# Lower bounds for the density of the law of locally elliptic Itô processes

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**Abstract:** We give lower and upper bounds for the denisty of the law of a locally elliptic Ito process. Locally elliptic means that the ellipticity assuption holds true only if the process lies in a tube around a deterministc given curve. This represents a generalization of the classical lower and upper bounds for the density of an uniformlly elliptic diffusion process but works for diffusions which may be locally degenerated, as the log-normal type diffusion for example. In this case the lower bound is no more Gaussian but has a log normal shape. This also works for Stochastic PDE's.

**Key-words:** Local density, lower bounds, local ellipticity, conditional Malliavin

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# Limites inférieures pour la densité de la loi d'un processus d'Itô localement elliptique

**Résumé :** On donne des minoration et des majoration de la densité de la loi marginale d'un processus d'Itô localement elliptique. Le caractère local de notre hypothèse consiste dans le fait qu'on suppose que les coefficients de diffusion ont la propriété d'ellipticité seulement si le processus d'Itô lui même ce trouve dans un voisinage d'une courbe détérministe donnée. C'est une généralisation du résulat classique concernnant les diffusions uniformement elliptiques mais il a l'avantage de s'appliquer aussi à des diffusions localement dég énèrés avec des coefficients non bornés. Par éxample pour des diffusions de type log-normal. Mais dans ce cas les minorations optimales ne sont plus de type Gaussien mais elles ont une forme compatible avec la loi log-normale. Le résultat s'applique aussi à des EDP Stochastiques.

**Mots-clés :** Densité local, minoration, ellipticité locale, Calcul de Malliavin conditionel.

# 1 Introduction

It is well known that under uniform ellipticity and boundedness assumptions for the diffusion matrix, the law of a diffusion process is absolutely continuous with respect to the Lebesgue's measure and one has Gaussian type lower and upper bounds for the density of the law. This classical result has been extended (see [6]; [5],[11]) to the more subtle case when instead of ellipticity one has Hörmander type hypothesis. In this paper we do not go in this sense. On the other hand, as an application of Malliavin's calculus, one has proved that, under appropriate hypothesis, a large variety of functionals on the Wiener space (for example solutions of Stochastic PDE's) have absolute continuous laws and the density is smooth (see Nualart [10]). Using already standard techniques one may also prove that some Gaussian upper bounds hold true. In number of cases one also succeed to prove that the density is strictly positive (see for example [1],[4],[2],[9] or [10]). But the techniques used in order to prove strict positivity are rather qualitative and do not provide lower bounds. So this remains a challenging problem. In a recent paper [7], A. Kohatzu-Higa developed a strategy which permits to attack this problem for abstract Wiener functionals. He gives a frame which essentially expresses the idea of uniform ellipticity for a Wiener functional and then develops a methodology for computing lower bounds. This paper was the starting point of our work and several important ideas come from there. But we do no more assume uniform ellipticity but only local ellipticity around a deterministic curve. This is much stronger and in particular permits to treat interesting new examples even in the classical frame of diffusion process. An outstanding example is that of log-normal type diffusions the coefficients of which are degenerated in zero and have unbounded coefficients. In this case the lower bound is no more Gaussian but has a shape which is compatible with the shape of the density of a log-normal distribution. Let us be more precise. We consider a

$q-$dimensional Ito process of the form

$$X_t^i = x_0^i + \sum_{j=1}^{\infty} \int_0^t U_s^{ij} dB_s^j + \int_0^t V_s^i ds, \quad i = 1, ..., q$$

and we are interested by the local density of $X_T$ in a point $y$. We assume that $U$ and $V$ are smooth in Malliavin's sense so that $X_T$ is smooth also. We give now the non-degeneracy assumption. We fix a differentiable curve $x_t, 0 \leq t \leq T$ such that $x_0 = X_0$ and $x_T = y$ and some $r_t > 0, 0 \leq t \leq T$. We also consider some $q \times q$ dimensional symmetric and positive definite matrixes $Q_t, 0 \leq t \leq T$ and denote by $\lambda_t > 0$ the lower eigenvalue of $Q_t$. Then our hypothesis are the following. For every $0 < t < T$ and $0 < \delta < T - t$

$$\begin{array}{rrcl} (H,i) & U_t U_t^* & \geq & Q_t \\ (H^\nu, ii) & \|\Gamma_\delta(t)\|_{k,p,t} & \leq & K(t)\delta^{\frac{1}{2}+\nu}, \nu > 0, \\ (H,iii) & \left( \sum_{i=1}^{q} \sum_{j=1}^{\infty} \left| U_t^{ij} \right|^2 \right)^{1/2} & \leq & K'(t) \end{array}$$

on the set $\left|Q_t^{-1/2}(X(t)-x(t))\right| \le r(t)$.

Let us explain this definition. One writes

$$X^i_{t+\delta} = X^i_t + \sum_{j=1}^{\infty}\int_t^{t+\delta} U^{ij}_s dB^j_s + \Gamma^i_\delta(t)$$
$$\text{with } \Gamma^i_\delta(t) := \sum_{j=1}^{\infty}\int_t^{t+\delta}(U_s-U_t)^{ij}dB^j_s + \int_t^{t+\delta} V^i_s ds.$$

The principal term is $\sum_{j=1}^{\infty}\int_t^{t+\delta} U^{ij}_t dB^j_s$ which is Gaussian conditionally to $F_t=\sigma(B^j_s, s\le t, j\in N)$. and has the covariance matrix $U_tU^*_t\times\delta$. So $(H,i)$ says that the principal term is non-degenerated. It represents the ellipticity assumption. $\Gamma^i_\delta(t)$ is a remainder and $(H^\nu, ii)$ says that this remainder may be ignored with respect to the principal term which is essentially of order $\delta^{1/2}$. The number $\nu$ is a strictly positive number which depends on the problem at hand - in the frame of diffusion processes $\nu=\frac{1}{2}$ and for Stochastic $PDE's, \nu=\frac{1}{4}$. The norm $\|\circ\|_{k,p,t}$ is a Sobolev norm which involves the $L^p$ norms of the first $k$ Malliavin derivatives where $p,k$ are some integers depending on the dimension $q$. The lower index $t$ signifies that we work with conditional expectations with respect to $F_t$ and not with usual expectations (we use a conditional Malliavin calculus).

Let us now comment the localization. Both $U_t=U_t(\omega)$ and $\|\Gamma_\delta(t)\|_{k,p,t}=\|\Gamma_\delta(t)\|_{k,p,t}(\omega)$ are random variables. So the properties $(H,i),(H^\nu,ii)$ and $(H,iii)$ hold true only for $\omega\in\{\left|Q_t^{-1/2}(X(t)-x(t))\right|\le r(t)\}$. Think for a moment to the example of the diffusion process $dX_t=\sigma(t,X_t)dB_t+b(t,X_t)dt$. Then $U_t=\sigma(t,X_t)$ and so $(H,i)$ says that $\sigma\sigma^*(t,x)\ge Q_t$ for $x$ such that $\left|Q_t^{-1/2}(x-x(t))\right|\le r(t)$. So we need the ellipticity assumption only on a tube around the curve $x_t$. The same is true for the boundedness assumption $(H,iii)$ : we need $|\sigma\sigma^*(t,x)|\le K'(t)$ only for $x$ in the tube. In particular this permits to handle unbounded coefficients (with linear growth for example).

We fix now some $h>0$ and define

$$\gamma_h(t) \;:\; = \min\{h, \frac{3}{2}r(t), \frac{1}{\sup_{t\le s\le t+h}\left|Q^{-1/2}(t)\partial_s x_s\right|}, \frac{\lambda^{(1+\nu)/2\nu}(t)}{C((K'(t)+\sqrt{\lambda(t)})K(t))^{1/2\nu}}\}$$
$$\mu_h(t) \;:\; = \sup_{0\vee(t-h)\le s\le t}\frac{1}{\gamma_h(s)}$$

where $C$ is an universal explicit constant. Our main result is the following. Under the above assumptions $(H,i),(H^\nu,ii),(H,iii)$ the random variable $X_T$ has a local density $p_T(x_0,y)$

which is continuous and moreover

$$\begin{aligned}\frac{C}{T^{q+2}\lambda_T^{q+2}}(P(|X_T-x_0| &\geq |y-x_0|))^{1/2} \\ &\geq p_T(x_0,y) \geq \frac{1}{4e(2\pi\gamma_h(0)\lambda(0))^{q/2}} \times \exp(-C'\int_0^T \mu_h^2(t)dt).\end{aligned}$$

Note that the above bounds hold true for every $h > 0$ and every deterministic path $x_t$ for which the ellipticity assumption holds true. So one may take the supremum over such paths in the lower bound. For example, in the case of log-normal type diffusions, if one takes $x_t$ to be the stright line which ilnks $x_0$ to $y$, the corresponding lower bounds are Gaussian and this is not optimal. At the contrary, if one takes exponential type curves then one obtains a lower bound which is of the same type as the density of a log normal law.

The idea of the proof is the following. One constructs recursively the time grid $t_0 = 0, \delta_0 = \gamma_h^2(t_0), t_1 = t_0 + \delta_0, ..., \delta_k = \gamma_h^2(t_{k-1}), t_k = t_{k-1} + \delta_k, ...$ and takes $N$ to be the fist integer such that $t_N \geq T$. Now, for each $k$ one writes

$$X_{t_k}^i = X_{t_{k-1}}^i + \sum_{j=1}^{\infty} \int_{t_{k-1}}^{t_{k-1}+\delta_k} U_{t_{k-1}}^{ij} dB_s^j + \Gamma_{\delta_k}^i(t_{k-1}). \tag{1}$$

Using the ellipticity assumption $(H, i)$ and the conditional Gaussian law ofthe random variable $\sum_{j=1}^{\infty} \int_{t_{k-1}}^{t_{k-1}+\delta_k} U_{t_{k-1}}^{ij} dB_s^j$ one derives a lower bound for the density of this term and then, using $(H^\nu, ii)$ one proves that the remainder $\Gamma_{\delta_k}^i(t_{k-1})$ may be ignored (it is in a lower scale). So one obtains a lower bound for conditional density of the type $P(X_{t_k} \in dy \mid X_{t_{k-1}} = x)$. The final lower bound will be obtained as the convolution of the $N$ lower bounds for the conditional density corresponding to each $t_k, k = 0, ..., N$. But note that the above evaluations hold true only if the difussion $X_t$ remains in a tube of radius $r_t$ around the curve $x_t$ (because our hypothesis are in force in this case only). The key problem is to find the good equilibrium between $\delta_k$ and the size of $X_{t_k} - X_{t_{k-1}}$ on one hand and to evaluate the probability that the process $X_t$ remains in a tube of radius $r_t$ around the curve $x_t$ on the other hand. This leads to the definition of the function $\gamma_h(t)$. Think for a moment that $X$ is a diffusion process so that $U_t = \sigma(t, X_t)$. Then there is a manifest analogy between the decomposition used in (1) and the one used in the parametrix method (compare with (4.1), (4.2) page 14 in [5]). In some way we use a stochastic version of the parametrix method on small intervals of time. There are two points on which the stochastic approch has some advantages: first of all it permits to localize on the set of trajectories which remain in a tube around the deterministic curve and this permits to deal with a certain class of diffusions which are not uniform elliptic and do not have bounded coefficients. In particular, the shape of the lower bound in such a case may be different from the Gaussian one. Moreover, it permits to deal with non-Markov problems.

The paper is organized as follows. In the first section we treat the problem in an abstract frame which is inspired from [7]. But the main evaluation, contained in Theorem 1 is stronger than Kohatsu Higa's result. In particular it permits to derive some tubes evaluations which are crucial in order to obtain local results. This section contains the main technical tools and results. In the second section we deal with Itô processes and we obtain the result presented above. In the third section we discuss the example of diffusion process. In order to illustrate our method we consider log-normal type diffusions (that is $\sigma(t,x)=\alpha(t,x)x$) and compare our lower bounds with the precise log-normal density. Finally in the last section we consider the example of Stochastic $PDE's$ (which has first been treated in [7]).

