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Selective Acoustic Focusing Using Time-harmonic Reversal Mirrors

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Abstract: A mathematical study of the focusing properties of acoustic fields obtained by a time-reversal process is presented. The case of time-harmonic waves propagating in a non-dissipative medium containing sound-soft obstacles is considered. In this context,the so-called D.O.R.T. method ("Decomposition of the Time-Reversal Operator" in french) was recently proposed to achieve selective focusing by computing the eigenelements of the *time-reversal operator*. The present paper describes a justification of this technique in the framework of the far field model, *i.e.*, for an ideal time-reversal mirror able to reverse the far field of a scattered wave. Both cases of closed and open mirrors, that is, surrounding completely or partially the scatterers, are dealt with. Selective focusing properties are established by an asymptotic analysis for small and distant obstacles.

Key-words: acoustic scattering, time-reversal, far field operator, small obstacles

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Focalisation acoustique sélective en régime harmonique par des miroirs à retournement temporel

Résumé : Nous présontons ici une étude mathématique des propriétés de focalisation acoustique obtenues à l'aide du processus de retournement temporel. On considère le cas d'ondes harmoniques en temps se propageant dans un milieu non dissipatif. Dans ce cadre, la méthode D.O.R.T. ("'Décomposition de l'Opérateur de Retournement Temporel") a été récemment proposée pour réaliser une focalisation acoustique sélective en calculant les vecteurs propres de *l'opérateur de retournement temporel.* Cet article fournit une justification mathématique de cette méthode dans le cadre d'une approche en champ lointain, *i.e.*, pour un miroir à retournement temporel idéal capable de rétropropager le champ lointain associé au champ diffracté. Nous traitons le cas d'un miroir fermé (entourant totalement les obstacles) et celui d'un miroir ouvert (entourant partiellement les obstacles). Les propriétés de focalisation sélective sont établies pour des obstacles diffractants petits et distants, à l'aide d'une analyse asymptotique du champ diffracté.

Mots-clés : focalisation acoustique, retournement temporel, opérateur de champ lointain, obstacles de petite taille

1 Introduction

Acoustic time-reversal has known in the last few years a significant growth of interest, covering a large number of applications (medical imaging, non destructive testing,...). The main idea of this phenomenon is to take advantage of the reversibility of the wave equation in a non dissipative unknown medium to back-propagate signals to the sources that emitted them. Today, the physical literature (cf. [9] for more details) on this topic is quite rich. Meanwhile, some mathematical works started to deal with different aspects of time-reversal phenomena: see for instance [2] and [4] for time-reversal in the time domain, [13] for time-reversal in the frequency domain, and [3] for time-reversal in random media.

In this work, we present a mathematical analysis of the so-called D.O.R.T method ("Decomposition of the Time-Reversal Operator" in french), introduced in [14, 15] to achieve selective focusing on diffracting obstacles using time-reversal mirrors (TRM) which are able to emit and receive acoustic waves. In the frequency domain, this method can be described as follows: the TRM first emits an acoustic wave in a homogeneous and non-dissipative medium containing some unknown obstacles, and then measures the diffracted field. The measured field is then conjugated (reversing time amounts to a conjugation when the time dependence is of the form $e^{i\omega t}$), and reemitted. The time-reversal operator T is the operator obtained by iterating this procedure twice. The experimental results obtained in [15] show that the number of non-zero (or significant) eigenvalues of T is exactly the number of obstacles contained in the propagation medium. Furthermore, the corresponding eigenvectors generate incident waves that focus selectively on the obstacles. Our aim here is to present a mathematical justification of these results related to selective focusing using time-reversal mirrors: we will show that these results are not true in general, but do hold for small and distant obstacles.

The paper is organized as follows. We first deal with a TRM which surrounds entirely the obstacles. In Section 2, we describe the mathematical model used to analyze time-reversal phenomena in the framework of time harmonic waves in the far field model, *i.e.*, for an ideal TRM able to reverse the asymptotic behavior at large distance of a scattered wave. This will in particular lead us to express the time-reversal operator by means of the far field operator, well-known in scattering theory. Section 3 recalls some results obtained in [13], concerning the global focusing properties of the eigenvectors of the time-reversal operator. The main result of the paper, which concerns selective focusing, is given in Section 4. It provides a mathematical justification of the D.O.R.T. method for the problem of scattering by several small and distant obstacles. In Section 5, we generalize the results obtained in the previous sections to the case of open mirrors (*i.e.*, mirrors which do not surround completely the scatterers). The main ingredient for the proof of our main result is formula (15), which provides the asymptotic behavior of the scattering amplitude for the diffraction by many small obstacles. This formula, which is of independent interest, is proved in the Appendix.

2 Mathematical setting of the problem and definition of the time-reversal operator

Consider a time-reversal mirror (TRM) surrounding completely a collection of sound-soft obstacles, located in a homogeneous medium of celerity c. During the emission step, the TRM illuminates the obstacles with an incident wave u_I which is supposed to be a Herglotz wave. Such waves are superpositions of planes waves $u_I^\alpha(x) = \exp(ik\alpha\cdot x)$ of direction $\alpha \in S^2$ (S^2 denotes the unit sphere in $\mathbb{R}^3$, $k = \omega/c$ is the wavenumber and ω is the frequency). More precisely, given a directional distribution $f \in L^2(S^2)$, we suppose that the incident field emitted by the TRM has the form

$$u_I(x) = \int_{S^2} f(\alpha)\, u_I^\alpha(x)\, d\alpha = \int_{S^2} f(\alpha)\, e^{ik\alpha\cdot x}\, d\alpha. \tag{1}$$

We assume that the TRM is located far enough from the obstacles, so that its influence on the diffracted field can be neglected. Morever, the TRM is supposed to measure the far field corresponding to the diffracted field.

Let Ω denote the propagation domain located outside the obstacles and let ν be the outgoing normal to Ω on its boundary $\Gamma = \partial\Omega$. When illuminated by the incident plane wave $u_I^\alpha(x) = e^{ik\alpha\cdot x}$ of direction $\alpha \in S^2$, the obstacles generate the diffracted field u_D^α that solves the classical Dirichlet exterior problem:

$$\left\{ \begin{array}{rcll} \Delta u_D^\alpha + k^2 u_D^\alpha & = & 0 & (\Omega) \\ u_D^\alpha & = & -u_I^\alpha & (\Gamma) \\ \displaystyle\lim_{R\longrightarrow+\infty} \int_{S_R} \left| \frac{\partial u_D^\alpha}{\partial\nu} - iku_D^\alpha \right|^2 dx & = & 0 & \end{array} \right.$$

where S_R is the sphere $\{x \in \mathbb{R}^3;\ |x| = R\}$ and where $\partial u_D^\alpha/\partial\nu$ denotes the radial derivative of u_D^α on S_R.

It is well-known (cf. [7]) that the far field asymptotics of the diffracted field in a given direction $\beta \in S^2$ is given by the formula

$$u_D^\alpha(\beta|x|) = \frac{e^{ik|x|}}{|x|} A(\alpha,\beta) + O(|x|^{-2})$$

where the bound $O(|x|^{-2})$ is uniform for all $\beta \in S^2$, and where $A(\alpha,\beta)$ is known as the *scattering amplitude*. This function satisfies some remarkable properties (cf. [7]), which are summarized in

Proposition 1 *The scattering amplitude $A(\cdot,\cdot)$ is given by the formula*

$$A(\alpha,\beta) = \frac{1}{4\pi} \int_\Gamma \frac{\partial u_T^\alpha}{\partial\nu}(y)\, \overline{u_I^\beta(y)}\, d\Gamma_y \tag{2}$$

where $u_T^\alpha = u_I^\alpha + u_D^\alpha$ *denotes the total field associated with the incident field* u_I^α. *Furthermore,* $A(\cdot,\cdot)$ *defines an analytic function on* $S^2 \times S^2$, *and satisfies the reciprocity relation*

$$A(\alpha,\beta) = A(-\beta,-\alpha). \tag{3}$$

Remark 1 This reciprocity relation simply states that the behavior of the diffracted field observed in the direction β when the scatterers are illuminated by a plane wave of direction α, is identical to its behavior in the direction $-\alpha$ under an incident plane wave with direction $-\beta$. This property is a direct consequence of the symmetry of the Green function of the diffraction problem (which follows itself from the self-adjointness of the Dirichlet Laplacian).

Note that in (2), the integral actually represents the duality product between $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$ since $\partial u_T^\alpha/\partial\nu$ belongs to the latter in general. We keep this simplified notation in the sequel.

By linearity, it follows from the results above that when illuminated by the Herglotz wave (1) associated with a given directional distribution $f \in L^2(S^2)$, the scattering obstacles generate the diffracted field u_D

$$u_D(x) = \int_{S^2} f(\alpha)\, u_D^\alpha(x)\, d\alpha.$$

Furthermore, the asymptotic behavior of u_D is given by the formula

$$u_D(\beta|x|) = \frac{e^{ik|x|}}{|x|} Ff(\beta) + O(|x|^{-2})$$

where the far field $Ff(\beta)$ in the direction $\beta \in S^2$ is simply given by the relation

$$Ff(\beta) = \int_{S^2} A(\alpha,\beta)\, f(\alpha)\, d\alpha. \tag{4}$$

The integral operator $F : L^2(S^2) \longrightarrow L^2(S^2)$ with kernel $A(\cdot,\cdot)$ is known in the literature as the *far field operator*. Its properties are given in

Proposition 2 *The far field operator* $F : L^2(S^2) \longrightarrow L^2(S^2)$ *defined by equation* (4) *is a compact and normal operator. Its adjoint is the operator* $F^* : L^2(S^2) \longrightarrow L^2(S^2)$ *defined by*

$$F^* f = \overline{RF\overline{Rf}}, \qquad \forall f \in L^2(S^2) \tag{5}$$

where R *is the symmetry operator defined by :* $Rf(\alpha) = f(-\alpha), \forall \alpha \in S^2$.

