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Abstract: We give a general Riemannian framework to the study of approximation of curvature measures, using the theory of the normal cycle. Moreover, we introduce a differential form which allows to define a new type of curvature measure encoding the second fundamental form of a hypersurface, and to extend this notion to geometric compact subsets of a Riemannian manifold . Finally, if a geometric compact subset is close to a smooth hypersurface of a Riemannian manifold, we compare their second fundamental form (in the previous sense), and give a bound of their difference in terms of geometric invariants and the mass of the involved normal cycles.

Key-words: Normal cycle, approximation, riemannian manifold, curvature, second fundamental form

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Approximation de la seconde forme fondamentale d'une hypersurface d'une variété Riemannienne

Résumé : Nous étudions l'approximation des mesures de courbures dans un cadre Riemannien, en utilisant la théorie du cycle normal. De plus, nous introduisons une forme différentielle qui permet de définir un nouveau type de mesure de courbure décrivant la seconde forme fondamentale d'une hypersurface. Enfin, nous comparons les mesures de courbures d'une hypersurface lisse et d'un compact géométrique qui lui est proche en bornant leur différence en fonction de certains invariants géométriques et de la masse des différents cycles normaux.

Mots-clés : Cycle normal, approximation, variete riemannienne, courbure,

1 Introduction

This report is a continuation of [2] and [3]. Its has a double goal: First of all it gives a general Riemannian framework to the study of approximation of curvature measures. In [2], we dealt with surfaces and geometric compact subsets in $\mathbb{E}^3$, while in [3] we considered only smooth surfaces and polyhedra. Here, we generalize the frame by working in any arbitrary Riemannian manifold $\tilde{M}$. Of course, our results depend in general on its curvature tensor, but its topology is not involved. As in $\mathbb{E}^n$, the main geometric objects lie in the tangent manifold $T\tilde{M}$ of $\tilde{M}$, whose geometry is rich.

The second goal of this report is to give a interpretation of the second fundamental form of a hypersurface M of $\tilde{M}$ in terms of curvature measures. Using this interpretation, we can compare the second fundamental form of a smooth hypersurface of $\tilde{M}$ with the second fundamental form of a geometric compact subset of $\tilde{M}$ strongly close to it. Here is the main result: we suppose that M bounds compact subset K and $\mathcal{C}$ is a geometric compact subset of $\tilde{M}$ whose boundary $\mathcal{B} = \partial\mathcal{C}$ is strongly close to M; B is any regular Borel subset of $\tilde{M}$ included in $\mathcal{B}$; $\mathcal{M}_k^{\mathcal{C}}$, *resp.* $\mathcal{M}_k^K$) denotes as usual the k-th measure of curvature associated to $\mathcal{C}$, (*resp.* K); (see also the notations below, in particular 3.2.1):

Theorem 1 *Let B be any regular Borel subset of $\mathcal{B}$. Then, for every $k, 0 \le k \le n-1$,*

$$|\mathcal{M}_k^{\mathcal{C}}(B) - \mathcal{M}_k^K(pr(B))| \le$$
$$C(n,k,R)\max(\delta_B,\alpha_B)\max(1,||\hbar_B||^{n-1})(\mathbf{M}(N(\mathcal{C})_{|T_B\tilde{M}}) + \mathbf{M}(\partial N(\mathcal{C})_{|T_B\tilde{M}}));$$

Moreover, if X, Y are vector fields of $\mathcal{H}$,

$$|\mathbf{h}_{\mathcal{C}}^{X,Y}(B) - \mathbf{h}_K^{X,Y}(pr(B))| \le$$
$$C_1(n,k,R)||X||_1||Y||_1\max(\delta_B,\alpha_B)\max(1,||\hbar_B||^{n-1})(\mathbf{M}(N(\mathcal{C})_{|T_B\tilde{M})} + \mathbf{M}(\partial N(\mathcal{C})_{|T_B\tilde{M}}));$$

where C and C_1 are constant depending on the dimension and on the curvature of the ambiant space.

Finally, we would like to mention the preprint of A. Bernig and L. Bröcker [1] in which some computations are close to ours.

2 A brief survey on the geometry of a tangent manifold

This paragraph summaries the geometry of the tangent bundle ($TM \xrightarrow{\pi} M$) of a smooth manifold M; (see [11] [12] for details). If the dimension of M is n, TM is a smooth manifold of dimension $2n$. We shall adopt the following notations: If a generic point m of M belongs to a domain U of coordinates, we write $m = (x^\alpha)$. If a vector z_m lies in the fiber T_mM, it will be expressed locally (x^α, y^α).

Consider the tangent bundle (TM, π, M) of a smooth manifold M, and the second tangent bundle $(TTM, \overset{\pi_{TM}}{\to} TM)$. We shall deal with the following diagram:

$$\begin{array}{ccc} TTM & \xrightarrow{\pi_*} & TM \\ \pi_{TM}\downarrow & & \downarrow\pi \\ TM & \xrightarrow{\pi} & M \end{array}$$

and the following exact sequence:

$$0 \to TM \times_M TM \xrightarrow{i} TTM \xrightarrow{j} TM \times_M TM \to 0,$$

where i is the natural injection defined by

$$i(v,w) = \frac{d}{dt}(v+tw)_{|t=0},$$

and $j = (\pi_{TM}, \pi)$.

The vertical bundle $V(M)$ is the kernel of j:

$$V(M) = \mathrm{Ker} j = \mathrm{Im} i.$$

If z is a point of TM, there is a natural isomorphism

$$i_z : T_{\pi(z)}M \to V_z(M),$$

defined by

$$i_z(v) = i(z,v).$$

We put $i_z(v) = v^z$, and v^z is called the **vertical lift** of v at z. We denote by ς_z the inverse of i_z.