# 2 The general theory

## 2.1 Conditional Malliavin calculus

In this section we will use partial Malliavin calculus so we have to introduce some notation. We consider an infinite dimensional Brownian motion $B=(B^j)_{j\in N}$ and denote by $F_t$ the filtration generated by $B$. We refer to Nualart [10] for the notation. In all this section we fix some $t\geq 0, \delta>0$. What is specific in our frame is that we employ the partial Malliavin calculus on the interval $[t,t+\delta]$. This means that we consider derivatives with respect to $B_s, s\in[t,t+\delta]$ only and we employ the corresponding Sobolev norms, under the conditional expectation with respect to $F_t$. For example, $D_t^{1,2}$ designes the space of the functionals which are one times differentiable in $L^2$, in Malliavin's sense, on the interval $[t,t+\delta]$. More precisely, this space is constructed in the following way. One considers the simple functionals $F=f(G,B_{s_1}-B_{s_0},...,B_{s_k}-B_{s_{k-1}})$ where $t=s_0<s_1<...<s_k=t+\delta$, $G$ is an $F_t$ measurable random variable and $f(G,y_1,...,y_k)$ is a smooth function with respect to $(y_1,...,y_k)$. For such a functional one defines the Malliavin derivative by

$$D_sF=\sum_{i=1}^k\frac{\partial f}{\partial y_i}(G,B_{s_1}-B_{s_0},...,B_{s_k}-B_{s_{k-1}})1_{[s_i,s_{i-1})}(s).$$

Then one considers the norm

$$\|F\|_{1,2,t}^2:=E_t(|F|^2)+E_t(\int_t^{t+\delta}|D_sF|^2\,ds).$$

Here and in the sequel $E_t$ designes the conditional expectation with respect to $F_t$ that is

$$E_t(\Phi):=E(\Phi\mid F_t).$$

So $\|F\|_{1,2,t}$ is an $F_t$ measurable random variable and not a constant. We define $D_t^{1,2}$ to be the closure of the simple functionals space with respect to this norm. So $F\in D_t^{1,2}$ means that there exists a sequence of simple functionals $F_n, n\in N$, such that $\|F_n-F_m\|_{1,2,t}\to 0$ and $E\left|F_n-F\right|^2\to 0$ almost surely as $n,m\to\infty$. In the same way one defines higher order derivatives and the Orenstein Ulemback operator $L$. For $F \in (D_t^{1,2})^q, F = (F^1, ..., F^q)$ the Malliavin covariance matrix is defined by

$$\phi_{F,t}^{ij} = \int_t^{t+\delta} D_s F^i D_s F^j ds.$$

We denote by $\underline{\lambda}_{F,t}$(respectively $\overline{\lambda}_{F,t}$) the lower (respectively the larger) eigenvalue of $\phi_{F,t.}$ They are given by

$$\underline{\lambda}_{F,t} = \inf_{|\xi|=1} \sum_{i,j=1}^{q} \xi_i \xi_j \phi_{F,t}^{ij}, \quad \overline{\lambda}_{F,t} = \sup_{|\xi|=1} \sum_{i,j=1}^{q} \xi_i \xi_j \phi_{F,t}^{ij}.$$

In order to define higher order derivatives we define the Sobolev norms

$$\|F\|_{k,p,t}^p := E_t(|F|^p) + \sum_{1 \leq |\alpha| \leq k} E_t((\int_t^{t+\delta} ... \int_t^{t+\delta} \left| D_{s_1,...,s_{|\alpha|}}^{\alpha} F \right|^2 ds_1 ... ds_{|\alpha|})^{p/2})$$

where, for a multi-index $\alpha = (\alpha_1, ..., \alpha_m)$ we denote $|\alpha| = m$. The derivatives $D_{s_1,...,s_{|\alpha|}}^{\alpha} F$ are first defined on simple functionals and then extended to general funcionals by passing to the limit in $\|\circ\|_{k,p,t}$. We denote by $D_t^{k,p}$ the closure of the simple functionals space with respect to $\|\circ\|_{k,p,t}$. The basic fact which we will use is the integration by parts formula which we give in the following. We consider $F = (F_1, ..., F_q)$, $F_i, G \in D_t^{k,p}$ and we assume that $F$ is non degenerated (that is $\underline{\lambda}_{F,t} > 0$ almost surely). For a function $f : R^q \to R$ which is infinitely differentiable and for a multi-index $\alpha = (\alpha_1, ..., \alpha_k)$ denote $D^{\alpha}$ the corresponding derivative that is $D^{\alpha} f = \partial^k f / \partial x_{\alpha_1} ... \partial x_{\alpha_k}$. Then

$$\begin{aligned} (IP_i) \quad E_t(\frac{\partial f}{\partial x_i}(F)G) &= E_t(f(F)H_i(F,G)) \quad with \\ H_i(F,G) &= -\sum_{j=1}^{q} (G\widehat{\phi}_{F,t}^{ji} L(F_j) + \int_t^{t+\delta} D_s F_j D_s (\widehat{\phi}_{F,t}^{ji} G) ds \end{aligned}$$

where $\widehat{\phi}_{F,t}$ designees the inverse of $\phi_{F,t}$ and $L$ designes the Ornstein Uhlambeck operator with respect to $B_s, s \in [t, t+\delta]$. This formula may be iterated in order to obtain

$$\begin{aligned} (IP_{\alpha}) \quad E_t(D^{\alpha} f(F)G) &= E_t(f(F)H_{\alpha}(F,G)) \quad with \\ H_{\alpha}(F,G) &= H_{\alpha_k}(F, H_{(\alpha_1,...,\alpha_{k-1})}(F,G)) \quad for \quad \alpha = (\alpha_1, ..., \alpha_k). \end{aligned}$$

We make two remarks concerning the above integration by parts formula. The first one concerns a localization procedure. We asked $\phi_{F,t}$ to be invertible. But taking a look to $(IP_i)$ we see that the matrix $\phi_{F,t}$ has to be invertible in the region where $G$ is not null, or put it otherwise, $G$ has to be null in the region in which $\phi_{F,t}$ is not invertible. This is true for the first term in $H_i(F,G)$. In the second term appears $D_s(\widehat{\phi}_{F,t}^{ji} G) = D_s \widehat{\phi}_{F,t}^{ji} G + \widehat{\phi}_{F,t}^{ji} D_s G$.

We need that $D_sG$ is also null in the region in which $\phi_{F,t}$ is degenerated. Roughly speaking this will be true if "$G$ is null if $\underline{\lambda}_{F,t}$ is null". We do not formalize more this idea which will be used in the sequel in a more precise frame in which the above rough description is much more clear. The second remark concerns the main evaluation produced by the integration by parts formula. There exist some constants $p_m, k_m$ such that for a multi-index $\alpha$ with $|\alpha| \leq m$

$$|E_t(D^\alpha f(F)G)| \leq \left\| \widehat{\phi}^{ji}_{F,t} \right\|_{p_m,t} \|F\|_{p_m,k_m,t} \|G\|_{p_m,k_m,t} .$$

If one takes a look to the recursive definition of $H_\alpha$ then one believes that the above evaluation holds true, but expects that the norm of $L$ comes on as well. In fact one may dominate the norms of the Orenstein Ulembeack operator $L$ by the norms of the derivatives (the Sobolev norms) using Meyer's inequalities. If one do not want to use these inequalities, one has to add in the definition of $\|F\|_{p,k,t}$ the norms of a certain number of operations $L$ as well (so this is not a really important point, but almost a metter of notation).

NOTATION: In all the following we will work with $q$-dimensional functionals where $q$ is a fixed integer and we will use integration by parts $q+1$ times. So we will employ the above evaluation with $m = q+1$. We denote $k = k_{q+1}$ and $p = p_{q+1}$ and use this notation throughout the whole paper. To finish we give the following simple fact concerning the Malliavin covariance matrix:

**Proposition 1** *Let $F, G \in (D^{1,2})^q$ . Then*

$$(\det \phi_{F+G,t})^{1/q} \geq \frac{1}{2}\underline{\lambda}_{F,t} - \overline{\lambda}_{G,t}.$$

**Proof.** Using the elementary inequality $(x+y)^2 \geq \frac{1}{2}x^2 - y^2$ one obtains

$$\begin{aligned}
(\det \phi_{F+G,t})^{1/q} &\geq \underline{\lambda}_{F+G,t} = \inf_{|\xi|=1} \int_t^{t+\delta} \left( \sum_{i=1}^q \xi_i D_s(F^i + G^i) \right)^2 ds \\
&\geq \frac{1}{2} \inf_{|\xi|=1} \int_t^{t+\delta} \left( \sum_{i=1}^q \xi_i D_s F^i \right)^2 ds - \sup_{|\xi|=1} \int_t^{t+\delta} \left( \sum_{i=1}^q \xi_i D_s G^i \right)^2 ds \\
&= \frac{1}{2}\underline{\lambda}_{F,t} - \overline{\lambda}_{G,t}.
\end{aligned}$$

■

## 2.2 Short time behavior and density evaluations

We consider now some measurable processes $h^{ij}(s), s \in [t, t+\delta], i = 1, ..., q, j \in N$ such that $h^{ij}(s)$ is $F_t$ measurable and denote

$$\|h\|_t^2 := \frac{1}{\delta} \sum_{i=1}^q \sum_{j=1}^\infty \int_t^{t+\delta} \left| h^{ij}(s) \right|^2 ds.$$

We assume that $\|h\|_t < \infty$ almost surely. Then we may define

$$J^i(h) = \sum_{j=1}^{\infty} \int_t^{t+\delta} h^{ij}(s) dB^j(s).$$

Since $h(s)$ is $F_t$ measurable, conditionally with respect to $F_t$, $J(h)$ is a Gaussian vector of covariance matrix

$$C^{ij}(J) = \sum_{k=1}^{\infty} \int_t^{t+\delta} h^{ik}(s) h^{jk}(s) ds.$$

Given some $F_t$ measurable random variable $V$ we define

$$G = V + \sum_{j=1}^{\infty} \int_t^{t+\delta} h^{ij}(s) dB^j(s) = V + J(h).$$

We consider a deterministic, symmetric, positive definite, $q \times q$ dimensional matrix $M$ which is invertible and we denote by $\Delta$ its smaller eigenvalue (which is strictly positive). We also denote $\Theta = M^{-1/2}$. This matrix is fixed in this section and our hypothesis are given in terms of $M$.

Given a point $z \in R^q$ and a set $A \subseteq \{\omega : |\Theta(V(\omega) - z)| \leq 1\}$ we consider the hypothesis

$$(H_1, A, z) \quad C(J) \geq M \quad \text{on the set } A.$$

Finally we consider a function $\phi \in C_b^\infty(R^q)$, $0 \leq \phi \leq 1$, $\int \phi = 1$ and $\phi(y) = 0$ for $|y| > 1$. This is an auxiliary function which is fixed in the sequel. We construct the sequence $\phi_\eta \to \delta_0$ defined by $\phi_\eta(y) = \eta^{-q}\phi(\eta^{-1}y)$.

We look now to $E_t(\phi_\eta(G - z))$ on the set $A$ where $V$ is closed to $z$ (in the sense given by $\Theta$). We think to this quantity as to an approximation of the density of $G - V = J(h)$ conditionally to $V = z$.

**Lemma 1** *Suppose that $(H_1, A, z)$ holds true. Then for every $\eta \in (0, \sqrt{\Delta})$*

$$E_t(\phi_\eta(G - z)) \geq \frac{1}{(2\pi\Delta)^{q/2} e^2} \quad \textit{on the set } A.$$

**Proof.** We denote by $< .,. >$ the scalar product in $R^q$ and we write

$$E_t(\phi_\eta(G-z)) = \int \phi_\eta(y) \frac{1}{(2\pi)^{q/2}\sqrt{\det C(J)}} \exp(-\frac{1}{2} < C(J)^{-1}(y-(V-z)), (y-(V-z)) >) dy.$$

By $(H_1, A, z)$, on the set $A$ we have $C(J) \geq M$ and consequently $C(J)^{-1} \leq M^{-1}$. For $|y| \leq \eta \leq \sqrt{\Delta}$ one has

$$\begin{aligned} &< C(J)^{-1}(y - (V - z)), (y - (V - z)) >^{1/2} \leq < M^{-1}(y - (V - z)), (y - (V - z)) >^{1/2} \\ \leq & < M^{-1}y, y >^{1/2} + < M^{-1}(V - z), (V - z) >^{1/2} \leq \frac{1}{\sqrt{\Delta}} |y| + |\Theta(V - z)| \leq 2 \end{aligned}$$

the last inequality being a consequence of the fact that, on the set $A, |\Theta(V-z)| \leq 1$. Since $\det C(J) \geq \Delta^q$ and $\int \phi_\eta = 1$ the proof is completed. ■

We consider now a $q-$dimensional random variable $R = (R^1, ..., R^q), R^i \in D_t^{k,p}$ (recall that $k = k_{q+1}, p = p_{q+1}$) and define

$$F = G + \sqrt{\Delta}R = V + \sum_{j=1}^{\infty} \int_t^{t+\delta} h^{ij}(s) dB^j(s) + \sqrt{\Delta}R.$$

Given a point $z \in R^q$ a set $A \subseteq \{|\Theta(V-z)| \leq 1\}$ and a number $\nu > 0$ we consider the hypothesis

$$(H_2^\nu, A, z) \quad 1_A \|R\|_{k,p',t} \leq C\Delta^\nu \tag{2}$$

with $p' = p'_{q+1} = 2^{k_{q+1}} p_{q+1} = 2^k p$.

**Remark 1** *In the frame of diffusion processes $\nu = \frac{1}{2}$ and in the frame of stochastic $PDE's, \nu = \frac{1}{4}$.*

We also denote

$$\Lambda := \int_t^{t+\delta} |D_s R|^2 \, ds = \int_t^{t+\delta} \sum_{i=1}^{q} \left|D_s R^i\right|^2 ds$$

and notice that

$$\overline{\lambda}_{R,t} = \sup_{|\xi|=1} \sum_{i,j=1}^{q} \xi_i \xi_j \int_t^{t+\delta} D_s R^i D_s R^j ds = \sup_{|\xi|=1} \int_t^{t+\delta} <\xi, D_s R>^2 \, ds \leq \Lambda.$$

As a consequence of $(H_2^\nu, A, z)$, we have

$$E_t \left|\Lambda\right|^{p/2} \leq C^p \Delta^{\nu p}.$$

The key evaluation in our approach is given by the following proposition.