Proof. The compactness of the integral operator F follows immediately from the analyticity of its kernel $A(\cdot,\cdot)$. The fact that F is a normal operator is a well-known result, which is proved for instance in [5] (see Corollary 2.5.). The adjoint F^* of F is the integral operator with kernel

$$A^*(\alpha,\beta) = \overline{A(\beta,\alpha)} = \overline{A(-\alpha,-\beta)},$$

where we have used the reciprocity relation (3). Formula (5) follows. ∎

Remark 2 In fact, in [5], it is proved more precisely that

$$FF^* = F^*F = \frac{2\pi}{ik}(F - F^*). \tag{6}$$

Since the far field operator F is related to the scattering matrix by the relation $S = I + (ik/2\pi)F$, formula (6) can be seen as an equivalent formulation of the fact that the scattering operator S is unitary, which is classical result in scattering theory (cf. [12]).

We are now able to give a rigorous definition of the time-reversal operator. During the time-reversal process, when a Herglotz wave associated with a density $f \in L^2(S^2)$ is emitted by the TRM, the far field corresponding to the diffracted field is measured, conjugated and then reemitted by the TRM. The new emission is characterized by the Herglotz wave associated with the density $g \in L^2(S^2)$ defined by

$$g = \overline{RFf}.$$

In this relation, the presence of the symmetry operator R is due to the fact that during the time-reversal process, the far field measured in a given direction $\beta \in S^2$ is used to define the new incident plane wave in the direction $-\beta$. The time-reversal operator T is then obtained by iterating this scheme once again, and thus, we have

$$Tf = \overline{RFg} = \overline{RF\overline{RFf}}.$$

Thanks to (5) and using the fact that F is a normal operator, we finally get

Proposition 3 *The time-reversal operator $T : L^2(S^2) \longrightarrow L^2(S^2)$ is given by*

$$T = F^*F = FF^*. \tag{7}$$

It is the integral operator with kernel

$$t(\alpha, \beta) = \frac{1}{4\pi} \int_{\Gamma\times\Gamma} j_0(k|y-z|) \, \frac{\partial u_T^\alpha}{\partial\nu}(y) \, \overline{\frac{\partial u_T^\beta}{\partial\nu}(z)} \; d\Gamma_y \, d\Gamma_z, \tag{8}$$

where $u_T^\alpha = u_I^\alpha + u_D^\alpha$ denotes the total field associated with the incident field u_I^α, and $j_0(\xi) = \sin(\xi)/\xi$ is the spherical Bessel function of order 0.

Proof. Since (7) has been already proved, we only have to show the second part of the proposition. From (7), it follows that T is the integral operator with kernel

$$t(\alpha, \beta) = \int_{S^2} A(\alpha, \gamma) \, \overline{A(\beta, \gamma)} \, d\gamma. \tag{9}$$

Substituting expression (2) of the scattering amplitude in the above relation sand inverting the integrals over S^2 with the integrals over Γ, we find that :

$$t(\alpha, \beta) = \frac{1}{(4\pi)^2} \int_{\Gamma\times\Gamma} \left(\int_{S^2} \overline{u_I^\gamma(y)} \, u_I^\gamma(z) \, d\gamma \right) \frac{\partial u_T^\alpha}{\partial\nu}(y) \, \overline{\frac{\partial u_T^\beta}{\partial\nu}(z)} \; d\Gamma_y \, d\Gamma_z.$$

Equation (8) follows then from the remarkable identity (cf. [1, p.155]):

$$\int_{S^2} \overline{u_I^\gamma(y)}\, u_I^\gamma(z)\, d\gamma = \int_{S^2} e^{ik\gamma\cdot(z-y)}\, d\gamma = 4\pi j_0(k|y-z|). \tag{10}$$

■

3 Global focusing

The time-reversal operator $T = F^*F : L^2(S^2) \longrightarrow L^2(S^2)$ is clearly a positive and self-adjoint operator. Moreover, by Proposition 2, it is also a compact operator. Besides the value 0, its spectrum is thus constituted of a finite or countable sequence of positive eigenvalues admitting 0 for only possible accumulation point. In this section, we will see how these eigenvectors can be used to generate incident waves that focus acoustic on the diffracting obstacles. These global focusing results have been obtained for the first time in [13]. First, we recall a classical result from linear operators theory (see for instance [19, p.442]):

Proposition 4 *Let N be a compact and normal on a Hilbert space H. If $\lambda_1, \lambda_2, \ldots$ is the sequence of all nonzero eigenvalues of N, arranged such that $|\lambda_1| \geq |\lambda_2| \geq \ldots$ and if $\varphi_1, \varphi_2, \ldots$ is a corresponding orthonormal sequence of eigenvectors, then $|\lambda_1|^2 \geq |\lambda_2|^2 \geq \ldots$ is the sequence of all nonzero eigenvalues of $N^*N = NN^*$, and $\varphi_1, \varphi_2, \ldots$ is a corresponding orthonormal sequence of eigenvectors.*

This result shows that the nonzero eigenvalues of the time reversal operator $T = F^*F = FF^*$ are exactly the positive numbers $|\lambda_1|^2 \geq |\lambda_2|^2 \geq \ldots$, where the complex numbers $(\lambda_p)_{p\geq 1}$ denote the nonzero eigenvalues of the normal compact far field operator F. Furthermore, the corresponding eigenvectors $(f_p)_{p\geq 1}$ of F are exactly the eigenvectors of $T = F^*F$. Consequently, it suffices to analyze the focusing properties of the eigenvectors of the far field F to obtain the same results for the time reversal operator T.

Let us first deal with the largest eigenvalue of the far field operator. Then, we have

Proposition 5 *Let λ_1 be the largest eigenvalue (in modulus) of F, and let $f_1 \in L^2(S^2)$ be an eigenvector of F associated with λ_1. Then,*

$$\sup_{f\in L^2(S^2),\, f\neq 0} \frac{\|Ff\|^2_{L^2(S^2)}}{\|f\|^2_{L^2(S^2)}} = \frac{\|Ff_1\|^2_{L^2(S^2)}}{\|f_1\|^2_{L^2(S^2)}} = |\lambda_1|^2.$$

In other words, the incident Herglotz wave $u_I^1(x) = \int_{S^2} f_1(\alpha)\, e^{ik\alpha\cdot x}\, d\alpha$ is, among all the possible Herglotz waves, the one that maximizes the energy scattered by the obstacles.

Proof. The Proposition is a straightforward consequence of the Min-Max principle. Indeed, applying this principle to the positive selfadjoint and bounded operator $T = F^*F$, we can write that the largest eigenvalue $|\lambda_1|^2$ of T satisfies

$$|\lambda_1|^2 = \sup_{f\in L^2(S^2),\, f\neq 0} \frac{(Tf, f)_{L^2(S^2)}}{\|f\|^2_{L^2(S^2)}} = \sup_{f\in L^2(S^2),\, f\neq 0} \frac{\|Ff\|^2_{L^2(S^2)}}{\|f\|^2_{L^2(S^2)}}.$$

■

Roughly speaking, this result says that the "best" way to illuminate a family of obstacles with Herglotz waves is to use a Herglotz wave u_I^1 corresponding to an eigenvector f_1 of F (or T) associated with its largest eigenvalue λ_1. The physical reason explaining this property is that the incident field generated by an eigenvector f_p associated with any eigenvalue $\lambda_p \neq 0$ of F, focuses on the obstacles. More precisely, the following result holds true.

Proposition 6 *Let $\lambda_p \neq 0$ be an eigenvalue of F and $f_p \in L^2(S^2)$, $f_p \neq 0$, an eigenvector of F associated with λ_p. Then, the Herglotz wave $u_{I,p}$ associated with f_p and defined by $u_{I,p}(x) = \int_{S^2} f_p(\alpha)\, u_I^\alpha(x)\, d\alpha = \int_{S^2} f_p(\alpha)\, e^{ik\alpha\cdot x}\, d\alpha$, has the following form*

$$u_{I,p}(x) = \frac{1}{\lambda_p} \int_\Gamma j_0(k\|x-y\|)\, \frac{\partial u_{T,p}}{\partial \nu}(y)\, d\Gamma_y, \tag{11}$$

where $u_{T,p} = u_{I,p} + u_{D,p}$ denotes the total field associated with the incident field $u_{I,p}$.