The vector field C associated to the one parameter group of homotheties with a positive ratio acting on the fibers of TM, is called the **canonical vertical vector field**. It can be defined as follows: Consider the diagonal morphism $\delta : TM \to TM \times_M TM$ given by:

$$\delta(z) = (z,z).$$

We put:

$$C = i \circ \delta : TM \to V(M).$$

The **vertical endomorphism** J is defined by

$$J = i \circ j.$$

It is an almost tangent structure of TM, $(J^2 = 0)$, such that

$$\mathrm{Ker}\ J = \mathrm{Im}\ J = V(M).$$

A **linear connexion** on M can be defined as a right splitting of the exact sequence, (also linear in the variable y), that is, a bundle morphism

$$\gamma : TM \times_M TM \to TTM,$$

such that

$$j \circ \gamma = Id_{E \times_M TM}.$$

If $w, z \in T_m M$ $z^h = \gamma_w(z)$ is called the **horizontal lift** of z at w. The subbundle

$$H_w(M) = \mathrm{Im}(\gamma(w, .))$$

is called the **horizontal bundle** at w.

Now,

$$h = \gamma \circ j : TTM \to TTM,$$

is called the **horizontal projection** on the horizontal bundle H. We denote by

$$v = \mathrm{Id} - h$$

the **vertical projection** on the vertical bundle $V(M)$.

Clearly, at every point z of TM, one has:

$$T_z TM = V_z(M) \oplus H_z(M).$$

Moreover, at each point z of TM, one has an isomorphism

$$j_z \times \varsigma_z : H_z(M) \times V_z(M) \to TM \times TM.$$

When M is endowed with a connexion, the bundle $(\pi_{TM} : TTM \to TM)$ is also endowed with an almost complex structure F, $(F^2 = -Id)$ defined by

$$\begin{cases} FJ &= h, \\ Fh &= -J. \end{cases}$$

One has

$$\begin{cases} TTM &= V(M) \oplus H(M), \\ FV(M) &= H(M), \\ FH(M) &= V(M). \end{cases}$$

The **covariant derivative** associated to the linear connexion γ is defined as follows: for every (local) vector field $y \in TM$ and every tangent vector w,

$$\nabla_w y = \varsigma_{z(x)} \circ v \circ z_* \circ w.$$

Suppose now that M is endowed with a Riemannian metric. Then, TM has a canonical Riemannian structure defined as follows

1. $V(M)$ is endowed with the metric defined on M transported by the identification induced by ς.
2. $H(M)$ is endowed with the metric defined on M transported by the identification induced by j.
3. $V(M)$ and $H(M)$ are orthogonal.

In the following, we shall systematically endow M with the Levi-Civita connexion associated to the metric of M.

Remarks

1. If C is the canonical vertical vector field, one has locally:

$$C = (x^\alpha, y^\alpha, 0, y^\alpha), \;\; FC = (x, y^\alpha, y^\alpha, -y^\beta y^\delta \Gamma^\alpha_{\beta\delta}),$$

   which implies that FC is a *semi-spray*, in the sense of [11], [12].

2. The couple $(<,>, F)$ induces a symplectic structure $\Omega =< F.,. >$ on the manifold TM. It is the image of the symplectic structure defined on the cotangent manifold T^*M, via the identification of TM and T^*M given by the metric.

3. It is easy to check that

$$\begin{cases} JF & = & v, \\ hF & = & Fv, \\ vF & = & Fh = -J. \end{cases}$$

In our context, it will be useful to deal with the connexion forms of the connexion, and to use the Maurer-Cartan formalism. In this context, we introduce the bundle $\mathcal{T}M = TM \backslash 0$, that is, the tangent bundle without the 0-section. Let $z_m \neq\neq 0$ be a point of $\mathcal{T}M$. Let $(e_1, ..., e_n)$ be any orthonormal frame $T_m M$, such that $e_n = \frac{z}{||z||}$. We denote by $(e_1^*, ..., e_n^*)$ the dual frame. If ω denotes the (Levi-Civita) connexion form on TM (considered as a vector valued one form taking its values in the Lie algebra of $SO(n)$), the Maurer-Cartan structure equations can be written as:

$$de_i^* = \omega_i^j \wedge e_j^*, d\omega_i^j = \omega_i^k \wedge \omega_k^j + \Omega_i^j, \tag{1}$$

where Ω_i^j are the curvature forms of the connexion, related to the curvature tensor R of M by:

$$\Omega_i^j(X, Y) =< R(X, Y)e_i, e_j >, \forall X, Y \in TM.$$

The Bianchy identity is

$$d\Omega_i^j = \omega_i^k \wedge \Omega_k^j.$$

Now we take the pullback of the one forms e_i^* by π. We get n covectors $(\theta_1, ..., \theta_n)$ on TM, null on the vertical bundle $V(M)$:

$$\theta_i = \pi^*(e_i^*), \forall i.$$

One has: $\forall X, Y \in TM$,

$$d\theta_i(X^v, Y^v) = 0, ;$$

$$d\theta_i(X^h, Y^h) == (\omega_i^j \wedge e_j^*)(X, Y);$$

$$d\theta_i(X^h, Y^v) = 0.$$

We define the 1-forms ϖ_i^j on $\mathcal{T}M$ by

$$\varpi_i^j(X^h) = \omega_i^j(X), \varpi_i^j(X^v) = 0, \forall X \in TM.$$

Finally, we obtain:

$$d\theta_i = \varpi_i^j \wedge \theta_j. \tag{2}$$

Associated to this frame, we define the n 1-forms

$$F^*(\theta_i) = \Theta_i, \forall i.$$

These forms are null on the horizontal bundle $H(M)$, and satisfy:

$$\Theta_i(e_j^v) = \delta_{ij}.$$

One has:

$$d\Theta_i = \varpi_i^k \wedge \Theta_k + \Re_i^n, \tag{3}$$

where $\Re_i^n$ is the 2-form on $\mathcal{T}M$ defined at z by:

$$\Re_i^n(X, Y) =< R(d\pi(X), d\pi(Y))z, e_i > .$$

3 Invariant $(n-1)$-forms on TM

3.1 The curvature forms

Let M be a n-dimensional Riemannian manifold. Let $z = (m, y_m), y_m \neq 0$ be a point of $\mathcal{T}M$. Let $(e_1, ..., e_n)$ be any orthonormal frame T_mM, such that $e_n = \frac{z}{||z||}$. With the notations of the previous section, we can build the $(n-1)$-form

$$(\theta_1 + t\Theta_1) \wedge ... \wedge (\theta_{n-1} + t\Theta_{n-1})_z.$$

When z varies, this defines a differential $(n-1)$-form on $\mathcal{T}M$. Consider this expression as a polynomial in the variable t; (remark that the coefficient of every t^i is a differential form which does not depend on the orthonormal frame $(e_1, ..., e_n = \frac{z}{||z||})$. We denote by ϕ_k the differential form which is the coefficient in t^k. Trivially one has:

$$\phi_k = \sum_\pi (-1)^\pi \theta_{\pi(1)} \wedge ... \wedge \theta_{\pi(k)} \wedge \Theta_{\pi(k+1)} \wedge ... \wedge \Theta_{\pi(n-1)}.$$