**Proposition 2** *We consider a point $z \in R^q$, a set $A \subseteq \{|\Theta(V-z)| \leq 1\}, \nu > 0$ and $\eta \in (0, \sqrt{\Delta})$. Suppose that $(H_1, A, z)$ and $(H_2, A, z)$ hold true, $a\delta \leq \Delta \leq \delta$ for some $a > 0$, and $\delta$ is sufficiently small in order that*

$$q 8^k \left(\frac{2\|h\|_t}{\sqrt{a}} + 1\right) C\delta^\nu \leq \frac{1}{4e^2(2\pi)^{q/2}} \quad \text{on the set } A. \tag{3}$$

*Then*

$$p_\eta(z) := E_t \phi_\eta(F-z) \geq \frac{1}{4e^2(2\pi\Delta)^{q/2}} \quad \text{on the set } A. \tag{4}$$

**Proof.**

- **Step 1. Renormalization.** We write

$$F = V + \sqrt{\Delta}(I + R)$$

where

$$I = \frac{1}{\sqrt{\Delta}} J(h).$$

- **Step 2. Localization.** In the sequel we assume (withought special mention) that we are on the set $A$ and in particular $|\Theta(V - z)| \leq 1$. Since $I$ is Gaussian its Malliavin covariance matrix coincides with its usual covariance matrix which is equal to $\Delta^{-1}C(J)$. So $\underline{\lambda}_{I,t} \geq 1$. Using our first proposition we obtain for every $\rho \in (0, 1)$

$$\det \phi_{I+\rho R,t} \geq (\frac{1}{2}\underline{\lambda}_{I,t} - \overline{\lambda}_{\rho R,t})^q = (\frac{1}{2} - \rho\overline{\lambda}_{R,t})^q \geq (\frac{1}{2} - \Lambda)^q.$$

The aim of this step is to localize on the set on which $\Lambda \leq 1/4$ and consequently $\det \phi_{I+\rho R,t} \geq 1/4^q$. We consider a localization function $\theta \in C_b^\infty(R_+; R_+)$ such that $0 \leq \theta \leq 1, \theta(x) = 1$ if $x < 1/8$ and $\theta(x) = 0$ if $x > 1/4$ and we denote $Q = \theta(\Lambda)$. We may choose $\theta$ such that for every multi-index $\alpha$ of length less or equal to $k$ one has $\|D^\alpha \theta\|_\infty \leq 8^{|\alpha|}$. Since $Q = 0$ on the set $\Lambda \geq \frac{1}{4}$ we have

$$Q \det \phi_{I+\rho R,t} \geq Q(\frac{1}{2} - \Lambda)^q \geq \frac{Q}{4^q}. \tag{5}$$

Since for every multi-index $\alpha = (\alpha_1, ..., \alpha_i)$, $D^\alpha \theta(x) = 0$ if $x > 1/4$ we also have $D^\alpha Q = 0$ on the set $\Lambda \geq \frac{1}{4}$ and so

$$\left|(D^{(\alpha_1,...,\alpha_i)}_{(s_1,...,s_i)} Q) \times \det \phi_{I+\rho R,t}\right| \geq \frac{1}{4^q}\left|D^{(\alpha_1,...,\alpha_i)}_{(s_1,...,s_i)} Q\right|. \tag{6}$$

- **Step 3. Sobolev norms.** Let us evaluate the Sobolev norm of $Q$ on the set $A$. First of all it is clear that $\|Q\|_{p,t} = (E_t |Q|^p)^{1/p} \leq 1$, and one can not hope to do better. Moreover, for $|\alpha| \leq k$ one has $\|D^\alpha \theta\|_\infty \leq 8^{|\alpha|}$ so that

$$\begin{aligned}(E_t(\int_t^{t+\delta} ... \int_t^{t+\delta} \left|D^\alpha_{s_1,...,s_k} Q\right|^2 ds_1...ds_k)^{p/2})^{1/p} &\leq 8^{|\alpha|} \times \|R\|_{k,p',t} \\ &\leq 8^k C\Delta^\nu \leq 8^k C\delta^\nu \leq 1\end{aligned}$$

the last inequality being a consequence of our hypothesis on $\delta$. So

$$\|Q\|_{k,p,t} \leq 1$$

We evaluate now the Sobolev norm of $1-Q$. Since $0\leq Q\leq 1$ and $Q=1$ on the set $\Lambda\leq 1/8$ we have

$$\begin{aligned}\|1-Q\|_{p,t}^{p} &\leq P_t(\Lambda\geq\frac{1}{8})=P_t(\Lambda^{p/2}\geq\frac{1}{8^{p/2}})\\ &\leq 8^{p/2}E_t(\Lambda^{p/2})\leq 8^{p/2}C^p\Delta^{p\nu}\leq 8^{p/2}C^p\delta^{p\nu}.\end{aligned}$$

For $\alpha$ with $|\alpha|=m\leq k$

$$\begin{aligned}&E_t(\int_t^{t+\delta}...\int_t^{t+\delta}\left|D^{\alpha}_{s_1,...,s_m}(1-Q)\right|^2 ds_1...ds_m)^{p/2}\\ =\ &E_t(\int_t^{t+\delta}...\int_t^{t+\delta}\left|D^{\alpha}_{s_1,...,s_m}Q\right|^2 ds_1...ds_m)^{p/2}\leq 8^{pk}C^p\delta^{p\nu}\end{aligned}$$

so that

$$\|1-Q\|_{k,p,t}\leq 8^kC\delta^{\nu}.$$

Finally we evaluate $\|I\|_{k,p,t}$. Note that $D_s^jI^i=\frac{1}{\sqrt{\Delta}}h_s^{ij}$ and $D^{\alpha}I=0$ for $|\alpha|>1$. We have

$$(E_t(\int_t^{t+\delta}|D_sI|^2 ds)^{p_{q+1}/2}))^{1/p_{q+1}}=\frac{1}{\sqrt{\Delta}}(\int_t^{t+\delta}\sum_{i=1}^{q}\sum_{j=1}^{\infty}\left|h_s^{ij}\right|^2 ds)^{1/2}\leq\frac{\|h\|_t}{\sqrt{a}}$$

The same is true for $(E_t(|I|^p))^{1/p}$ so that

$$\|I\|_{k,p,t}\leq\frac{2\,\|h\|_t}{\sqrt{a}}.$$

- **Step 4. Development in Taylor series of order one.** We localize first (multiply by $Q$) and use then a development in Taylor series with respect to $R$ in order to obtain (note that $\phi_\eta\geq 0$)

$$\begin{aligned}p_\eta(z) &\geq E_t(\phi_\eta(V-z+\sqrt{\Delta}(I+R))Q)\\ &= E_t(\phi_\eta(G-z)Q)+\sum_{i=1}^{q}\int_0^1E_t(\frac{\partial\phi_\eta}{\partial x^i}(V-z+\sqrt{\Delta}(I+\rho R))\sqrt{\Delta}R^iQ)d\rho\\ &= :J+J'.\end{aligned}$$

Let us evaluate the reminder $J'$. We define

$$\Phi_\eta(x)=:\int_{-\infty}^{x_1}dy_1...\int_{-\infty}^{x_q}dy_q\phi_\eta(V-z+\sqrt{\Delta}y)$$

so that

$$\phi_\eta(V - z + \sqrt{\Delta}x) = \frac{\partial^q \Phi_\eta}{\partial x_1 ... \partial x_q}(x)$$

and

$$\frac{\partial \phi_\eta}{\partial x^i}(V - z + \sqrt{\Delta}x) = \frac{\partial^{q+1}\Phi_\eta}{\partial x^i \partial x_1 ... \partial x_q}(x) \quad \textit{and} \quad 0 \leq \Phi_\eta(x) \leq \frac{1}{\Delta^{q/2}}$$

the last inequality being obtained using the change of variable $x = V - z + \sqrt{\Delta}y$ and of the fact that $\int \phi_\eta = 1$. We use the integration by parts formula $q+1$ times (with respect to the random variable $I + \rho R$) and we obtain

$$\begin{aligned} & E_t(\frac{\partial \phi_\eta}{\partial x_i}(V - z + \sqrt{\Delta}(I + \rho R))R^i Q) \\ = \ & E_t(\frac{\partial^{q+1}\Phi_\eta}{\partial x_i \partial x_1 ... \partial x_q}(I + \rho R) R^i Q) \\ = \ & E_t(\Phi_\eta(I + \rho R) H_{(1,2,\ldots,q,i)}(I + \rho R, R^i Q)). \end{aligned}$$

The random variable $H_{(1,2,\ldots,q,i)}(I + \rho R, R^i Q)$ is the weight which appears in the Malliavin's integration by parts formula $q+1$ times. By (5), (6) we know that we may minored the determinant of the Malliavin covariance matrix which appears in $H_{(1,2,\ldots,q,i)}(I + \rho R, R^i Q)$ by $1/4^q$. This is because $\widehat{\phi}_{I+\rho R}, t$ is always multiplied by $Q$ or one of its derivatives. Finally we dominate $\Phi_\eta$ by $\Delta^{-q/2}$ and we obtain

$$\begin{aligned} & \left| E_t(\frac{\partial \phi_\eta}{\partial x^i}(V - z + \sqrt{\Delta}(I + \rho R) R^i Q) \right| \\ \leq \ & \frac{1}{\Delta^{q/2}} \left\| H_{(1,2,\ldots,q,i)}(I + \rho R, R^i Q) \right\|_{k,p,t} \\ \leq \ & \frac{4^q}{\Delta^{q/2}} \left\| I + \rho R \right\|_{k,p,t} \left\| R^i Q \right\|_{k,p,t} \\ \leq \ & \frac{4^q}{\Delta^{q/2}} (\left\| I \right\|_{k,p,t} + \left\| R \right\|_{k,p,t}) \left\| R \right\|_{k,2p,t} \left\| Q \right\|_{k,2p,t} \\ \leq \ & \frac{4^q}{\Delta^{q/2}} (\frac{2\left\| h \right\|_t}{\sqrt{a}} + C\Delta^\nu) C\Delta^\nu \leq \frac{4^q}{\Delta^{q/2}} (\frac{2\left\| h \right\|_t}{\sqrt{a}} + 1) C\delta^\nu. \end{aligned}$$

We have proved

$$|J'| \leq \frac{q4^q}{\Delta^{q/2}} (\frac{2\left\| h \right\|_t}{\sqrt{a}} + 1) C\delta^\nu \sqrt{\Delta} \leq \frac{q4^q}{\Delta^{q/2}} (\frac{2\left\| h \right\|_t}{\sqrt{a}} + 1) C\delta^{\frac{1}{2}+\nu}.$$

We evaluate now $J$. We use the previous lemma and obtain

$$E_t(\phi_\eta(G - z)) \geq \frac{1}{(2\pi\Delta)^{q/2} e^2}.$$

In order to evaluate $E_t(\phi_\eta(G-z)(1-Q))$ we integrate by parts $q$ times with respect to $I$ and obtain

$$\begin{aligned} E_t(\phi_\eta(G-z)(1-Q)) &= E_t(\phi_\eta(V-z+\sqrt{\Delta}I)(1-Q)) \\ &= E_t(\frac{\partial^q\Phi_\eta}{\partial x_1...\partial x_q}(I)(1-Q)) \\ &= E_t(\Phi_\eta(I)H_{(1,2,...,q)}(I,(1-Q)). \end{aligned}$$

Since $\underline{\lambda}_{I,t}\geq 1$ and $0\leq\Phi_\eta(x)\leq\Delta^{-q/2}$ we obtain

$$|E_t(\phi_\eta(G-z)(1-Q))|\leq\frac{1}{\Delta^{q/2}}\left\|I\right\|_{k,p,t}\left\|1-Q\right\|_{k,p,t}\leq\frac{8^kC\delta^\nu}{\Delta^{q/2}}\frac{2\left\|h\right\|_t}{\sqrt{a}}.$$

It follows that

$$\begin{aligned} E_t(\phi_\eta(G-z))Q) &= E_t(\phi_\eta(G-z))-E_t(\phi_\eta(G-z))(1-Q)) \\ &\geq \frac{1}{e^2(2\pi\Delta)^{q/2}}-\frac{8^kC\delta^\nu}{\Delta^{q/2}}\frac{2\left\|h\right\|_t}{\sqrt{a}}\geq\frac{1}{2e^2(2\pi\Delta)^{q/2}} \end{aligned}$$

the last inequality being a consequence of our hypothesis on $\delta$. Finally we obtain

$$\begin{aligned} p_\eta(z) &\geq \frac{1}{2e^2(2\pi\Delta)^{q/2}}-|J'| \\ &\geq \frac{1}{2e^2(2\pi\Delta)^{q/2}}-\frac{q4^q}{\Delta^{q/2}}(\frac{2\left\|h\right\|_t}{\sqrt{a}}+1)C\delta^{\frac{1}{2}+\nu}\geq\frac{1}{4e^2(2\pi\Delta)^{q/2}} \end{aligned}$$

the last inequality being a consequence of our hypothesis (3). ■

## 2.3 Evolution sequences

The following objects are given.

◇ A time grid $\Pi_N=(t_0,...,t_N)$ with $0=t_0<t_1<...<t_N=T$. We denote $\delta_k=t_k-t_{k-1}$.

◇ A sequence of matrixes $M_k,k=0,...,N$ which are symmetric, positive definite and invertible. We denote by $\Delta_k$ the lower eigenvalue of $M_k$ and $\Theta_k=M_k^{-1/2}$. We suppose that $a_k\delta_k\leq\Delta_k\leq\delta_k$ for some given $a_k>0$.