Proof. Since $f_p(\beta) = \lambda_p^{-1} F f_p(\beta) = \lambda_p^{-1} \int_{S^2} A(\alpha,\beta)\, f_p(\alpha)\, d\alpha$, we obtain by using expression (2) of $A(\alpha,\beta)$ that

$$\begin{aligned} f_p(\beta) &= (4\pi\lambda_p)^{-1} \int_{S^2}\int_\Gamma \frac{\partial u_T^\alpha}{\partial \nu}\, \overline{u_I^\beta}\, d\Gamma\; f_p(\alpha)\, d\alpha \\ &= (4\pi\lambda_p)^{-1} \int_\Gamma\int_{S^2} \frac{\partial u_T^\alpha}{\partial \nu}\, f_p(\alpha)\, d\alpha\; \overline{u_I^\beta}\, d\Gamma. \end{aligned}$$

But by superposition, the integral $\int_{S^2} \partial u_T^\alpha / \partial\nu\, f_p(\alpha)\, d\alpha$ is nothing but the normal derivative of the total field $u_{T,p}$ associated with the incident field $u_{I,p}$, and thus:

$$f_p(\beta) = (4\pi\lambda_p)^{-1} \int_\Gamma \frac{\partial u_{T,p}}{\partial \nu}\, \overline{u_I^\beta}\, d\Gamma. \tag{12}$$

We can now obtain the expression of the incident field generated by the eigenvector f_p. From (12), we have:

$$\begin{aligned} u_{I,p}(x) &= \int_{S^2} f_p(\beta)\, u_I^\beta(x)\, d\beta \\ &= (4\pi\lambda_p)^{-1} \int_{S^2}\int_\Gamma \frac{\partial u_{T,p}}{\partial \nu}\, \overline{u_I^\beta}\, d\Gamma\; u_I^\beta(x)\, d\beta \\ &= (4\pi\lambda_p)^{-1} \int_\Gamma \left(\int_{S^2} u_I^\beta(x)\, \overline{u_I^\beta(y)}\, d\beta \right) \frac{\partial u_{T,p}}{\partial \nu}(y)\, d\Gamma_y. \end{aligned}$$

Formula (11) follows then from identity (10). ■

Since $j_0(\xi) = \sin(\xi)/\xi$, formula (11) shows that as expected, the incident field $u_{I,p}(x)$ generated by an eigenvector f_p of F (or T) decreases like $r(x)^{-1}$, if $r(x)$ denotes the distance of x to the obstacles. In this sense, one can say that $u_{I,p}$ focuses on the obstacles located in the propagation medium. Furthermore, the quality of this focusing (given by the amplitude of the far field) is exactly given by the magnitude of the eigenvalue λ_p, since $|\lambda_p| = \dfrac{\|Ff_p\|_{L^2(S^2)}}{\|f_p\|_{L^2(S^2)}}$.

4 Selective focusing

The aim of this section is to propose a mathematical justification of the so-called D.O.R.T. method presented in [15] and briefly described in the introduction of this paper. Roughly speaking, we answer the two following questions:

- Is the number of obstacles contained in a homogeneous medium equal to the number of "significant" eigenvalues of the far field operator F (or, equivalently, to those of the time-reversal operator $T = F^*F = FF^*$)?
- If so, do the associated eigenvectors selectively focus on the obstacles?

As it can be seen from the numerical experiments presented in [6], the answer to the first question is in general negative (there can be several "significant" eigenvalues even when there is just one scatterer). We will confirm this result by studying in subsection 4.1 the special case of a single spherical obstacle. Nevertheless, we will show that the answer becomes positive provided the obstacles considered are small enough. Under this assumption, we show in subsection 4.2 that selective focusing can be achieved using the eigenvectors of the far field operator.

4.1 Diffraction by a single spherical obstacle

In this subsection, we deal with the case where the scatterer is a sphere of radius $a > 0$. For this particular geometry, an explicit formula can be obtained for the eigenvalues of the far field mapping, and thus for those of the time-reversal operator. Most of the results of this subsection are classical, and can be found for instance in [7].

The Jacobi-Anger formula (cf. [7, p.31]) shows that for $x = \beta|x|$, the plane wave $u_I^\alpha(x)$ has the following expansion

$$u_I^\alpha(x) = \sum_{n=0}^{+\infty} i^n(2n+1)j_n(k|x|)P_n(\cos\theta),$$

where θ denotes the angle between α and β, and where j_n and P_n are respectively the spherical Bessel function and the Legendre polynomial of order n. Furthemore, the convergence of this sum is uniform on bounded subsets of $\mathbb{R}^3$. The corresponding diffracted field for $|x| > a$ reads then

$$u_D^\alpha(x) = -\sum_{n=0}^{+\infty} i^n(2n+1)\frac{j_n(ka)}{h_n^1(ka)}h_n^1(k|x|)P_n(\cos\theta),$$

where h_n^1 denotes the spherical Hankel function of the first kind of order n, and where the convergence is once again uniform on bounded subsets of the propagation domain $\{x \in \mathbb{R}^3;\, |x| > a\}$. One can then obtain the following expression of the scattering amplitude (cf. [7, p.52]), where the sum converges uniformly on $S^2 \times S^2$:

$$A(\alpha,\beta) = \frac{i}{k}\sum_{n=0}^{+\infty}(2n+1)\frac{j_n(ka)}{h_n^1(ka)}P_n(\cos\theta).$$

On the other hand, P_n can be written in terms of the spherical harmonics $(Y_n^m)_{|m|\le n}$ thanks to the addition formula

$$(2n+1)P_n(\cos\theta) = 4\pi\sum_{m=-n}^{n} Y_n^m(\alpha)\,\overline{Y_n^m(\beta)}.$$

Consequently, we finally get that

$$A(\alpha,\beta) = \sum_{n=0}^{+\infty}\sum_{m=-n}^{n}\frac{4i\pi}{k}\frac{j_n(ka)}{h_n^1(ka)}Y_n^m(\alpha)\,\overline{Y_n^m(\beta)}. \tag{13}$$

Now, since the far field mapping F is the integral operator of $L^2(S^2)$ with kernel $A(\alpha,\beta)$, and since $(Y_n^m)_{n\ge0,|m|\le n}$ defines an orthonormal basis of $L^2(S^2)$, then for any given

$$f = \sum_{n=0}^{+\infty}\sum_{m=-n}^{n} a_n^m Y_n^m \in L^2(S^2),$$

we have

$$Ff(\beta) = \sum_{n=0}^{+\infty}\sum_{m=-n}^{n}\frac{4i\pi}{k}\,\frac{j_n(ka)}{h_n^1(ka)}\,a_n^m\,Y_n^m(\beta).$$

Since $(Y_n^m)_{n\ge0,|m|\le n}$ is an orthonormal basis of $L^2(S^2)$, this formula shows that the following result holds.

Proposition 7 *The eigenvalues of the far field operator F in the case of a single sound-soft spherical scatterer of radius a are given by*

$$\lambda_n = \frac{4i\pi}{k}\,\frac{j_n(ka)}{h_n^1(ka)}, \qquad \forall n \ge 1. \tag{14}$$

The eigenspace associated with the eigenvalue λ_n is the vector space of dimension $2n+1$ with basis Y_n^m, for $|m| \le n$.

Remark 3 Equation (14) shows in particular that the eigenvalues λ_n of the far field operator F satisfy $|\lambda_n| \leq 4\pi/k$ (recall that $h_n^1 = j_n + iy_n$). This property can also be obtained from the fact that the scattering operator $S = I + (ik/2\pi)F$ is unitary. Indeed, this property implies that the eigenvalues λ_n lie on the circle of radius $4\pi/k$ and centered at $(0, 2\pi/k)$.

Proposition 7 shows in particular that **the number of non-zero eigenvalues is not necessarily equal to the number of obstacles**. However, in the case of a point scatterer or in the case of the low-frequency scattering (both cases which correspond to the asymptotic limit $ka \longrightarrow 0$), this result becomes true. Indeed, using the asymptotic behavior of Bessel and Hankel functions, we easily see that the eigenvalues λ_n given by (14) satisfy

$$\lambda_n \sim -\frac{4\pi^2}{k} \frac{(ka/2)^{2n+1}}{\Gamma(n+1/2)\Gamma(n+3/2)}$$

when ka goes to zero (and n is fixed). Thus, λ_{n+1}/λ_n decreases like $(ka)^2$, and hence, one can consider that the only significant eigenvalue in the limit case $ka \longrightarrow 0$ is the largest one λ_1. This observation suggests that the number of non zero eigenvalues can be related to the number of obstacles when the obstacles are small. The next subsection provides a justification of this statement.

4.2 Diffraction by several small obstacles

Consider a family of obstacles $\{\mathcal{O}_p^\varepsilon;\ p = 1, N\}$ depending on a small parameter ε, where each $\mathcal{O}_p^\varepsilon$ is the image of a reference open set $\mathcal{O}_p$ (which is assumed to contain the origin) by a dilation of ratio ε centered at a given point $s_p \in \mathbb{R}^3$ (see figure 1):

$$\mathcal{O}_p^\varepsilon = \left\{ x \in \mathbb{R}^3;\ \xi = \frac{x - s_p}{\varepsilon} \in \mathcal{O}_p \right\}.$$

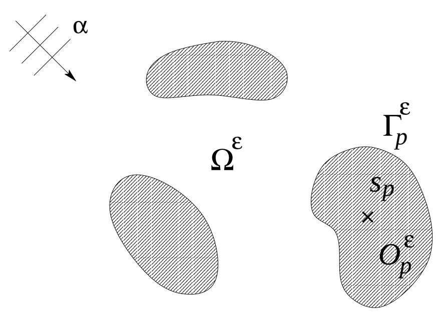

Figure 1:

Of course the "centers" s_p are chosen different so that for small enough ε, the obstacles do not intersect.