As usual, we give the following

Definition 1 *For $1 \leq i \leq n-1$, the $(n-1)$-form ϕ_k on $\mathcal{T}M$ is called the k^{th}-Lipschitz-Killing curvature form.*

Proposition 1 *Let M be a n-dimensional Riemannian manifold. Then,*

- *each invariant form ϕ_k satisfies $||\phi_k|| = 1$;*
- *moreover, if the norm of the curvature form Ω of M is bounded by a positive constant R, then*

$$||d\phi_k|| \leq C(k, n, R),$$

where $C(k, n, R)$ is a positive constant depending on the dimension and on the bound R.

Proof of Proposition 1:

- The first item is trivial.
- One has:

$$d\phi_k = \sum \pm d[\theta_{\pi(1)} \wedge ... \wedge \theta_{\pi(k)} \wedge \Theta_{\pi(k+1)} \wedge ... \wedge \Theta_{\pi(n-1)}] =$$

$$\sum \pm \theta_{\pi(1)} \wedge ... \wedge d\theta_{\pi(j)} \wedge ... \wedge \theta_{\pi(k)} \wedge \Theta_{\pi(k+1)} \wedge ... \wedge \Theta_{\pi(n-1)} +$$

$$\sum \pm \theta_{\pi(1)} \wedge ... \wedge ... \wedge \theta_{\pi(k)} \wedge \Theta_{\pi(k+1)} \wedge ... \wedge d\Theta_{\pi(k+l)} \wedge ... \wedge \Theta_{\pi(n-1)}.$$

  To bound $||d\phi_k||$, we use equations 3, 2, replacing the terms $d\theta_i$ and $d\Theta_i$ by their values in terms of ϖ_i^j and $\Re_i^n$. We get a sum of indecomposable forms which are the wedge products of θ_i, Θ_i and $\Re_i^n$, (the terms involving $\varpi_i^j, 1 \leq i, j \leq n-1$ cancel). The conclusion follows.

Remarks:

1. In particular, if M is flat, $(R = 0)$, then we deduce that $d\phi_k$ has an expression of the type $\sum \pm \theta_{i_1} \wedge ... \wedge \omega_{i_{n-k-1}}^j \wedge \Theta_{i_{n-k}} \wedge ... \wedge \Theta_{n-1}$. The norm of each decomposable term of this sum is 1, each term appearing at most $k+1$ times, and the terms of type $\sum \pm \theta_{i_1} \wedge ... \wedge \Theta_{i_{n-k-1}} \wedge \Theta_{i_{n-k}} \wedge ... \wedge \Theta_{n-1}$ appearing $k+1$ times. We deduce that

$$||d\phi_k|| = k+1, \forall k \leq n-2;$$

$$||d\phi_{n-1}|| = 0.$$

2. In the case where the manifold is the three dimensional Euclidean space $\mathbb{E}^3$, we get three 2 forms on $\mathcal{T}\mathbb{E}^3$. We give here their explicit expressions in the standard frame $(x_1, x_2, x_3, y_1, y_2, y_3)$, at any point (m, y) such that $||y|| = 1$:

$$\phi_{\mathcal{A}} = y_1 dx_2 \wedge dx_3 + y_2 dx_3 \wedge dx_1 + y_3 dx_1 \wedge dx_2;$$

$$\phi_{\mathcal{G}} = y_1 dy_2 \wedge dy_3 + y_2 dy_3 \wedge dy_1 + y_3 dy_1 \wedge dy_2;$$

$$\phi_{\mathcal{H}} = y_1(dx_2 \wedge dy_3 + dx_2 \wedge dy_3) + y_2(dx_3 \wedge dy_1 + dy_3 \wedge dx_1) + y_3(dx_1 \wedge dy_2 + dy_1 \wedge dx_2).$$

   Since these forms are invariant when we replace y by λy, we deduce that

$$||\phi_{\mathcal{A}}|| = ||\phi_{\mathcal{G}}|| = ||\phi_{\mathcal{H}}|| = 1,$$

$$||d\phi_{\mathcal{A}}|| = 1, ||d\phi_{\mathcal{G}}|| = 0, ||d\phi_{\mathcal{H}}|| = 2.$$

3. Suppose now that the manifold is the unit sphere $\mathbb{S}^3$. In this case, the curvature form is non zero, and satisfies:

$$\Omega_i^j = \theta_i \wedge \theta_j.$$

   We deduce by a direct computation:

$$||\phi_{\mathcal{A}}|| = ||\phi_{\mathcal{G}}|| = ||\phi_{\mathcal{H}}|| = 1,$$

$$||d\phi_{\mathcal{A}}|| = 1, ||d\phi_{\mathcal{G}}|| = 2, ||d\phi_{\mathcal{H}}|| = 2.$$

3.2 The fundamental $(n-1)$-form on $\mathcal{T}M$ associated to a couple of horizontal vector fields

In this subsection, we introduce a $(n-1)$ form depending (bi)linearly on horizontal vector fields. Let M be a n-dimensional Riemannian manifold. We introduce the $(n-1)$ dimensional subbundle $\mathcal{H}$ of the horizontal bundle $H(M)$ orthogonal to FC:

$$\mathcal{H} = FC^{\perp_H}.$$

We shall build a tensor field of type $(0, 2)$

$$\mathbf{h} : \mathcal{H} \times \mathcal{H} \to \Lambda^{n-1}\mathcal{T}M,$$

acting $\mathcal{H}$ and taking its values in the space of differential $(n-1)$-forms on TTM. We give two equivalent constructions. The first one uses local orthonormal frames, the second one is more geometric.