◇ A sequence of points $x_k\in R^q,k=0,...,N$ such that

$$|\Theta_1(x_1-x_0)|\leq 1.$$

◇ A sequence of measurable processes $h_k^{ij}(s),s\in[t_{k-1},t_k],i=1,...,q,j\in N$ such that $h_k^{ij}(s)$ is $F_{t_{k-1}}$ measurable and

$$\left\|h_k\right\|_{t_{k-1}}^2:=\frac{1}{t_k-t_{k-1}}\sum_{i=1}^q\sum_{j=1}^\infty\int_{t_{k-1}}^{t_k}\left|h_k^{ij}(s)\right|^2ds<\infty$$

almost surely. Then we may define

$$J_k^i = \sum_{j=1}^{\infty} \int_{t_{k-1}}^{t_k} h_k^{ij}(s) dB^j(s).$$

Conditionally with respect to $F_{t_{k-1}}$, $J_k$ is a Gaussian vector of covariance matrix

$$C^{ij}(J_k) = \sum_{l=1}^{\infty} \int_{t_{k-1}}^{t_k} h_k^{il}(s) h_k^{jl}(s) ds.$$

◇ We introduce now the main object in this section, the evolution sequences. We consider a sequence of $R^q$ valued random variables $F_0, ..., F_N$ of the form

$$F_k = F_{k-1} + \sum_{j=1}^{\infty} \int_{t_{k-1}}^{t_k} h_k^j(s) dB^j(s) + \sqrt{\Delta_k} R_k = F_{k-1} + J_k + \sqrt{\Delta_k} R_k$$

where $R_k$ are $q-$dimensional $F_{t_k}$ measurable random variables. We are interested in the density of the conditional law of $F_k$ with respect to $F_{t_{k-1}}$. Since we do not know that a conditional density exists we are obliged to work with the following "regularization of the conditional density":

$$p_{\eta,k}(z) = E_{t_{k-1}} \phi_\eta(F_k - z).$$

This quantity makes sense independently of any non-degeneracy assumptions. We think tat it as an approximation of $E_{t_{k-1}} \delta_z(F_k)$ that is of the conditional density of $F_k$ computed in $z$.

◇ Finally we define the sets

$$A_k = \{\omega : |\Theta_i(F_{i-1}(\omega) - x_i)| < \frac{1}{2}, i = 1, ..., k\} \in F_{t_{k-1}}.$$

**Definition 1** *We say that $F_0, ..., F_N$ is an elliptic evolution sequence if $R_i \in D_{k,p,t_{k-1}}$ (recall that $k = k_{q+1}, p = p_{q+1}$) and*

$$\begin{array}{rrcl} (H, i) & C(J_i) & \geq & M_i \quad on \quad A_i, \\ (H^\nu, ii) & 1_{A_i} \left\| R_i \right\|_{k,p',t_{k-1}} & \leq & C_i \Delta_i^\nu \end{array}$$

*where $C_i, i = 1, ..., N$ is a sequence of positive numbers and $\nu > 0$.*

The frame given in the above definition has been introduced by A. Kohatzu Higa in [7].

The time grid $\Pi_N$, the path $x = (x_0, ..., x_N), \nu, M_k, \Delta_k, \delta_k, a_k, C_k, k = 1, ..., N$ appear as the parameters of the evolution sequence. The evaluations related to an evolution sequence are given in terms of these parameters.

As a consequence of the proposition from the previous section we have the following result.

**Proposition 3** *Let $F_1,...,F_N$ be an elliptic evolution sequence. We fix $k\in\{1,...,N\}$. Let $z\in R^q$ and $\eta>0$ be such that $|\Theta_k(x_k-z)|\leq\frac{1}{2}$ and $0<\eta\leq\sqrt{\Delta_k}$. Suppose that*

$$q8^k(\frac{\|h_k\|_{t_{k-1}}}{\sqrt{a_k}}+1)C_k\delta_k^\nu\leq\frac{1}{4e^2(2\pi)^{q/2}}\quad \textit{on the set } A_k. \tag{7}$$

*Then*

$$p_{\eta,k}(z)\geq\frac{1}{4e^2(2\pi\Delta_k)^{q/2}}\quad \textit{on the set } A_k.$$

**Proof.** Suppose that we are on the set $A_k$. Since $|\Theta_k(x_k-z)|\leq\frac{1}{2}$ we have $|\Theta_k(F_{k-1}-z)|\leq|\Theta_k(F_{k-1}-x_k)|+|\Theta_k(z-x_k)|\leq\frac{1}{2}+\frac{1}{2}$ and so $A_k\subseteq\{|\Theta_k(F_{k-1}-z)|\leq 1\}$ and, since we have an elliptic sequence, the hypothesis $(H_1,A_k,z)$ and $(H_2^\nu,A_k,z)$ hold true. So we may employ our previous result. ∎

## 2.4 Tubes evaluations

The aim of this section is to give lower bounds for $P(A_N)$. We prove first the following lemma.

**Lemma 2** *For every $\eta$ such that $\eta\in(0,\frac{1}{2}\sqrt{\Delta_k})$ one has*

$$P(A_k)\geq E(1_{A_{k-1}}\int_{|\Theta_k(y-x_k)|\leq(1/2)-\eta/\sqrt{\Delta_k}}p_{\eta,k-1}(y)dy). \tag{8}$$

**Proof.** We assume without loss of generality that the conditional law of $F_{k-1}$ with respect to $F_{t_{k-2}}$ is absolutely continuous with respect to the Lebesgue measure and has a density $\pi$ (if not we replace in a first stage $F_{k-1}$ by $F_{k-1}+\gamma_n$ where $\gamma_n$ is a centered Gaussian random variable of variance $\frac{1}{n}$, which is independent of $F_{t_{k-1}}$. We prove the inequality for $F_{k-1}+\gamma_n$ and then we pass to the limit and obtain the inequality for $F_{k-1}$). Since $A_k=A_{k-1}\cap\{|\Theta_k(F_{k-1}-x_k)|\leq\frac{1}{2}\}$ we have

$$P(A_k)=E(1_{A_{k-1}}\int\pi(y)1_{|\Theta_k(y-x_k)|\leq 1/2}dy)=E(1_{A_{k-1}}\int\pi(y+z)1_{|\Theta_k(y+z-x_k)|\leq 1/2}dy)$$

the last equality being true for every $z\in R^q$ (just because the Lebesgue measure is invariant to translations). We integrate in the above equality against $\phi_\eta(z)dz$ and we obtain

$$\begin{aligned}
P(A_k) &= E(1_{A_{k-1}}\int\int dz\phi_\eta(z)\pi(y+z)1_{|\Theta_k(y+z-x_k)|\leq 1/2}dy)\\
&= E(1_{A_{k-1}}\int\int du\phi_\eta(u-y)\pi(u)1_{|\Theta_k(u-x_k)|\leq 1/2}dy)\\
&= E(1_{A_{k-1}}\int E_{t_{k-2}}(\phi_\eta(F_{k-1}-y)1_{|\Theta_k(F_{k-1}-x_k)|\leq 1/2})dy)\\
&\geq E(1_{A_{k-1}}\int_{|\Theta_k(y-x_k)|\leq(1/2)-\eta/\sqrt{\Delta_k}}E_{t_{k-2}}(\phi_\eta(F_{k-1}-y)1_{|\Theta_k(F_{k-1}-x_k)|\leq 1/2})dy).
\end{aligned}$$

Note that, if $\phi_\eta(F_{k-1}-y)\neq 0$, then $|F_{k-1}-y|\leq \eta$. If we also have $|\Theta_k(y-x_k)|\leq (1/2)-\eta/\sqrt{\Delta_k}$ then

$$|\Theta_k(F_{k-1}-x_k)|\leq |\Theta_k(F_{k-1}-y)|+|\Theta_k(y-x_k)|\leq \frac{\eta}{\sqrt{\Delta_k}}+\frac{1}{2}-\frac{\eta}{\sqrt{\Delta_k}}=\frac{1}{2}$$

and so the above term is equal to

$$E(1_{A_{k-1}}\int_{|\Theta_k(y-x_k)|\leq(1/2)-\eta/\sqrt{\Delta_k}}E_{t_{k-2}}(\phi_\eta(F_{k-1}-y))dy)$$

and the proof is completed. ∎

**Corollary 1** *Let $F_k, k=0,...,N$ be an elliptic evolution sequence such that (7) holds true. For every $k=1,...,N$*

$$P(A_k)\geq \frac{c(q)}{4^{q+1}e^2(2\pi)^{q/2}}\frac{\Delta_k^{q/2}}{\Delta_{k-1}^{q/2}}P(A_{k-1}) \tag{9}$$

*where $c(q)$ is a constant such that $m(B_r(0))=c(q)r^q$ with $m$ the Lebesgue measure. In particular*

$$P(A_N)\geq (\frac{\Delta_N}{\Delta_1})^{\frac{q}{2}}(\frac{c(q)}{4^{q+1}e^2(2\pi)^{q/2}})^{N-1}\geq (\frac{\Delta_N}{\Delta_1})^{\frac{q}{2}}e^{-N\theta}. \tag{10}$$

*with*

$$\theta=\ln(\frac{4^{q+1}e^2(2\pi)^{q/2}}{c(q)}).$$

**Proof.** We take $\eta=\frac{1}{4}\sqrt{\Delta_k}$ so that $(1/2)-\eta/\sqrt{\Delta_k}=1/4$ and employ the above lemma and the density minoration in order to obtain

$$\begin{aligned}
P(A_k) &\geq E(1_{A_{k-1}}\int_{|\Theta_k(y-x_k)|\leq 1/4}p_{\eta,k-1}(y)dy)\\
&\geq \frac{1}{4e^2(2\pi\Delta_{k-1})^{q/2}}m(|\Theta_k(y-x_k)|\leq\frac{1}{4})P(A_{k-1})\\
&\geq \frac{1}{4e^2(2\pi\Delta_{k-1})^{q/2}}m(|y-x_k|\leq\frac{\sqrt{\Delta_k}}{4})P(A_{k-1})\\
&= \frac{c(q)}{4^{q+1}e^2(2\pi)^{q/2}}\frac{\Delta_k^{q/2}}{\Delta_{k-1}^{q/2}}P(A_{k-1})
\end{aligned}$$

and the proof of (9) is completed. In order to prove (10) we employ recurrence and obtain

$$P(A_N)\geq (\frac{c(q)}{4^{q+1}e^2(2\pi)^{q/2}})^{N-1}\prod_{k=2}^{N}(\frac{\Delta_k}{\Delta_{k-1}})^{\frac{q}{2}}P(|\Theta_1(F_0-x_1)|\leq\frac{1}{2}).$$

Since $|\Theta_1(F_0-x_1)|=|\Theta_1(x_0-x_1)|\leq\frac{1}{2}$, (10) is proved. ∎

## 2.5 The main result

Our final result is the following. We look for lower bounds for the density of $F_N$. We say that the law of $F_N$ has a local density $p_{F_N}$ with respect to the Lebesgue measure on $R^q$, in a neighborhood of $x_N$, if there exists some $\delta > 0$ such that for every smooth function $\psi$ with the support included in $B_\delta(x_N)$ one has

$$E\psi(F_N) = \int \psi(x) p_{F_N}(x) dx.$$

**Theorem 1** *Let $F_k, k = 0, ..., N$ be an elliptic evolution sequence which satisfies (7). Suppose that the law of $F_N$ has a continuous local density $p_{F_N}$ with respect to the Lebesgue measure on $R^q$, in a neighborhood of $x_N$. Then*

$$p_{F_N}(x_N) \geq \frac{1}{4e^2(2\pi\Delta_1)^{q/2}} e^{-N\theta}.$$

**Proof.** We use Proposition 5 in order to get

$$\begin{aligned}
\int_{R^q} p_{F_N}(x)\phi_\eta(x - x_N)dx &= E(\phi_\eta(F_N - x_N)) = E(E_{t_{N-1}}\phi_\eta(F_N - x_N)) \\
&\geq E(E_{t_{N-1}}\phi_\eta(F_N - x_N)1_{A_N}) \\
&\geq \frac{1}{4e^2(2\pi\Delta_N)^{q/2}} P(A_N) \\
&\geq \frac{1}{4e^2(2\pi\Delta_1)^{q/2}} e^{-N\theta}.
\end{aligned}$$

Now we use the continuity of $p_{F_N}$ and pass to the limit with $\eta \to 0$ in order to obtain the result. ■

# 3 Elliptic Itô processes

We consider a $q-$dimensional Itô process of the form

$$X_t^i = x_0^i + \sum_{j=1}^{\infty} \int_0^t U_s^{ij} dB_s^j + \int_0^t V_s^i ds, \quad i = 1, ..., q.$$

We assume that for every $T > 0$

$$\begin{aligned}
i) \quad E(\int_0^T (\|U_s\|^2 + |V_s|)ds) &< \infty \qquad (11) \\
ii) \quad U_s, V_s &\in D_t^{k,p} \quad \forall 0 \leq t \leq s \leq T
\end{aligned}$$

where

$$\|U_s\|^2 = \sum_{i=1}^{q}\sum_{j=1}^{\infty}\left|U_s^{ij}\right|^2 \quad and \quad |V_s|^2 = \sum_{i=1}^{q}\left|V_s^i\right|^2.$$