The main ingredient to show that selective focusing can be achieved using the eigenvectors of the far field operator when ε is small enough is given by the following result, which provides the asymptotic behavior of the scattering amplitude $A^\varepsilon(\alpha,\beta)$ associated with the family of obstacles $\{\mathcal{O}_p^\varepsilon\}$:

Proposition 8 *There exist N positive constants $C_1,\dots,C_N$ depending only on the geometry of the reference obstacles $\mathcal{O}_1,\dots,\mathcal{O}_N$ (called the "capacities" of these obstacles) such that*

$$\frac{A^\varepsilon(\alpha,\beta)}{\varepsilon} = A^{(1)}(\alpha,\beta) + O(\varepsilon) \quad \textit{with} \quad A^{(1)}(\alpha,\beta) = \frac{-1}{4\pi} \sum_{p=1,N} C_p \, u_I^\alpha(s_p)\, \overline{u_I^\beta(s_p)}, \tag{15}$$

where the bound $O(\varepsilon)$ is uniform for all $\alpha,\beta \in S^2$.

For the sake of clarity, the – rather technical – proof of this Proposition is given in the Appendix.

Remark 4 The capacity of a spherical soft obstacle of radius a is $C = 4\pi a$ (since the solution to (36) is simply given in this case by $V(x) = a/\|x\|$).

Thanks to Proposition 8, we know that the far field operator F^ε of the family of obstacles $\{\mathcal{O}_p^\varepsilon;\ p = 1, N\}$ satisfies

$$\sup_{f \in L^2(S^2)\setminus\{0\}} \frac{\|(\varepsilon^{-1}F^\varepsilon - F^{(1)})f\|_{L^2(S^2)}}{\|f\|_{L^2(S^2)}} = O(\varepsilon),$$

where $F^{(1)}$ is the integral operator on $L^2(S^2)$ with kernel $A^{(1)}$:

$$F^{(1)} f\,(\beta) = \int_{S^2} A^{(1)}(\alpha,\beta)\, f(\alpha)\, d\alpha.$$

Since F^ε is compact and normal, perturbation theory [11] ascertains the continuity of any finite system of eigenvalues as well as of the associated total eigenprojection. More precisely, assume that $\lambda^{(1)}$ is an isolated eigenvalue of $F^{(1)}$ with finite multiplicity m, which implies that $\lambda^{(1)} \neq 0$.

- Then for small enough ε, the spectrum of $\varepsilon^{-1}F^\varepsilon$ can be separated into two parts. On one hand the so-called $\lambda^{(1)}$-*group* consists of $m' \leq m$ eigenvalues λ_j^ε, with $j = 1$ to m', having a constant multiplicity m_j for $\varepsilon \neq 0$, and which are continuous near $\varepsilon = 0$, namely
$$|\lambda_j^\varepsilon - \lambda^{(1)}| = O(\varepsilon).$$
Moreover the total multiplicity $\sum_{j=1,m'} m_j$ of the $\lambda^{(1)}$-group coincide with the multiplicity m of $\lambda^{(1)}$. On the other hand, the complementary of the $\lambda^{(1)}$-group in the spectrum of $\varepsilon^{-1}F^\varepsilon$ lies outside a vicinity of $\lambda^{(1)}$.

- The total projection P^ε for the $\lambda^{(1)}$-group, *i.e.*, the sum of the orthogonal projections on the eigenspaces associated with the λ_j^ε, is continuous at $\varepsilon = 0$, and

$$\sup_{f \in L^2(S^2)\setminus\{0\}} \frac{\|(P^\varepsilon - P^{(1)})f\|_{L^2(S^2)}}{\|f\|_{L^2(S^2)}} = O(\varepsilon),$$

  where $P^{(1)}$ is the eigenprojection associated with $\lambda^{(1)}$.

Notice that in general, one cannot assert the existence of a continuous family of eigenvectors associated respectively with the λ_j^ε. However, for our particular choice of geometric perturbation ($\varepsilon-$dilation), such a result holds, since the perturbation actually is analytic with respect to ε (which is easily deduced from the Appendix). But this result is of poor practical interest.

An eigenvalue of $\varepsilon^{-1}F^\varepsilon$ either belongs to some $\lambda^{(1)}$-group for a nonzero eigenvalue $\lambda^{(1)}$ of $F^{(1)}$, or vanishes as ε tends to 0. In the latter case, the above result do not apply: perturbation theory only provides the continuity of non-stationary eigenelements. So it remains to study the spectral properties of $F^{(1)}$, whose *degenerate kernel* will be rewritten in the form

$$A^{(1)}(\alpha,\beta) = -\sum_{p=1,N} C_p\, \overline{e_p(\alpha)}\, e_p(\beta) \quad \text{where} \quad e_p(\alpha) = \frac{e^{-ik\alpha\cdot s_p}}{2\sqrt{\pi}} \;\; (p = 1, N). \tag{16}$$

Remark 5 Each e_p appears as a unit function of $L^2(S^2)$ and corresponds to an incident Herglotz wave $u_{I,p}$ which focuses on the p-th obstacle, for

$$u_{I,p}(x) = \int_{S^2} e_p(\alpha)\, u_I^\alpha(x)\, d\alpha = 2\sqrt{\pi}\, j_0(k\|x - s_p\|),$$

by virtue of (10).

The above expression of $A^{(1)}$ then yields

$$F^{(1)}f = -\sum_{p=1,N} C_p\; (f, e_p)_{L^2(S^2)}\; e_p. \tag{17}$$

Proposition 9 *The limit far field operator* (17) *is a negative selfadjoint operator with finite rank N (the number of obstacles) and whose spectral radius cannot be smaller than the greatest capacity C_p of the obstacles.*

In the case where the wavelength $\ell = 2\pi/k$ is small compared with the minimum distance $d = \min_{1\le p\neq q\le N} \|s_p - s_q\|$ between the obstacles, the family $\{e_p;\; p = 1, N\}$ defined in (16) provides an approximate basis of eigenvectors associated with the approximate eigenvalues $\{-C_p;\; p = 1, N\}$:

$$F^{(1)}e_p = -C_p\, e_p + O\left(\frac{\ell}{d}\right). \tag{18}$$

Proof. The bilinear form associated with $F^{(1)}$:

$$(F^{(1)}f, f')_{L^2(S^2)} = -\sum_{p=1,N} C_p \ (f, e_p)_{L^2(S^2)} \ \overline{(f', e_p)_{L^2(S^2)}}$$

is clearly negative and selfadjoint, and so is $F^{(1)}$. The range of $F^{(1)}$ is spanned by $\{e_p;\ p = 1, N\}$. To see that this family is linearly independent, suppose that

$$\sum_{p=1,N} z_p \, e_p = 0 \text{ with } z_p \in \mathbb{C}.$$

Choose $\alpha_0 \in S^2$ and τ_0 a tangent vector to S^2 at point α_0 such that the real numbers $\tau_0 \cdot s_p$ are all distinct. Taking successive tangential derivatives of the above dependence relation then yields

$$\sum_{p=1,N} (-ik\,\tau_0 \cdot s_p)^j \, z_p \, e_p(\alpha_0) = 0 \quad \text{for } j = 0, N-1,$$

which amounts to an invertible Vandermonde system. Hence all the z_p must vanish: the rank of $F^{(1)}$ is exactly the number of obstacles.

The lower bound for the spectral radius follows from the fact that

$$\left|(F^{(1)}e_q, e_q)_{L^2(S^2)}\right| = \sum_{p=1,N} C_p \left| \ (e_p, e_q)_{L^2(S^2)}\right|^2 \geq C_q \quad \text{for } q = 1, N,$$

since the e_p are unit functions. On the other hand, nothing can be said in general about the gap between 0 and the other eigenvalues, which may be arbitrarily close to the former. This actually depends on the constructive or destructive interactions between the different obstacles, which are measured by the following scalar products (see (10)):

$$(e_p, e_q)_{L^2(S^2)} = j_0(k\|s_p - s_q\|) = \frac{\sin(k\|s_p - s_q\|)}{k\|s_p - s_q\|}.$$

These relations show in particular that

$$(e_p, e_q)_{L^2(S^2)} = \begin{cases} 1 & \text{for } q = p, \\ O\left(\dfrac{\ell}{d}\right) & \text{for } q \neq p, \end{cases}$$

which means that $\{e_p;\ p = 1, N\}$ is close to an orthogonal basis of the range of $F^{(1)}$ when $\ell \ll d$. The estimate (18) follows: each e_p is an approximate eigenvector. ∎

What are the practical consequences of the above results as regards selective focusing? Mainly that the eigenvectors of the time-reversal operator (or the far field operator) will produce selective focusing acoustic waves if

- the obstacles are small enough, compared to the wavelength,

- the smallest distance between them is large, compared again to the wavelength,
- their capacities are all distinct.

Indeed in this case all the nonzero eigenvalues of $F^{(1)}$ will be simple: the diagonalization of the time-reversal operator will then yields approximations of the focusing densities e_p.