3.2.1 Two constructions of the tensor h

Let U be an open neighborhood of a point $(m, \xi_m) \in \mathcal{T}M$. Let $(e_1, ..., e_n)$ be a local orthonormal frame on U, such that $e_n = \frac{\xi}{||\xi||}$. Let $\theta_i = \pi^*(e_i^*), 1 \leq i \leq n$. For two fixed vectors $e_{i_0} \neq e_n, e_{j_0} \neq e_n$, we define the $(n-1)$-form

$$\mathbf{h}_{i_0,j_0} = \theta^1 \wedge ... \wedge \hat{\theta^{i_0}} \wedge ... \wedge \theta^j \wedge ... \wedge \hat{\theta^n} \wedge \Theta_{j_0}.$$

When the two indices i_0, j_0 vary, the previous formula defines a tensor of type $(0,2)$, independent of the orthonormal local frame $(e_1, ..., e_n)$: if $X, Y \in \mathcal{H}, X = \sum_{i=1}^{n-1} X^i e_i^h, Y = \sum_{i=1}^{n-1} Y^i e_i^h$, then

$$\mathbf{h}(X,Y) = \sum_{i,j} X^i Y^j \mathbf{h}_{i,j}.$$

Here is a global construction of the same tensor $\mathbf{h}$: Let $X, Y \in \mathcal{H}$.

- Let τ_X be the $(n-2)$-form on $\mathcal{H}$ defined by

$$\tau_X(u_1, ..., u_{n-2}) = \det(FC, X, u_1, ..., u_{n-2});$$

- Let Y^* be the dual form to FY in $V(M)$;
- define

$$\mathbf{h} : \mathcal{H} \times \mathcal{H} \to \Lambda^{n-1}\mathcal{T}M,$$

  by

$$\mathbf{h}(X,Y) = \tau_X \wedge Y^*.$$

Both constructions are equivalent. To simplify the notations, we put $\mathbf{h}(X,Y) = \mathbf{h}^{X,Y}$.

Definition 2 *The form* $\mathbf{h}^{X,Y}$ *is called the* **fundamental form associated to the couple** (X,Y). *Moreover, if* $\mathcal{C}$ *is a geometric subset of* M, *we denote by* $\mathbf{h}_{\mathcal{C}}^{X,Y}$ *the corresponding curvature measure.*

The following proposition gives a bound on the norm of $\mathbf{h}$. The norm $||.||_1$ used here in $\mathcal{H}$ is defined by $||X||_1 = \sup(||X||, ||\nabla X||)$, where ∇ is the covariant derivative in TM.

Proposition 2 *For all* $X, Y \in \mathcal{H}$, *one has:*

- $||\mathbf{h}^{X,Y}|| \leq C(n,R)||X||||Y||$, *where* $C(n,R)$ *is a real constant depending on the dimension of* M, *and on the norm of its curvature tensor;*
- $||d\mathbf{h}^{X,Y}|| \leq C_1(n,R)||X||_1||Y||_1$, *where* $C_1(n,R)$ *is a real constant depending on the dimension of* M, *on the norm of its curvature tensor.*

Sketch of proof of Proposition 2: The map

$$\mathbf{h} : \mathcal{H} \times \mathcal{H} \to (X, Y) \to \Lambda(TM),$$

defined by $\mathbf{h}(X, Y) = \mathbf{h}^{X,Y}$ is bilinear and C^∞; the differential

$$d : (\Lambda(TM), ||.||_1) \to (\Lambda(TM), ||.||)$$

is linear and continuous. Using Maurer-Cartan equations, we see in the local expression of $\mathbf{h}$ that the differential of terms of type $d\Theta_i$ involves the curvature tensor of M. The conclusion follows by simple computations.

3.2.2 Generalization

It must be noticed that this construction can be generalized as follows: instead of taking two indices i_0, j_0, it is possible to take an arbitrary number of indices $l_1, ..., l_p$ different to n and to consider the tensor $\mathbf{h}_{l_1,...,l_p}$ obtained by taking off the 1-forms θ_{l_i} and adding the corresponding Θ_{l_i}. In such a way, one constructs new tensors

$$\mathbf{h}_{l_1,...,l_p} = \theta^1 \wedge ... \wedge \theta^{\hat{l_1}} \wedge ... \wedge \theta^{\hat{l_p}} \wedge ... \wedge \hat{\theta^n} \wedge \Theta_{l_1} \wedge ... \wedge \Theta_{l_p}.$$

For instance, if we take four indices, and if we plug the resulting tensor on the unit normal bundle of a hypersurface, (see 5.1), one gets an expression involving its curvature tensor, by Gauss equation. The computation is left to the reader.

4 Background on currents and normal cycles of geometric subsets

4.1 Rectifiable currents and integral currents

For this section, see [5], [13] for details. Let M^n be a C^∞ n-dimensional manifold. We denote by $\mathcal{D}^m$ the $\mathbb{E}$-vector space of C^∞ differential m-forms with compact support on M^n. The algebraic dual of $\mathcal{D}^m$ is the R-vector space $\mathcal{D}_m$ of *currents* on M^n. We can endow $\mathcal{D}_m$ with the weak topology:

$$T_j \to T \iff T_j(\phi) \to T(\phi), \forall \phi \in \mathcal{D}^m.$$

We shall also deal with the mass and the flat semi-norm on defined $\mathcal{D}_m$ as follows: for every T in $\mathcal{D}_m$,

$$\mathbf{M}(T) = \sup\{T(\phi), \phi \in \mathcal{D}^m, ||\phi|| \le 1\};$$

$$\mathcal{F}(T) = \sup\{T(\phi), \phi \in \mathcal{D}^m, ||\phi|| \le 1, ||d\phi|| \le 1.\}$$

One can associate a m-current to any oriented rectifiable subset S of dimension m of M: let $\vec{S}$ be the unit m-vector associated to almost every point x of S. For every $\phi \in \mathcal{D}^m(M)$, we define a current (still denoted by S) by

$$< S, \phi >= \int_S < \vec{S}, \phi > dS,$$

and more generally,

$$< \alpha S, \phi >= \alpha \int_S < \vec{S}, \phi > dS, \forall \alpha \in Z.$$

If the support of S is compact, we say that S is *rectifiable.* We denote by $\mathcal{R}_m$ the space of rectifiable currents.