We fix $T$ and $y \in R^q$ and we study the density of the law of $X_T$ in $y$. In order to do it we have to give a non degeneracy assumption on $X_T$ in $y$, and this assumption is related with a path from $x_0$ to $y$. So we consider a continuous function $x:[0,T] \to R^q$ such that $x(0) = x_0$ and $x(T) = y$. We consider also some continuous, strictly positive functions $r, K, K' : [0,T] \to R_+$. The significance of these function is the following. We work on a tube around the path $x(t)$ and $r(t)$ represents the radius of this tube, $K$ controls the small increments of our process and $K'$ controls the norm of $U$ on the tube. Finally we consider $Q_t, t \in [0,T]$ which are symmetric positive definite and invertible matrixes. We denote by $\lambda_t$ the smaller eigenvalue of $Q_t$ and we assume that $\lambda_t \leq 1$. We also consider some $\nu > 0$. We fix now $1 \geq h > 0$ and define

$$\begin{aligned}
\gamma_h(t) \quad &: \quad = \min\{h, \frac{3}{2}r(t), \frac{1}{\sup_{t\leq s\leq t+h}\left|Q^{-1/2}(t)\partial_s x_s\right|}, \\
&\qquad \frac{\lambda^{(1+\nu)/2\nu}(t)}{(q8^{k_{q+1}+1}e^2(2\pi)^{q/2}(K'(t)+\sqrt{\lambda(t)})K(t))^{1/2\nu}}\} \\
\mu_h(t) \quad &: \quad = \sup_{0\vee(t-h)\leq s\leq t}\frac{1}{\gamma_h(s)}
\end{aligned}$$

We will use the function $\gamma_h$ in order to construct an evolution sequence for $X_T$. We construct by recurrence the time grid $t_k, k = 1, ..., N$ in the following way. We put $t_0 = 0$ and $\delta_1 = \gamma_h^2(t_0)$. Suppose now that $t_k$ is known. Then we define $\delta_{k+1} = \gamma_h^2(t_k)$ and put $t_{k+1} = t_k + \delta_{k+1}$. We define $N$ to be the first integer such that $t_N \geq T$. This supposes that such an $N$ exists so we have to assume that $\sum_{k=1}^{\infty}\gamma_h^2(t_k) \geq T$. Generally $t_N \neq T$ so we have to modify the last increment in an autoritharian way. We re-define $\delta_{N-1} = T - t_{N-1}$ and put $t_N = t_{N-1} + \delta_{N-1} = T$. This will be the time net that we will use in the following. For the moment we want to evaluate $N$. We note that $t_k - t_{k-1} = \delta_k = \gamma_h^2(t_{k-1}) \leq h^2 \leq h$ and so that for $t \in [t_{k-1}, t_k]$ one has $t - h \leq t_{k-1} \leq t$ and consequently $\mu_h(t) \geq 1/\gamma_h(t_{k-1})$. It follows that

$$\int_0^T \mu_h^2(t)dt = \sum_{k=1}^{N}\int_{t_{k-1}}^{t_k}\mu_h^2(t)dt \geq \sum_{k=1}^{N}\frac{1}{\gamma^2(t_{k-1})} \times \delta_k = N.$$

We denote

$$C^{ij}(U_t) = \sum_{l=1}^{\infty}U^{il}(t)U^{jl}(t)$$

and, for $\delta > 0$

$$\Gamma_\delta^i(t) := \sum_{j=1}^{\infty}\int_t^{t+\delta}(U_s - U_t)^{ij}dB_s^j + \int_t^{t+\delta}V_s^i ds.$$

**Definition 2** *We say that the path $x$ is $(r, K, K', Q, \nu)-$elliptic for $X$ if for every $0 < t < T$ and $0 < \delta < T - t$*

$$
\begin{array}{rrcl}
(H, i) & C(U_t) & \geq & Q_t \\
(H^\nu, ii) & \left\| \Gamma_\delta(t) \right\|_{k,p,t} & \leq & K(t)\delta^{\frac{1}{2}+\nu} \\
(H, iii) & \left( \sum_{i=1}^{q} \sum_{j=1}^{\infty} \left| U_t^{ij} \right|^2 \right)^{1/2} & \leq & K'(t)
\end{array}
$$

*on the set $\left| Q_t^{-1/2}(X(t) - x(t)) \right| \leq r(t)$.*

**Theorem 2** *Suppose that the law of $X_T$ admits a continuous local density in $y$. Suppose that there exists a path $x(t), t \in [0, T]$ such that $x(0) = x, x(T) = y$ and which is $(K, K', r, Q, \nu)-$elliptic for $X$. Then one has*

$$p_T(x_0, y) \geq \frac{1}{4e(2\pi\gamma(0)\lambda(0))^{q/2}} \times \exp(-\theta \int_0^T \mu_h^2(t)dt)$$

*where*

$$\theta = \ln(\frac{4^{q+1}e^2(2\pi)^{q/2}}{c(q)}).$$

**Remark 2** *Note that if $\int_0^T \mu_h^2(t)dt = \infty$ our statement is void. So there are some hypothesis hidden in the assumption $\int_0^T \mu_h^2(t)dt < \infty$. In particular, if for some $t \in [0, T]$ we have $\lambda_t = 0$, then $1/\gamma_h(t) = \infty$ and consequently $\mu_h(s) = \infty$ on an interval of length $h$. It follows that $\int_0^T \mu_h^2(t)dt = \infty$. We conclude that in order to obtain a significant result it is necessary that $\lambda_t > 0$ for every $t \in [0, T]$, and this is the ellipticity assumption along the path $x_t$.*

**Proof.** We will use the result from the previous section so we construct an evolution sequence for $X_T$. We consider the time grid $0 = t_0 < t_1 < ... < t_N = T$ defined by $\delta_k = t_k - t_{k-1} = \gamma^2(t_{k-1})$ (see above). We also denote $\Delta_k = \lambda_{t_{k-1}}\delta_k$. Moreover we define $F_k = X(t_k)$ and write

$$
\begin{aligned}
F_k &= F_{k-1} + \int_{t_{k-1}}^{t_k} U_s dB_s + \int_{t_{k-1}}^{t_k} V_s ds \\
&= F_{k-1} + \int_{t_{k-1}}^{t_k} U_{t_{k-1}} dB_s + \int_{t_{k-1}}^{t_k} (U_s - U_{t_{k-1}}) dB_s + \int_{t_{k-1}}^{t_k} V_s ds \\
&= F_{k-1} + J_k + \sqrt{\Delta_k} R_k
\end{aligned}
$$

with

$$J_k^i = \sum_{j=1}^{\infty}\int_{t_{k-1}}^{t_k} U_{t_{k-1}}^{ij} dB_s^j,$$
$$R_k = \frac{1}{\sqrt{\Delta_k}}\Big(\int_{t_{k-1}}^{t_k}(U_s - U_{t_{k-1}})dB_s + \int_{t_{k-1}}^{t_k} V_s ds\Big) = \frac{1}{\sqrt{\Delta_k}}\Gamma_{t_k - t_{k-1}}(t_{k-1}).$$

Coming back to the notation concerning the evolution sequences we have $h_k(s) = U(t_{k-1})$ for $s \in [t_{k-1}, t_k)$ and so $C(J_k) = \delta_k C(U_{t_{k-1}})$ and $M_k = \delta_k Q(t_{k-1})$. Consequently $\Theta_k = M_k^{-1/2} = \delta_k^{-1/2} Q^{-1/2}(t_{k-1})$ and the smaller eigenvalue of $M_k$ is $\Delta_k = \lambda(t_{k-1})\delta_k$. Our aim is to check that this is an evolution sequence which satisfies (7). We will first check the space-time relation $|\Theta_k(x(t_k) - x(t_{k-1}))| \leq 1$ which amounts to $\left|Q^{-1/2}(t_{k-1})(x(t_k) - x(t_{k-1}))\right| \leq \sqrt{\delta_k}$. We write

$$\begin{aligned}
& \left|Q^{-1/2}(t_{k-1})(x(t_k) - x(t_{k-1}))\right| \\
= & \left|\int_{t_{k-1}}^{t_k} Q^{-1/2}(t_{k-1})\partial_t x_t dt\right| \leq \int_{t_{k-1}}^{t_k}\left|Q^{-1/2}(t_{k-1})\partial_t x_t\right| dt \\
\leq & \ \delta_k \sup_{t_{k-1}\leq t\leq t_{k-1}+h}\left|Q^{-1/2}(t_{k-1})\partial_t x_t\right|
\end{aligned}$$

So a sufficient condition in order that our time-space relation holds true is

$$\gamma(t_{k-1}) = \sqrt{\delta_k} \leq \frac{1}{\sup_{t_{k-1}\leq t\leq t_{k-1}+h}\left|Q^{-1/2}(t_{k-1})\partial_t x_t\right|}$$

which enters in the definition of $\gamma_h$.

Recall that (by the very definition) on the set $A_k$ one has

$$\left|Q^{-1/2}(t_{k-1})(X(t_{k-1}) - x(t_k))\right| = \sqrt{\delta_k}\,|\Theta_k(X(t_{k-1}) - x(t_k))| \leq \frac{1}{2}\sqrt{\delta_k}$$

so that $\left|Q^{-1/2}(t_{k-1})(X(t_{k-1}) - x(t_{k-1})\right| \leq \frac{1}{2}\sqrt{\delta_k} + \left|Q^{-1/2}(t_{k-1})(x(t_k) - x(t_{k-1}))\right| \leq \frac{3}{2}\sqrt{\delta_k} = \frac{3}{2}\gamma(t_{k-1}) \leq r(t_{k-1})$. So $A_k \subseteq \{\left|Q^{-1/2}(t_{k-1})(X(t_{k-1}) - x(t_{k-1})\right| \leq r(t_{k-1})\}$ and we may use the hypothesis $(H, i), (H^\nu, ii), (H, iii)$.

Using $(H, i)$ we have $C(J_k) = \delta_k C(U_{t_{k-1}}) \geq \delta_k Q(t_{k-1}) = M_k$. Moreover, by $(H^\nu, ii)$

$$\begin{aligned}
\|R_k\|_{k,p,t_{k-1}} &= \frac{1}{\sqrt{\Delta_k}}\left\|\Gamma_{t_k - t_{k-1}}(t_{k-1})\right\|_{k,p,t_{k-1}} \leq \frac{K(t_{k-1})\delta_k^{\frac{1}{2}+\nu}}{\sqrt{\Delta_k}} = C_k\Delta_k^\nu \\
\textit{with} \quad C_k &= \frac{K(t_{k-1})}{\lambda(t_{k-1})^{\frac{1}{2}+\nu}}.
\end{aligned}$$

Let us check (7). Recall that $h_k(s) = U(t_{k-1})$ so that, using the hypothesis $(H, iii)$

$$\|h_k\|_{t_{k-1}} = \left( \sum_{i=1}^{q} \sum_{j=1}^{\infty} \left| U^{ij}_{t_{k-1}} \right|^2 \right)^{1/2} \leq K'(t_{k-1}).$$

So the hypothesis (7) reads

$$q8^k \left( \frac{2K'(t_{k-1})}{\sqrt{\lambda(t_{k-1})}} + 1 \right) \frac{K(t_{k-1})}{\lambda(t_{k-1})^{\frac{1}{2}+\nu}} \delta_k^\nu = q8^k (2K'(t_{k-1}) + \sqrt{\lambda(t_{k-1})}) \frac{K(t_{k-1})}{\lambda(t_{k-1})^{1+\nu}} \delta_k^\nu \leq \frac{1}{4e^2(2\pi)^{q/2}}$$

and since $\sqrt{\delta_k} = \gamma(t_{k-1})$, this is true by the definition of $\gamma_h$. So we may use our density evaluation and obtain

$$p_T(x_0, y) \geq \frac{1}{4e^2(2\pi\Delta_1)^{q/2}} e^{-N\theta}.$$

One has $\Delta_1 = \lambda(0)\gamma(0)$ and $N \leq \int_0^T \mu_h^2(t)dt$ so the proof is completed. ■

## 3.1 Upper bounds

We will now give sufficient conditions in order that $X_T$ has a continuous local density in $y$ and we establish upper bounds for this density. First of all we assume that

$$(H, iv) \quad U^{il} \in \bigcap_{l=1}^{\infty} L^p([0,T]; D^{k,l})$$

so that $X_T \in \bigcap_{l=1}^{\infty} D^{k,l}$. Note that here our notation corresponds to the classical Malliavin calculus (in our previous notation we have to take $t = 0$). We denote

$$L_s^{il} = U_s^{il} - U_T^{il} + \int_s^T \left( \sum_{p=1}^{\infty} D_s^l U_r^{ip} dB_r^p + D_s^l V_r^i dr \right)$$

so that

$$D_s^l X_T^i = U_T^{il} + L_s^{il}.$$

In our computations $s$ will be closed to $T$ so we look to $U_T^{il}$ as to the principal term and to $L_s^{il}$ as to a remainder. We assume that there exists some $\mu > 0, r(T) > \eta > 0$ and $C_p, p \in N$, such that for every $\varepsilon > 0$ and $p \in N$

$$(H^\mu, v) \quad E\left( \left( \int_{T-\varepsilon}^{T} \sum_{i=1}^{q} \sum_{l=1}^{\infty} \left| L_s^{il} \right|^2 ds \right)^{p/2} 1_{(|X_T - y| \leq \eta)} \leq C_p \varepsilon^{p(\frac{1}{2}+\mu)}.$$

We denote by $\sigma_{X_T}$ the Malliavin covariance matrix of $X_T$ (that is $\sigma_{X_T} = \phi_{0,X_T}$ in our previous notation) and denote by $\lambda_{X_T}$ its smaller eigenvalue.

**Lemma 3** *Suppose that $(H,i),(H,iv)$ and $(H^\mu,v)$ hold true. Then for every $l\in N$*

$$E((\det\sigma_{X_T})^{-l};|X_T-y|\leq\eta)\leq\left(\frac{4}{T\lambda_T}\right)^{ql}+2\left(\frac{4}{\lambda_T}\right)^{p(\frac{1}{2}+\mu)}C_p$$

*with* $p=\frac{ql+2}{\mu}$.