But if one of these assumptions is missing, the nice focusing properties will disappear, at least for some groups of eigenvectors.

On one hand, if the interactions between the obstacles become significant, *i.e.*, when $d/\ell = O(1)$, these properties may reduce to the purely global focusing presented in section 3. In particular, for very low frequencies, the situation $\varepsilon \ll d \ll \ell$ may occur. In this case we have

$$e_p = \tilde{e} + O\left(\frac{d}{\ell}\right) \quad \text{with } \tilde{e}(\alpha) = \frac{e^{-ik\,\alpha\cdot\tilde{s}}}{2\sqrt{\pi}},$$

where $\tilde{s}$ may be chosen as a convex combination of the s_p. As a consequence

$$F^{(1)}f = -\left(\sum_{p=1,N} C_p\right)\ (f,\tilde{e})_{L^2(S^2)}\ \tilde{e} + O\left(\frac{d}{\ell}\right),$$

which shows that the cluster of obstacles behaves like a unique obstacle which accumulates their respective capacities: only one significant eigenvalue of the time-reversal operator may be observed. Of course, for several distant clusters, we shall recover selective focusing on each cluster.

On the other hand, if some of the obstacles have neighbouring capacities, the time-reversal operator may admit non simple eigenvalues. In this situation, the diagonalization of the latter cannot choose the selective focusing densities among all their linear combinations which compose the corresponding eigenspace.

5 Open time-reversal mirrors

In this section, we consider the case of a TRM that does not entirely surround the obstacle. Given a subset $\widehat{S}$ of S^2, we assume that the TRM can emit plane waves of directions $\alpha \in \widehat{S}$, and measures the far field in the opposite directions $\beta \in (-\widehat{S})$. One emission-diffraction-reception cycle is described by the directional far field operator

$$\widehat{F} = \widehat{P}_-\, F\, \widehat{P}_+^* : L^2(+\widehat{S}) \longrightarrow L^2(-\widehat{S}),$$

where $\widehat{P}_\pm$ are the restriction operators from $L^2(S^2)$ to $L^2(\pm\widehat{S})$, and thus their respective adjoints $\widehat{P}_\pm^* : L^2(\pm\widehat{S}) \longrightarrow L^2(S^2)$ are the operators of continuation by 0 outside $\pm\widehat{S}$. Note here that $\widehat{F}$ appears as the integral operator

$$\widehat{F}f(\beta) = \int_{+\widehat{S}} A(\alpha,\beta)\, f(\alpha)\, d\alpha \quad \text{for } \beta \in -\widehat{S}.$$

The time-reversal operator $\widehat{T}$ in the case of an open TRM is then defined by

$$\widehat{T}f = \overline{\widehat{R}\,\widehat{F}\overline{\widehat{R}\,\widehat{F}f}},$$

where $\widehat{R} : L^2(-\widehat{S}) \longrightarrow L^2(+\widehat{S})$ is the restriction of the symmetry operator defined in section 2 (*i.e.*, $\widehat{R}f(\alpha) = f(-\alpha)$ for $\alpha \in \widehat{S}$).

But one can easily check that $\widehat{R}\,\widehat{P}_- = \widehat{P}_+\,R$ and $\widehat{P}_+^*\,\widehat{R} = R\,\widehat{P}_-^*$, and since these operators commute with the conjugation, we have by virtue of (5)

$$\widehat{F}^* f = \widehat{P}_+ F^* \widehat{P}_-^* f = \overline{\widehat{P}_+ RF\overline{R\widehat{P}_-^* f}} = \overline{\widehat{R}\,\widehat{P}_- F\widehat{P}_+^*\overline{\widehat{R}\,f}} = \overline{\widehat{R}\,\widehat{F}\,\overline{\widehat{R}\,f}}.$$

Hence, we can state the following result

Proposition 10 *The time-reversal operator $\widehat{T}$ for an open TRM is given by*

$$\widehat{T} = \widehat{F}^*\widehat{F} : L^2(\widehat{S}) \longrightarrow L^2(\widehat{S}).$$

Thus, it is the integral operator with kernel

$$\widehat{t}(\alpha,\beta) = \int_{-\widehat{S}} A(\alpha,\gamma)\,\overline{A(\beta,\gamma)}\,d\gamma \quad \textit{for } \alpha,\beta \in \widehat{S}. \tag{19}$$

Moreover, $\widehat{T}$ defines a compact positive and selfadjoint operator.

Besides the value 0, the spectrum $\widehat{T}$ is thus constituted of a finite or countable sequence of positive eigenvalues $(\widehat{\mu}_p)_{p\geq 1}$ admitting 0 for only possible accumulation point. The largest eigenvalue $\widehat{\mu}_1$ of $\widehat{T}$ is thus given by

$$\widehat{\mu}_1 = \sup_{f\in L^2(\widehat{S})\,,\,f\neq 0} \frac{\left(\widehat{T}f, f\right)_{L^2(\widehat{S})}}{\|f\|^2_{L^2(S^2)}} = \sup_{f\in L^2(\widehat{S})\,,\,f\neq 0} \frac{\|\widehat{F}f\|^2_{L^2(-\widehat{S})}}{\|f\|^2_{L^2(+\widehat{S})}}.$$

This expression shows in particular that the incident field corresponding to an eigenvector associated with this eigenvalue maximizes the diffracted field in the direction of the TRM. Our goal now is to see if the global and selective properties proved respectively in sections 3 and 4 for closed mirrors still hold in the case of an open TRM. The main difference between both situations is that in the latter one, the directional far field operator $\widehat{F}$ is not anymore normal (the range of $\widehat{F}^*\widehat{F}$ is contained in $L^2(\widehat{S})$, when that of $\widehat{F}\widehat{F}^*$ is contained in $L^2(-\widehat{S})$). Consequently, the eigenelements of $\widehat{T} = \widehat{F}^*\widehat{F}$ can not be directly related to those of $\widehat{F}$. Contrary to the case of a closed TRM, the spectral analysis need thus to be carried on the time reversal operator and not on the far field one. Nevertheless, as we are going to see now, all the focusing results obtained previously still hold.

5.1 Global focusing

In this subsection, we prove a global focusing property similar to the one given in Proposition 6. More precisely, we have the following result

Proposition 11 *Let $\widehat{\mu}_p \neq 0$ be an eigenvalue of $\widehat{T}$ and $\widehat{f}_p \in L^2(\widehat{S})$ a corresponding eigenvector. Then, the Herglotz wave $\widehat{u}_{I,p}$ associated with $\widehat{f}_p$ and defined by $\widehat{u}_{I,p}(x) = \int_{\widehat{S}} \widehat{f}_p(\alpha)\, u_I^\alpha(x)\, d\alpha$ can be written in the form*

$$\widehat{u}_{I,p}(x) = \int_\Gamma \widehat{\jmath}\,(k(x-y))\, h_p(y)\, d\Gamma, \tag{20}$$

for some density h_p, where

$$\widehat{\jmath}\,(k(x-y)) = \int_{\widehat{S}} u_I^\beta(x)\, \overline{u_I^\beta(y)}\, d\beta = \int_{\widehat{S}} e^{ik\beta\cdot(x-y)}\, d\beta. \tag{21}$$

Proof. Like in the proof of Proposition 6, formula (20) will be proved if we can write $\widehat{f}_p$ in the form

$$\widehat{f}_p(\beta) = \int_\Gamma h_p\, \overline{u_I^\beta}\, d\Gamma, \tag{22}$$

for a given density h_p. Indeed, if such a relation holds, then

$$\widehat{u}_{I,p}(x) = \int_{\widehat{S}} \widehat{f}_p(\beta)\, u_I^\beta(x)\, d\beta = \int_{\widehat{S}} \int_\Gamma h_p\, \overline{u_I^\beta}\, d\Gamma\ u_I^\beta(x)\, d\beta.$$

Equation (20) follows then by inverting the integrals over $\widehat{S}$ and Γ.

Thus, it only remains to prove (22). We first write that for all $\beta \in \widehat{S}$,

$$\widehat{f}_p(\beta) = \frac{1}{\widehat{\mu}_p} \widehat{T}\widehat{f}_p(\beta) = \frac{1}{\widehat{\mu}_p} \int_{\widehat{S}} \widehat{t}(\alpha,\beta)\, \widehat{f}_p(\alpha)\, d\alpha. \tag{23}$$

Thanks to the reciprocity relation (3), formula (19) can be written

$$\widehat{t}(\alpha,\beta) = \int_{-\widehat{S}} A(\alpha,\gamma)\, \overline{A(-\gamma,-\beta)}\, d\gamma.$$

Using in the above relation the integral representation (2), we get after some simple computations that

$$\widehat{t}(\alpha,\beta) = \int_\Gamma h_p^\alpha\, \overline{u_I^\beta}\, d\Gamma, \tag{24}$$

where the density h_p^α is given by

$$h_p^\alpha(x) = \frac{1}{(4\pi)^2} \int_{-\widehat{S}} \int_\Gamma \frac{\partial u_T^\alpha}{\partial \nu}\, \overline{u_I^\gamma}\, d\Gamma\ \overline{\frac{\partial u_T^{-\gamma}}{\partial \nu}(x)}\ d\gamma.$$

Combining (23) and (24), one obtains the claimed relation (22), with the density $h_p(x) = \widehat{\mu}_p^{-1} \int_{\widehat{S}} h_p^\alpha(x)\, \widehat{f}_p(\alpha)\, d\alpha$. ∎

It is well known in oscillatory integrals theory that the function $\widehat{\jmath}(x)$ defined by (21) satisfies (one can use the stationary phase theorem, see for instance [18])

$$\widehat{\jmath}(x) = \mathcal{O}\left(\|x\|^{-1}\right). \tag{25}$$

In the directions which are not covered by the TRM (*i.e.* when $x/\|x\| \notin \pm S$), one can in fact obtain a faster decay for $\widehat{\jmath}(x)$, since we have then $\widehat{\jmath}(x) = \mathcal{O}\left(\|x\|^{-3/2}\right)$.