A current is said *integral* if it is rectifiable and if its boundary is rectifiable. The space of integral m-currents is denoted by I_m. We mention the constancy theorem for integral currents, ([5], 4.1.31):

Theorem 2 *Let M be an oriented compact submanifold of $\mathbb{E}^N$, and T be an integral current (of the same dimension), whose support lies in M, and such that the support of ∂T lies in ∂M. Then, there exists an integer k such that $T = kM$.*

4.2 Normal cycle associated to geometric compact subsets

It is well known that the integral of curvatures of a hypersurface M of a Riemannian manifold, can be considered as the integral of curvature forms on the unit normal bundle of M, (see the important work on [16], [17], [18], [19], [6], [6], [7], [8], [9], [10]). In this context, this bundle appears as an integral (closed) current, called the **normal cycle** associated to M. Given a singular subset, it is interesting to find an analogous to the unit normal bundle. The good category in which this research can be done seems to be the category of subanalytic sets, which contains in particular the class of smooth submanifolds, the class of subsets of positive reach, and the class of polyhedra. When it exists, a characterization of the normal cycle is due to [8]. Following [8], a compact subset of M is said to be **geometric** if it admits a normal cycle.

5 The fundamental $(n-1)$-form on normal cycles

5.1 The case of a smooth hypersurface

Let $x : (M, g) \hookrightarrow (\tilde{M}, \tilde{g})$ be a codimension one isometric immersion of a Riemannian submanifold M into a Riemannian manifold $\tilde{M}$. We will use the following notations: $\hbar$ denotes the second fundamental form of the immersion i, (that is the symmetric tensor with values in the normal bundle $T^{\perp}M$), A denotes the Weingarten endomorphism. One has $\forall X, Y \in TM, \forall \xi \in T^{\perp}M$,

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y),$$
$$\tilde{\nabla}_X \xi = \nabla^{\perp}_X \xi - A_\xi X.$$

Let ξ be a unit normal vector field on the hypersurface M. We denote by G the Gauss map associated to the immersion of M:

$$G : M \hookrightarrow T\tilde{M}$$

is defined by

$$G(m) = (m, \xi_m).$$

Using the isomorphism $j_{\xi_m} \times \varsigma_{\xi_m}$ between $H_{\xi_m}(\tilde{M}) \times V_{\xi_m}(\tilde{M})$ and $T_m\tilde{M} \times T_m\tilde{M}$, we get:

$$(j_{\xi_m} \times \varsigma_{\xi_m}) \circ dG(X_m) = ((m, X_m), (m, (\tilde{\nabla}_X \xi)_m)),$$

that is

$$(j_{\xi_m} \times \varsigma_{\xi_m}) \circ dG(X_m) = ((m, X_m), (m, -A_{\xi_m}(X)),$$

since ξ_m is a unit vector.

Proposition 3 *Let M be an (oriented) hypersurface of an oriented n-dimensional manifold $\tilde{M}$. Let $\hbar$ be its second fundamental form. Let dv be the volume form of $\tilde{M}$. Then $\forall X, Y \in TM$,*

$$\hbar(X,Y)dv = G^*\mathbf{h}(X^h, Y^h).$$

Proof of Proposition 3: Let $e_1, ..., e_n$ be a local frame of $\tilde{M}$ such that $e_1, ..., e_{n-1}$ are tangent to M and e_n is normal to M. Let (e_{i_0}, e_{j_0}) be two vectors of this frame, different to e_n. One has

$$G^*\mathbf{h}(e^h_{i_0}, e^h_{j_0})(e_1, ..., e_{n-1}) =$$
$$\mathbf{h}(e^h_{i_0}, e^h_{j_0})(dG(e_1), ..., dG(e_{n-1})) =$$
$$\Theta_{j_0}(dG(e_{i_0})) = \hbar(e_{i_0}, e_{j_0}).$$

Remarks:

- A direct consequence of Proposition 3 is that, for $X, Y \in \mathcal{H}$, one has

  $$G^*(\mathbf{h}^{X,Y}) = \hbar(\pi_* X, \pi_* Y)dv.$$

  An immediate global corollary can be state as follows: let U be a domain of M and denote by $ST^{\perp}U$ the portion of the unit normal bundle over U. We have

  Corollary 1 *Forall $X, Y \in \mathcal{H}$, on has:*

  $$\int_{ST^{\perp}U} \mathbf{h}^{X,Y} = \int_U \hbar(\pi_* X, \pi_* Y)dv.$$

- Proposition 3 implies the symmetry of the tensor $G^*\mathbf{h}(.^h, .^h)$, since the second fundamental form h is symmetric on TM; (this last property can be seen directly by using the fact that the normal bundle of the hypersurface is Lagrangian in $TT\tilde{M}$).