**Proof.**

**Step 1.** For every $\varepsilon>0$

$$\begin{aligned}\lambda_{X_T} &= \inf_{|\xi|=1}\int_0^T\sum_{l=1}^\infty\left|\sum_{i=1}^q D_s^l X_T^i\xi^i\right|^2 ds\geq\inf_{|\xi|=1}\int_{T-\varepsilon}^T\sum_{l=1}^\infty\left|\sum_{i=1}^q(U_T^{il}+L_s^{il})\xi^i\right|^2 ds\\ &\geq \frac{\varepsilon}{2}\inf_{|\xi|=1}\sum_{l=1}^\infty\left|\sum_{i=1}^q U_T^{il}\xi^i\right|^2 ds-\sup_{|\xi|=1}\int_{T-\varepsilon}^T\sum_{l=1}^\infty\left|\sum_{i=1}^q L_s^{il}\xi^i\right|^2 ds\end{aligned}$$

the last inequality being a consequence of the elementary inequality $(a+b)^2\geq\frac{1}{2}a^2-b^2$. Using $(H,i)$ we obtain that on the set $|X_T-y|\leq\eta\leq r(T)$

$$\lambda_{X_T}\geq\frac{\varepsilon}{2}\lambda_T-\int_{T-\varepsilon}^T\sum_{l=1}^\infty\sum_{i=1}^q\left|L_s^{il}\right|^2 ds$$

where $\lambda_T$ is the smaller eigenvalue of $Q_T$.

**Step 2.** We write

$$\begin{aligned}P(\lambda_{X_T} &\leq \frac{\varepsilon\lambda_T}{4},|X_T-y|\leq\eta)\leq P(\frac{\varepsilon}{2}\lambda_T-\int_{T-\varepsilon}^T\sum_{l=1}^\infty\sum_{i=1}^q\left|L_s^{il}\right|^2 ds\leq\frac{\varepsilon\lambda_T}{4},|X_T-y|\leq\eta)\\ &\leq P(\int_{T-\varepsilon}^T\sum_{l=1}^\infty\sum_{i=1}^q\left|L_s^{il}\right|^2 ds\geq\frac{\varepsilon\lambda_T}{4},|X_T-y|\leq\eta)\\ &\leq \left(\frac{4}{\varepsilon\lambda_T}\right)^{p/2}E(\left(\int_{T-\varepsilon}^T\sum_{l=1}^\infty\sum_{i=1}^q\left|L_s^{il}\right|^2 ds\right)^{p/2},|X_T-y|\leq\eta)\\ &\leq \left(\frac{4}{\varepsilon\lambda_T}\right)^{p/2}C_p\varepsilon^{p(\frac{1}{2}+\mu)}=\left(\frac{4}{\lambda_T}\right)^{p/2}C_p\varepsilon^{p\mu}.\end{aligned}$$

**Step 3.** We want to use the above inequality for $\varepsilon = \frac{4}{k\lambda_T}$, but we can not do it for every $k$ because we need that $\varepsilon < T$. So we need that $k \geq k_1 := [\frac{4}{T\lambda_T}]$. This is the significance of $k_1$ which appears in the sequel. Since $\det \sigma_{X_T} \geq \lambda_{X_T}^q$ we have

$$\begin{aligned} E((\det \sigma_{X_T})^{-l}; |X_T - y| &\leq \eta) \leq E((\lambda_{X_T})^{-ql}; |X_T - y| \leq \eta) \\ &\leq 1 + \sum_{k=1}^{\infty} E((\lambda_{X_T})^{-ql}; \lambda_{X_T} \in [\frac{1}{k+1}, \frac{1}{k}), |X_T - y| \leq \eta) \\ &\leq k_1^{lq} + \sum_{k=k_1}^{\infty} (k+1)^{ql} P(\lambda_{X_T} \leq \frac{1}{k}, |X_T - y| \leq \eta). \end{aligned}$$

We use now the inequality from the previous step with $\varepsilon = \frac{4}{k\lambda_T}$ and $p = \frac{ql+2}{\mu}$ so that the above quantity is dominated by

$$k_1^{lq} + \sum_{k=k_1}^{\infty} (k+1)^{ql} \left(\frac{4}{\lambda_T}\right)^{p(\frac{1}{2}+\mu)} C_p \frac{1}{k^{ql+2}}$$

and the proof is completed. ■

**Theorem 3** *Suppose that there exists a path $x(t), t \in [0,T]$ such that $x(0) = x, x(T) = y$ and which is $(K, K', r, Q)-$elliptic for $X$. Suppose also that $(H, iv)$ and $(H^{\mu}, v)$ hold true. Then the law of $X_T$ admits a continuous local density in $y$ and*

$$\begin{aligned} \left(\frac{4}{(T \wedge 1)\lambda_T}\right)^p C(P(|X_T - x_0| &\geq |y - x_0|))^{1/2} \\ &\geq p_T(x_0, y) \geq \frac{1}{4e(2\pi\gamma(0)\lambda(0))^{q/2}} \times \exp(-\theta\beta^2 \int_0^T \frac{dt}{\gamma^2(t)}) \end{aligned}$$

*where $C$ and $p$ are constant depending on the constants appearing in our hypothesis..*

**Remark 3** *The evaluation of $P(|X_T - x_0| \geq |y - x_0|)$ is subject to some large deviation type evaluations and this is the reason for which some exponential upper bounds appear in the frame of uniform elliptic diffusion processes with bounded coefficients. But in a general frame these evaluations may be different and give other shapes then exponential ones. This depends on the hypothesis on $U$ and $V$. Note that our assumptions are just local and this is not sufficient in order to determine the above probability.*

**Remark 4** *In the case of uniform elliptic diffusion processes the time appears in the upper bound as $T^{-q/2}$ so our evaluation is not optimal. This is due to the fact that we have not precise hypothesis on the Sobolev norms (in Malliavin sense) of $X_T$. In particular situations one may reduce the power of $1/T$ using a precise analysis of these norms. But in the general situation we are not able to do it because we have just local evaluations on $U$ and $V$.*

**Proof.** In order to simplifie notation we suppose that $x_0^i < y^i, i = 1, ..., q$ (the other cases are completely analogues) and we denote $D(x_0, y) = \{x : y^i \leq x^i, i = 1, ..., q\}$. We also denote by $\Psi_\eta(x)$ the regularization by convolution of $1_{D(x_0,y)}(x)$ and $\psi_\eta = \partial^q \Psi_\eta / \partial x^1, ..., \partial x^q$. Clearly $\psi_\eta$ is a smooth function with the support contained in ball of radius $\eta$ around $y$ and $\psi_\eta \to \delta_y$ and $\Psi_\eta \to 1_{D(x_0,y)}$. Under the hypothesis of the above lemma there is some $\eta > 0$ such that the Malliavin covariance matrix is invertible on $|X_T - y| \leq \eta$ so we may use the integration by parts formula and obtain

$$E(\psi_\eta(X_T)) = E(\Psi_\eta(X_T)H)$$

where $H$ is the weight which appears in Malliavin's integration by parts formula. Since the support of $\psi_\eta$ and of all its derivatives are included in $|X_T - y| \leq \eta$ we obtain $E(\Psi_\eta(X_T)H) = E(\Psi_\eta(X_T)H1_{\{|X_T - y| \leq \eta\}})$ . Passing to the limit with $\eta \to 0$ in the above inequality proves that $X_T$ admits a local density in $y$ and moreover

$$\begin{aligned} p_T(x_0, y) &\leq \overline{\lim}_{\eta \to 0} \left| E(\Psi_\eta(X_T)H1_{\{|X_T - y| \leq \eta\}}) \right| = \left| E(1_{D(x_0,y)}(X_T)H1_{\{|X_T - y| \leq \eta\}}) \right| \\ &\leq (P(X_T \in D(x_0, y)))^{1/2} \left| E(H^2 1_{\{|X_T - y| \leq \eta\}}) \right|^{1/2}. \end{aligned}$$

Clearly $P(X_T \in D(x_0, y)) \leq P(|X_T - x_0| \geq |y - x_0|)$. Moreover $H^2 1_{\{|X_T - y| \leq \eta\}} \leq (\det \sigma_{x_T})^{-l}$ $\widehat{H}^2 1_{\{|X_T - y| \leq \eta\}}$ for some $l$. Here $\widehat{H}$ is a complex expression involving Malliavin derivatives of $X_T$. So, using the evaluation from the previous lemma

$$\begin{aligned} \left| E(H^2 1_{\{|X_T - y| \leq \eta\}}) \right|^{1/2} &\leq E((\det \sigma_{X_T})^{-2l}; |X_T - y| \leq \eta)^{1/2} \left| E(\widehat{H}^4) \right|^{1/2} \\ &\leq \left( \frac{4}{(T \wedge 1)\lambda_T} \right)^p C \end{aligned}$$

for some constants $C$ and $p$. ■

# 4 Diffusion processes

We considered now the central example of diffusion processes. More precisely

$$dX_t^i = \sum_{j=1}^m \sigma_j^i(t, X_t, \theta_t) dB_t^j + b^i(t, X_t, \theta_t) dt, \quad i = 1, ..., q, \quad X_0 = x_0$$

where $\sigma_j^i, b^i : [0, \infty) \times R^q \times R^p \to R$ and $\theta$ is a $p-$dimensional adapted process. The $p-$dimensional process $\theta$ may be regarded as a parameter. But one may think to a different situation as well. Say that $\sigma$ and $b$ does not depend on $X$ but just on $\theta$ and $\theta$ itself is a diffusion process. Then the above frame covers the case $X_t = f(\theta_t)$ where $\theta$ is a diffusion process (it suffice to employ Ito's formula in order to obtain the semimartingale decomposition of $f(X_t)$). All the evaluations will be uniform with respect to this parameter.

We recall that we use the notation $p = p_{q+1}$ and $k = k_{q+1}$ where $p_{q+1}$ and $k_{q+1}$ are some constants depending on $q$ which are fixed in the first section. Our hypothesis are the following. We begin with $\theta$. We assume that for each $t \in [0,T]$

$$(H_1) \quad \theta_t \in D_t^{p,k} \quad and \quad \|\theta_s - \theta_t\|_{p,t} \le C\sqrt{s-t}.$$

We also assume that there exists some $h > 0$ such that for every multi-index $\alpha$ with $|\alpha| = l \le k$

$$(H_2) \qquad \sup_{t \le s_1 \le \ldots s_l \le u \le t+h} \left| D^{\alpha}_{s_1,\ldots,s_l} \theta_u \right| \le C(1 + |X_t|).$$

Moreover

$$(H_3) \quad \sigma^i_j, b^i \in C^k$$

and, for every $1 \le i \le q, 1 \le j \le m$ and every multi-index $\eta, \eta'$ with $0 \le |\eta|, |\eta'| \le k$

$$\begin{aligned} (H_4) \quad i) \quad \left|\sigma^i_j(t,x,\theta)\right|, \left|b^i(t,x,\theta)\right| &\le C_0(1+|x|) \\ ii) \quad \left|D^{\eta'}_{\theta} D^{\eta}_x \sigma^i_j(t,x,\theta)\right|, \left|D^{\eta'}_{\theta} D^{\eta}_x b^i(t,x,\theta)\right| &\le C_0. \end{aligned}$$

We consider also an $q \times m$ dimensional matrix $\rho = \rho(t,x)$ and assume that

$$(H_5) \quad |\rho\rho^*(t,x)| \le C_1(1+|x|).$$

Of course, in the case when $\sigma$ does not depend on a supplementary parameter $\theta$ we may just take $\rho = \sigma$. Our aim is to study the density of $X_T$ in $y \in R^q$. We fix a differentiable path $x : [0,T] \to R^q$ with $x(0) = x_0$ and $x(T) = y$ . We denote by $\lambda(t)$ the smaller eighenvalue of $\rho\rho^*(t,x_t)$ and assume that $1 \ge \lambda(t) > 0$. We also assume that

$$(H_6) \quad \sigma\sigma^*(t,x_t,\theta) \ge \rho\rho^*(t,x_t) \quad \forall \theta \in R^p.$$

We will use the result from the previous section with $Q(t) = \frac{1}{2}\rho\rho^*(t,x_t)$ and $r(t) = \frac{\sqrt{\lambda_t}}{2C_0}$. Let us check that the assumptions there hold true here. We are on the set $\left|Q_t^{-1/2}(X(t)-x(t))\right| \le r(t)$ and so in particulat we have $|X(t) - x(t)| \le \sqrt{\lambda_t} r(t) = \frac{\lambda_t}{2C_0}$. First of all we check $(H,i)$ that is $C(U_t) \ge Q_t = \frac{1}{2}\rho\rho^*(t)$ on $\left|Q_t^{-1/2}(X(t)-x(t))\right| \le r(t)$. We write

$$C(U_t) = \sigma\sigma^*(t,X_t,\theta_t) \ge \sigma\sigma^*(t,x_t,\theta_t) - C_0 |X_t - x_t| \ge \rho\rho^*(t,x_t) - \frac{\lambda_t}{2}.$$

So $(H,i)$ amounts to $\frac{1}{2}\rho\rho^*(t,x_t) \ge \frac{\lambda_t}{2}$ which is true by the very definition of $\lambda_t$. Then we need to control the norm of the matrix $\sigma\sigma^*(t,X_t,\theta_t)$ in order to determine $K'(t)$. For the same reason as above, we have

$$|\sigma\sigma^*(t,X_t,\theta_t)| \le |\sigma\sigma^*(t,x_t,\theta_t)| + C_0 |X(t) - x(t)| \le C_1(1+|x_t|) + \frac{\lambda_t}{2} =: K'(t)^2.$$

Finally we need to control the reminder and to compute $K(t)$. In order to do it we prove the following lemma.