Thanks to (25), formula (20) shows thus that the incident field generated by an eigenvector of $\widehat{T}$ focuses on the obstacles located in the propagation medium.

5.2 Diffraction by several small obstacles

Now we turn to the analysis of the selective focusing in the case of a TRM partially surrounding several small obstacles. The assumptions made on the geometry of the small scatterers are identical to those made in section 4. Let us recall that the main difference with section 4 is that since $\widehat{F}$ is not normal, the spectral analysis can not anymore be achieved on $\widehat{F}$, but has to be carried out directly on the time reversal operator $\widehat{T}$. In this subsection, we are going to see that the selective focusing results obtained in section 4 can be extended to the case of an open mirror.

Using the asymptotic formula (15) of the scattering amplitude, one easily gets that the kernel $\widehat{t}^{\varepsilon}(\alpha,\beta)$ of the time reversal operator $\widehat{T}^{\varepsilon} = (\widehat{F}^{\varepsilon})^{*}\widehat{F}^{\varepsilon}$ satisfies

$$\frac{\widehat{t}^{\varepsilon}(\alpha,\beta)}{\varepsilon^2} = \widehat{t}^{(1)}(\alpha,\beta) + O(\varepsilon),$$

where

$$\widehat{t}^{(1)}(\alpha,\beta) = \int_{-\widehat{S}} A^{(1)}(\alpha,\gamma)\,\overline{A^{(1)}(\beta,\gamma)}\,d\gamma, \qquad \forall\alpha,\beta\in\widehat{S}$$

and where $A^{(1)}(\cdot,\cdot)$ is the degenerate kernel defined in (15). Since $\widehat{T}^{\varepsilon}$ is compact and selfadjoint, classical results of perturbation theory show again that for small ε, the spectral elements of $\varepsilon^{-2}\widehat{T}^{\varepsilon}$ can be approximated by those of the integral operator $\widehat{T}^{(1)}$ with kernel $\widehat{t}^{(1)}(\cdot,\cdot)$, which also reads $\widehat{T}^{(1)} = (\widehat{F}^{(1)})^{*}\widehat{F}^{(1)}$ where the operator $\widehat{F}^{(1)} : L^2(\widehat{S}) \longrightarrow L^2(-\widehat{S})$ is defined by

$$\widehat{F}^{(1)} f(\beta) = -\sum_{p=1,N} C_p\,(f,e_p)_{L^2(\widehat{S})}\,e_p(\beta), \qquad \text{for } \beta\in -\widehat{S}.$$

If we define the normalized functions $\{\widehat{e}_p;\ p=1,N\}$ in $L^2(\widehat{S})$ and $L^2(-\widehat{S})$ by

$$\widehat{e}_p(\alpha) = (4\pi\widehat{r})^{-1/2}\,e^{-ik\alpha\cdot s_p}, \tag{26}$$

where $\widehat{r} = \mathrm{mes}(\widehat{S})/(4\pi)$ is the *opening ratio* of the T.R.M., then

$$\widehat{F}^{(1)} f = -\widehat{r}\sum_{p=1,N} C_p\,(f,\widehat{e}_p)_{L^2(\widehat{S})}\,\widehat{e}_p \ \text{ in } L^2(-\widehat{S}). \tag{27}$$

Hence, for all $f \in L^2(\widehat{S})$, we have

$$\widehat{T}^{(1)} f = \widehat{r}^{\,2} \sum_{q=1,N} C_q \left(\sum_{p=1,N} C_p \, (f, \widehat{e}_p)_{L^2(\widehat{S})} (\widehat{e}_p, \widehat{e}_q)_{L^2(-\widehat{S})} \right) \widehat{e}_q. \tag{28}$$

We can now state the main result of this subsection.

Proposition 12 *The limit time reversal operator $\widehat{T}^{(1)} : L^2(\widehat{S}) \longrightarrow L^2(\widehat{S})$ defined by (28) is a selfadjoint operator with finite rank N (the number of obstacles).*

Furthermore, if the wavelength $\ell = 2\pi/k$ is small compared with the minimum distance between the obstacles, the family $\{\widehat{e}_p;\ p = 1, N\}$ defined in (26) provides an approximate basis of eigenvectors of $\widehat{T}^{(1)}$ associated with the approximate eigenvalues $(\widehat{r}\, C_p)^2$:

$$\widehat{T}^{(1)} \widehat{e}_p = (\widehat{r}\, C_p)^2 \, \widehat{e}_p + O\left(\frac{\ell}{d}\right). \tag{29}$$

Proof. The fact that $\widehat{T}^{(1)}$ is of rank N follows from the fact that the family $\{\widehat{e}_p; p = 1, N\}$ is linearly independent in $L^2(\widehat{S})$(see the proof of Proposition 12). Equation (29) follows from (28) combined with the fact that

$$(\widehat{e}_p, \widehat{e}_q)_{L^2(\widehat{S})} = \overline{(\widehat{e}_p, \widehat{e}_q)_{L^2(-\widehat{S})}} = \begin{cases} 1 & \text{for } p = q, \\ O\left(\dfrac{\ell}{d}\right) & \text{for } p \neq q. \end{cases}$$

The last estimate follows from the relation

$$(\widehat{e}_p, \widehat{e}_q)_{L^2(\widehat{S})} = (4\pi\hat{r})^{-1} \int_{\widehat{S}} e^{ik\beta\cdot(s_p - s_q)} \, d\beta = (4\pi\hat{r})^{-1} \jmath(k(s_p - s_q)),$$

and from the decay property (25) of $\jmath$, for $p \neq q$. ∎

Remark 6 *Contrary to the case of a closed mirror (compare propositions 9 and 12), we have not been able to compare the spectral radius of $\hat{T}^{(1)}$ with the greatest value taken by the quantities $(\hat{r} C_p)^2$.*

As in the case of a closed mirror, Proposition 12 shows that the eigenvectors of the time-reversal operator for an open mirror will produce selective focusing acoustic waves if

- the obstacles are small enough, compared to the wavelength,
- the smallest distance between them is large, compared to the wavelength,
- their capacities are all distinct.

Indeed in this case all the nonzero eigenvalues of $\widehat{T}^{(1)}$ will be simple: the diagonalization of the time-reversal operator will then yields approximations of the focusing densities $\widehat{e}_p$ since each $\widehat{e}_p$ generates an incident Herglotz wave $\widehat{u}_{I,p}$ which focuses on the p-th obstacle, for

$$\widehat{u}_{I,p}(x) = \int_{\widehat{S}} \widehat{e}_p(\alpha) \, u_I^\alpha(x) \, d\alpha = \frac{1}{\sqrt{4\pi\widehat{r}}} \, \jmath(k(x - s_p)) = O\left(\frac{\ell}{\|x - s_p\|}\right).$$

A Asymptotics for small obstacles

We detail here a constructive proof of the asymptotic behavior (15) of the scattering amplitude for small obstacles, claimed in Proposition 8. This result is formally derived in other papers (see e.g. [16, 17]). A more abstract proof based on potential theory was recently proposed in [8].

The idea of our proof is to rewrite the scattering problem as a *regular* perturbation of a Fredholm equation in a *fixed* Hilbert space, in the sense that it does not depend on the size, say ε, of the obstacles:

$$(I + \mathbb{K}^\varepsilon)\varphi^\varepsilon = g^\varepsilon. \tag{30}$$

We obtain such a formulation by means of a variant of the integral method introduced by Jami and Lenoir [10], which has the advantage to involve non singular kernels, contrary to usual integral equations (for which perturbation theory requires more complicated arguments).

Consider the family of obstacles $\{\mathcal{O}_p^\varepsilon;\ p = 1, N\}$ introduced in subsection 4.2. We denote by Γ_p^ε (respectively, Γ_p) the boundary of $\mathcal{O}_p^\varepsilon$ (respectively, of $\mathcal{O}_p$), $\Gamma^\varepsilon = \bigcup_{p=1,N} \Gamma_p^\varepsilon$ and $\mathcal{O}^\varepsilon = \bigcup_{p=1,N} \mathcal{O}_p^\varepsilon$. Our exterior Dirichlet problem for the diffracted field u^ε reads

$$\left\{ \begin{array}{l} \Delta u^\varepsilon + k^2 u^\varepsilon = 0 \ \text{ in } \mathbb{R}^3 \setminus \overline{\mathcal{O}^\varepsilon}, \\ u^\varepsilon = f \ \text{ on } \Gamma^\varepsilon, \\ R.C., \end{array} \right. \tag{31}$$

where $R.C.$ stands for the outgoing radiation condition, and $f = -u_I^\alpha$ is the Dirichlet datum associated with an incident plane wave $u_I^\alpha(x) = \exp(ik\alpha \cdot x)$ of direction $\alpha \in S^2$.