5.2 The case of a hypersurface of an Euclidean space

In this paragraph, we assume that the ambiant space is $\mathbb{E}^n$. The canonical parallelism of $\mathbb{E}^n$ and $T\mathbb{E}^n$ allows to identify at each point m, $T_m\mathbb{E}^n$ and $\mathbb{E}^n$, and at each vector $\xi_m \in T_m\mathbb{E}^n$, $H_{\xi_m}(\mathbb{E}^n)$, $V_{\xi_m}(\mathbb{E}^n)$ and $\mathbb{E}^n$. If X, Y are parallel vector fields on $\mathbb{E}^n$, X and Y can be considered as horizontal vector fields, and one can evaluate their projection X', Y' on $\mathcal{H}$. If M is a hypersurface of $\mathbb{E}^n$, remark that the restriction of X', Y' to $ST^\perp M$ is nothing but their (orthogonal) projection on TM. Consequently, Corollary 1 can be state as follows: let U be a domain of M and denote by $ST^\perp U$ the portion of the unit normal bundle over U. We have

$$\int_{ST^\perp U} \mathbf{h}^{\mathrm{pr}_{TM}X, \mathrm{pr}_{TM}Y} = \int_U \hbar(\mathrm{pr}_{TM}X, \mathrm{pr}_{TM}Y)dv.$$

Remark also that $\hbar(\mathrm{pr}_{TM}X, \mathrm{pr}_{TM}Y)$ has a particular expression. Indeed, since Y is constant, we have on M $\tilde{\nabla}_{\mathrm{pr}_{TM}}Y = 0$, (where $\tilde{\nabla}$ denotes the Levi-Civita connexion on $\mathbb{E}^n$). If we decompose the restriction of Y to M in its tangent and normal component,

$$Y = \mathrm{pr}_{TM}Y \oplus \alpha_Y \xi,$$

we have:

$$\hbar(\mathrm{pr}_{TM}X, \mathrm{pr}_{TM}Y) = -\mathrm{pr}_{TM}X(\alpha_Y).$$

In other words,

$$\int_U G^*(\mathbf{h}^{X,Y})$$

measures the average on U of the variation of the gradient of α_Y in the direction $\mathrm{pr}_{TM}X$.

5.3 The case of a polyhedron

Let P be a polyhedron of $\mathbb{E}^n$, and let $N(P)$ its normal cycle. We shall evaluate

$$< N(P), \mathbf{h}^{X,Y} >,$$

for any vector field $X, Y \in \mathcal{H}$. Since the cycle $N(P)$ can be decomposed as a sum of elementary currents, the support of which lies above each simplex of dimension $i, 1 \leq i \leq n-1$, we shall evaluate $\mathbf{h}^{X,Y}$ above each simplex. We need he following notations: let σ_k be any k-dimensional simplex of a polyhedron P. The support of $N(P)_{|\sigma_k}$ is the product of σ_k by a portion of $(n-k-1)$-sphere. In particular, the support of $N(P)_{|\sigma_{n-2}}$ is the product $\sigma_{n-2} \times C$, where C is a portion of circle. Let $(e_1, ..., e_{n-2})$ be an orthonormal frame field tangent to σ_{n-2}. Any point of $\sigma_{n-2} \times C$ is a couple (m, e_{n-1}), where m is a point of σ_{n-2} and e_{n-1} is a unit vector orthogonal to σ_{n-2}. With these notations, we have the following

Proposition 4 *Let σ_k be a k-dimensional simplex of a polyhedron P, and X, Y be any vector fields lying in $\mathcal{H}$. Then,*

- *if* $k \neq n-2, < N(P)_{|\sigma_{n-2}}, \mathbf{h}^{X,Y} >= 0$,
- *and*
$$< N(P)_{|\sigma_{n-2}}, \mathbf{h}^{X,Y} >= \int_{\sigma_{n-2}\times C} < X, e_{(n-1)_h} >< Y, e_{(n-1)_h} >,$$
where $e_{(n-1)_h}$ *denotes the horizontal lift of* $e_{(n-1)}$.

6 An approximation result

In this section, we shall compare the second fundamental form of a smooth hypersurface of a Riemannian manifold $\tilde{M}$ and the second fundamental form (as defined in the previous paragraph) of a geometric compact subset *close to it.* The result we obtain can be considered as a quantitative version of the *approximation theorem* of J.Fu, [9]. Remark that we shall not use the compacity theorem, which is a crucial tool in [9]. For simplicity, we restrict ourselves to hypersurfaces. In the following, M is a smooth closed (oriented) hypersurface of $\tilde{M}$ bounding a compact subset K and $\mathcal{C}$ be a geometric compact subset of $\tilde{M}$ whose boundary $\mathcal{B} = \partial\mathcal{C}$ is strongly close to M. Finally, B is any regular Borel subset of $\tilde{M}$ included in $\mathcal{B}$.

6.1 Fine tubular neighborhood of a hypersurface

Let M be a closed (oriented embedded) hypersurface of $\tilde{M}^n$, bounding a compact subset K. We denote by ξ the outward unit normal vector field to K on M. One can define the orthogonal projection of a (small) tubular neighborhood U of M onto M as follows: If p is a point of U, one associates to p the only point m of M which realizes the distance between m and M. We put $m = \mathrm{pr}(p)$. Remark that p and $m = \mathrm{pr}(p)$ lie on the (unique) geodesic orthogonal to M, based at m and throwing p; (it is tangent to ξ_m).
For our purpose, this neighborhood will still be to large. We shall shrink it so that the differential of the exponential map (restricted to M) will be large enough: let m be a point of M, and γ^m be the geodesic whose base point is m, and which is tangent to ξ_m. Then the exponential map is a diffeomorphism of a neighborhood of $(m,0) \in T\tilde{M}$ onto a neighborhood of $m \in \tilde{M}$, and the norm of its differential at $(m,0)$ is 1. Then, for t "small enough", the norm of the differential of exp at $(m, t\xi)$ is "large enough".

This allows us to give the following

Definition 3 *A tubular neighborhood* U_r *of* M *(of radius* r*) is said to be* **fine** *if*

- *the projection of* U_r *onto* M *is well defined;*
- *the exponential map restricted to* M *satisfies*
$$||D\exp_m rX|| \geq \frac{1}{2},$$
for all $m \in M$ *and all unit vector* X_m *tangent to* $\tilde{M}$.