**Lemma 4** *Under the above hypothesis*

$$\|\Gamma_\delta(t)\|_{k,p,t} \leq C_2(1+|X_t|)\delta$$

*for some constant $C_2$ which does not depend on $t, X_t$ and $\theta_t$.*

Once this lemma is proved we have

$$\begin{aligned}\|\Gamma_\delta(t)\|_{k,p,t} &\leq C_2(1+|X_t|)\delta \leq C_2(1+|x_t|+|X(t)-x(t)|)\delta \\ &= C_2(1+|x_t|+\frac{\lambda_t}{2})\delta =: K(t)\delta\end{aligned}$$

so $(H^\nu, ii)$ holds true with $\nu = \frac{1}{2}$.

**Proof of the lemma.** In order to simplify the notation we assume that $b=0$ and $q=1$. The proof is analogues in the general case. We first evaluate

$$\begin{aligned}\|\Gamma_\delta(t)\|_{p,t}^p &= E_t\left(\int_t^{t+\delta}(\sigma(s,X_s,\theta_s)-\sigma(t,X_t,\theta_t))dB_s\right)^p \\ &\leq CE_t\left(\int_t^{t+\delta}(\sigma(s,X_s,\theta_s)-\sigma(t,X_t,\theta_t))^2ds\right)^{p/2} \\ &\leq C\delta^{\frac{p}{2}-1}\int_t^{t+\delta}E_t(\sigma(s,X_s,\theta_s)-\sigma(t,X_t,\theta_t))^p ds \\ &\leq CC_0\delta^{\frac{p}{2}-1}\int_t^{t+\delta}(\delta^p+E_t(X_s-X_t)^p+E_t(\theta_s-\theta_t)^p)ds.\end{aligned}$$

Using the hypothesis $(H_1)$ we obtain $E_t(\theta_s-\theta_t)^p \leq C\delta^{p/2}$. We use now the equation verified by $X$ and obtain

$$\begin{aligned}E_t(X_s-X_t)^p &\leq CE_t\left(\int_t^{t+\delta}\sigma(s,X_s,\theta_s)^2ds\right)^{p/2} \leq CE_t\left(\int_t^{t+\delta}C(1+|X_s|)^2ds\right)^{p/2} \\ &\leq C\delta^{\frac{p}{2}-1}\int_t^{t+\delta}C(1+E_t|X_s|)^p ds.\end{aligned}$$

Standard arguments based on the equation verified by $X$ and Gronwall's lemma give $E_t|X_s|)^p \leq |X_t|^p e^{C(s-t)} \leq C|X_t|^p e^{C\delta}$. So we have $E_t(X_s-X_t)^p \leq \delta^{\frac{p}{2}}Ce^{Ch}(1+|X_t|)^p$. Finally we have proved that

$$\|\Gamma_\delta(t)\|_{p,t}^p \leq C(1+|X_t|)^p\delta^p.$$

We turn now to the derivatives of $\Gamma_\delta(t)$. For $u \in [t, t+\delta]$ we have

$$\begin{aligned} D_u\Gamma_\delta(t) &= (\sigma(u, X_u, \theta_u) - \sigma(t, X_t, \theta_t)) \\ &+ \int_u^{t+\delta} \partial_x\sigma(s, X_s, \theta_s) D_u X_s dB_s + \int_u^{t+\delta} \partial_\theta\sigma(s, X_s, \theta_s) D_u\theta_s dB_s \\ : &= I_1(u) + I_2(u) + I_3(u). \end{aligned}$$

The same arguments as above show that

$$E_t \left| \int_t^{t+\delta} |I_1(u)|^2 du \right|^{p/2} \leq C(1 + |X_t|)^p \delta^p.$$

Moreover, using the fact that $\partial_\theta\sigma$ is bounded and $D_u\theta_s$ satisfies $(H_2)$ we obtain

$$\begin{aligned} E_t \left| \int_t^{t+\delta} |I_3(u)|^2 du \right|^{p/2} &\leq C\delta^{\frac{p}{2}-1} \int_t^{t+\delta} E_t \left| \int_u^{t+\delta} \partial_\theta\sigma(s, X_s, \theta_s) D_u\theta_s dB_s \right|^p du \\ &\leq C\delta^{\frac{p}{2}-1} \int_t^{t+\delta} E_t \left| \int_u^{t+\delta} |\partial_\theta\sigma(s, X_s, \theta_s) D_u\theta_s|^2 ds \right|^{p/2} du \\ &\leq C\delta^{p-2} \int_t^{t+\delta} \int_u^{t+\delta} E_t |D_u\theta_s|^p dsdu \leq C(1 + |X_t|)^p \delta^p. \end{aligned}$$

It remains to treat $I_2(u)$. We write down the equation of $D_u X_s$ :

$$D_u X_s = \sigma(u, X_u, \theta_u) + \int_u^s \partial_x\sigma(r, X_r, \theta_s) D_u X_r dB_r + \int_u^s \partial_x\sigma(r, X_r, \theta_s) D_u\theta_r dB_r.$$

Using the fact that $D_u\theta_r$ satisfies $(H_2)$ and Gronwall's lemma one may check that $D_u X_s$ satisfies $(H_2)$ also and so, the same reasoning as for $I_3$ applies to $I_2$ and the proof is completed for the first order derivatives. For the higher order derivatives the proof is similar. ■

We resume: we have proved that under the above hypothesis $X_t$ is an elliptic Itô process associated to the path $x_t$ and with the parameters $\nu = \frac{1}{2}, Q_t = \frac{1}{2}\rho\rho^*(t, x_t)$ and

$$K_t = C_2(1 + |x_t| + \frac{\lambda_t}{2}), \quad K'_t = \sqrt{C_1(1 + |x_t|) + \frac{\lambda_t}{2}}, \quad r_t = \frac{R}{2C_0}$$

where $\lambda_t$ is the smaller eigenvalues of $\rho\rho^*(t, x_t)$. We have

$$\begin{aligned} \gamma_h(t) &= \min\{h, \frac{\sqrt{\lambda_t}}{2C_0}, \frac{1}{\sup_{t\leq s\leq t+h} \left| Q_t^{-1/2}\partial_s x(s) \right|}, \frac{C\lambda_t^{3/2}}{(1 + |x_t| + \lambda_t)^{3/2}}\} \\ &= \min\{h, \frac{\sqrt{\lambda_t}}{2C_0}, \frac{1}{\sup_{t\leq s\leq t+h} \left| Q_t^{-1/2}\partial_s x(s) \right|}, \frac{C}{(\frac{1+|x_t|}{\lambda_t} + 1)^{3/2}}\} \end{aligned}$$

where $C$ is some constant. If we assume that $\lambda_t \leq 1$ this gives

$$C\max\{\frac{1}{h}, \sup_{t\leq s\leq t+h}\left|Q_t^{-1/2}\partial_s x(s)\right|, (\frac{1+|x_t|}{\lambda_t})^{3/2}\}$$
$$\leq \quad \frac{1}{\gamma_h(t)} \leq C'\max\{\frac{1}{h}, \sup_{t\leq s\leq t+h}\left|Q_t^{-1/2}\partial_s x(s)\right|, (\frac{1+|x_t|}{\lambda_t})^{3/2}\}$$

where $C, C'$ are some constants. So we have

$$\mu_h(t) = C\sup_{t-h\leq t'\leq t}\max\{\frac{1}{h}, \sup_{t'\leq s\leq t'+h}\left|Q_{t'}^{-1/2}\partial_s x(s)\right|, (\frac{1+|x_{t'}|}{\lambda_{t'}})^{3/2}\}.$$

Using the result from the previous section we obtain

**Theorem 4** *Let $x(t), t\in[0,T]$ be a path such that $x(0)=x_0, x(T)=y$ and $(H_i), i=1,...,6$ hold true. Then the law of $X_T$ has a continuous local density in $y$ which verifies (for some constants $C$ and $l$)*

$$\frac{C}{((T\wedge 1)\lambda_T)^l}(P(|X_T-x_0| \quad \geq \quad |y-x_0|))^{1/2} \geq$$
$$p_T(x_0,y) \quad \geq \quad \frac{1}{4e(2\pi\gamma(0)\lambda(0))^{q/2}}\times\exp(-\theta\int_0^T \mu_h^2(t)dt).$$

## 4.1 Log-normal type diffusions

In this section we will consider diffusion processes which are not uniformly elliptic but have a singularity in the origin. They represent a generalization of the log-normal distribution and so the shape of the density is different from the Gaussian density. We will taste our method in this frame in order to see that we may capture such a shape. So we consider the equation

$$dX_t^i = \sum_{j=1}^q \sigma_j^i(t,X_t)dB_t^j + b^i(t,X_t)dt, \quad i=1,...,q, \quad X_0=x_0$$

where $\sigma_j^i, b^i : [0,\infty)\times R^q \to R$. We assume that $\sigma_j^i, b^i \in C^{k_{q+1}}$and, for every $1\leq i,j\leq q$ and every multi-index $\eta$ with $0\leq|\eta|\leq k_{q+1}$

$$(A_1) \quad \left|D_x^\eta\sigma_j^i(t,x)\right|, \left|D_x^\eta b^i(t,x)\right| \leq C.$$

Moreover we make the non-degeneracy hypothesis

$$(A_2) \quad A\left|x\right|^2 \geq \sigma\sigma^*(t,x) \geq a\left|x\right|^2.$$

The example that we have in mind is $\sigma(t,x)=\alpha(t,x)x$ where $\alpha$ is a matrix which verifies

$$(A_1') \quad \nabla\alpha(t,x) \quad = \quad 0 \quad if \quad |x|\geq R$$
$$(A_2') \quad A \quad \geq \quad \alpha\alpha^*(t,x) \geq a.$$

We come now back to $\sigma(t,x)$ which verifies $(A_1)$ and $(A_2)$. We take the matrix $\rho$ defined by

$$\rho(t,x)=\sqrt{a}\,|x|\times I$$

where $I$ is the identity $q\times q$ dimensional matrix. We fix a path $x_t$ and then $Q_t=\frac{1}{2}a\,|x_t|^2\times I$ and $\lambda_t=\frac{1}{2}a\,|x_t|^2$. We also have $\left|Q_t^{-1/2}\partial_s x_s\right|=\frac{2}{\sqrt{a}|x_t|}\,|\partial_s x_s|$. Then for every $h>0$ we have

$$\mu_h(t)=C\sup_{t-h\leq t'\leq t}\max\{\frac{1}{h},\frac{2}{\sqrt{a}\,|x_{t'}|}\sup_{t'\leq s\leq t'+h}|\partial_s x(s)|\,,(\frac{2(1+|x_{t'}|)}{a\,|x_{t'}|^2})^{3/2}\}.$$

We will now compare the lower bounds given by our evaluations with the precise form of the density of a log-normal density. So we consider the simplest possible situation, that is $q=1$ and $b=0$. So we look to the diffusion $dX_t=\sigma(t,X_t)dB_t$ with $X_0=x_0$ and we look to the density in the point $y$. We take the exponential path $x_t=x_0e^{\pi t}$ and in order to obtain $x_T=y$ we have to take $\pi=\frac{1}{T}\ln\frac{y}{x_0}$. We have $\partial_s x_s=\pi x_s$ so that

$$\frac{\sup_{t'\leq s\leq t'+h}|\partial_s x(s)|}{|x_{t'}|}=\pi\frac{\sup_{t'\leq s\leq t'+h}|x(s)|}{|x_{t'}|}=\pi e^{\pi h}.$$

We take $h=\frac{1}{\pi}$ so that

$$\mu_h(t)=C\sup_{t-h\leq t'\leq t}\max\{\pi,\frac{2\pi e}{\sqrt{a}},(\frac{2(1+|x_{t'}|)}{a\,|x_{t'}|^2})^{3/2}\}\leq C'\max\{\frac{\pi}{\sqrt{a}},(\frac{1+x_0\vee y}{a\,|x_0\wedge y|^2})^{3/2}\}.$$

It follows that

$$\int_0^T\mu_h(t)^2dt\leq CT\max\{\frac{(\ln\frac{y}{x_0})^2}{Ta},(\frac{1+x_0\vee y}{a\,|x_0\wedge y|^2})^3\}.$$

On the other hand

$$\begin{aligned}\gamma_h(0)&\geq C'\max\{\frac{\pi}{\sqrt{a}},(\frac{2(1+|x_0|)}{a\,|x_0|^2})^{3/2}\}=C'\max\{\frac{\ln\frac{y}{x_0}}{T\sqrt{a}},(\frac{2(1+|x_0|)}{a\,|x_0|^2})^{3/2}\}\\ \lambda_0&=\frac{1}{2}a\,|x_0|^2.\end{aligned}$$

So we obtain

$$p_T(x_0,y)\geq\frac{C}{\sqrt{a}x_0(\max\{\frac{\ln\frac{y}{x_0}}{T\sqrt{a}},(\frac{2(1+|x_0|)}{a|x_0|^2})^{3/2}\})^{1/2}}\exp(-C'T\max\{\frac{(\ln\frac{y}{x_0})^2}{Ta},(\frac{1+x_0\vee y}{a\,|x_0\wedge y|^2})^3\}.$$

We have to compare this with the log-normal density which is

$$p_T(x_0,y)=\frac{1}{ay\sqrt{2\pi T}}\exp(-\frac{(\ln\frac{y}{x_0}+\frac{\sigma^2}{2}T)^2}{2Ta^2}).$$

This looks rather compatible to the exponent level but different to the level of the coefficient. But note that we have chosen the exponential path and maybe this is not the optimal choice.