Reduction to a bounded domain

Around each reference obstacle $\mathcal{O}_p$, we delimit a bounded part D_p of its exterior by a fictitious boundary Σ_p which does not intersect Γ_p. We denote by D_p^ε and Σ_p^ε the images of D_p and Σ_p by the same dilation as for $\mathcal{O}_p^\varepsilon$, as well as $D^\varepsilon = \bigcup_{p=1,N} D_p^\varepsilon$ and $\Sigma^\varepsilon = \bigcup_{p=1,N} \Sigma_p^\varepsilon$.

The Jami–Lenoir method consists in introducing a transparent boundary condition on Σ^ε which is derived from the usual integral representation of u^ε. Here, in order to get rid of the normal derivative of u^ε on Γ^ε, the single-layer potential is re-expressed as a volume potential by Green's formula. Indeed it is easy to see that near Σ^ε we have,

$$u^\varepsilon = f \overset{\Gamma^\varepsilon}{*} \frac{\partial G_k}{\partial \nu} + k^2 u^\varepsilon \overset{D^\varepsilon}{*} (\chi^\varepsilon G_k) - \nabla u^\varepsilon \overset{D^\varepsilon}{*} \nabla(\chi^\varepsilon G_k),$$

where the different "convolutions" represent respectively the surface double-layer potential

$$\left\{ f \overset{\Gamma^\varepsilon}{*} \frac{\partial G_k}{\partial \nu} \right\}(x) = \int_{\Gamma^\varepsilon} f(y) \frac{\partial G_k}{\partial \nu_y}(x - y)\, d\gamma_y,$$

and the volume potentials

$$\begin{aligned}\left\{u^\varepsilon \overset{D^\varepsilon}{*} (\chi^\varepsilon G_k)\right\}(x) &= \int_{D^\varepsilon} u^\varepsilon(y)\,\chi^\varepsilon(y) G_k(x-y)\,dy,\\ \left\{\nabla u^\varepsilon \overset{D^\varepsilon}{*} \nabla(\chi^\varepsilon G_k)\right\}(x) &= \int_{D^\varepsilon} \nabla u^\varepsilon(y)\cdot\nabla_y(\chi^\varepsilon(y)G_k(x-y))\,dy.\end{aligned}$$

In the above expressions, G_k stands for the outgoing Green function of $\Delta+k^2$, *i.e.*, $G_k(x) = -\exp(ik|x|)/(4\pi|x|)$, and χ^ε denotes a family of regular cutoff functions $(\chi_p^\varepsilon)_{p=1,\dots,N}$ defined by the ε-dilation: $\chi_p^\varepsilon(x) = \chi_p((x-s_p)/\varepsilon)$ if $x\in D_p^\varepsilon$, where each χ_p is equal to 1 in a vicinity of Γ_p and 0 in a vicinity of Σ_p. Note that these integrals involve regular kernels when x is near Σ^ε.

As a consequence if u^ε solves (31), its restriction v^ε to D^ε satisfies

$$\left\{\begin{array}{l}\Delta v^\varepsilon + k^2 v^\varepsilon = 0 \;\text{ in } D^\varepsilon,\\ v^\varepsilon = f \;\text{ on } \Gamma^\varepsilon,\\ Z^\varepsilon v^\varepsilon = Z^\varepsilon\left\{f \overset{\Gamma^\varepsilon}{*} \dfrac{\partial G_k}{\partial\nu} + k^2 v^\varepsilon \overset{D^\varepsilon}{*} (\chi^\varepsilon G_k) - \nabla v^\varepsilon \overset{D^\varepsilon}{*} \nabla(\chi^\varepsilon G_k)\right\} \;\text{ on } \Sigma^\varepsilon,\end{array}\right. \tag{32}$$

where Z^ε stands for the boundary operator $(\partial/\partial\nu + i/\varepsilon)$ on Σ^ε.

Conversely, the solution to this problem extends outside Σ^ε (by the integral representation) to the solution to (31) (thanks to the term involving i/ε which prevents the so-called *irregular frequencies* from being real, see [10]).

The limiting process

In order to work in a functional framework independent of ε, we perform in each subdomain D_p^ε the change of variable $\xi = (x-s_p)/\varepsilon$. By denoting $\varphi_p^\varepsilon(\xi) = v^\varepsilon(x)$ and $f_p^\varepsilon(\xi) = f(x)$, for $x\in D_p^\varepsilon$, as well as

$$G_{pq}^\varepsilon(\xi,\eta) = G_k(s_p - s_q + \varepsilon(\xi-\eta)) \text{ for } \xi\in\overline{D_p} \text{ and } \eta\in\overline{D_q},$$

problem (32) amounts to a family of N problems set on the domains D_p coupled by the transparent boundary conditions written on Σ_p:

$$\left\{\begin{array}{l}\Delta\varphi_p^\varepsilon + (\varepsilon k)^2\varphi_p^\varepsilon = 0 \;\text{ in } D_p,\\ \varphi_p^\varepsilon = f_p^\varepsilon \;\text{ on } \Gamma_p,\\ Z\varphi_p^\varepsilon = Z\displaystyle\sum_{q=1,N}\{\varepsilon f_q^\varepsilon \overset{\Gamma_q}{*} \dfrac{\partial G_{pq}^\varepsilon}{\partial\nu}\\ \qquad\qquad + \varepsilon^3 k^2 \varphi_q^\varepsilon \overset{D_q}{*} (\chi_q G_{pq}^\varepsilon) - \varepsilon\nabla\varphi_q^\varepsilon \overset{D_q}{*} \nabla(\chi_q G_{pq}^\varepsilon)\} \;\text{ on } \Sigma_p,\end{array}\right. \tag{33}$$

where $Z = (\partial/\partial\nu + i)$ on Σ_p.

We are now able to define the formal limit of the latter problem. Let G_0 be the limit of $G_{\varepsilon k}$ when ε tends to 0, *i.e.*, $G_0(x) = -1/(4\pi|x|)$. Notice that

$$\begin{array}{ll} G^\varepsilon_{pq}(\xi,\eta) = G_k(s_p - s_q) + O(\varepsilon) & \text{if } p \neq q, \\ G^\varepsilon_{pp}(\xi,\eta) = \varepsilon^{-1} G_0(\xi - \eta) + O(1) & \text{if } p = q. \end{array} \tag{34}$$

where these formulas hold uniformly in any compact subset of $\overline{D_p} \times \overline{D_q}$ which does not contain points of the diagonal when $p = q$, and can be derived with respect to ξ or η. Hence the formal limit of problem (33) reads

$$\left\{ \begin{array}{l} \Delta \varphi^0_p = 0 \ \text{ in } D_p, \\ \varphi^0_p = f^0_p = -e^{ik\alpha \cdot s_p} \ \text{ on } \Gamma_p, \\ Z\varphi^0_p = Z\left\{ f^0_p \overset{\Gamma_p}{*} \dfrac{\partial G_0}{\partial \nu} - \nabla \varphi^0_p \overset{D_p}{*} \nabla(\chi_p G_0) \right\} \ \text{ on } \Sigma_p, \end{array} \right. \tag{35}$$

which correspond to a family of uncoupled problems. Each of them amounts to solving an exterior Laplace equation. More precisely, we can write that $\varphi^0_p = -u^\alpha_I(s_p)\, V_p$ where V_p is the *static* potential solution to

$$\left\{ \begin{array}{l} \Delta V_p = 0 \ \text{ in } \mathbb{R}^3 \setminus \mathcal{O}_p, \\ V_p = 1 \ \text{ on } \Gamma_p, \\ V_p = O(1/x) \ \text{ as } |x| \to \infty. \end{array} \right. \tag{36}$$

Convergence

Consider the closed subspace of the usual Sobolev space $H^1(D_p)$ given by $\mathcal{H}_p = \{\psi_p \in H^1(D_p);\ \psi_p = 0 \text{ on } \Gamma_p\}$. The variational formulation of (33) appears as a coupled system of variational equations:

$$\begin{array}{l} \text{Find } \varphi^\varepsilon_p \in f^\varepsilon_p + \mathcal{H}_p,\ p = 1, N,\ \text{such that} \\ \displaystyle\int_{D_p} \nabla\varphi^\varepsilon_p \cdot \overline{\nabla\psi_p} - (\varepsilon k)^2 \int_{D_p} \varphi^\varepsilon_p \overline{\psi_p} + i \int_{\Sigma_p} \varphi^\varepsilon_p \overline{\psi_p}\, d\sigma \\ \displaystyle + \int_{\Sigma_p} Z \left\{ \sum_{q=1,N} \varepsilon^3 k^2 \varphi^\varepsilon_q \overset{D_q}{*} (\chi_q G^\varepsilon_{pq}) - \varepsilon \nabla \varphi^\varepsilon_q \overset{D_q}{*} \nabla(\chi_q G^\varepsilon_{pq}) \right\} \overline{\psi_p}\, d\sigma \\ \displaystyle = \int_{\Sigma_p} Z \left\{ \sum_{q=1,N} \varepsilon f^\varepsilon_q \overset{\Gamma_q}{*} \frac{\partial G^\varepsilon_{pq}}{\partial \nu} \right\} \overline{\psi_p}\, d\sigma \quad \forall \psi_p \in \mathcal{H}_p,\ p = 1, N. \end{array}$$