- *A geometric compact subset $\mathcal{C}$ of such a tubular neighborhood U of M in $\tilde{M}$ is said to be* **strongly close** *to M if it lies in a nice tubular neighborhood of M, and if the orthogonal projection pr defines an homeomorphism from G onto M.*

We shall prove the following

Proposition 5 *Let M be a closed (oriented embedded) hypersurface of $\tilde{M}$, and U_r a fine tubular neighborhood of radius r. Then, the orthogonal projection*

$$pr : U \to M$$

is differentiable and satisfies

$$||D\ pr_{|U}|| \leq 2.$$

Proof of Proposition 5: Let $0 < \rho < r$. The differentiability of pr is a classical result. Denote by S_ρ the smooth hypersurface of points at distance ρ to M. We have

$$\mathrm{pr}_{|S_\rho} \circ \exp \rho\xi = Id_M.$$

Since $||D \exp_m \rho\xi|| \geq \frac{1}{2}$, we deduce that $||D\ \mathrm{pr}_{|S_\rho}|| \leq \frac{1}{2}$. Since pr is constant on the geodesic orthogonal to M, we conclude that $||D\ \mathrm{pr}_{|U}|| \leq 2$.

6.2 The deviation of a geometric subset with respect to a hypersurface

Let M be a closed (oriented embedded) hypersurface of $\tilde{M}^n$, bounding a compact subset K, and $\mathcal{C}$ be a geometric compact subset of $\tilde{M}^n$ whose boundary $\mathcal{B}$ is strongly close to M. The main invariant involved in the study of the couple M and $\mathcal{C}$ is the *angular deviation.* We give now a precise definition. We need some notations: let p be any point of $\mathcal{B}$, and γ^p be the geodesic whose end points are p and $\mathrm{pr}(p)$, (this geodesic is tangent to the normal vector field ξ of M at $\mathrm{pr}(p)$). If (p, n_p) is a point of spt $\mathrm{N}(\mathcal{C})$, we denote by $\mathbf{n}$ the vector field parallel along γ^p and whose initial value is n_p.

Definition 4 *Let p be any point of $\mathcal{B}$. The* **angular deviation** *between p and $pr(p)$ is the maximal angle α_p between $\mathbf{n}_{pr(p)}$ and $\xi_{pr(p)}$, when n_p describes $sptN(\mathcal{C})_{|p}$. If B is any Borel subset of $\mathcal{B}$, the* **angular deviation** *between B and $pr(B)$ is the real number $\alpha_B = \sup_{p\in B} \alpha_p$.*

7 The 3-dimensional case

In this paragraph, we shall simplify the previous construction when the ambiant space is $\mathbb{E}^3$. The identification of $T\mathbb{E}^3$ with $\mathbb{E}^3 \times \mathbb{E}^3$, the existence of the cross product, and the existence of a global parallelism implies simplifications.

For each couple of 3-vectors in $\mathbb{E}^3$, we define two 2-differential forms on $T\mathbb{E}^3 \simeq \mathbb{E}^3 \times \mathbb{E}^3$ from which we shall recover the fundamental form studied in the previous sections.

Definition 5 *Let X, Y two (constant) vectors in $\mathbb{E}^3$. Given a point $(p,n) \in \mathbb{E}^3 \times \mathbb{E}^3$, we set*

$$\omega^{X,Y}_{(p,n)} = (n \times X) \wedge Y;$$
$$\tilde{\omega}^{X,Y}_{(p,n)} = X \wedge (n \times Y).$$

Note that these 2-differential forms are bilinear in X and Y but they are not symmetric in general. The two following theorem relate these two forms with the fundamental forms, and compute them explicitly on a smooth surface and on a polyhedron.

Let M is a smooth closed (oriented) surface of $\mathbb{E}^3$ bounding a compact subset K and P be a polyhedron of $\mathbb{E}^3$ with boundary $\mathcal{B}$. Let B be any regular Borel subset of $\mathbb{E}^3$ included in $\mathcal{B}$.

Theorem 3 *Let M be a closed (oriented embedded) hypersurface of $\tilde{M}$, bounding a compact subset K, and $\mathcal{C}$ be a geometric compact subset of $\tilde{M}^n$ whose boundary is a triangulation P. Let B be a Borel subset of M, (resp. P. Then,*

- *over M,*

$$< N(K)_{|B}, \omega^{X,Y} >= \int_B \hbar(pr_{TM}X, pr_{TM}Y)dv,$$

- *and over P,*

$$< N(\mathcal{C})_{|B}, \omega^{X,Y} >=$$
$$\sum_{e \, edge \; of \; P} \frac{\beta(e)}{l(e \cap B)} (< X, \overrightarrow{e \cap B} >< Y, \overrightarrow{e \cap B} >),$$

where $\overrightarrow{e \cap B}$ denotes the 3-vector with the same direction as the edge e, the same length as $e \cap B$.

To evaluate $\tilde{\omega}$ in the smooth case, we need to introduce the tensor $\tilde{h}$ on a smooth surface, which is deduced from h by inverting the eigenvalues of h: if e_1, e_2 are the two eigenvectors of h with eigenvalues λ_1, λ_2 at a point m of M, then e_1, e_2 are the two eigenvectors of $\tilde{h}$ with eigenvalues λ_2, λ_1 at the same point m. In other words, $\tilde{h}(.,.) = h(j.,j.)$, where j is the almost complex structure of M compatible with the metric.

Theorem 4 *One has:*

$$< N(K)_{|B}, \tilde{\omega}^{X,Y} >= \int_B \tilde{\hbar}(pr_{TM}X, pr_{TM}Y)dv,$$

Moreover,

$$< N(\mathcal{B})_{|B}, \omega^{X,Y} >=$$
$$\sum_{e\,edge\ of\ P} \frac{l(e \cap B)}{2}[(\beta(e) - \sin\beta(e)) < e^+, X >< e^+, Y > +(\beta(e) + \sin\beta(e)) < e^-, X >< e^-, Y >].$$

where e^+ (resp. e^-) denote the normalized sum (resp. difference) of the unit normal vector to triangles incident on e.

The proof of these two theorems is left to the reader.

8 An approximation theorem

Let M be a closed (oriented embedded) hypersurface of $\tilde{M}^n$, bounding a compact subset K, and $\mathcal{C}$ be a geometric compact subset of $\tilde{M}^n$ whose boundary $\mathcal{B}$ is strongly close to M. The purpose of this section is to prove the following

Theorem 5

$$\mathcal{F}(N(\mathcal{C})_{|T_B\tilde{M}} - N(K)_{|T_{pr(B)\tilde{M}}}) \leq$$
$$2^{n-1}\max(\delta_B, \alpha_B)\max(1, ||h_B||^{n-1})(\mathbf{M}(N(\mathcal{C})_{|T_B\tilde{M})} + \mathbf{M}(\partial N(\mathcal{C})_{|T_B\tilde{M}})).$$

The rest of this section is devoted to the proof of Theorem 5.

