delta \in K,\ G(x^*(\delta),\delta) \leq_{\mathcal{C}} -\alpha <_{\mathcal{C}} 0_n .$$

This achieves the proof of Theorem 1.

□

Theorem 1 is now applied to the issue of finding the worst-case optimal value of a convex objective under generalized inequality constraints.

Corollary 2. *Let $G : \mathbb{R}^p \times K \to \mathbb{R}^n$, $g : \mathbb{R}^p \times K \to \mathbb{R}$ be continuous functions, $\mathcal{C}$-convex with respect to the first variable.*

Then

$$\sup_{\delta \in K} \inf \{g(x,\delta) \ :\ x \in \mathbb{R}^p,\ G(x,\delta) <_{\mathcal{C}} 0_n\}$$
$$= \sup_{\delta \in K} \inf \{g(x^*(\delta),\delta) \ :\ x^* \textit{ polynomial},\ \forall \delta' \in K, G(x^*(\delta'),\delta') <_{\mathcal{C}} 0_n\} .$$

Proof. Let $\gamma \overset{\text{def}}{=} \sup_{\delta \in K} \inf\{g(x,\delta) \ :\ x \in \mathbb{R}^p,\ G(x,\delta) <_{\mathcal{C}} 0_n\}$, and assume that $\gamma < +\infty$ (the case $\gamma = +\infty$, which requires straightforward adaptations, is left to the reader). First, one has, for every $\delta \in K$: $\inf\{g(x^*(\delta),\delta) \ :\ x^* \text{ polynomial},\ \forall \delta' \in K, G(x^*(\delta'),\delta') <_{\mathcal{C}} 0_n\} \geq \inf\{g(x,\delta) \ :\ x \in \mathbb{R}^p,\ G(x,\delta) <_{\mathcal{C}} 0_n\}$, due to the inclusion of the first set involved in the second one. Thus,

$$\gamma \leq \sup_{\delta \in K} \inf \{g(x^*(\delta),\delta) \ :\ x^* \text{ polynomial},\ \forall \delta' \in K, G(x^*(\delta'),\delta') <_{\mathcal{C}} 0_n\} .$$

On the other hand, by definition, $\inf\{g(x,\delta) \ : \ x \in \mathbb{R}^p, \ G(x,\delta) <_{\mathcal{C}} 0_n\} \leq \gamma$ for every $\delta \in K$. Thus, for any $\varepsilon > 0$, for any $\delta \in K$, there exists $x \in \mathbb{R}^p$ such that $G(x,\delta) <_{\mathcal{C}} 0_n$ and $g(x,\delta) < \gamma + \varepsilon$. In other words, for any $\varepsilon > 0$, the following parameter-dependent LMI is feasible:

$$\forall \delta \in K, \exists x \in \mathbb{R}^p, \ \begin{pmatrix} g(x,\delta) - \gamma - \varepsilon & 0_{1\times n} \\ 0_{n\times 1} & G(x,\delta) \end{pmatrix} <_{\mathbb{R}^+ \times \mathcal{C}} 0_{n+1} \ .$$

Here, we denote by $<_{\mathbb{R}^+ \times \mathcal{C}}$ the product order relation, defined on $\mathbb{R}^{n+1}$ by: $(a,\alpha) <_{\mathbb{R}^+ \times \mathcal{C}} \Leftrightarrow a < 0$ and $\alpha <_{\mathcal{C}} 0_n$. The cone $\mathbb{R}^+ \times \mathcal{C}$ is proper in $\mathbb{R}^{n+1}$, and, by use of Theorem 1, for any $\varepsilon > 0$, there exists a *polynomial* map $x^*_\varepsilon : K \to \mathbb{R}^p$ such that

$$\forall \delta \in K, \ G(x^*_\varepsilon(\delta),\delta) <_{\mathcal{C}} 0_n \text{ and } g(x^*_\varepsilon(\delta),\delta) < \gamma + \varepsilon \ .$$

Thus, for any $\varepsilon > 0$, for every $\delta \in K$,

$$\inf\{g(x^*(\delta),\delta) \ : \ x^* \text{ polynomial}, \ \forall \delta' \in K, G(x^*(\delta'),\delta') <_{\mathcal{C}} 0_n\} \leq g(x^*_\varepsilon(\delta),\delta) < \gamma + \varepsilon \ ,$$

so, for any $\varepsilon > 0$,

$$\sup_{\delta \in K} \inf\{g(x^*(\delta),\delta) \ : \ x^* \text{ polynomial}, \ \forall \delta' \in K, G(x^*(\delta'),\delta') <_{\mathcal{C}} 0_n\} \leq \max_{\delta \in K} g(x^*_\varepsilon(\delta),\delta) < \gamma + \varepsilon \ .$$

This results finally in:

$$\sup_{\delta \in K} \inf\{g(x^*(\delta),\delta) \ : \ x^* \text{ polynomial}, \ \forall \delta' \in K, G(x^*(\delta'),\delta') <_{\mathcal{C}} 0_n\} \leq \gamma \ ,$$

whence the claimed equality. This ends the proof of Corollary 2. □

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