Adding these equations yields the announced Fredholm equation (30) in the Hilbert space $\mathcal{H} = \mathcal{H}_1 \times \ldots \times \mathcal{H}_N$ which can be equipped with the scalar product

$$(\varphi, \psi) = \sum_{p=1,N} \int_{D_p} \nabla\varphi_p \cdot \overline{\nabla\psi_p}.$$

Indeed, define $\varphi^\varepsilon = (\varphi_1^\varepsilon, \dots, \varphi_N^\varepsilon)$, $f^\varepsilon = (f_1^\varepsilon, \dots, f_N^\varepsilon)$ and respectively the operator $\mathbb{K}^\varepsilon$ defined in $\mathcal{H}$ and $g^\varepsilon \in \mathcal{H}$ by

$$\begin{aligned}
(\mathbb{K}^\varepsilon \varphi, \psi) &= \sum_{p=1,N} -(\varepsilon k)^2 \int_{D_p} \varphi_p \overline{\psi_p} + i \int_{\Sigma_p} \varphi_p \overline{\psi_p}\, d\sigma \\
&+ \int_{\Sigma_p} Z \left\{ \sum_{q=1,N} \varepsilon^3 k^2 \varphi_q \overset{D_q}{*} (\chi_q G_{pq}^\varepsilon) - \varepsilon \nabla \varphi_q \overset{D_q}{*} \nabla(\chi_q G_{pq}^\varepsilon) \right\} \overline{\psi_p}\, d\sigma \\
(g^\varepsilon, \psi) &= \sum_{p=1,N} \int_{\Sigma_p} Z \left\{ \sum_{q=1,N} \varepsilon f_q^\varepsilon \overset{\Gamma_q}{*} \frac{\partial G_{pq}^\varepsilon}{\partial \nu} \right\} \overline{\psi_p}\, d\sigma,
\end{aligned}$$

for all $\varphi = (\varphi_1, \dots, \varphi_N)$ and $\psi = (\psi_1, \dots, \psi_N)$ in $\mathcal{H}$. Then our coupled system reads

$$\text{Find } \varphi^\varepsilon \in f^\varepsilon + \mathcal{H} \text{ such that } (I + \mathbb{K}^\varepsilon)\varphi^\varepsilon = g^\varepsilon. \tag{37}$$

And of course we have a similar expression of the limit problem (35) with

$$\begin{aligned}
(\mathbb{K}^0 \varphi, \psi) &= \sum_{p=1,N} i \int_{\Sigma_p} \varphi_p \overline{\psi_p}\, d\sigma - \int_{\Sigma_p} Z \left\{ \nabla \varphi_p \overset{D_p}{*} \nabla(\chi_q G_0) \right\} \overline{\psi_p}\, d\sigma \\
(g^0, \psi) &= \sum_{p=1,N} \int_{\Sigma_p} Z \left\{ f_p^0 \overset{\Gamma_p}{*} \frac{\partial G_0}{\partial \nu} \right\} \overline{\psi_p}\, d\sigma.
\end{aligned}$$

Note that the uniqueness of the solution to (31) (respectively, (36)) implies that $I + \mathbb{K}^\varepsilon$ (respectively, $I + \mathbb{K}^0$) is injective, and thus bijective thanks to the following

Lemma 1 $\mathbb{K}^\varepsilon$ *defines a family of compact operators in* $\mathcal{H}$ *which satisfies*

$$\|\mathbb{K}^\varepsilon - \mathbb{K}^0\| = \sup_{\varphi, \psi \in \mathcal{H} \setminus \{0\}} \frac{(\mathbb{K}^\varepsilon \varphi, \psi)}{\|\varphi\| \|\psi\|} = O(\varepsilon). \tag{38}$$

Proof. Consider for instance the part of $\mathbb{K}^\varepsilon$ corresponding to the operator $\mathbb{T}_{pq}^\varepsilon$ given by

$$\begin{aligned}
(\mathbb{T}_{pq}^\varepsilon \varphi, \psi) &= \int_{\Sigma_p} Z \left\{ \varepsilon \nabla \varphi_q \overset{D_q}{*} \nabla(\chi_q G_{pq}^\varepsilon) \right\} \overline{\psi_p}\, d\sigma, \\
&= \int_{\Sigma_p} \int_{D_q} \nabla \varphi_q(\eta) \cdot \nabla_\eta (\varepsilon \chi_q(\eta) Z_\xi G_{pq}^\varepsilon(\xi - \eta))\, d\eta\, \overline{\psi_p}\, d\sigma_\xi.
\end{aligned}$$

We detail the proof only for the latter: similar arguments can be used for the other terms involved in the definition of $\mathbb{K}^\varepsilon$.

The compactness of $\mathbb{T}^\varepsilon_{pq}$ can be easily deduced from that of its adjoint. Indeed using Schwarz inequality yields

$$\begin{aligned}
\|(\mathbb{T}^\varepsilon_{pq})^*\psi\| &= \sup_{\varphi\in\mathcal{H}\setminus\{0\}} \frac{(\mathbb{T}^\varepsilon_{pq}\varphi,\psi)}{\|\varphi\|} \leq C^\varepsilon_{pq}\,\|\psi\|_{L^2(\Sigma_p)} \quad \text{where} \\
C^\varepsilon_{pq} &= \left(\int_{\Sigma_p}\int_{D_q} \left\|\nabla_\eta\{\varepsilon\chi_q(\eta)Z_\xi G^\varepsilon_{pq}(\xi-\eta)\}\right\|^2 d\eta\, d\sigma_\xi\right)^{1/2}.
\end{aligned}$$

But the trace operator is compact from $H^1(D_q)$ to $L^2(\Sigma_p)$, which implies the compactness of $(\mathbb{T}^\varepsilon_{pq})^*$ in $\mathcal{H}$.

If $p\neq q$, formula (34) shows that $C^\varepsilon_{pq}=O(\varepsilon)$, and consequently the same holds for $\|\mathbb{T}^\varepsilon_{pq}\|=\|(\mathbb{T}^\varepsilon_{pq})^*\|$. If $p=q$, the limit operator is given by

$$(\mathbb{T}^0_{pp}\varphi,\psi)=\int_{\Sigma_p} Z\left\{\nabla\varphi_p \overset{D_p}{*} \nabla(\chi_p G_0)\right\}\overline{\psi_p}\,d\sigma,$$

since (34) shows in this case (again by Schwarz inequality) that

$$\left|((\mathbb{T}^\varepsilon_{pp}-\mathbb{T}^0_{pp})\varphi,\psi)\right| \leq \varepsilon\, C\,\|\nabla\varphi\|_{L^2(D_p)}\,\|\psi\|_{L^2(\Sigma_p)}.$$

Hence $\|\mathbb{T}^\varepsilon_{pp}-\mathbb{T}^0_{pp}\|=O(\varepsilon)$. ■

Lemma 1 turns our problem into one of the simplest situation of perturbation theory [11]: the use of the Neumann series readily shows that

$$\|(I+\mathbb{K}^\varepsilon)^{-1}-(I+\mathbb{K}^0)^{-1}\|=O(\varepsilon).$$

It remains to notice that $\|f^\varepsilon-f^0\|$ and $\|g^\varepsilon-g^0\|$ are also of order ε, from which we conclude that

$$\|\varphi^\varepsilon-\varphi^0\|=O(\varepsilon). \tag{39}$$

The scattering amplitude

Thanks to formula (2), the local convergence expressed by the latter result also provides the far field asymptotics. Here, using our homothetic changes of variables, the scattering amplitude reads

$$A^\varepsilon(\alpha,\beta)=\frac{-\varepsilon}{4\pi}\sum_{p=1,N}\int_{\Gamma_p}\frac{\partial}{\partial\nu}(\varphi^\varepsilon_p(\alpha)-f^\varepsilon_p(\alpha))\,\overline{f^\varepsilon_p(\beta)}\,d\gamma.$$

On one hand (39) implies that $\partial\varphi^\varepsilon_p(\alpha)/\partial\nu$ tends to $\partial\varphi^0_p(\alpha)/\partial\nu=-u^\alpha_I(s_p)\,\partial V_p/\partial\nu$ in $H^{-1/2}(\Gamma_p)$ (recall that V_p is defined in (36)). On the other hand $f^\varepsilon_p(\alpha)$ tends to the constant function

$f_p^0(\alpha) = -u_I^\alpha(s_p)$. Hence

$$\begin{aligned} A^\varepsilon(\alpha,\beta) &= \frac{-\varepsilon}{4\pi} \sum_{p=1,N} C_p \, u_I^\alpha(s_p) \, \overline{u_I^\beta(s_p)} \; + \; O(\varepsilon^2) \quad \text{where} \\ C_p &= \int_{\Gamma_p} \frac{\partial V_p}{\partial \nu} \, d\gamma \; = \; \int_{\mathbb{R}^3 \setminus \mathcal{O}_p} |\nabla V_p|^2 \end{aligned}$$

is referred to as the *capacity* of the obstacle $\mathcal{O}_p$. Proposition 8 is thus proved.

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