- Let K be the compact domain whose boundary is M. Consider the map f defined by the following diagram:

$$\begin{array}{ccc} TU & \xrightarrow{f} & \text{spt } N(K) \\ \pi\downarrow & & \uparrow G \\ U & \xrightarrow{\text{pr}} & M^{n-1} \end{array}$$

For further use, we need the following

Lemma 1

$$||Df|| \leq 2\sup(1, ||h||).$$

Proof of Lemma 1: One has $Df = DG \circ D\text{pr} \circ D\pi$. On the other hand,

$$||DG|| \leq \sup(1, ||h||), ||D\text{pr}|| \leq 2, ||D\pi|| \leq 1,$$

from which we deduce Lemma 1.

Let B be any Borel subset of $T\tilde{M}$. To simplify the notations, we define the $(n-1)$-current D by $D = N(\mathcal{C})_{|T_B\tilde{M}}$. Define the $(n-1)$-current E by $E = N(K)_{|T\text{pr}(B)\tilde{M}}$ and the n-current:

$$C = h_\sharp(D \times [0, 1]).$$

Lemma 2 *One has:*

$$f_\sharp(D) = E.$$

Proof of Lemma 2:
We apply the constancy theorem, (2): the support of the image of D (resp. ∂D) by f is included in the support of E, (resp. ∂E). Consequently, there exists an integer c such that $f_\sharp(D) = cE$. To prove that $c = 1$, we evaluate $f_\sharp(D)$ on the form ω_0. One has

$$\mathrm{vol}(\mathrm{pr}(B)) = < f_\sharp(D), \omega_0 > = < E, \omega_0 >,$$

since f is one-one from B to $\mathrm{pr}(B)$.

- Now, we define a homotopy g between f and the identity: to every point p in U, we can associate the (unique) geodesic

$$\gamma^p : [0,1] \to \tilde{M}^n$$

from p to $m = \mathrm{pr}_M(p)$ satisfying $\gamma^p(0) = p, \gamma^p(1) = m$. We shall also use the geodesic $\tilde{\gamma}$ from m to p, (tangent at p to the normal vector field ξ), but with the reverse orientation: $\tilde{\gamma}^m(0) = m, \tilde{\gamma}^m(1) = p$.

Let τ_{γ^p}, (*resp.* $\tau_{\tilde{\gamma}^m}$) be the parallel transport with respect to γ^p, (*resp.* $\tilde{\gamma}^m$). Let X be the vector field over γ obtained by transporting X_p by parallelism:

$$X_{\gamma^p(t)} = \tau_{\gamma^p(t)}(X_p).$$

In the same way, let $\xi_{\tilde{\gamma}^m(u)}$ be the vector field over $\tilde{\gamma}^m$ obtained by transporting the normal vector ξ_m by parallelism:

$$\xi_{\tilde{\gamma}^m(u)} = \tau_{\tilde{\gamma}^m(u)}(\xi_m).$$

We define the homotopy

$$g : TU \times [0,1] \to \mathrm{spt} N(K) \subset T\tilde{M}^n,$$

by

$$g(x, X_x, t) = (\gamma^x(t), (1-t)X_{\gamma^x(t)} + t\xi_{\tilde{\gamma}^{\mathrm{pr}(x)}(1-t)}).$$

Let B be any Borel subset of $\mathcal{B}$. One defines

  - the $(n-1)$-current D by $D = N(C)_{|T\mathrm{pr}\tilde{M}}$;
  - the $(n-1)$-current E by $E = N(K)_{|(\mathrm{pr}(B)\times\mathbb{E}^n)}$;

– the n-current:

$$C = g_\sharp([0,1] \times D).$$

The homotopy formula for currents (cf. [5], 4.1.9.), gives immediately

$$\partial C = f_\sharp(D) - D - g_\sharp([0,1] \times \partial D).$$

Theorem 6

$$\mathcal{F}(D-E) \leq (M(D) - M(\partial D)) \sup_{spt\ D} (||\frac{dg}{dt}||) \sup_{spt\ D} (||Df||^{n-2}, ||Df||^{n-1}, 1).$$

Proof of Theorem 6:
To evaluate the flat norm of $D-E$, we decompose $D-E$ in a sum of a $(n-1)$-current and the boundary of a n current, by writing:

$$D - E = \partial C - g_\sharp([0,1] \times \partial D). \tag{4}$$

By definition of the flat norm,

$$\mathcal{F}(D-E) \leq M(C) + M(h_\sharp([0,1] \times \partial D)). \tag{5}$$

To evaluate $M(C)$, we use the fact that D is representable by integration. In a local frame parallel along a geodesic γ from x to $\mathrm{pr}(x)$, one can express the homotopy g by

$$g(x, X_x, t) = ((1-t)x + t\mathrm{pr}(x), (1-t)X_x + t\xi_{\mathrm{pr}(x)} = ((1-t)\mathrm{Id} + tf)(x, X_x).$$

By a computation similar to ([4] 4.1.9.), we deduce that:

$$M(C) = M(g_\sharp(D \times [0,1])) \leq M(D) \sup_{\mathrm{spt}\ D} (||\frac{dh}{dt}||) \sup_{\mathrm{spt}\ D} (||Df||^{n-1}, ||Id||^{n-1}), \tag{6}$$

and

$$M(h_\sharp(\partial D \times [0,1])) \leq M(\partial D) \sup_{\mathrm{spt}\ D} ||\frac{dg}{dt}|| \sup_{\mathrm{spt}\ D} (||Df||^{n-2}, ||Id||^{n-2}). \tag{7}$$

Using Lemma 1, we deduce Theorem 6. Theorem 1 is an obvious consequence of Theorem 6 and Propositions 1 and 2.

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