# 5 Stochastic Partial Differential Equations

In this section we discuss the solution of the Stochastic $PDE$

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(t,x) + \sigma(u(t,x))W'(t,x) + b(u(t,x)), \quad (t,x) \in [0,\infty) \times [0,1] \\ u(0,x) &= f(x), u(t,0) = u(t,1) = 0. \end{aligned}$$

This equation has been introduced by Walsh in [12]. $W'(t,x)$ is a time-space white noise and the above equation is just formal - we give in a moment the rigorous weak formulation. In [2] one proves that if the coefficients $\sigma$ and $b$ are smooth then for each fixed $(t,x), \omega \to u(t,x,\omega)$ is a smooth functional of the white noise so that the Malliavin calculus with respect to $W$ applies. One also proves that for every $t > 0$ and every $0 < x_1 < ... < x_q < 1$ the law of $\omega \to (u(t,x_1,\omega),...,u(t,x_q,\omega))$ has a local smooth density with respect to the Lebesgue's measure in any point $(y_1,...y_n)$ such that $\sigma(y_i) > 0, i = 1,...,n$ and this density is strictly positive. Finally in [7] one gives lower bonds for the density under the uniform ellipticity assumption $\sigma(x) \geq c > 0, \forall x \in [0,1]$. Our aim is to show how the results presented in this paper give Kohatzu-Higa's result (since our result is not new we will just skech the poofs). Note that the ellipticity assumption considered here is uniform - although our approach supposes only local ellipticity assumptions, we do not see how to localize in this frame and so the only result under local assumptions remains the strict posititvity proved in [2]. The weak formulation of the above stochastic $PDE$ is the following. We consider the fundamental solution of the linear problem $\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}, u(t,0) = u(t,1) = 0$ which we denote by $G_t(x,y)$. This solution may be expressed as an explicit sery of Gaussian kernels and many useful evaluations related to it are available (see the above papers). Then one considers the mild form of the above Stochastic $PDE$ :

$$u(t,x) = G_t(f)(x) + \int_0^t \int_0^1 G_{t-s}(x,y)\sigma(u(s,y))dW(s,y) + \int_0^t \int_0^1 G_{t-s}(x,y)b(u(s,y))dyds$$

where $G_t(f)(x) = \int_0^1 G_{t-s}(x,y)f(y)dy$ and $dW(s,y)$ designees the stochastic integral with respect to $W$. See [12] for the rigorous definition of this integral - anyway here we express this integral in an alternative form which involves a sery of usual stochastic integrals. More precisely we consider an orthonormal basis $e_k, k \in N$ of $L^2[0,1]$ and define

$$B_t^k := \int_0^t \int_0^1 e_k(y)dW(s,y)$$

where $dW$ still represents Walsh's space-time stochastic integral. Then one may easily prove that $B^k, k \in N$ are independent standard Brownian motions and for every $\phi \in L^2(R_+ \times [0,1])$ one has

$$\int_0^t \int_0^1 \phi(s,y)dW(s,y) = \sum_{k=1}^{\infty} \int_0^t \int_0^1 \phi(s,y)e_k(y)dB_s^k = \sum_{k=1}^{\infty} \int_0^t (\phi(s,.),e_k)_{L^2}dB_s^k.$$

So our stochastic $PDE$ reads

$$u(t,x)=G_t(f)(x)+\sum_{k=1}^{\infty}\int_0^t (G_{t-s}(x,.)\sigma(u(s,.)),e_k)_{L^2}dB_s^k+\int_0^t\int_0^1 G_{t-s}(x,y)b(u(s,y))dyds.$$

We fix now $T>0$ and $0<x<1$ and look to density of the random variable $X_T(\omega):=u(T,x,\omega)$ in a fixed point $y$. We want to express this random variable as the value at time $T$ of an Ito process so we define

$$X_t=G_T(f)(x)+\sum_{k=1}^{\infty}\int_0^t (G_{T-s}(x,.)\sigma(u(s,.)),e_k)_{L^2}dB_s^k+\int_0^t\int_0^1 G_{T-s}(x,y)b(u(s,y))dyds.$$

Note that $X_t\neq u(t,x)$ so the evolution of the Ito process is essentially different from the evolution of the solution of the stochastic $PDE$. This is normal because $X_t$ is a one dimensional process while the solution of the Stochastic $PDE$ is an infinite dimensional object. But one has $X_T=u(T,x)$. So we have

$$\begin{aligned} x_0 &= G_T(f)(x) \\ U_s^k &= (G_{T-s}(x,.)\sigma(u(s,.)),e_k)_{L^2}, \quad V_s=(G_{T-s}(x,.),b(u(s,.)))_{L^2}. \end{aligned}$$

We assume that

$$\begin{aligned} (H,i) \quad C &\geq \sigma(x)\geq c>0, \\ (H,ii) \quad \sigma,b &\in C_b^{\infty}. \end{aligned}$$

Under the above assuption $u(t,x)$ is smoth in Malliavin's sense (see [2]) and

$$\begin{aligned} C(U_s) &= \sum_{k=1}^{\infty}U_s^kU_s^k=\sum_{k=1}^{\infty}(G_{T-s}(x,.)\sigma(u(s,.)),e_k)_{L^2}^2=\int_0^1 G_{T-s}^2(x,y)\sigma^2(u(s,y))dy \\ &\geq c^2\int_0^1 G_{T-s}^2(x,y)dy=:\lambda_s. \end{aligned}$$

At this stage it is clear why we are not able to use our localization procedure in this frame: because we may localize according to the position of $X_t$ but $X_t$ has almost nothing to do with $u(s,y)$ which appears as the argument of $\sigma$. So we define $\lambda_s$ as above and we take $Q_s=:\lambda_sI$. We have to check that $\lambda_s>0$ for every $s\in(0,T)$.

We consider the straight line which links the initial value $x_0=G_T(f)(x)$ to the point $y$ that is $x_t^i=f(x_i)\frac{T-t}{T}+y_i\frac{t}{T}$. So $\partial_tx_t=\frac{1}{T}(y-G_T(f)(x))$ and $\left|Q^{-1/2}(t)\partial_sx_s\right|=\frac{1}{T\sqrt{\lambda_t}}\left|y-G_T(f)(x)\right|$. We will take $h>0$ and $r_t=\infty$. We have to compute now $K_t$ and $K_t'$. Since $\sigma$ is bounded we have $K_t'=\left\|\sigma\right\|_\infty(\int_0^1 G_{T-s}^2(x,y)dy)^{1/2}=c\left\|\sigma\right\|_\infty\sqrt{\lambda_t}$. We want to verify now $(H^\nu,ii)$ with $\nu=\frac{1}{4}$. We have

**Lemma 5** *Under the hypothesis* $(H,i),(H,ii)$

$$\left\|\Gamma_\delta(t)\right\|_{k_1,p_1,t}\leq K_t\delta^{1/4}$$

*with* $K_t=C\sqrt{\lambda_t}$.

**Proof.** The proof of this lemma is rather long and technical. It follows the same ideas as that in [2] and [3]. So we will not give it in extenso but just point out the main argument. We use Burkholder's inequality for Hilbert space valued martingales and obtain

$$
\begin{aligned}
& E_t(\int_t^{t+\delta}\sum_{k=1}^{\infty}(U_s^k-U_t^k)dB_s^k)^p \\
\le\ & CE_t(\int_t^{t+\delta}\sum_{k=1}^{\infty}(U_s^k-U_t^k)^2ds)^{p/2} \\
=\ & CE_t(\int_t^{t+\delta}\int_0^1(G_{T-s}(x,y)\sigma(u(s,y))-G_{T-t}(x,y)\sigma(u(t,y)))^2ds)^{p/2} \\
\le\ & C'E_t(\int_t^{t+\delta}\int_0^1(G_{T-s}(x,y)-G_{T-t}(x,y))^2ds)^{p/2}+ \\
& CE_t(\int_t^{t+\delta}\int_0^1 G_{T-t}^2(x,y)(u(s,y))-u(t,y))^2ds)^{p/2}.
\end{aligned}
$$

Using standard evaluations on the Green function $G$ (see [3] Appendix B) and the fact that $E_t\left|u(s,y))-u(t,y)\right|^p\le C(s-t)^{p/4}$ (see [12]) one dominates the above term by

$$C(\int_0^1 G_{T-t}^2(x,y)dy)^{p/2}\delta^{p(\frac{1}{2}+\frac{1}{4})}=C\lambda_t^{p/2}\delta^{p(\frac{1}{2}+\frac{1}{4})}.$$

The proof concerning the Malliavin derivatives are more involved but of the same nature, so we live it out. ■

We fix now $h>0$ and we compute

$$
\begin{aligned}
\gamma_h(t) &= C\min\{h,\frac{T\sqrt{\lambda_t}}{|y-G_T(f)(x)|},\frac{\lambda^{(1+\nu)/2\nu}(t)}{(K'(t)+\sqrt{\lambda(t)})K(t))^{1/2\nu}}\} \\
&= C\min\{h,\frac{T\sqrt{\lambda_t}}{|y-G_T(f)(x)|},\sqrt{\lambda_t}\}.
\end{aligned}
$$

It is easy to check that there exists two constants $\alpha,\beta>0$ such that

$$\frac{\alpha}{\sqrt{T-t}}\le\int_0^1 G_{T-t}^2(x,y)dy\le\frac{\beta}{\sqrt{T-t}}$$

for ever $0\le t<T$. It follows that for every $t$ and $s$ one has $\frac{\alpha}{\beta}\lambda_t\le\lambda_s\le\frac{\beta}{\alpha}\lambda_t$. So we may take $h=\infty$ and obtain

$$\mu(t)=\frac{1}{\gamma(t)}=(\frac{|y-G_T(f)(x)|}{T}\vee 1)\frac{1}{\sqrt{\lambda_t}}$$

and consequently

$$\int_0^T \mu^2(t)dt = (\frac{|y-G_T(f)(x)|}{T} \vee 1)^2 \int_0^T \frac{dt}{\lambda_t}$$
$$= (\frac{|y-G_T(f)(x)|}{T} \vee 1)^2 \int_0^T (\int_0^1 G_{T-t}^2(x,y)dy)^{-1}dt.$$

We are now able to use our result on the Ito process and we obtain

**Lemma 6**

$$p_T(x_0,y) \geq \frac{C(\frac{|y-G_T(f)(x)|}{T} \vee 1)^{1/2}}{(\int_0^1 G_T^2(x,y)dy)^{3/8}} \exp(-C'(\frac{|y-G_T(f)(x)|}{T} \vee 1)^2 \int_0^T (\int_0^1 G_{T-t}^2(x,y)dy)^{-1}dt).$$

Let us compare the above lower bound with the precise lower bound obtained for a constant $\sigma$. In this case $u(T,x)$ is a Gaussian random variable

$$u(T,x) = G_T f(x) + \int_0^T \int_0^1 G_{T-t}(x,y)dW(t,y)$$

and we know explicitelly the density. The mean of this random variable is $G_T f(x)$ and the variance is $\int_0^T \int_0^1 G_{T-t}^2(x,y)dydt$ so that the precise density is

$$g_T(y) = \frac{1}{\sqrt{2\pi \int_0^T \int_0^1 G_{T-t}^2(x,y)dydt}} \exp(-\frac{|y-G_T(f)(x)|^2}{2\int_0^T \int_0^1 G_{T-t}^2(x,y)dydt}).$$

The first thing to be noted is that in the exponent of our density we have $\int_0^T (\int_0^1 G_{T-t}^2(x,y)dy)^{-1}dt$ while in the precise density one has $(\int_0^T \int_0^1 G_{T-t}^2(x,y)dydt)^{-1}$. Since $x \to \frac{1}{x}$ is a convex function $\int_0^T (\int_0^1 G_{T-t}^2(x,y)dy)^{-1}dt \geq (\int_0^T \int_0^1 G_{T-t}^2(x,y)dydt)^{-1}$ so we loose something in our evaluations. But if we look to these evaluations up to some multiplicative constant (and anyway such constants appear) we see that $\int_0^T (\int_0^1 G_{T-t}^2(x,y)dy)^{-1}dt \leq C\int_0^T \sqrt{T-t}dt = CT\sqrt{T}$, and $\int_0^1 G_T^2(x,y)dy \geq \frac{C'}{\sqrt{T}}$ so our evaluation reads

$$p_T(x_0,y) \geq CT^{3/16}(\frac{|y-G_T(f)(x)|}{T} \vee 1)^{1/2} \exp(-C'(\frac{|y-G_T(f)(x)|}{T} \vee 1)^2 T\sqrt{T}).$$

Think for a moment that $|y-G_T(f)(x)| \geq T$. Then we obtain

$$p_T(x_0,y) \geq \frac{C}{T^{13/16}} \sqrt{|y-G_T(f)(x)|} \exp(-C'\frac{|y-G_T(f)(x)|^2}{\sqrt{T}}).$$

On the other hand $\int_0^T \int_0^1 G_{T-t}^2(x,y)dydt \sim \int_0^T \frac{1}{\sqrt{T-t}}dt = C\sqrt{T}$ so

$$g_T(y) = \frac{c}{\sqrt{2\pi\sqrt{T}}} \exp(-\frac{c'\,|y-G_T(f)(x)|^2}{\sqrt{T}}).$$

So our evaluation remains correct at list at the level of the exponent.

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